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Some Observations on Cohomologically P-Ample Bundles (*) (**).

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Sunto. – *Usando un recente risultato di Deligne e Illusie si prova un teorema di annullamento per la coomologia di un fibrato coomologicamente p-ampio. Si studiano le proprietà dei gruppi di coomologia dei fibrati vettoriali nel caso in cui il morfismo di Frobenius si estenda a un sollevamento della varietà su $W_2(k)$, anello dei vettori di Witt di lunghezza 2 del campo. Si studiano, infine, a titolo di esempio, i fibrati coomologicamente p-ampi su P^d.*

0. – Introduction.

On of the main features of characteristic p geometry is the existence of the Frobenius morphism; in particular, using this morphism, one can define an operation on vector bundles and, more generally, on coherent sheaves.

One possible description of this operation is the following: if the bundle E on the variety X is given, with respect to an affine open covering $\mathfrak{U} = \{U_i\}$ by transition functions $g_{ij} \in GL(\mathcal{O}_X(U_i \cap U_j))$, the bundle $E^{(p)}$ associated to it is defined by the new transition functions $g'_{ij} \in GL(\mathcal{O}_X(U_i \cap U_j))$ obtained raising all the entries of the matrices to the p-th power (the point being that the map $(a_{ij}) \to (a_{ij}^p)$ *is a representation of* $GL(n, k)$ *if* $\operatorname{char} k = p$).

This operation permits the introduction of two generalizations of the notion of ampleness of a vector bundle, namely *p-ampleness and cohomological p-ampleness* (see § 1 for definitions).

In this paper we show how a recent result of DELIGNE and ILLUSIE ([4]) can be used to study these notions, especially cohomological p-ampleness, *in the case that the variety X can be lifted over $W_2(k)$, the ring of Witt vectors of length 2 of the field k.*

The content of the paper is:

in § 1 we give the definitions relevant to the paper and recall some of the properties of the objects which are being considered;

in § 2 we recall the main theorem of the already quoted paper of DELIGNE and

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ILLUSIE ([4]) and we prove a very strong vanishing theorem for cohomologically p-ample vector bundles; as a corollary we get a «Weak Lefschetz Theorem» for codimension 2 zero-loci of sections of such bundles;

in § 3 we find strong consequences of the hypothesis that the Frobenius morphism extends to a morphism of a lifting of X to $W_2(k)$.

in § 4 we consider, as an example, cohomologically p-ample bundles on $X = \boldsymbol{P}^d_{F_p}$. In this case one can easily compute $F_* \mathcal{O}_X$ (F being the Frobenius morphism) and thus the map $H^i(X, E) \to H^i(X, E^{(p)})$ is quite easy to understand. An argument using Beilinson's theorem about coherent sheaves on $\boldsymbol{P}^d$ ([2]) allows us to give a resolution of a cohomologically p-ample bundle and a characterization of cohomologically p-ample bundles on $\boldsymbol{P}^d$.

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1. – Terminology and basic properties of p-ample and cohomologically p-ample bundles.

Let X be a smooth projective scheme over an algebraically closed field k of positive characteristic p.

Let $X' = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k)$ where k is acted on by itself by the p-th power map. In others words, X' is the scheme $(X, \mathcal{O}_X)$ but the k-algebra structure of $\mathcal{O}_{X'}$ is given by multiplication with the p-th roots of the elements of k.

Let $F: X \to X'$ be the k-linear Frobenius, *i.e. F is the identity map on the underlying topological space of X and the p-th power map on the structure sheaf.*

DEFINITION 1. – *Let E be a vector bundle on X. We define $E^{(p)}$ to be the vector bundle F^*E', E' being the vector bundle on X' obtained from E by change of base. We will denote $E^{(p^n)}$ the bundle obtained applying this operation n times, i.e. $E^{(p^n)} = (E^{(p^{n-1})})^{(p)}$.*

DEFINITION 2 (cfr. [7]). – *A vector bundle E on X is said to be p-ample if, given any coherent sheaf $\mathcal{F}$ on X, there exists an integer N such that:*

$$E^{(p^n)} \otimes \mathcal{F} \quad \text{is generated by global sections } \forall n \geq N.$$

DEFINITION 3 (cfr. [8]). – *A vector bundle E on X is said to be cohomologically p-ample if, given any coherent sheaf $\mathcal{F}$ on X, there exists an integer N such that:*

$$H^i(X, E^{(p^n)} \otimes \mathcal{F}) = 0 \qquad \forall i > 0 \ \forall n \geq N.$$

REMARKS. – 1) The definition of the operation $E \to E^{(p)}$ given here agrees with the one given in the introduction.

2) If rank $E = 1$, $E^{(p^n)} = E^{\otimes p^n}$, hence

E is ample $\Leftrightarrow E$ is p-ample $\Leftrightarrow E$ is cohomologically p-ample.

3) The functor:

$$(\cdot)^{(p^n)}: \{\text{Vector bundles on } X\} \to \{\text{Vector bundles on } X\}$$

is, under our assumptions, exact.

Thus extensions of ample line bundles provide examples of cohomologically p-ample bundles.

4) If E is p-ample, $E^{(p^n)}$ is, for $n \gg 0$, a quotient of a direct sum of ample line bundles. Again by exactness it follows that for any coherent sheaf $\mathcal{F}$

$$H^d(X, E^{(p^n)} \otimes \mathcal{F}) = 0 \qquad \text{for } n \gg 0, \ d = \text{dimension of } X.$$

5) In general

E is cohomologically p-ample $\Rightarrow E$ is p-ample $\Rightarrow E$ is ample

but the reverse implications are false.

6) If $\dim X = 1$ ampleness, p-ampleness and cohomological p-ampleness coincide.

7) If $\dim X = 2$ and rank $E = 2$, it is quite easy to see, using Riemann Roch theorem, that

$$E \text{ is cohomologically } p\text{-ample} \Leftrightarrow c_1^2(E) - 2c_2(E) > 0$$

while this is not, in general, a positive polynomial for ample vector bundles (cfr. FULTON [5, Chap. 12]).

8) One can give a definition of cohomological k-p-ampleness as follows (cfr. [12]):

DEFINITION 3′. – *A vector bundle E on X is said to be cohomologically k-p-ample if, given any coherent sheaf $\mathcal{F}$ on X, there exists an integer N such that:*

$$H^i(X, E^{(p^n)} \otimes \mathcal{F}) = 0 \qquad \forall i > k, \quad \forall n \geq N.$$

In the sequel of this paper we will prove, or rather observe, that cohomologically p-ample bundles have very strong vanishing properties on schemes which can be lifted to smooth schemes over $W_2(k)$.

Examples of such liftable schemes are reductions at a generic prime p of smooth schemes over a field of characteristic 0.

2. – A strong vanishing theorem for cohomologically p-ample bundles.

Recall that, given a perfect field k of characteristic p, $W_2(k)$ is the unique, up to isomorphism, ring which is flat over $\mathbf{Z}/p^2\mathbf{Z}$ and such that $W_2(k)/pW_2(k)$ is isomorphic to k.

In their beautiful paper ([4]) DELIGNE and ILLUSIE prove the degeneration of the Hodge to de Rham spectral sequence at E_1 for schemes (whose dimension is smaller then the characteristic) liftable over $W_2(k)$.

This enables them, among other things, to give an algebraic proof of the Kodaira-Akizuki-Nakano vanishing theorem. We will now recall some of their results and observe that the same arguments give a strong vanishing theorem for cohomologically p-ample bundles:

THEOREM ([4, p. 250]). – *If X is a smooth proper scheme over a perfect field k of characteristic p, to every smooth scheme*

$$\tilde{X} \to \operatorname{Spec}(W_2(k)),$$

such that

$$\tilde{X} \times_{\operatorname{Spec}(W_2(k))} \operatorname{Spec}(k) \simeq X,$$

there corresponds an isomorphism in the derived category D(X′) of coherent sheaves on X′,

$$\Phi_{\tilde{X}} : \bigoplus_{i<p} \Omega^i_{X'/k}[-i] \to \tau_{<p} F_* \Omega^\bullet_{X/k},$$

where

— $\Omega^i_{X'/k} = \Lambda^i \Omega^1_{X'/k}$ is the sheaf of differential forms on X′, of degree i;

— $F_* \Omega^\bullet_{X/k}$ is the direct image by Frobenius of the de Rham complex $(\Omega^\bullet_{X/k}, d)$ of X; $F_* \Omega^\bullet_{X/k}$ is a complex of $\mathcal{O}_{X'}$-modules, i.e. d is $\mathcal{O}_{X'}$-linear, since $d(f^p \omega) = f^p d\omega$;

— $\tau_{<p}$ indicates truncation of a complex at p, i.e.

$$\tau_{<p} F_* \Omega^\bullet_{X/k} =$$

$$= 0 \to F_* \mathcal{O}_{X/k} \to F_* \Omega^1_{X/k} \to \dots \to F_* \Omega^{p-2}_{X/k} \to \operatorname{Ker}\{d : F_* \Omega^{p-1}_{X/k} \to F_* \Omega^p_{X/k}\} \to 0;$$

— $[-i]$ denotes the i-th shift of a complex.

Isomorphism in the derived category just means that $\Phi_{\tilde{X}}$ is not a «real» morphism of complexes, but there exists a third complex of sheaves on X', $K^\bullet$, and quasi-isomorphisms Ψ and Ξ:

$$\bigoplus_{i<p} \Omega^i_{X'/k}[-i] \xrightarrow{\Psi} K^\bullet \xleftarrow{\Xi} \tau_{<p} F_* \Omega^\bullet_{X/k}.$$

In this case $K^\bullet$ is the single complex of the sheafified Čech double complex associated to an affine covering. (For basic facts about derived categories, cfr. GRIVEL [6] and the very clear short introduction by BOREL [3, pp. 97-112]).

Let $\tilde{X}$ be a lifting of X to $W_2(k)$; it results from well known examples of Igusa, Mumford, Serre that $\tilde{X}$ may not exist.

In the (rare) case that the Frobenius morphism $F: X \to X'$ lifts to a morphism

$$\tilde{F}: \tilde{X} \to \tilde{X}'$$

($\tilde{X}'$ is deduced from $\tilde{X}$ in much the same way as X' is deduced from X: by functoriality $W_2(k)$ inherits the Frobenius morphism of k) the results can be considerably sharpened. In fact *in this case the arrow in the derived category is induced by a real morphism of complexes* which is constructed as follows (this goes back to MAZUR [9]):

For simplicity assume that X is defined over $\boldsymbol{F}_p$.

Let $\{\mathrm{Spec}(\tilde{A}_\alpha)\}$ be an affine covering of $\tilde{X}$: $\tilde{A}_\alpha$ are $\boldsymbol{Z}/p^2\boldsymbol{Z}$-algebras.

Let A_α be their reductions mod. p: $\{\mathrm{Spec}(A_\alpha)\}$ is an affine covering of X.

$\tilde{F}$ is then given by

$$\tilde{F}_\alpha(s) = s^p + pu(s).$$

Where $u(s)$ is a well defined element in A_α.

Denote

$$\delta s = s^{p-1}ds + du(\tilde{s}), \qquad s \in A_\alpha,$$

where $\tilde{s}$ is a lifting of s to $\tilde{A}_\alpha$ and $\delta s \in \Omega^1_{A_\alpha}$ is independent of the choice of $\tilde{s}$. Since $\delta(st) = s^p \delta t + t^p \delta s$ and $d(\delta s) = 0$ the map Φ sending

$$s_0 ds_1 \wedge \ldots \wedge ds_k \to s_0^p \delta s_1 \wedge \ldots \wedge \delta s_k = s_0^p s_1^{p-1} \ldots s_k^{p-1} ds_1 \wedge \ldots \wedge ds_k + (\text{exact form}),$$

defines a quasi-isomorphism

$$\Phi_{\tilde{X}}: \bigoplus_{i=0}^{d} \Omega^i_{X'/k}[-i] \to F_* \Omega^\bullet_{X/k}.$$

In this case no truncation is required.

REMARK. – In general the obstruction to lift F to a morphism on $\tilde{X}$ is a class in $H^1(H', \Theta_{X'/k} \otimes F_X \mathcal{O}_X)$, $\Theta_{X'/k}$ being the tangent sheaf of X'.

THEOREM. – *Let X be a smooth projective scheme over k liftable over $W_2(k)$. Assume* $\dim X < p$, *and let E be a cohomologically p-ample bundle. Then*

$$H^i(X, \Omega^j_{X/k} \otimes E) = 0 \quad \textit{for } i + j > \dim X.$$

PROOF. – We repeat the argument given by DELIGNE and ILLUSIE for a line bundle (cfr. [4, p. 258]). We prove that, for any vector bundle G,

$H^i(X, \Omega^j_{X/k} \otimes G^{(p)}) = 0,$

$$\text{for } i+j > \dim X \text{ implies } H^i(X, \Omega^j_{X/k} \otimes G) = 0, \quad \text{for } i+j > \dim X$$

and conclude by descending induction on the Frobenius operation, and the assumption

$$H^i(X, \Omega^j_{X/k} \otimes E^{(p^n)}) = 0, \qquad \text{for } n \gg 0 \text{ and } i > 0.$$

The projection formula gives

$$0 = H^i(X, \Omega^j_{X/k} \otimes G^{(p)}) = H^i(X', F_* \Omega^j_{X/k} \otimes G'), \qquad \text{for } i+j > \dim X$$

(recall that the Frobenius morphism is affine).

$(F_* \Omega^{\bullet}_{X/k} \otimes G', d \otimes 1)$ is an $\mathcal{O}_{X'}$-linear complex of sheaves and $H^i(X', F_* \Omega^j_{X/k} \otimes G')$ is the E_1^{ji} term of a hypercohomology spectral sequence abutting to

$$\mathbf{H}^{i+j}(X', F_* \Omega^{\bullet}_{X/k} \otimes G') = \mathbf{H}^{i+j}(X', \oplus \Omega^l_{X'/k}[-l] \otimes G') = \bigoplus_l H^{i+j-l}(X', \Omega^l_{X'/k} \otimes G'),$$

by the theorem of Deligne and Illusie.

So $H^i(X', \Omega^j_{X'/k} \otimes G') = 0$ for $i+j > \dim X$ and $H^i(X, \Omega^j_{X/k} \otimes G) = 0$ for $i+j > \dim X$ by base change. ■

REMARKS. – 1) The same argument gives the following:

THEOREM. – *Let X be a smooth projective scheme over k liftable over $W_2(k)$. Assume* $\dim X < p$, *and let E be a cohomologically k-p-ample bundle. Then*

$$H^i(X, \Omega^j_{X/k} \otimes E) = 0, \qquad \textit{for } i+j > \dim X + k.$$

2) The Kodaira-Akizuki-Nakano vanishing theorem gives

$$H^i(X, \Omega^j_{X/k} \otimes E) = 0, \qquad \text{for } i+j \geqslant \dim X + \operatorname{rank} E.$$

Thus cohomologically p-ample bundles have much stronger vanishing properties.

Usually vanishing properties of bundles are reflected in «topological restrictions» of the zero-loci of their sections; however we can prove a result of this kind only in the case that rank $E = 2$.

The reason is that it is unclear, at least to the author, how cohomological p-ampleness behaves with respect to linear algebra operations, in particular to exterior powers.

THEOREM. – *Let X be a smooth projective scheme over k liftable over $W_2(k)$. Assume* $\dim X < p$, *and let E be a cohomologically p-ample bundle of rank* 2. *Let* *s be a section of E whose zero-locus is a smooth irreducible codimension 2 subscheme Y of X and assume that also Y is liftable to a scheme over* Spec $(W_2(k))$.

Then the restriction maps

$$H^n_{DR}(X/k) \to H^n_{DR}(Y/k),$$

are isomorphisms if $n < \dim Y$ *and injective if* $n = \dim Y$.

(*Recall that* $H^n_{DR}(X/k)$ *is the n-th hypercohomology group of the k-linear de Rham complex* $(\Omega^{\bullet}_{X/k}, d)$.)

PROOF. - Note first that restriction of a cohomologically p-ample vector bundle to a subscheme is still cohomologically p-ample.

$\Lambda^2 E = \det E$ is an ample line bundle (a weak Lefschetz theorem without the assumption on the codimension would be proved if one knew that E cohomologically p-ample $\Rightarrow \Lambda^i E$ cohomologically p-ample $\forall i$).

Consider now the Koszul complex of Y tensorized by $\Omega^i_{X/k}$:

$$0 \to \Lambda^2 E^\vee \otimes \Omega^i_{X/k} \to E^\vee \otimes \Omega^i_{X/k} \to \Omega^i_{X/k} \to \Omega^i_{X/k|Y} \to 0,$$

to conclude, applying the vanishing theorem just proved and Serre duality, that

$$H^k(X, \Omega^i_{X/k}) \to H^k(Y, \Omega^i_{X/k|Y}),$$

is an isomorphism if $i + k < \dim Y$ *and injective if* $i + k = \dim Y$.

Consider now the adjunction exact sequence

$$0 \to E^\vee_{|Y} \to \Omega^1_{X/k|Y} \to \Omega^1_{Y/k} \to 0$$

and its consequences

$$0 \to F_i \to \Omega^i_{X/k|Y} \to \Omega^i_{Y/k} \to 0,$$

$$0 \to \Omega^{i-2}_{Y/k} \otimes \Lambda^2 E^\vee_{|Y} \to F_i \to \Omega^{i-1}_{Y/k} \otimes E^\vee_{|Y} \to 0.$$

By induction, and using the vanishing theorem for $E_{|Y}$ and $\Lambda^2 E_{|Y}$, we get

$$H^k(Y, \Omega^i_{X/k|Y}) \to H^k(Y, \Omega^i_{Y/k})$$

is an isomorphism if $i + k < \dim Y$ *and injective if* $i + k = \dim Y$.

Thus, combining the two statements

$$H^k(X, \Omega^i_{X/k}) \to H^k(Y, \Omega^i_{Y/k}),$$

is an isomorphism if $i + k < \dim Y$ *and injective if* $i + k = \dim Y$.

Using the degeneration at E_1 of the Hodge to de Rham spectral sequences for X and Y, which is another consequence of the theorem of Deligne and Illusie,

$$H^n_{DR}(X/k) = \bigoplus_{i+j=n} H^i(X, \Omega^j_{X/k}),$$

$$H^n_{DR}(Y/k) = \bigoplus_{i+j=n} H^i(Y, \Omega^j_{Y/k}),$$

we get the result. ■

REMARKS. – 1) Compare this result with the Lefschetz theorem for zero-loci of sections of an ample vector bundle (cfr. for instance [1, pp. 306-307]).

2) The topological restrictions just found are the same verified by a codimension 2 complete intersection in X.

3. – Case that the Frobenius morphism lifts.

Suppose now that the Frobenius morphism F extends to $\tilde{F}: \tilde{X} \to \tilde{X}$. Then one has the quasi-isomorphism

$$\Phi_{\tilde{X}}: \bigoplus_{i=0}^{d} \Omega^i_{X'/k}[-i] \to F_* \Omega^\bullet_{X/k},$$

constructed in § 2.

Consider the following diagram:

$$\begin{array}{ccccc}
H^i(X', F_*\Omega^j_{X/k} \otimes E') & \times & H^{d-i}(X', F_*\Omega^{d-j}_{X/k} \otimes (E')^\vee) & \xrightarrow{\tilde{\cup}} & H^d(X', \Omega^d_{X'/k}) \\
\uparrow{\scriptstyle \Phi\otimes 1} & & \uparrow{\scriptstyle \Phi\otimes 1} & \nearrow{\scriptstyle \cup} & \\
H^i(H', \Omega^j_{X'/k} \otimes E') & \times & H^{d-i}(X', \Omega^{d-j}_{X'/k} \otimes (E')^\vee). & &
\end{array}$$

The maps $\Phi_{\tilde{X}} \otimes 1$ are induced by the morphism $\Phi_{\tilde{X}}$.

The pairing $\cup$ is given by cup product and is, by the Serre duality theorem, a perfect pairing.

The other horizontal pairing $\tilde{\cup}$ is also a perfect pairing: in fact $F_*\Omega^{d-j}_{X/k}$ is dual to $F_*\Omega^j_{X/k}$ with values in $\Omega^d_{X'/k}$ and the pairing

$$F_*\Omega^{d-j}_{X/k} \times F_*\Omega^j_{X/k} \to \Omega^d_{X'/k},$$

is given by

$$F_*\Omega^{d-j}_{X/k} \times F_*\Omega^j_{X/k} \xrightarrow{\wedge} F_*\Omega^d_{X/k} \xrightarrow{p} \frac{F_*\Omega^d_{X/k}}{dF_*\Omega^{d-1}_{X/k}} \xrightarrow{C} \Omega^d_{X'/k},$$

where

$\wedge$ is the wedge product,

p is the canonical projection,

C is the Cartier isomorphism.

Let's quickly recall the definition of this latter:

Let $\mathcal{H}^i F_* \Omega^\bullet_{X/k}$ denote the cohomology sheaves of the complex $(F_* \Omega^\bullet_{X/k}, d)$. Observe that cup product makes $\bigoplus_{i=0}^{d} \mathcal{H}^i F_* \Omega^\bullet_{X/k}$ into an $\mathcal{O}_{X'}$-graded algebra: then C *is the unique morphism of* $\mathcal{O}_{X'}$*-graded algebras*

$$C: \bigoplus_{i=0}^{d} \mathcal{H}^i F_* \Omega^\bullet_{X/k} \to \bigoplus_{i=0}^{d} \Omega^i_{X'/k}$$

(*i.e.* $C(\omega_1 \wedge \omega_2) = C(\omega_1) \wedge C(\omega_2)$ *and* $C(f^p \omega) = fC(\omega)$), *such that* $C(s^{p-1} ds) = ds$.

It turns out that C is an isomorphism as one can easily check in the case $\mathcal{O}_X$ is a polynomial ring, to which one is reduced by étale localization.

One can proceed similarly to prove that

$$F_* \Omega^{d-j}_{X/k} \times F_* \Omega^j_{X/k} \to \Omega^d_{X'/k},$$

is a perfect pairing.

Using the explicit description of $\Phi_{\tilde{X}}$ given in § 2 and these properties of the Cartier isomorphism it is easy to check commutativity of the diagram.

From this we easily conclude:

PROPOSITION. – *If the Frobenius morphism extends to some lifting* $\tilde{X}$ *of* X, *for any vector bundle* E *on* X

$$\dim_k H^i(X, \Omega^j_{X/k} \otimes E) \leq \dim_k H^i(X, \Omega^j_{X/k} \otimes E^{(p)}).$$

PROOF. – By base change and Leray spectral sequence it is enough to prove that the maps

$$\Phi_{\tilde{X}} \otimes 1: H^i(X', \Omega^j_{X'/k} \otimes E') \to H^i(X', F_* \Omega^j_{X/k} \otimes E'),$$

are injective for all i and j.

Suppose

$$a \in H^i(X', \Omega^j_{X'/k} \otimes E'), \qquad a \neq 0, \qquad \Phi_{\tilde{X}} \otimes 1(a) = 0,$$

since the pairing $\cup$ is perfect, there exists an element $\tilde{a} \in H^{d-i}(X', \Omega^{d-j}_{X'/k} \otimes (E')^\vee)$ such that $a \cup \tilde{a} \neq 0$.

But this contradicts $\Phi_{\tilde{X}} \otimes 1(a) \,\tilde{\cup}\, \Phi_{\tilde{X}} \otimes 1(\tilde{a}) = 0$. ■

Thus, if Frobenius lifts and E is a p-ample vector bundle,

$$H^d(X, \Omega^j_{X/k} \otimes E) = 0, \qquad d = \dim X, \qquad \forall j,$$

by the Remark 3 of § 1.

If E is a cohomologically p-ample vector bundle we get the very strong:

$$H^i(X, \Omega^j_{X/k} \otimes E) = 0, \qquad \forall i > 0, \qquad \forall j$$

and, more generally, if E is cohomologically k-p-ample,

$$H^i(X, \Omega^j_{X/k} \otimes E) = 0, \qquad \forall i > k, \qquad \forall j.$$

COROLLARY. – *No projective smooth scheme of dimension greater than* 1, *liftable over* $W_2(k)$, *has cohomologically p-ample cotangent bundle.*

PROOF. – Suppose X satisfies the hypothesis and $\Omega^1_{X/k}$ is cohomologically p-ample:

$$\dim H^1(X', \Theta_{X'/k} \otimes F_* \mathcal{O}_X) = \dim H^1(X, \Theta^{(p)}_{X/k}) = \dim H^{d-1}(X, (\Omega^1_{X/k})^{(p)} \otimes \Omega^d_{X/k}) = 0$$

by the vanishing theorem for cohomologically p-ample bundles in § 2.

Thus there is no obstruction to lift Frobenius.

But $\Omega^d_{X/k} = \Lambda^d \Omega^1_{X/k}$ is ample and $\dim H^d(X, (\Omega^d_{X/k})^{(p)}) = 0$ for $n \gg 0$; $\dim H^d(X, \Omega^d_{X/k}) = 1$ because X is complete and we get a contradiction with the proposition just proved. ■

REMARK. – One can observe that a projective scheme cannot have cohomologically p-ample tangent bundle as well, using Mori's solution to Hartshorne conjecture ([10]).

A very similar argument gives the following:

COROLLARY. – *Suppose that X is projective, no multiple of the canonical bundle* $K_X = \Omega^d_{X/k}$ *is trivial and there is a non zero plurigenus, i.e.* $H^0(X, K_X^{\otimes m}) \neq 0$ *for some m. Then the Frobenius morphism cannot be extended to any lifting of X to* $W_2(k)$.

PROOF. – Observe first that $H^0(X, K_X^{\otimes -n}) = 0 \ \forall n$.

In fact if $s \in H^0(X, K_X^{\otimes -n})$ and $t \in H^0(X, K_X^{\otimes m})$, $s^m t^n \in H^0(X, \mathcal{O}_X)$ is a non zero, and thus never vanishing, section. This implies that s and t don't vanish at any point, and give trivialization of $K_X^{\otimes -n}$ and $K_X^{\otimes m}$ respectively, against the hypothesis.

If Frobenius lifted we would have

$$1 = \dim H^d(X, K_{X/k}) \leq \dim H^d(X, K_X^{\otimes p}) = \dim H^d(X, K_X^{\otimes 1-p}) = 0. \qquad ■$$

So, if $k(X)$ denotes the Kodaira dimension of X, and the Frobenius morphism lifts, $k(X) = -\infty$ or $k(X) = 0$.

Actually, the few examples known to the author of schemes for which Frobenius lifts are rational.

Such are, for instance, besides the obvious $\boldsymbol{P}^{d_1}_k \times \boldsymbol{P}^{d_2}_k \times \ldots \times \boldsymbol{P}^{d_r}_k$, the Hirzebruch surfaces Σ_n, defined in $\boldsymbol{P}^2_k \times \boldsymbol{P}^1_k$ with bihomogeneous coordinates $(X_0, X_1, X_2)(Y_0, Y_1)$ by the equation $X_1 Y_0^n - X_2 Y_1^n = 0$.

In the case of surfaces the corollary rules out the possibility that Frobenius can be lifted if the surface is of general type or elliptic.

4. – An example: cohomologically p-ample vector bundles on $P^d_{F_p}$.

NOTE. – *In this section we will purposely forget to indicate changes of base to keep notation simple.*

Of course this is one of the few cases in which the Frobenius morphism lifts:

Furthermore, $F_* \mathcal{O}_{P^d}$ can be computed directly:

$$F_* \mathcal{O}_{P^d} = \mathcal{O}_{P^d} \oplus \mathcal{O}_{P^d}(-1)^{\oplus \phi_1(p,\, d)} \oplus \ldots \oplus \mathcal{O}_{P^d}(-d)^{\oplus \phi_d(p,\, d)},$$

where $\phi_1(p, d), \ldots, \phi_d(p, d)$ are numerical functions which we are not interested in. Thus

$$H^i(X, E^{(p)}) = H^i(X, E) \oplus H^i(X, E(-1))^{\oplus \phi_1(p,\, d)} \oplus \ldots \oplus H^i(X, E(-d))^{\oplus \phi_d(p,\, d)}.$$

Therefore if E is cohomologically p-ample

$$H^i(X, E(-k)) = 0 \qquad \forall i \geq 0, \;\; \forall k \leq d.$$

EXAMPLE. – $\Theta_{P^d}(1)$ is cohomologically p-ample: just use the sequence

$$0 \to \mathcal{O}_{P^d}(1) \to \mathcal{O}_{P^d}(2)^{\oplus d+1} \to \Theta_{P^d}(1) \to 0$$

and exactness of $(\cdot)^{(p^n)}$.

To verify cohomological p-ampleness it is enough to prove that

$$H^i(\boldsymbol{P}^d, (\Theta_{P^d}(1))^{(p^n)} \otimes \mathcal{O}_{P^d}(-k)) = 0, \qquad \forall i \geq 0, \;\; \forall k, \;\; \forall n \gg 0,$$

since any sheaf can be resolved by line bundles. Thus it is enough to take $p^n \geq k$, and the exact sequence

$$0 \to \mathcal{O}_{P^d}(p^n - k) \to \mathcal{O}_{P^d}(2p^n - k)^{\oplus d+1} \to (\Theta_{P^d}(1))^{(p^n)} \otimes \mathcal{O}_{P^d}(-k) \to 0,$$

gives the vanishing of the higher cohomology groups.

REMARK. – If E is cohomologically p-ample $H^i(\boldsymbol{P}^d, \Omega^j_{P^d/k} \otimes E) = 0$ $\forall i > 0$, $\forall j$ so that the dimension of the only nonzero group $H^0(\boldsymbol{P}^d, \Omega^j_{P^d/k} \otimes E)$ can be computed by the Riemann Roch formula:

$$H^0(\boldsymbol{P}^d, \Omega^j_{P^d/k} \otimes E) = ch(E \otimes \Omega^j_{P^d/k})\, Td(\Theta_{P^d}).$$

Recall now the following theorem of Beilinson ([2]):

For any coherent sheaf $\mathcal{F}$ *on* $\boldsymbol{P}^d$ *there exists a complex* $F^\bullet$ *of locally free sheaves on* $\boldsymbol{P}^d$ *such that:*

$$a)\ \mathcal{H}^k(F^\bullet) = \begin{cases} \mathcal{F} & \text{if } k = 0, \\ 0 & \text{if } k \neq 0, \end{cases}$$

$$b)\ F^k = \bigoplus_{i+j=k} H^i(\boldsymbol{P}^d, \mathcal{F} \otimes \Omega_{P^d}^{-j}(-j)) \otimes \mathcal{O}_{P^d}(j).$$

Using this theorem we get the following:

PROPOSITION. – *Let* E *be a vector bundle on* $\boldsymbol{P}^d$ *such that*

$$H^i(\boldsymbol{P}^d, E(k)) = 0 \quad \text{for } k \geq -d-1 \text{ and } i > 0$$

and

$$H^i(\boldsymbol{P}^d, E \otimes \Omega_{P^d}^j) = 0 \quad \text{for } i > 0 \text{ and any } j,$$

then E *has a resolution*

$$0 \to \mathcal{O}_{P^d}^{\oplus n_{d,0}} \to \bigoplus_{i=1}^{d} \mathcal{O}_{P^d}^{\oplus n_{d-1,i}}(i) \to \ldots \to \bigoplus_{i=1}^{d} \mathcal{O}_{P^d}^{\oplus n_{0,i}}(i) \to E \to 0$$

where

$$n_{i,j} = \dim H^{d-i-j}(\boldsymbol{P}^d, E \otimes \Omega_{P^d}^{d-j}(-j))$$

in particular E *is cohomologically p-ample.*

PROOF. – Let's first prove the following: Under our assumptions

$$H^i(\boldsymbol{P}^d, E \otimes \Omega_{P^d}^j(k)) = 0 \quad \text{for } i \geq j+1 \text{ and } k \geq -d+j-1.$$

Consider the so called Euler sequences

$$0 \to \Omega_{P^d}^j \to \Lambda^j V \otimes \mathcal{O}_{P^d}(-j) \to \Omega_{P^d}^{j-1} \to 0 \quad \text{for } j \geq 1,\ V = H^0(\boldsymbol{P}^d, \mathcal{O}_{P^d}(1))$$

and tensor with $E(k)$.

The long exact cohomology sequence of the first Euler sequence

$$0 \to \Omega_{P^d}^1 \otimes E(k) \to E(k-1) \otimes V \to E(k) \to 0,$$

gives

$$\ldots \to H^{i-1}(\boldsymbol{P}^d, E(k)) \to H^i(\boldsymbol{P}^d, \Omega_{P^d}^1 \otimes E(k)) \to H^i(\boldsymbol{P}^d, E(k-1)) \otimes V \to \ldots$$

and our assumptions imply

$$H^i(\boldsymbol{P}^d, \Omega_{P^d}^1 \otimes E(k)) = 0 \quad \text{for } i \geq 2,\ k \geq -d.$$

Assume now that

$$H^i(\boldsymbol{P}^d, \Omega_{P^d}^{j-1} \otimes E(k)) = 0 \quad \text{for } i \geqslant j, \ k \geqslant -d+j-2$$

and take the long exact cohomology sequence associated to

$$0 \to \Omega_{P^d}^j \otimes E(k) \to \Lambda^j V \otimes E(k-j) \to \Omega_{P^d}^{j-1} \otimes E(k) \to 0,$$

to conclude the induction.

Thus, in particular,

$$H^i(\boldsymbol{P}^d, \Omega_{P^d}^j(j) \otimes E(-d)) = 0, \qquad \text{for } i \geqslant j+1$$

and the complex $F^\bullet$ associated to $E(-d)$ is zero in strictly positive degrees, that is $F^\bullet$ gives a left resolution of $E(-d)$.

We also have that the only piece in $F^\bullet$ with $\mathcal{O}_{P^d}(-d)$ is in F^{-d} since

$$H^i(\boldsymbol{P}^d, \Omega_{P^d}^d(d) \otimes E(-d)) = H^i(\boldsymbol{P}^d, \Omega_{P^d}^d \otimes E) = 0, \qquad \text{for } i > 0.$$

Thus $F^\bullet(d)$ is a resolution of E of the form

$$0 \to \mathcal{O}_{P^d}^{\oplus n_{d,0}} \to \bigoplus_{i=1}^{d} \mathcal{O}_{P^d}^{\oplus n_{d-1,i}}(i) \to \ldots \to \bigoplus_{i=1}^{d} \mathcal{O}_{P^d}^{\oplus n_{0,i}}(i) \to E \to 0.$$

It is elementary to show that such E is cohomologically p-ample. ∎

Since, on the other hand, cohomologically p-ample bundles satisfy the assumptions of the proposition, as observed at the beginning of this section, we have

COROLLARY. – *A vector bundle E on $\boldsymbol{P}^d$ is cohomologically p-ample if and only if*

$$H^i(\boldsymbol{P}^d, E(k)) = 0 \quad \textit{for } i > 0, \ k \geqslant -d-1$$

and

$$H^i(\boldsymbol{P}^d, \Omega_{P^d}^j \otimes E) = 0 \quad \textit{for } i > 0, \ \forall j.$$

REFERENCES

[1] E. ARBARELLO - M. CORNALBA - J. HARRIS - PH. GRIFFITHS, *Geometry of Algebraic Curves*, Grundl. Math. Wiss., **267**, Springer-Verlag (1984).

[2] A. BEILINSON, *Coherent sheaves on P^n and a problem of linear algebra*, Funct. Anal. Appl., **12** (1978), pp. 214-216.

[3] A. BOREL, *Sheaf theoretic intersection cohomology*, in *Intersection Cohomology*, PM 50, Birkhauser (1984).

[4] P. DELIGNE - L. ILLUSIE, *Relèvements mod p^2 et décomposition du complexe de de Rham*, Invent. Math., **89** (1987), pp. 247-270.

[5] W. FULTON, *Intersection Theory*, Ergeb. Math., 2 Springer-Verlag (1984).
[6] P. P. GRIVEL, *Categories derivées et foncteurs derivées*, in *Algebraic D-modules*, Persp. Math., **2**, Academic Press (1986).
[7] R. HARTSHORNE, *Ample vector bundles*, Publ. Math. IHES, **29** (1966), pp. 63-94.
[8] S. KLEIMAN, *Ample vector bundles on algebraic surfaces*, P.A.M.S., **21**, n. 3 (1969), pp. 673-676.
[9] B. MAZUR, *Frobenius and the Hodge filtration (estimates)*, Ann. Math., **98** (1973), pp. 58-95.
[10] S. MORI, *Projective manifolds with ample tangent bundle*, Ann. Math., **110** (1979), pp. 593-606.
[11] J. OESTERLÉ, *Dégénérescence de la suite spectrale de Hodge vers de Rham*, Sém. Bourbaki, **673**.
[12] B. SHIFFMAN - A. J. SOMMESE, *Vanishing theorems on complex manifolds*, PM 56, Birkhauser (1985).