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Regularity of Minimizers of Non-Isotropic Integrals of the Calculus of Variations (*).

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Abstract. – *The regularity of the minimizers of a special type of non-isotropic variational minimization problem is studied. The particularity of the potential of energy is that it has different growth rate with respect to different parts of the derivatives of the function. In particular, the model treated in this paper can be described as*

$$\Phi(Du) = |\partial_1 u|^2 + |\partial_2 u|^2 + |\partial_3 u|^2 + |\partial_3 u_-|^p.$$

By using a result of P. MARCELLINI (cf. [4]) *and perturbation method, it is proved that the minimizer of the Dirichlet boundary value problem is a function of* $W^{1,\infty}_{\mathrm{loc}}$. *This result can also be extended to Neumann boundary value problems.*

1. – Introduction.

For certain reinforced material, strong anisotropic behavior is exhibited. For example, let $\Omega = Q \times [0, L]$ be a cylinder in R^3 with Q a bounded open piecewise C^1 subset of R^2. Suppose that Ω is occupied by a reinforced material. Let u be the displacement function in the axial direction of the cylinder. Neglect the other deformation factors, the potential of energy Φ can be estimated as follows:

$$(1) \quad C_1(|\partial_1 u|^2 + |\partial_2 u|^2 + |\partial_3 u|^2 + |\partial_3 u_-|^p - 1) \leq$$
$$\leq \Phi(Du) \leq C_2(|\partial_1 u|^2 + |\partial_2 u|^2 + |\partial_3 u|^2 + |\partial_3 u_-|^p + 1).$$

For technical reasons, we assume in the following that

$$(2) \quad \exists\delta > 0, \text{ such that } \Phi(Du) - \delta(|\partial_1 u|^2 + |\partial_2 u|^2 + |\partial_3 u|^2 + |\partial_3 u_-|^p) \text{ is convex},$$

where $\partial_3 u_+ = \max\{\partial_3 u, 0\}$, $\partial_3 u_- = \max\{-\partial_3 u, 0\}$, $2 < p < 6$. The potential of energy of this type says that the material is linear elastic with respect to extension in

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the axial direction. Whereas for compression along the axial direction, when the deformation is relatively (note that we already work in the linearized version) small, it behaves linearly elastic, but it becomes much harder to deform the body in this direction to have a relatively bigger deformation due to the reinforcement of the material by added fibres. For the convenience of the later discussion, we make the following assumptions on Φ:

$$\Phi \text{ is } C^2, \tag{3}$$

$$c_3 |b|^2 \leq \Phi_{ij}(a) b_i b_j \leq c_4 (1 + |a_{3-}|^{p-2}) |b|^2. \tag{4}$$

Here $\Phi_{ij} = \partial_i \partial_j \Phi$. More complicated examples arise from anelasticity problems where the plasticity criterion is described by

$$\sigma(x) \in C \text{ a.e.}$$

with C a complicated unbounded convex set in $R^{n \times n}_{\text{sym}}$, σ the Cauchy stress tensor. We do not study such problems here.

The mathematical problem is then given as follows

$$\operatorname{Inf}\left\{ \int_\Omega \Phi(Du(x))\, dx \colon u \in A \text{ with suitable boundary conditions} \right\}$$

where A is the set of all kinematically admissible functions. To avoid the possibility that $\{u \in A$ with suitable boundary conditions$\}$ is empty, we usually suppose that the boundary condition is good enough so that the admissible set contains at least some functions from $W^{1,p}$. We will also use frequently the statement that Ω is a bounded regular open subset where «regular» stands for «piecewise C^1».

We are interested in studying the minimization problems with potential of energy Φ satisfying (1)-(4). Under some further assumptions on the shape and regularity of Ω, we give the existence, approximation and regularity of the solution.

2. – Kinematically admissible function set and existence.

To deal with the existence result, the first important thing is to define a good kinematically admissible function set on which it is hopeful to prove that a solution exists. By (1), it is clear that the most natural choice is

$$A(\Omega) = \{u \in H^1(\Omega),\ \partial_3 u_- \in L^p(\Omega)\}.$$

We claim the following result:

PROPOSITION 2.1. – *Let $A(\Omega)$ be the set given above, Ω be a bounded regular open subset of R^3, $2 < p < 6$, we have*

(i) *$A(\Omega)$ is a subset of $H^1(\Omega)$ and $W^{1,p}(\Omega)$ is a subset of $A(\Omega)$.*

(ii) *$A(\Omega)$ is a convex cone but is not a linear space.*

(iii) *For any sequence $\{u_n\} \subset A(\Omega)$ such that the set of their «natural norm» is bounded, i.e.,*

$$\|u_n\|_{A(\Omega)} = \|u_n\|_{L^2(\Omega)} + \sum_{\alpha=1}^{2} \|\partial_\alpha u_n\|_{L^2(\Omega)} + \|\partial_3 u_{n+}\|_{L^2(\Omega)} + \|\partial_3 u_{n-}\|_{L^p(\Omega)} < \text{const},$$

there exists a subsequence $\{u_{n_k}\}$ and a function u in $A(\Omega)$ such that

$$u_{n_k} \text{ converges weeakly to } u \text{ in } H^1(\Omega).$$

PROOF. – (i) and (ii) are obvious. (iii) holds due to the lower semicontinuity of convex functionals.

Now, we show the existence and some preliminary approximation results.

PROPOSITION 2.2. – *Let Ω be a regular open subset of R^3, u be a function in $A(\Omega)$, then there exists a sequence $\{u_n\}$ in $C_0^\infty(\Omega)$ such that $u_n \to u$ in $A(\Omega')$ for any $\Omega' \subset\subset \Omega$. i.e.,*

$$|u_n - u|_{A(\Omega')} \to 0 \quad \text{as } n \to \infty$$

with

$$|u_n - u|_{A(\Omega')} = \|u_n - u\|_{L^2(\Omega')} + \|\partial_1(u_n - u)\|_{L^2(\Omega')} + \|\partial_2(u_n - u)\|_{L^2(\Omega')} + \\ + \|\partial_3 u_{n+} - \partial_3 u_+\|_{L^2(\Omega')} + \|\partial_3 u_{n-} - \partial_3 u_-\|_{L^p(\Omega')}.$$

Further, if $u|_{\partial\Omega} = u_0$ and u_0 is the trace of a $W^{1,p}$ function, then there exists a sequence $\{u_n\} \subset\subset C^\infty(\bar{\Omega})$ such that $u_n \to u$ in $A(\Omega)$.

PROOF. – For the first conclusion, we notify that any function u in $A(\Omega)$ is a function in $H^1(\Omega)$. As Ω is a regular open subset of R^3, we can extend u to a neighborhood of Ω as a H^1 function. We use the standard mollifying sequence restricted on Ω: let $\varphi(x)$ be the standard mollifier, $\varphi_n(x) = n^3\varphi(nx)$, $u_n = \varphi_n * \tilde{u}|_\Omega$ with $\tilde{u}$ an extension of u. Noting the facts that

$$(f * \varphi_n)_+ \leq f_+ * \varphi_n,$$

$$(f * \varphi_n)_- \leq f_- * \varphi_n,$$

and using Jensen's Inequality, we have the following estimates

$$\|u_n - u\|_{A(\Omega')} \leq \sup_{\{|y| \leq 1/n\}} \|u(\cdot + y) - u(\cdot)\|_{A(\Omega' + B(0, 1/n))}$$

which converges to zero as n tends to infinity.

For the second conclusion, the extension of u can be made in $W^{1,p}$, therefore we can have the global convergence.

THEOREM 2.3. – *Let Φ be a function satisfying* (1)-(4), u_0 *be the trace of a* $W^{1,p}(\Omega)$ *function on* $\partial\Omega$ *which is also noted as* u_0. *The problem*

$$P \qquad \operatorname{Inf}\left\{ \int_\Omega \Phi(Du(x))\,dx \colon u \in A,\ u|_{\partial\Omega} = u_0 \right\}$$

admits one and only one solution.

PROOF. – By (4), Φ is a strictly convex function on R^3, therefore, the existence and uniqueness are straightforward.

3. – Regularization of the problem.

MARCELLINI has studied (cf. [4]) the regularity of minimizers for the following kind of potential of energy:

$$C_1\left(\sum_{i=1}^{3} \|\partial_i u\|_{L^{p_i}}^{p_i} - 1\right) \leqslant \Phi(Du) \leqslant C_2\left(\sum_{i=1}^{n} \|\partial_i u\|_{L^{p_i}}^{p_i} + 1\right). \tag{5}$$

He used the finite difference method in the weak form of the Euler-Lagrange equation. The difficulty for us to use a similar approach here is that the quantities $\partial_3 u_+$ and $\partial_3 u_-$ should satisfy different growth conditions to make $\Phi(Du)$ integrable and therefore, should be in different function spaces. Consequently, we do not know if the weak form of Euler-Lagrange equation holds for appropriate truncation of u nor how to get the right convergence when the step length of the finit difference tends to zero. So we have to introduce a regularized version of the problem and prove suitable convergence to apply existing theory. In this section, we are going to regularize our problem to satisfy (5) depending on a small parameter ε, and therefore, Marcellini's results can be applied. In the next section, we prove that when $\varepsilon \to 0$, we can get necessary estimates to establish the result of regularity.

We define the following sequence of problems:

$$P_\varepsilon \qquad \operatorname{Inf}\left\{ \int_\Omega \Phi_\varepsilon(Du(x))\,dx \colon u \in W^{1,p},\ u|_{\partial\Omega} = u_0 \right\} \tag{6}$$

where $\Phi_\varepsilon(Du) = \Phi(Du) + \varepsilon|\partial_3 u_+|^p$ and we have the following results:

PROPOSITION 3.1. – *When u_0 is the trace of a $W^{1,p}$ function on the boundary, the solution u_ε of Problem P_ε belongs to* $W^{1,\infty}_{\mathrm{loc}}(\Omega)$.

PROOF. – First, the kinematically admissible set is nonempty because u_0 is a candidate. Then the existence and uniqueness of a solution are straight forward by convex analysis (see also Theorem 2.3). As for the regularity result that $u_\varepsilon \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$, cf. [4].

Next, we want to prove the

PROPOSITION 3.2. – *Under all the previous assumptions on Φ, and suppose that Ω is star-shaped, bounded regular open subset of R^3. Let u_ε be a solution of P_ε, then as ε tends to zero, there exists a subsequence still noted as $\{u_\varepsilon\}$ which converges weakly to u in H^1 and*

$$\lim_{\varepsilon\to 0} \operatorname{Inf} P_\varepsilon = \operatorname{Inf} P, \tag{7}$$

$$\lim_{\varepsilon\to 0} \int_\Omega \Phi(Du_\varepsilon)\,dx = \int_\Omega \Phi(Du)\,dx, \tag{8}$$

$$\lim_{\varepsilon\to 0} \int_\Omega \varepsilon\,|\partial_3 u_{\varepsilon +}|^p\,dx = 0. \tag{9}$$

Moreover, u is the solution of Problem P.

It is easy to see that

$$\operatorname{Inf} P_\varepsilon \geqslant \operatorname{Inf} P. \tag{10}$$

Therefore, to give a full proof of (7), we need to prove the inverse inequality of (10). (8) and (9) would then follow easily. Before being able to do this, we have to establish an approximation result and it is here we need the assumption that Ω is of particular form—star-shaped.

LEMMA 3.3. – *Let Ω be a star-shaped, bounded regular open subset of R^3. Let u (u_ε resp.) be a solution of Problem P (Problem P_ε resp.). For any $\delta > 0$, there exists a function u_δ ($u_{\varepsilon\delta}$ resp.) $\in W^{1,p}(\Omega)$ such that*

$$|u_\delta - u|_{A(\Omega)} < \delta \qquad (|u_{\varepsilon\delta} - u_\varepsilon| < \delta \text{ resp.}) \tag{11}$$

$$u_\delta|_{\partial\Omega} = u_0 \qquad (u_{\varepsilon\delta}|_{\partial\Omega} = u_0 \text{ resp.}) \tag{12}$$

and

$$\int_\Omega \Phi(Du_\delta)\,dx < \operatorname{Inf} P + \delta \left(\int_\Omega \Phi_\varepsilon(Du_{\varepsilon\delta})\,dx < \operatorname{Inf} P_\varepsilon + \delta \text{ resp.} \right). \tag{13}$$

PROOF. – We only give the proof for the case of Problem P. The proof for the Problem P_ε is exactly the same. Also to simplify the discussion, we suppose that Ω is star-shaped with respect to the origin. Let m be a positive integer, and $v_m = u_0 + w_m(x)$ where

$$w_m(x) = \begin{cases} (u - u_0)\left(\dfrac{m+1}{m}x\right) & \text{when } x \in \dfrac{m}{m+1}\Omega, \\ 0 & \text{elsewhere}. \end{cases}$$

Then it is clear that $w_m \in H_0^1(\Omega) \cap A(\Omega)$. Let $\gamma_m \leqslant (1/2)\,\mathrm{dist}\,(R^3 - \overline{\Omega}; (m/(m+1)\Omega)$, φ be the standard mollifier,

$$u^m(x) = u_0(x) + w_m * \varphi_{\gamma_m}(x).$$

It is easy to verify that $u^m(x) \in W^{1,p}$, $u^m(x)|_{\partial\Omega} = u_0$. Further, by Jensen's inequality, we have

$$\int_\Omega \Phi(Du^m(x))\,dx = \int_\Omega \Phi(Du_0 + (Dw_m) * \varphi_{\gamma_m}(x))\,dx \leqslant$$

$$\leqslant \int_\Omega \int_{R^3} \Phi(Du_0(x) + Dw_m(x-y))\,\varphi_{\gamma_m}(y)\,dy\,dx \leqslant \operatorname*{Sup}_{\{y \leqslant \gamma_m\}} \int_\Omega \Phi(Du_0(x) + Dw_m(x-y))\,dx.$$

By the continuity of Lebesgue integrals, letting $m \to \infty$, we have

$$\lim_{m\to\infty} \int_\Omega \Phi(Du_m)\,dx \leqslant \int_\Omega \Phi(Du)\,dx.$$

As $\lim\limits_{m\to\infty} \int_\Omega \Phi(Du_m)\,dx \geqslant \int_\Omega \Phi(Du)\,dx$ is clear, we have

$$\lim_{m\to\infty} \int_\Omega \Phi(Du_m)\,dx = \int_\Omega \Phi(Du)\,dx.$$

Using a similar argument as in the proof of Proposition 2.2, we can conclude that

$$\lim_{m\to\infty} |u_m - u|_{A(\Omega)} = 0.$$

For each $\delta > 0$, choose m sufficiently large so that (11) and (13) are satisfied and let this u^m be u_δ, the lemma is proved.

REMARK 3.4. – *As a matter of fact, our approximation result holds when Ω satisfies the following assumptions: a) there exists a sequence of open sets $\{\Omega_\varepsilon\}$ such that*

$\Omega_\varepsilon \subset\subset \Omega$, $\Omega_\varepsilon \uparrow \Omega$, *b)* Ω_ε *is the image of* Ω *under a one-to-one mapping* $\psi_\varepsilon = (\psi_{\varepsilon 1}(x_1, x_2), \psi_{\varepsilon 2}(x_1, x_2), \psi_{\varepsilon 3}(x_1, x_2, x_3)) \in C^1(R^3)$, *with* $\psi_\varepsilon \to \mathrm{Id}$ *in* C_1 *as* $\varepsilon \to 0$.

Now we accomplish the proof of Proposition 3.2: by Lemma 3.3, for any $\delta > 0$, there exists a $u_\delta \in W^{1,p}(\Omega)$ which is P admissible and

$$\operatorname{Inf} P \geqslant \int_\Omega \Phi(Du_\delta)\, dx - \delta .$$

As

$$(14) \qquad \operatorname{Inf} P \geqslant \int_\Omega \Phi(Du_\delta)\, dx - \delta = \int_\Omega \Phi_\varepsilon(Du_\delta)\, dx - \int_\Omega \varepsilon |\partial_3 u_{\delta +}|^p\, dx - \delta \geqslant$$

$$\geqslant \operatorname{Inf} P_\varepsilon - \varepsilon \int_\Omega |\partial_3 u_{\delta +}|^p\, dx - \delta .$$

Take the limit $\varepsilon \to 0$ in (14), as δ is arbitrary, we get (7).

Next, let u_ε be a solution of P_ε, we know that $\{u_\varepsilon\}$ is bounded in $H^1(\Omega)$. Therefore, there exists a subsequence still noted as u_ε such that

$$u_\varepsilon \rightharpoonup u \quad \text{in } H^1(\Omega)$$

and we have the following estimates

$$\lim_{\varepsilon \to 0} \operatorname{Inf} P_\varepsilon = \lim_{\varepsilon \to 0} \int_\Omega \Phi(Du_\varepsilon) + \varepsilon |\partial_3 u_{\varepsilon +}|^p\, dx \geqslant \lim_{\varepsilon \to 0} \int_\Omega \Phi(Du_\varepsilon)\, dx \geqslant \int_\Omega \Phi(Du)\, dx \geqslant \operatorname{Inf} P .$$

As $\lim_{\varepsilon \to 0} \operatorname{Inf} P_\varepsilon = \operatorname{Inf} P$, u must be a solution of P for it is obviously admissible. So (8) and (9) hold automatically.

REMARK 3.6. – *The proof of Proposition 3.2 is a use of Γ-convergence argument. We do not get into more details here about the Γ-convergence theory, interested readers may refer to* [2], [5], [6] *and many others that I can not give a full liste here.*

In the following, we indicate that the Euler-Lagrange equation of our problem P can only be satisfied by using test functions in $W^{1,p}$. For general kinematically admissible functions, it is not obvious how to prove this.

LEMMA 3.7. – *Let* $\Phi = \Phi(\xi)$ *be a function defined on* R^3 *such that*

$$(15) \qquad \Phi(\xi) \leqslant M\left(1 + \sum_{i=1}^{2} |\xi_i|^2 + |\xi_{3+}|^2 + |\xi_{3-}|^p\right), \qquad \textit{for any } \xi \in R^3$$

where $M > 0$ *and* $p > 1$. *If* Φ *is convex, then the partial derivatives of* Φ *satisfy*

$$(16)\qquad |\Phi_{\xi_3}(\xi)| \leqslant C\left(1 + \sum_{j=1}^{2} |\xi_j|^2 + |\xi_3|^2\right)^{1/2} \quad \textit{for all } \xi \in R^3 \quad \textit{s.t.} \quad \xi_3 > 0,$$

$$(17)\qquad |\Phi_{\xi_3}(\xi)| \leqslant C\left(1 + \sum_{j=1}^{2} |\xi_j|^2 + |\xi_n|^p\right)^{1-1/p} \quad \textit{for all } \xi \in R^3 \quad \textit{s.t.} \quad \xi_3 < 0,$$

$$(18)\qquad |\Phi_{\xi_i}(\xi)| \leqslant C\left(1 + \sum_{j=1}^{2} |\xi_j|^2 + |\xi_{3+}|^2 + |\xi_-|^p\right)^{1/2} \quad \textit{for all } \xi \in R^3 \quad \textit{and } i = 1, 2.$$

The proof of this lemma is based on the standard convex analysis. Interested readers may refer to [4].

PROPOSITION 3.8. – *Let* Φ *be a convex function of class* $C^1(R^3)$ *satisfying* (15) *with* $2 < p < 6$. *Let* u *be a minimizer of Problem P. Then for any function* $\psi \in W_0^{1,p}$, *we have*

$$(19)\qquad \int_\Omega \sum_{i=1}^{3} \Phi_{\xi_i}(Du)\, \psi_{x_i}\, dx = 0\,.$$

PROOF. – Use the Lebesgue Dominated Convergence Theorem and the estimates obtained in Lemma 3.7 in the following expression when $\psi \in W_0^{1,p}$:

$$(20)\qquad \int_\Omega \frac{\Phi(Du + tD\psi) - \Phi(Du)}{t}\, dx \geqslant 0 \quad \text{for any } t > 0\,,$$

we get

$$\int_\Omega \sum_{i=1}^{3} \Phi_{\xi_i}(Du)\, \psi_{x_i}\, dx \geqslant 0\,.$$

As $-\psi$ also satisfies the same inequality, (19) follows naturally.

We notice that in general, for perturbations like $\psi = \varphi u$ with $\varphi \in C_0^\infty$, $\varphi(x) \geqslant 0$ and u admissible, if we use the Lebesgue Dominated Convergence Theorem, we could only conclude that

$$\int_\Omega \Phi_{\xi_i}(Du)\, \psi_{x_i}\, dx \geqslant 0$$

but not (10) because $-\psi$ is not admissible a priori.

4. – Regularity.

We study the problem of regularity in this section. Our result is that if u is a solution of Problem P then it belongs to $W^{1,\infty}_{\mathrm{loc}}(\Omega)$. One of the most direct approaches to prove this result is to use finite difference of solutions as test function in the weak form of Euler-Lagrange equation. But unfortunately, in our context, it has not yet been proved that the weak form of Euler-Lagrange equation holds with truncation of u as test function. However, we can use the result of regularization developed in Section 3 and the existing result of Marcellini to prove such regularity in our case.

PROPOSITION 4.1. – *Let u_ε be a solution of Problem P_ε, then u_ε satisfies*

$$\left(\int_\Omega (\eta|\Delta_h^i u_\varepsilon|)^6\,dx\right)^{1/3} \leq c\int_0^1 dt\int_\Omega |\Delta_h^i u_\varepsilon|^2[|D\eta|^2 + \Phi_{\varepsilon\xi_i\xi_j}(Du_\varepsilon + th\Delta_h^i Du_\varepsilon)\eta_{x_i}\eta_{x_j}]\,dx \tag{21}$$

where $\eta \in C_0^\infty(\Omega)$, $\eta(x) \geq 0$, $0 < h < \operatorname{dist}(\operatorname{supp}\eta, \partial\Omega)$, Δ_h^i is the following finite difference operator:

$$\Delta_h^i u = \frac{u(x+te_i) - u(x)}{h}, \qquad (e_i)_j = \delta_{ij},$$

c is a constant independent of h and ε.

PROOF. – This can be derived by following the proof in [4] with the following remark: in deriving the left hand side of the inequality, we can neglect the penelization term because it is a convex functional, and therefore, the constant c can be chosen independent of ε.

PROPOSITION 4.2. – *For each $\varepsilon > 0$, we have the following inequality:*

$$\left(\int_\Omega |\eta\partial_3 u_\varepsilon|^6\,dx\right)^{1/3} \leq c\int_\Omega |D\eta|^2[|\partial_3 u_\varepsilon|^2 + |D_3 u_{\varepsilon -}|^p + \varepsilon|D_3 u_{\varepsilon +}|^p]\,dx. \tag{22}$$

PROOF. – From (21), we have

$$\left(\int_\Omega |\eta\Delta_h^3 u_\varepsilon|^6\,dx\right)^{1/3} \leq \tag{23}$$

$$\leq c\int_\Omega |\Delta_h^3 u_\varepsilon|^2\{|D\eta|^2 + [1 + |((1-t)\partial_3 u_\varepsilon + t\partial_3 u_\varepsilon(x+he_3))_-|^{p-2} +$$

$$+\varepsilon|((1-t)D_3 u_\varepsilon + t\partial_3 u_\varepsilon(x+he_3))_+|^{p-2}]|D\eta|^2\}\,dx.$$

As

$$\Delta_h^3 u_\varepsilon \to \partial_3 u_\varepsilon \text{ in } L^p, \qquad \text{as } h \to 0,$$

$$(1-t)\,\partial_3 u_\varepsilon(x) + t\partial_3 u_\varepsilon(x+he_3) \to \partial_3 u_\varepsilon(x) \text{ in } L^p, \qquad \text{as } h \to 0,$$

we can apply Fatou's Lemma on the left hand side of (23) to get (22).

Let $\eta(x)=1$ in $B_r(x_0) \subset\subset \Omega$, $\eta(x)=0$ in $\Omega - B_R(x_0)$, $B_R(x_0)\subset\subset\Omega$ and $R>r$, $|D\eta(x)| \leqslant c/(R-r)$, we have

$$\left[\int_{B_r(x)} |\partial_3 u_\varepsilon|^6\,dx\right]^{1/3} \leqslant \frac{c}{(R-r)^2}\int_{B_R(x)} \{|\partial_3 u_\varepsilon|^2 + |\partial_3 u_{\varepsilon-}|^p + \varepsilon|\partial_3 u_{\varepsilon+}|^p\}\,dx. \tag{24}$$

LEMMA 4.3. – *Let Ω be a star-shaped bounded regular open subset, if u is a solution of P, then for any $\eta \in C_0^\infty(\Omega)$, $\eta \geqslant 0$ satisfying all the above requirements, we have*

$$\left[\int_{B_r(x)} |\partial_3 u|^6\,dx\right]^{1/3} \leqslant \frac{c}{(R-r)^2}\int_{B_R(x)} (|\partial_3 u|^2 + |\partial_3 u_-|^p)\,dx. \tag{25}$$

PROOF. – By hypothesis (2), we have

$$\lim_{\varepsilon\to 0}\int_\Omega |\partial_3 u_\varepsilon|^2 + |\partial_3 u_{\varepsilon-}|^p\,dx \geqslant \int_\Omega |\partial_3 u|^2 + |\partial_3 u_-|^p\,dx,$$

$$\lim_{\varepsilon\to 0}\int_\Omega \Phi_\varepsilon(Du_\varepsilon) - \delta(|\partial_3 u_\varepsilon|^2 + |\partial_3 u_{\varepsilon-}|^p)\,dx \geqslant \int_\Omega \Phi(Du) - \delta(|\partial_3 u|^2 + |\partial_3 u_-|^p)\,dx.$$

As $\lim\limits_{\varepsilon\to 0}\int_\Omega \Phi(Du_\varepsilon)\,dx = \int_\Omega \Phi(Du)\,dx$, we have

$$\lim_{\varepsilon\to 0}\int_\Omega (|\partial_3 u_\varepsilon|^2 + |\partial_3 u_{\varepsilon-}|^p)\,dx = \int_\Omega \{|\partial_3 u|^2 + |\partial_3 u_-|^p\}\,dx. \tag{26}$$

Using (9), we know that

$$\lim_{\varepsilon\to 0}\int_\Omega \varepsilon|\partial_3 u_{\varepsilon+}|^p\,dx = 0.$$

Finally, taking the limit $\varepsilon \to 0$ in the two sides of (24), we get

$$\left[\int_{B_r(x)} |\partial_3 u|^6\,dx\right]^{1/3} \leqslant \frac{c}{(R-r)^2}\int_{B_R(x)} (|\partial_3 u|^2 + |\partial_3 u_-|^p)\,dx. \tag{27}$$

THEOREM 4.4. – *If u is a solution of P, Ω is a star-shaped bounded regular open subset of R^3, then $\partial_3 u \in L^q_{\mathrm{loc}}(\Omega)$ for any $6 \geq q \geq 2$ and for any $\Omega' \subset\subset \Omega$, we have*

$$\left[\int_{\Omega'} |\partial_3 u|^6 dx\right]^{1/3} \leq \frac{c(\Omega')}{\operatorname{dist}(\partial\Omega, \Omega')^2} \int_{\Omega} (|\partial_3 u|^2 + |\partial_3 u_-|^p)\, dx. \tag{28}$$

PROOF. – For any $x \in \bar{\Omega}'$, take $R = \operatorname{dist}(x, \partial\Omega)$, $r = R/2$, choose a finite cover of $\bar{\Omega}'$ of type $B(x, R)$. On each ball, use (27). Add together, it is then easy to see that (28) holds.

Till now, we showed that if u is a solution of Problem P, then $u \in H^1(\Omega)$, $\partial_3 u \in L^6_{\mathrm{loc}}$. Therefore, $\partial_3 u \in L^p_{\mathrm{loc}}$. It is now easy to follows Marcellini's argument [4] to conclude the

THEOREM 4.5. – *If u is a solution of P, Ω is a star-shaped bounded open regular subset of R^3, then $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ and for every $\Omega' \subset\subset \Omega$, there is an increasing function $\psi: [0, \infty) \to [0, \infty)$ such that*

$$|Du|_{L^\infty(\Omega')} \leq \psi(|\partial_1 u|_{L^2} + |\partial_2 u|_{L^2} + |\partial_3 u|_{L^2} + |\partial_3 u_-|_{L^p}). \tag{29}$$

The proof of this result is left to interested readers.

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