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Surfaces of a Euclidean Space with Helical or Planar Geodesics Through a Point (*).

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0. – Introduction.

Helical submanifolds were first introduced in [Be.A]. Helical submanifolds in a Euclidean space or a unit sphere have been studied by K. SAKAMOTO [S-1], [S-5], [S-6], since 1982. He proved that such submanifolds are either Blaschke manifolds or Euclidean planes.

On the other hand, in 1981, B.-Y. CHEN and P. VERHEYEN [Ch.B-V-1], [Ch.B-V-2], introduced the notion of submanifolds with geodesic normal sections and classified surfaces with geodesic normal sections in a Euclidean space. They also proved that helical submanifolds have geodesic normal sections. Later, P. VERHEYEN [V], proved that a submanifold M in a Euclidean space E^m of dimension m with geodesic normal sections are helical. So the concept of submanifolds with geodesic normal sections coincides with the concept of helical submanifold if the ambient space is a Euclidean space.

In [Ho.S], S. L. HONG introduced the notion of planar geodesic immersions. Such immersions were later classified by J. A. LITTLE [L], and K. SAKAMOTO [S-1], independently, who proved that M^n is a compact symmetric space of rank one and the second fundamental form is parallel. The Veronese surface can be considered as one of the best examples determined by the planar geodesic immersion if the ambient space is a 5-dimensional Euclidean space E^5.

However, there has been no research on a submanifold M in a Euclidean space E^m with the property that for a fixed point p in M every geodesic passing through p is a helix of the same curvatures or every geodesic through p is planar or a normal section at the point p. From this point of view, we are going to study surfaces in a Euclidean space which have such properties and to characterize such surfaces.

In § 1, we introduce some fundamental definitions and concepts which are the necessary background for the study throughout this paper.

In § 2, we study compact connected surfaces in a Euclidean space with helical geodesics through a point. If the ambient space is a 3-dimensional Euclidean space E^3 then such surfaces are characterized as standard spheres. If the ambient manifold is a

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4-dimensional Euclidean space E^4, then we obtain that geodesics through the point must be of rank 2, i.e., they are planar curves, and surfaces are characterized as standard spheres which lie in E^3 or pointed Blaschke surfaces which fully lie in E^4. If the ambient manifold is a 5-dimensional Euclidean space E^5, then geodesics of the given surface through the point may be of rank 4. In this case, using some fondamental equations obtained from the helices through the point, we set up a system of ordinary differential equations. By solving this system of differential equations, we have examples of pointed Blaschke surfaces which are diffeomorphic to a real projective space and lies fully in E^5. So we can characterize such surfaces as standard spheres which lie in E^3 or pointed Blaschke surfaces which lie fully in E^4 or E^5. By means of this characterization, we have a new characterization of the Veronese surface, namely, a Veronese surface is characterized as a compact connected surface with constant Gaussian curvature and a nonumbilical point through which every geodesic is a helix.

In § 3, we study a surface M in a three-dimensional Euclidean space E^3 with geodesic normal sections at a point p. By adopting geodesic polar coordinates about the base point p, geodesics are proved to depend only on the arc length and thus M is characterized as locally a surface of revolution around the point p. So, if M is complete and connected, then M is a surface of revolution if and only if there is a point p through which every geodesic is a normal section.

In § 4, we study a surface M in a Euclidean space E^m with planar geodesics through a point p. We prove that planar geodesics through a point p are normal sections of M at p. We also prove that geodesics through p only depends on the arc length and thus Frenet curvatures are independent of the choice of the direction. So, we can precisely determine how the surface looks like in a neighborhood by means of the Frenet curvature of a fixed geodesic through p. We also observe that a surface in a Euclidean space E^m with planar geodesic through p is possible to lie fully in a considerably higher dimensional Euclidean space $E^n \subset E^m$ if p is an isolated flat point.

This paper is taken a part of my doctoral dissertation at Michigan State University. I wish to extend my deep appreciation to my thesis advisor Professor BANG-YEN CHEN for his guidance and constant encouragement in the preparation of this study.

1. - Some fundamental concepts.

Let x: $M \to \tilde{M}$ an isometric immersion of M into a Riemannian manifold $\tilde{M}$. Let ∇ and $\tilde{\nabla}$ be the covariant differential operators of M and $\tilde{M}$ respectively. Then the Gauss equation is given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y) \tag{1.1}$$

for vector fields X and Y tangent to M, where σ denotes the second fundamental form.

The Weingarten equation is given by

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi \tag{1.2}$$

for a vector field ξ normal to M, where A_ξ denotes the Weingarten map associated with ξ and the normal connection D. A_ξ is related to σ as $\langle A_\xi X, Y\rangle = \langle \sigma(X, Y), \xi\rangle$, where $\langle , \rangle$ is the Riemannian metric tensor. We now assume the ambient manifold $\tilde{M}$ is a Euclidean space E^m of dimension m. Let R be the curvature tensor of M. Then the structure equations of Gauss and Codazzi are given by

$$\langle R(X, Y)Z, W\rangle = \langle \sigma(X, W), \sigma(Y, Z)\rangle - \langle \sigma(Y, W), \sigma(X, Z)\rangle, \tag{1.3}$$

$$(\overline{\nabla}_X \sigma)(Y, Z) - (\overline{\nabla}_Y \sigma)(X, Z) = 0 \tag{1.4}$$

for all vector fields X, Y, Z, W tangent to M, where $\overline{\nabla}_X \sigma$ is the covariant derivative of σ defined by

$$(\overline{\nabla}_X \sigma)(Y, Z) = D_X \sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z).$$

We now define the Frenet curvatures, Frenet curvature vectors and Frenet frame of a curve γ: $I \to \tilde{M}$ parametrized by the arc lengths s. Let $\gamma'(s) = T_1(s)$ be the unit tangent vector and put $\kappa_1 = \|\tilde{\nabla}_{T_1} T_1\|$. If κ_1 is identically zero on I, then γ is said to be of *rank* 1. If κ_1 is not identically zero, then one can define T_2 by $\tilde{\nabla}_{T_1} T_1 = \kappa_1 T_2$ on $I_1 = \{s \in I \,|\, \kappa_1(s) \neq 0\}$. Set $\kappa_2 = \|\tilde{\nabla}_{T_1} T_2 + \kappa_1 T_1\|$. If κ_2 is identically zero on I_1, then γ is said to be of *rank* 2. If κ_2 is not identically zero on I_1, then we define T_3 by $\tilde{\nabla}_{T_1} T_2 = -\kappa_1 T_1 + \kappa_2 T_3$. Inductively, we can define T_d and $\kappa_d = \|\tilde{\nabla}_{T_1} T_d + \kappa_{d-1} T_{d-1}\|$ and if $\kappa_d = 0$ identically on $I_{d-1} = \{s \in I \,|\, \kappa_{d-1}(s) \neq 0\}$, then γ is said to be of *rank* d. If γ is of rank d, then we have a matrix equation

$$\tilde{\nabla}_{T_1}(T_1, T_2, \ldots, T_d) = (T_1, T_2, \ldots, T_d)\Lambda,$$

on I_{d-1}, where Λ is a $d \times d$-matrix defined by

$$\Lambda = \begin{pmatrix} 0 & -\kappa_1 & 0 & \cdots & \\ \kappa_1 & 0 & -\kappa_2 & 0 & \mathbf{0} \\ & \kappa_2 & 0 & & \\ & & \cdots & & -\kappa_{d-1} \\ \mathbf{0} & & & \kappa_{d-1} & 0 \end{pmatrix}.$$

The matrix Λ, $\{T_1, T_2, \ldots, T_d\}$ and $\kappa_1, \ldots \kappa_d$ are called the Frenet formula, Frenet frame and Frenet curvatures of γ respectively. Here, we recall the definition of the helical immersion. Let M be a connected Riemannian manifold and x: $M \to \tilde{M}$ an isometric immersion of M into a Riemannian manifold $\tilde{M}$. If the image $x \circ \gamma$ of each geodesic γ in M has constant Frenet curvatures which are independent on the choice of the geodesic γ, then x is called a *helical immersion.*

We now recall the definition of a normal section. Let M be an n-dimensional submanifold of a Euclidean space E^m of dimension m. Let p be a point of M and t be a nonzero vector tangent to M at p. Let $E(p; t)$ be the affine space generated by t and normal space $T_p^\perp M$ at p. Then the dimension of $E(p; t)$ is $m - n + 1$. The intersection of M and $E(p; t)$ gives rise to a curve on a neighborhood of p. Such a curve is called the *normal section* of M at p in the direction of t. We say that the submanifold M has *geodesic normal sections* if every normal section is a geodesic.

On the other hand, it is well-known that a helical immersion is λ-isotropic. We now recall the definition of isotropy. An isometric immersion x: $M \to E^m$ is said to be *λ-isotropic at a point p* if $\lambda = \|\sigma(X, X)\|$ does not depend upon the choice of the unit vector X tangent to M at p. If λ is also independent of the choice of point, then x is said to be *constant isotropic*. It is easily seen that M is λ-isotropic at p if and only if

$$\sum A_{\sigma(X, Y)} Z = \lambda^2 \sum \langle X, Y\rangle Z$$

for every $X, Y, Z \in T_p(M)$, where $\sum$ denotes the cyclic sum with respect to X, Y, Z.

B. O'NEILL [On], proved that M is isotropic at p if and only if

$$\langle \sigma(X, X), \sigma(X, Y)\rangle = 0 \tag{1.5}$$

for any two orthonormal vectors X and Y in $T_p(M)$.

Using O'Neill's idea, we can prove the following.

LEMMA 1.1. – Let T be a symmetric tensor of type $(0, r)$ defined on E^m. Then $\|T(u^r)\|$ does not depend on the choice of the unit vector u if and only if

$$\langle T(u^r), T(u^{r-1}, u^\perp)\rangle = 0 \tag{1.6}$$

for any vector $u^\perp$ which is perpendicular to u, where $T(u^r) = T(u, u, u, \ldots, u)$ and $T(u^{r-1}, u^\perp) = T(u, u, \ldots, u, u^\perp)$.

PROOF. – Since all the unit vectors in E^m form a unit sphere, a vector $u^\perp$ which is perpendicular to a unit vector u is tangent to the unit sphere. So,

$$\langle T(u^r), T(u^r)\rangle = C\,(\text{constant}) \quad \text{on the unit sphere}$$

if and only if

$$u^\perp \langle T(u^r), T(u^r)\rangle = 0\,.$$

Since u is a position vector on the unit sphere,

$$\langle T(u^r), T(u^{r-1}, u^\perp)\rangle = 0\,. \quad \text{(Q.E.D.)}$$

Throughout this paper, $t^\perp$ always means a unit vector perpendicular to t for some vector t unless it is stated otherwise.

LEMMA 1.2. – Let M be a submanifold in a Riemannian manifold $\tilde{M}$ such that M is isotropic at a point p in M. Then we have

$$\|\sigma(e_1, e_1)\|^2 = \langle\sigma(e_1, e_1), \sigma(e_2, e_2)\rangle + 2\|\sigma(e_1, e_2)\|^2 \tag{1.7}$$

for every pair of orthonormal vectors e_1 and e_2 tangent to M at p.

PROOF. – Let e_1 and e_2 be orthonormal vectors tangent to M at p. Set $X = (e_1 + e_2)/\sqrt{2}$, $Y = (e_1 - e_2)/\sqrt{2}$. Then X and Y are orthonormal. Since M is isotropic at p,

$$\langle\sigma(e_1, e_1), \sigma(e_1, e_1)\rangle = \langle\sigma(X, X), \sigma(X, X)\rangle.$$

Using (1.5), we obtain (1.7). (Q.E.D.)

For later use, we define a Blaschke manifold M_1. Let $p \in M_1$ and $\mathrm{Cut}(p)$ be the cut locus of p in M_1. If for every $q \in \mathrm{Cut}(p)$ the link $L_{p,q} = \{(d\gamma/ds)(q) \,|\, \gamma$ is a minimal geodesic from p to $q\}$ is a great sphere of $U_q M_1$, then M_1 is said to be a *Blaschke manifold at the point* p, where $U_q M_1$ is the unit tangent space of M_1 at q. If M_1 is a Blaschke manifold at every point of M_1, then M_1 is said to be a *Blaschke manifold.* It is well known that M_1 is a Blaschke manifold at p if and only if the $\mathrm{Cut}(p)$ is spherical (see [Be.A], p. 137).

Throughout this paper, all the manifolds and geometric quantities are C^∞ unless it is sated otherwise.

2. – Surfaces in E^m with property $(*_1)$.

Let M be a complete connected surface in E^m $(m \geq 3)$ with Riemannian connection ∇. We also denote the normal connection by D and the Weingarten map associated to a normal vector ξ by A_ξ and the second fundamental form by σ as usual.

We now define the property $(*_1)$.

$(*_1)$ There is a point p in M such that every geodesic through p, which is regarded as a curve in E^m, is a helix of the same constant Frenet curvatures.

Clearly, every helical immersion satisfies the property $(*_1)$.

Suppose M has the property $(*_1)$. Since every geodesic has the constant curvatures, $\langle\sigma(t, t), \sigma(t, t)\rangle$ does not depend on the choice of the unit vector $t \in T_p M$.

LEMMA 2.1. – Let M be a surface in a Euclidean space E^m $(m \geq 3)$. Suppose that M satisfies the property $(*_1)$. Then M is isotropic at p.

We now prove

THEOREM 2.2. – Let M be a complete connected surface in E^3. Then M satisfies the property $(*_1)$ if and only if M is a standard sphere or a plane E^2.

PROOF. – Suppose that M satisfies $(*_1)$. By Lemma 2.1 we see that M is isotropic at p. In this case, p is an umbilical point. Choose a geodesic γ through p. Suppose γ is of rank 1. It is clear that M is a plane E^2 in E^3. Suppose that γ is of rank 2. Since every geodesic is a circle of the same radius and the same center, M is a standard sphere. Suppose that γ is of rank 3. We assume that γ is parametrized by the arc length s. Let $\gamma'(s) = T$. Then $\gamma''(s) = \sigma(T, T)$ and $\gamma'''(s) = -A_{\sigma(T,T)}T + (\overline{\nabla}_T \sigma)(T, T)$ since γ is a geodesic. Since γ is of rank 3, $\gamma'(s) \wedge \gamma''(s) \wedge \gamma'''(0) \neq 0$, and so $T \wedge \sigma(T, T) \wedge \wedge A_{\sigma(T,T)} T \neq 0$ along γ. It follows that $T \wedge A_{\sigma(T,T)} T \neq 0$. Since M is isotropic at p, $\langle \sigma(t, t), \sigma(t, t^\perp)\rangle = 0$, where $t = T(0)$ and $\gamma(0) = p$. Accordingly, $A_{\sigma(t,t)} t \perp t^\perp$, i.e., $A_{\sigma(t,t)} t \wedge t = 0$. So, this case cannot occur. The converse is clear. (Q.E.D.)

We now assume that a surface M which lies in E^m $(m \geq 4)$ is compact and suppose that M satisfies the property $(*_1)$. By Lemma 2.1 M is isotropic at p. The equation (1.7) implies that only two cases may occur:

Case 1) $\sigma(e_1, e_2) \equiv 0$ for any orthonormal vectors e_1 and e_2. In this case, $\dim(\operatorname{Im}\sigma)_p = 1$ since M is compact, where $(\operatorname{Im}\sigma)_p = \{\sigma(X, Y) \mid X, Y \in T_pM\}$ is called the *first normal space at the point p*.

Case 2) $\sigma(e_1, e_2) \neq 0$ for an orthonormal basis $\{e_1, e_2\}$ of T_pM. In this case, $\dim(\operatorname{Im}\sigma)_p \geq 2$.

LEMMA 2.3. – *Let M be a compact connected surface in E^4 satisfying the property $(*_1)$. If the dimension of the first normal space at p is one, then M is a standard sphere lying in E^3.*

PROOF. – Suppose $\dim(\operatorname{Im}\sigma)_p = 1$, i.e., $\sigma(e_1, e_2) = 0$ for any orthonormal basis $\{e_1, e_2\}$ of T_pM. So, p is an umbilical point of M. Choose a geodesic γ through p such that $\gamma(0) = p$ and $\gamma'(s) = T$. Then we have

$$\gamma''(s) = \sigma(T, T), \tag{2.1}$$

$$\gamma'''(s) = -A_{\sigma(T,T)}T + (\overline{\nabla}_T \sigma)(T, T), \tag{2.2}$$

$$\gamma^{(4)}(s) = -(\overline{\nabla}_T A)_{\sigma(T,T)} T - \sigma(T, A_{\sigma(T,T)} T) - 2A_{(\overline{\nabla}_T \sigma)(T,T)} T + D_T((\overline{\nabla}_T \sigma)(T, T)), \tag{2.3}$$

where $(\overline{\nabla}_T A)_\eta Y = \nabla_X(A_\eta Y) - A_{D_X \eta} Y - A_\eta \nabla_X Y$ for $X, Y \in TM$ and $\eta \in T^\perp M$.

We are going to show that γ is of rank 2. It is enough to show that $(\overline{\nabla}_t \sigma)(t, t) = 0$, where $T(0) = t$.

Suppose that $(\overline{\nabla}_t \sigma)(t, t) \neq 0$. We may put

$$\sigma(T, T) = \kappa_1 \xi,$$

where κ_1 is the first Frenet curvature of γ and ξ a unit normal vector field along γ. As a matter of fact ξ is in the direction of the mean curvature vector H at p. Since κ_1 is

$$\langle (\overline{\nabla}_T \sigma)(T, T), \sigma(T, T)\rangle = 0 \quad \text{along } \gamma$$

and since $\sigma(t, t^{\perp}) = 0$, we get

$$A_{(\overline{\nabla}_t \sigma)(t,\, t)}\, t = 0\,.$$

On the other hand, $\kappa_1 = \langle \sigma(T, T), \sigma(T, T)\rangle = \langle A_{\sigma(T,\, T)} T, T\rangle$. Covariant differentiation of this equation along the geodesic γ leads to

$$\langle (\overline{\nabla}_T A)_{\sigma(T,\, T)} T, T\rangle = 0$$

because γ is a geodesic. Evaluate this at p and then we have

$$\langle (\overline{\nabla}_t A)_\xi t, t\rangle = 0$$

since $\kappa_1 \neq 0$. Since this holds for any direction, linearization and the Codazzi equation (1.4) give

$$(\overline{\nabla} A)_\xi = 0\,.$$

So, we can obtain the following:

$$\begin{aligned}
\gamma'(0) &= t\,,\\
\gamma''(0) &= \sigma(t, t)\,,\\
\gamma'''(0) &= -A_{\sigma(t,\, t)}\, t + (\overline{\nabla}_t \sigma)(t, t)\,,\\
\gamma^{(4)}(0) &= -\sigma(A_{\sigma(t,\, t)}\, t, t) + D_t((\overline{\nabla}_T \sigma)(T, T))\,.
\end{aligned}$$

Since the curvatures of γ are constant, $\langle \gamma''(0), \gamma'''(0)\rangle = 0$ and $\langle \gamma'''(0), \gamma^{(4)}(0)\rangle = 0$. Since $\gamma^{(4)}(0)$ is a normal vector to M, $\gamma''(0) \wedge \gamma^{(4)}(0) = 0$. Thus, γ is of rank 3. Since M is compact, this is impossible. Therefore, we have

$$(\overline{\nabla}_t \sigma)(t, t) = 0\,.$$

Since the curvatures are constant, $(\overline{\nabla}_T \sigma)(T, T) = 0$ along γ. So, γ is of rank 2. Thus every geodesic through p is of rank 2. Moreover, every geodesic through p is a circle of radius $1/\kappa_1$ and centered at $p - (1/\kappa_1)\,\xi$ and so M is a standard sphere $S^2(1/\kappa_1)$. (Q.E.D.)

Suppose $\dim(\operatorname{Im}\sigma)_p = 2$. Then there is an orthonormal basis $\{e_1, e_2\}$ of T_pM such that $\sigma(e_1, e_2) \neq 0$.

LEMMA 2.4. – Let M be a surface in E^m $(m \geqslant 4)$ such that M is isotropic at p, where $\dim(\operatorname{Im}\sigma)_p = 2$. Then $\|\sigma(e_1, e_2)\|$ does not depend on the choice of the orthonormal basis $\{e_1, e_2\}$ of T_pM.

PROOF. – Let $\{X, Y\}$ be an orthonormal basis of T_pM. Then there exists θ $(0 \leqslant \theta < 2\pi)$ such that

$$\begin{aligned}
X &= \cos\theta\, e_1 - \sin\theta\, e_2\,,\\
Y &= \sin\theta\, e_1 + \cos\theta\, e_2\,,
\end{aligned}$$

for the orthonormal basis $\{e_1, e_2\}$ of T_pM.

Since M is isotropic at p, $\sigma(e_1, e_2) \perp \sigma(e_1, e_2)$ and $\sigma(e_2, e_2) \perp \sigma(e_1, e_2)$. So, $\sigma(e_1, e_1) \wedge \sigma(e_2, e_2) = 0$ because $\dim(\operatorname{Im}\sigma)_p = 2$. Since $\|\sigma(e_1, e_1)\| = \|\sigma(e_2, e_2)\|$, $\sigma(e_1, e_1) = \pm\sigma(e_2, e_2)$. If we observe (1.7), then we obtain

$$\sigma(e_1, e_1) + \sigma(e_2, e_2) = 0, \tag{2.4}$$

that is, the mean curvature vector H vanishes at p. Therefore, we get

$$\sigma(X, Y) = \cos 2\theta\,\sigma(e_1, e_2) + \sin 2\theta\,\sigma(e_1, e_1),$$

If we compute the length of $\sigma(X, Y)$ and make use of (1.7), then we see that

$$\|\sigma(X, Y)\| = \|\sigma(e_1, e_2)\|. \quad \text{(Q.E.D.)}$$

In this case, we are also going to prove that every geodesic through p is of rank 2. Suppose $(\overline{\nabla}_t\sigma)(t, t) \neq 0$ for $t \in U_pM$.

Let $\{e_1, e_2\}$ be an orthonormal basis for T_pM. Consider a geodesic γ_1 such that $\gamma_1(0) = p$, $\gamma_1'(0) = (e_1 + e_2)/\sqrt{2}$. Since γ_1 has its constant first Frenet curvature, we have

$$\langle(\overline{\nabla}_{T_1}\sigma)(T_1, T_1), \sigma(T_1, T_1)\rangle = 0,$$

where $T_1 = d\gamma_1/ds$. Since the mean curvature vector is zero at p, we see that

$$\begin{aligned}\langle(\overline{\nabla}_{e_1}\sigma)(e_1, e_1), \sigma(e_1, e_1)\rangle &+ 3\langle(\overline{\nabla}_{e_1}\sigma)(e_1, e_2), \sigma(e_1, e_2)\rangle \\ &+ 3\langle(\overline{\nabla}_{e_1}\sigma)(e_2, e_2), \sigma(e_1, e_2)\rangle + \langle(\overline{\nabla}_{e_2}\sigma)(e_2, e_2), \sigma(e_1, e_2)\rangle = 0.\end{aligned} \tag{2.5}$$

Consider another geodesic γ_2 such that $\gamma_2(0) = p$, $\gamma_2'(0) = (e_1 - e_2)/\sqrt{2}$. Then we get

$$\langle(\overline{\nabla}_{T_2}\sigma)(T_2, T_2), \sigma(T_2, T_2)\rangle = 0,$$

where $\gamma_2'(s) = T_2$. This implies

$$\begin{aligned}-\langle(\overline{\nabla}_{e_1}\sigma)(e_1, e_1), \sigma(e_1, e_1)\rangle &+ 3\langle(\overline{\nabla}_{e_1}\sigma)(e_1, e_2), \sigma(e_1, e_2)\rangle \\ &- 3\langle(\overline{\nabla}_{e_1}\sigma)(e_2, e_2), \sigma(e_1, e_2)\rangle + \langle(\overline{\nabla}_{e_2}\sigma)(e_2, e_2), \sigma(e_1, e_2)\rangle = 0.\end{aligned} \tag{2.6}$$

Putting, (2.5) and (2.6) together, we obtain

$$3\langle(\overline{\nabla}_{e_1}\sigma)(e_1, e_2), \sigma(e_1, e_2)\rangle + \langle(\overline{\nabla}_{e_2}\sigma)(e_2, e_2), \sigma(e_1, e_2)\rangle = 0. \tag{2.7}$$

On the other hand, since geodesic through p have the same constant curvatures, $\langle(\overline{\nabla}\sigma)(X, X, X), (\overline{\nabla}\sigma)(X, X, X)\rangle$ is independent of the choice of the unit vector X. By Lemma 1.2, we have

$$\langle(\overline{\nabla}\sigma)(X, X, X), (\overline{\nabla}\sigma)(X, X, X^{\perp})\rangle = 0$$

for every unit vector X tangent to M at p. So, $(\overline{\nabla}\sigma)(e_1, e_1, e_1) \perp (\overline{\nabla}\sigma)(e_1, e_1, e_2)$. Since

$(\overline{\nabla}\sigma)(e_1, e_1, e_1) \perp \sigma(e_1, e_1)$ we get

$$(\overline{\nabla}\sigma)(e_1, e_1, e_2) \perp \sigma(e_1, e_2).$$

Therefore, (2.7) implies that

$$\langle(\overline{\nabla}_{e_2}\sigma)(e_2, e_2), \sigma(e_1, e_2)\rangle = 0.$$

Since $(\overline{\nabla}_{e_2}\sigma)(e_2, e_2) \wedge \sigma(e_1, e_2) = 0$, we obtain

$$(\overline{\nabla}_{e_2}\sigma)(e_2, e_2) = 0.$$

But, this contradicts $(\overline{\nabla}_t\sigma)(t, t) \neq 0$ for $t \in U_pM$. Thus, it follows that $(\overline{\nabla}_t\sigma)(t, t) = 0$ for every $t \in U_pM$, i.e., every geodesic through p is of rank 2. By using the fundamental theorem of curves, we can write the immersion x: $M \to E^4$ with respect to the geodesic polar coordinate system (s, θ) as

$$x(s, \theta) = C(\theta) + \frac{1}{\kappa}(\cos\kappa s)\,\mathbf{f}_1(\theta) + \frac{1}{\kappa}(\sin\kappa s)\,\boldsymbol{f}_2(\theta), \tag{2.8}$$

where $C(\theta)$ is a vector function depending upon θ, $\boldsymbol{f}_1(\theta)$ and $\boldsymbol{f}_2(\theta)$ are orthonormal vectors in E^4 at p depending on θ and κ is the Frenet curvature of each geodesic through p.

Without loss of generality we may assume the point p is the origin o of E^4. Then

$$o = x(0, \theta) = C(\theta) + \frac{1}{\kappa}\boldsymbol{f}_1(\theta) \quad \text{for all } \theta. \tag{2.9}$$

Let e_1 and e_2 be orthonormal vectors tangent to M at o which generate the geodesic polar coordinates (s, θ).

Since $x_*(\partial/\partial s)(0, \theta) = \boldsymbol{f}_2(\theta) \in T_oM$,

$$\boldsymbol{f}_2(\theta) = \cos\theta\, e_1 + \sin\theta\, e_2. \tag{2.10}$$

Since $(\tilde{\nabla}_{x_*(\partial/\partial s)}x_*(\partial/\partial s))(0, \theta) = (\partial^2 x/\partial s^2)(0, \theta) = \sigma(\boldsymbol{f}_2(\theta), \boldsymbol{f}_2(\theta))$,

$$\sigma(\boldsymbol{f}_2(\theta), \boldsymbol{f}_2(\theta)) = -\frac{1}{\kappa}\boldsymbol{f}_1(\theta), \tag{2.11}$$

where $\tilde{\nabla}$ is the Riemannian connection in E^4.

Combining (2.9), (2.10) and (2.11), we obtain

$$\begin{aligned} x(s, \theta) &= \frac{1}{\kappa}(\sin\kappa s)\,\boldsymbol{f}_2(\theta) + \frac{1}{\kappa^2}(1 - \cos\kappa s)\,\sigma(\boldsymbol{f}_2(\theta), \boldsymbol{f}_2(\theta)) \\ &= \frac{1}{\kappa}\sin\kappa s\,\cos\theta e_1 + \frac{1}{\kappa}\sin\kappa s\,\cos\theta e_2 + \\ &\qquad \frac{1}{\kappa^2}(1 - \cos\kappa s)\,\cos 2\theta\sigma(e_1, e_1) + \frac{1}{\kappa^2}(1 - \cos\kappa s)\,\sin 2\theta\sigma(e_1, e_2) \end{aligned}$$

since $\sigma(e_1, e_1) + \sigma(e_2, e_2) = 0$.

Since $\sigma(e_1, e_1) \perp \sigma(e_1, e_2)$, choose e_3 as $\sigma(e_1, e_1)/\|\sigma(e_1, e_1)\| = \sigma(e_1, e_1)/\kappa$ and e_4 as $\sigma(e_1, e_2)/\|\sigma(e_1, e_2)\| = \sigma(e_1, e_2)/\kappa$.

If we use the coordinate system with respect to e_1, e_2, e_3 and e_4, then $x(s, \theta)$ is of the form

$$(2.12) \quad x(s,\theta)=\left(\frac{1}{\kappa}\sin\kappa s\cos\theta,\ \frac{1}{\kappa}\sin\kappa s\sin\theta,\ \frac{1}{\kappa}(1-\cos\kappa s)\cos 2\theta,\ \frac{1}{\kappa}(1-\cos\kappa s)\sin 2\theta\right).$$

We now prove

LEMMA 2.5. – Let M be a compact connected surface in E^4. Suppose that M satisfies $(*_1)$ and that $\dim(\operatorname{Im}\sigma)_p$ is maximal. Then M is a Blaschke surface at p and M is diffeomorphic to a real projective space RP^2 but not isometric to RP^2 with the standard metric.

PROOF. – In order to prove that M is a Blaschke surface at p, it is enough to show that the cut-locus $\operatorname{Cut}(p)$ of the point p is spherical.

We may assume that p is the origin o of E^4. Since each geodesic through o is a circle of radius $1/\kappa$, we have to show that two distinct geodesics through o do not intersect on the open interval $(0, \pi/\kappa)$. Suppose $x(s, \theta) = x(s_o, \theta_o)$ for $0 < s$, $s_o < \pi/\kappa$ and $0 < |\theta - \theta_o| < \pi/2$. By using (2.12), we can easily derive a contradiction. Thus $\operatorname{Cut}(o)$ is spherical. $\operatorname{Cut}(o)$ is indeed the set of all antipodal points of o with respect to each geodesic through o.

According to BOTT-SAMELSON [Bo], [Sa], (or 7.33 Theorem of BESSE [Be.A]), we see that M is diffeomorphic to RP^2. We now prove that M is not isometric to RP^2. It is sufficient to show that the Gaussian curvature $\boldsymbol{K}$ cannot be constant.

Suppose that the Gaussian curvature $\boldsymbol{K}$ is a constant > 0. Then we can easily get $G = 1/\boldsymbol{K} \sin^2(\sqrt{\boldsymbol{K}}s)$, where $G = \langle x_*(\partial/\partial\theta), x_*(\partial/\partial\theta)\rangle$. On the other hand, G can be directly computed from (2.12) as

$$G = \frac{1}{\kappa^2}\{\sin^2\kappa s + 4(1-\cos\kappa s)^2\}.$$

If we compare these two equations, we have a contradiction. So, even though the surface is diffeomorphic to RP^2 it is not a standard real projective space RP^2. (Q.E.D.)

Conversely, if a compact connected surface M is a standard sphere S^2 or has the form of (2.12), then it is easily proved that M satisfies the property $(*_1)$.

Thus by combining Lemma 2.3, Lemma 2.5 and the statement above we can conclude the following.

THEOREM 2.6 (Classification). – Let M be a compact connected surface in E^4. Then M satisfies the property $(*_1)$ if and only if M is a standard sphere which lies in E^3 or a Blaschke surface at a point which lies in E^4 of the form (2.12). In the second case M is diffeomorphic to RP^2.

REMARK. – In the case of the Blaschke surface at the point o in the above theorem, we see that the locus of the centers of geodesics through the point o is a circle with radius κ whose points go around the circumference twice while points on a geodesic circle centered at o on M go around the geodesic circle once. On the other hand, if we compute the torsion of the cut-locus $\mathrm{Cut}(o) = x(\pi, \theta)$ of the point o, then we see that the torsion is zero and $\mathrm{Cut}(o)$ is indeed a circle.

Let us consider a compact connected surface M in E^5. We shall characterize surfaces in E^5 satisfying the property $(*_1)$.

We now suppose that M satisfies the property $(*_1)$. As usual, we may assume the base point p of $(*_1)$ as the origin o of E^5. Then the immersion $x\colon M \to E^5$ can be expressed in terms of the geodesic polar coordinates (s, θ) as

$$x(s, \theta) = r_1(\cos\beta s - 1)\boldsymbol{f}_1(\theta) + r_1 \sin\beta s\, \boldsymbol{f}_2(\theta) + r_2(\cos\delta s - 1)\boldsymbol{f}_3(\theta) + r_2 \sin\delta s\, \boldsymbol{f}_4(\theta), \tag{2.13}$$

where r_1 and r_2 are nonnegative numbers, β and δ are some positive constants and $\boldsymbol{f}_1(\theta)$, $\boldsymbol{f}_2(\theta)$, $\boldsymbol{f}_3(\theta)$, and $\boldsymbol{f}_4(\theta)$ are orthonormal vectors in T_oE^5 depending on θ.

Without loss of generality, we may assume

$$\boldsymbol{f}_1(0) = (1, 0, 0, 0, 0), \quad \boldsymbol{f}_2(0) = (0, 1, 0, 0, 0), \quad \boldsymbol{f}_3(0) = (0, 0, 1, 0, 0), \quad \boldsymbol{f}_4(0) = (0, 0, 0, 1, 0), \tag{2.14}$$

Let $\boldsymbol{f}_5(\theta)$ be a unit vector in T_oE^5 such that $\{\boldsymbol{f}_1(\theta), \boldsymbol{f}_2(\theta), \boldsymbol{f}_3(\theta), \boldsymbol{f}_4(\theta), \boldsymbol{f}_5(\theta)\}$ forms an orthonormal basis for T_oE^5. Automatically we may set

$$\boldsymbol{f}_5(0) = (0, 0, 0, 0, 1). \tag{2.15}$$

Differentiating (2.13) with respect to s for a fixed θ, we obtain

$$x_*(\partial/\partial s) = -r_1\beta \sin\beta s\, \boldsymbol{f}_1(\theta) + r_1\beta \cos\beta s\, \boldsymbol{f}_2(\theta) - r_2\delta \sin\delta s\, \boldsymbol{f}_3(\theta) + r_2\delta \cos\delta s\, \boldsymbol{f}_4(\theta). \tag{2.16}$$

Set

$$e(\theta) = x_*(\partial/\partial s)(0, \theta) = r_1\beta \boldsymbol{f}_2(\theta) + r_2\delta \boldsymbol{f}_4(\theta), \tag{2.17}$$

which implies

$$(r_1\beta)^2 + (r_2\delta)^2 = 1. \tag{2.18}$$

Let

$$e_1 = e(0) = r_1\beta \boldsymbol{f}_2(0) + r_2\delta \boldsymbol{f}_4(0) = (0, r_1\beta, 0, r_2\delta, 0)$$

and

$$e_2 = e\left(\frac{\pi}{2}\right) = r_1\beta \boldsymbol{f}_2\left(\frac{\pi}{2}\right) + r_2\delta \boldsymbol{f}_4\left(\frac{\pi}{2}\right).$$

Then $e(\theta)$ can be expressed as

$$e(\theta) = \cos\theta\, e_1 + \sin\theta\, e_2 . \tag{2.19}$$

For a fixed θ, $x(s, \theta)$ is a geodesic and thus $\langle \partial^2 x/\partial s^2, \partial x/\partial\theta\rangle = 0$, which gives

$$\begin{aligned}(2.20)\quad & r_1^2\beta\langle f_1'(\theta), f_2(\theta)\rangle + r_2^2\delta\langle f_3'(\theta), f_4(\theta)\rangle + \\ & +(-r_1^2\beta\langle f_1'(\theta), f_2(\theta)\rangle + r_1r_2\beta\langle f_2'(\theta), f_3(\theta)\rangle)\cos\beta s \\ & -(-r_2^2\delta\langle f_3'(\theta), f_4(\theta)\rangle + r_1r_2\delta\langle f_1'(\theta), f_4(\theta)\rangle)\cos\delta s \\ & -r_1r_2\beta\langle f_1'(\theta), f_3(\theta)\rangle\sin\beta s + r_1r_2\delta\langle f_1'(\theta), f_3(\theta)\rangle\sin\delta s \\ & -\frac{r_1r_2}{2}(\beta-\delta)(\langle f_1'(\theta), f_4(\theta)\rangle + \langle f_2'(\theta), f_3(\theta)\rangle)\cos(\beta+\delta)s \\ & +\frac{r_1r_2}{2}(\beta+\delta)(\langle f_1'(\theta), f_4(\theta)\rangle - \langle f_2'(\theta), f_3(\theta)\rangle)\cos(\beta-\delta)s \\ & +\frac{r_1r_2}{2}(\beta-\delta)(\langle f_1'(\theta), f_3(\theta)\rangle - \langle f_2'(\theta), f_4(\theta)\rangle)\sin(\beta+\delta)s \\ & +\frac{r_1r_2}{2}(\beta+\delta)(\langle f_1'(\theta), f_3(\theta)\rangle + \langle f_2'(\theta), f_4(\theta)\rangle)\sin(\beta-\delta)s = 0 .\end{aligned}$$

LEMMA 2.7. – $\beta \neq \delta$ provided the geodesics through o are of rank 4.

PROOF. – Suppose $\beta = \delta$. Let κ_1, κ_2 and κ_3 be the first, second and third Frenet curvatures of the geodesic $x(s, \theta)$ for a fixed θ respectively. Then

$$\kappa_1^2 = \left\langle \frac{\partial^2 x}{\partial s^2}(0, \theta),\ \frac{\partial^2 x}{\partial s^2}(0, \theta)\right\rangle = (r_1^2 + r_2^2)\beta^4 .$$

Since $\langle(\partial x/\partial s)(0, \theta), (\partial x/\partial s)(0, \theta)\rangle = 1$, we obtain

$$(r_1^2 + r_2^2)\beta^2 = 1 ,$$

that is, $\beta^2 = 1/(r_1^2 + r_2^2)$. Therefore, $\kappa_1 = \beta$.

On the other hand, the curvatures κ_i's and the frequencies β and δ have the following relations:

$$\kappa_1^2 + \kappa_2^2 + \kappa_3^2 = \beta^2 + \delta^2 = 2\beta^2 ,$$

$$\kappa_1^2\kappa_3^2 = \beta^2\delta^2 = \beta^4 .$$

Since $\kappa_1 = \beta$, $\kappa_3^2 = \beta^2$. The first equation gives $\kappa_2 = 0$. This contradicts the fact that $x(s, \theta)$ is of rank 4. Thus, we have $\beta \neq \delta$. (Q.E.D.)

LEMMA 2.8. – For very θ, the geodesic $x(s, \theta)$ is periodic.

PROOF. – If $x(s, \theta)$ is of rank 2, then this is obvious. We now assume that $x(s, \theta)$ is of rank 4. Suppose $x(s, \theta)$ is not periodic. Then β and δ are independent over the rational numbers, that is $\overline{x(s, \theta)} = S^1(r_1) \times S^2(r_2)$, a torus denoted by T, where $\overline{x(s, \theta)}$ is the closure of $x(s, \theta)$ in E^5. Certainly, T is contained in $x(M)$. But T does not satisfies the property $(*_1)$. Thus $x(s, \theta)$ must be periodic for every θ. (Q.E.D.)

We now suppose that $r_1 \neq 0$ and $r_2 \neq 0$, that is, every geodesic through o is of rank 4. Combining (2.20), Lemma 2.7 and Lemma 2.8, we obtain

$$\text{(2.21)} \qquad \langle \boldsymbol{f}_1'(\theta), \boldsymbol{f}_2(\theta)\rangle = \langle \boldsymbol{f}_1'(\theta), \boldsymbol{f}_3(\theta)\rangle = \langle \boldsymbol{f}_1'(\theta), \boldsymbol{f}_4(\theta)\rangle =$$

$$= \langle \boldsymbol{f}_2'(\theta), \boldsymbol{f}_3(\theta)\rangle = \langle \boldsymbol{f}_2'(\theta), \boldsymbol{f}_4(\theta)\rangle = \langle \boldsymbol{f}_3'(\theta), \boldsymbol{f}_4(\theta)\rangle = 0$$

for all θ. So we have the following system of differential equations:

$$\text{(2.22)} \qquad \boldsymbol{f}_i'(\theta) = \lambda_i(\theta)\boldsymbol{f}_5(\theta), \qquad \boldsymbol{f}_5'(\theta) = -\sum_{i=1}^{4} \lambda_i(\theta)\boldsymbol{f}_i(\theta)$$

for $i = 1, 2, 3$ and 4, in other words,

$$\begin{pmatrix} \boldsymbol{f}_1'(\theta) \\ \boldsymbol{f}_2'(\theta) \\ \boldsymbol{f}_3'(\theta) \\ \boldsymbol{f}_4'(\theta) \\ \boldsymbol{f}_5'(\theta) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & \lambda_1(\theta) \\ 0 & 0 & 0 & 0 & \lambda_2(\theta) \\ 0 & 0 & 0 & 0 & \lambda_3(\theta) \\ 0 & 0 & 0 & 0 & \lambda_4(\theta) \\ -\lambda_1(\theta) & -\lambda_2(\theta) & -\lambda_3(\theta) & -\lambda_4(\theta) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{f}_1(\theta) \\ \boldsymbol{f}_2'(\theta) \\ \boldsymbol{f}_3'(\theta) \\ \boldsymbol{f}_4'(\theta) \\ \boldsymbol{f}_5'(\theta) \end{pmatrix},$$

where the λ_i's are periodic functions with period 2π.

Differentiating (2.17) with respect to θ and making use of (2.19) and (2.22), we get

$$\text{(2.23)} \qquad (r_1\beta\lambda_2(\theta) + r_2\delta\lambda_4(\theta))\boldsymbol{f}_5(\theta) = -\sin\theta\, e_1 + \cos\theta\, e_2\,,$$

from which, we obtain that $r_1\beta\lambda_2(\theta) + r_2\delta\lambda_4(\theta) = \pm 1$ and we may assume that $r_1\beta\lambda_2(\theta) + r_2\delta\lambda_4(\theta) = 1$. If we differentiate (2.23) twice and use (2.22), then we obtain

$$\boldsymbol{f}_5''(\theta) + \boldsymbol{f}_5(\theta) = 0\,.$$

Since $\boldsymbol{f}_5(0) = (0, 0, 0, 0, 1)$ and $\boldsymbol{f}_5(\pi/2) = -e_1 = (0, -r_1\beta, 0, -r_2\delta, 0)$, we have

$$\text{(2.24)} \qquad \boldsymbol{f}_5(\theta) = (0, -r_1\beta\sin\theta, 0, -r_2\delta\sin\theta, \cos\theta)\,.$$

Since $\boldsymbol{f}_i'(\theta) = \lambda_i(\theta)\boldsymbol{f}_5(\theta)$ $(1 \leq i \leq 4)$, $\boldsymbol{f}_1(0) = (1, 0, 0, 0, 0)$, $\boldsymbol{f}_2(0) = (0, 1, 0, 0, 0)$, $\boldsymbol{f}_3(0) = (0, 0, 1, 0, 0)$ and $\boldsymbol{f}_4(0) = (0, 0, 0, 1, 0)$, we get

$$\text{(2.25)} \qquad \boldsymbol{f}_1(\theta) = \left(1, -r_1\beta\int_0^\theta \lambda_1(t)\sin t\,dt, 0, -r_2\delta\int_0^\theta \lambda_1(t)\sin t\,dt, \int_0^\theta \lambda_1(t)\cos t\,dt\right),$$

$$f_2(\theta) = \left(0,\ -r_1\beta\int_0^\theta \lambda_2(t)\sin t\,dt + 1,\ 0,\ -r_2\delta\int_0^\theta \lambda_2(t)\sin t\,dt,\ \int_0^\theta \lambda_2(t)\cos t\,dt\right), \tag{2.26}$$

$$f_3(\theta) = \left(0,\ -r_1\beta\int_0^\theta \lambda_3(t)\sin t\,dt,\ 1,\ -r_2\delta\int_0^\theta \lambda_3(t)\sin t\,dt,\ \int_0^\theta \lambda_3(t)\cos t\,dt\right), \tag{2.27}$$

$$f_4(\theta) = \left(0,\ -r_1\beta\int_0^\theta \lambda_4(t)\sin t\,dt,\ 0,\ -r_2\delta\int_0^\theta \lambda_4(t)\sin t\,dt + 1,\ \int_0^\theta \lambda_4(t)\cos t\,dt\right). \tag{2.28}$$

Now let us compute $\lambda_1(\theta)$. Since $\langle f_1(\theta), f_5(\theta)\rangle = 0$ for all θ, (2.24) and (2.25) imply

$$(r_1\beta)^2 \sin\theta\int_0^\theta \lambda_1(t)\sin t\,dt + (r_2\delta)^2 \sin\theta\int_0^\theta \lambda_1(t)\sin t\,dt + \cos\theta\int_0^\theta \lambda_1(t)\cos t\,dt = 0.$$

It follows that

$$\sin\theta\int_0^\theta \lambda_1(t)\sin t\,dt + \cos\theta\int_0^\theta \lambda_1(t)\cos t\,dt = 0$$

because $(r_1\beta)^2 + (r_2\delta)^2 = 1$. By differentiating this, we obtain

$$\lambda_1(\theta) = -\cos\theta\int_0^\theta \lambda_1(t)\sin t\,dt + \sin\theta\int_0^\theta \lambda_1(t)\cos t\,dt,$$

which gives $\lambda_1'(\theta) = 0$ for all θ, that is λ_1 is a constant. In fact, $\lambda_1(\theta) = 0$ for all θ. Thus, $f_1(\theta)$ is completely determined

$$f_1(\theta) = (1, 0, 0, 0, 0).$$

Similarly, we can compute λ_2, λ_3 and λ_4:

$$\lambda_2(\theta) = r_1\beta, \quad \lambda_3(\theta) = 0, \quad \lambda_4(\theta) = r_2\delta \qquad \text{for all } \theta.$$

Consequently (2.25) ~ (2.28) are precisely determined as follows:

$$f_1(\theta) = (1, 0, 0, 0, 0),$$

$$f_2(\theta) = (0,\ 1-(r_1\beta)^2(1-\cos\theta),\ 0,\ -r_1r_2\beta\delta(1-\cos\theta),\ r_1\beta\sin\theta),$$

$$f_3(\theta) = (0, 0, 1, 0, 0),$$

$$f_4(\theta) = (1,\ -r_1r_2\beta\delta(1-\cos\theta),\ 0,\ 1-(r_2\delta)^2(1-\cos\theta),\ r_2\delta\sin\theta).$$

These, together with (2.24), show that the immersion x has the representation

$$\text{(2.29)}\quad \begin{cases} x(s,\theta)=(r_1(\cos\beta s-1), r_1\sin\beta s-r_1\beta(1-\cos\theta)(r_1^2\beta\sin\beta s+r_2^2\delta\sin\delta s), \\ r_2(\cos\delta s-1), r_2\sin\delta s-r_2\delta(1-\cos\theta)(r_2^2\beta\sin\beta s+r_2^2\delta\sin\delta s), \\ (r_1^2\beta\sin\beta s+r_2^2\delta\sin\delta s)\sin\theta). \end{cases}$$

In this case, each geodesic through o is periodic with period $L=2\pi p/\beta=2\pi q/\delta$ for some integers p and q. Using a similar argument to that in Lemma 2.5, we see that the cut-locus, Cut(o), of the point o is spherical and thus the surface M is a Blaschke manifold at o which is diffeomorphic to RP^2. Thus, we have

PROPOSITION 2.9. – Let M be a compact connected surface in E^5 with the property $(*_1)$ relative to the origin o. If every geodesic through the point o is of rank 4, then M is a Blaschke manifold at o which is diffeomorphic to RP^2 and has the form (2.29).

We now suppose that $x(s,\theta)$ is of rank 2 for every θ. Then the immersion x can be written with respect to the geodesic polar coordinate (s,θ) as

$$\text{(2.30)}\qquad x(s,\theta)=\frac{1}{\kappa}(\cos\kappa s-1)\boldsymbol{f}_1(\theta)+\frac{1}{\kappa}\sin\kappa s\,\boldsymbol{f}_2(\theta),$$

where κ is the Frenet curvature of the planar geodesic $x(s,\theta)$ for every θ and $\boldsymbol{f}_1(\theta)$ and $\boldsymbol{f}_2(\theta)$ are orthonormal vectors in E^5 at the point o. From (1.33) we obtain

$$\text{(2.31)}\qquad x_*(\partial/\partial s)(0,\theta)=\boldsymbol{f}_2(\theta)\in T_oM$$

and

$$\text{(2.32)}\qquad (\tilde{\nabla}_{x_*(\partial/\partial s)}x_*(\partial/\partial s))(0,\theta)=\frac{\partial^2 x}{\partial s^2}(0,\theta)=\sigma(\boldsymbol{f}_2(\theta),\boldsymbol{f}_2(\theta))=-\kappa\boldsymbol{f}_1(\theta),$$

where $\tilde{\nabla}$ is the Riemannian connection in E^5.

Let $\{e_1, e_2\}$ be an orthonormal basis of T_oM such that

$$\text{(2.33)}\qquad \boldsymbol{f}_2(\theta)=\cos\theta\, e_1+\sin\theta\, e_2.$$

Suppose that $\dim(\operatorname{Im}\sigma)_o=1$, that is, the point o is an umbilical point of M. In this case, every geodesic through o is a circle of radius $1/\kappa$ and centered at $-(1/\kappa)H$. Thus, M is a standard sphere $S^2(1/\kappa)$ which lies in E^3.

Suppose that $\dim(\operatorname{Im}\sigma)_o=2$. In this case, exactly the same proof used to derive (2.12) is applied and thus the immersion x is of the form

$$\text{(2.34)}\quad x(s,\theta)=\frac{1}{\kappa}(\sin\kappa s\cos\theta,\ \sin\kappa s\sin\theta,\ (1-\cos\kappa s)\cos 2\theta,\ (1-\cos\kappa s)\sin 2\theta,\ 0)$$

for a suitable choice of Euclidean coordinates in E^5. Clearly, M lies in E^4.

We now assume that $\dim(\operatorname{Im}\sigma)_o=3$, that is, the dimension of the first normal space at the point o is maximal. Then

$$\sigma(e_1, e_1) \wedge \sigma(e_1, e_2) \wedge \sigma(e_2, e_2) \neq 0.$$

Let

$$e_3 = \frac{\sigma(e_1, e_1)}{\kappa}, \qquad e_4 = \frac{\sigma(e_1, e_2)}{a} \quad \text{and} \quad e_5 = \frac{\tilde{e}_5}{\|\tilde{e}_5\|}, \tag{2.35}$$

where

$$a = \|\sigma(e_1, e_2)\| \quad \text{and} \quad \tilde{e}_5 = \sigma(e_2, e_2) - \frac{g(\sigma(e_1, e_1), \sigma(e_2, e_2))}{\kappa^2}\sigma(e_1, e_1).$$

Set $b = \|\tilde{e}_5\|$. Then we have from Lemma 1.2 that

$$b^2 + \frac{(\kappa^2 - 2a^2)^2}{\kappa^2} = \kappa^2. \tag{2.36}$$

Using (2.31), (2.32), (2.35) and (2.36), we can write the immersion x in the form

$$x(s, \theta) = \left(\frac{1}{\kappa}\sin\kappa s\cos\theta,\ \frac{1}{\kappa}\sin\kappa s\sin\theta,\ \frac{1}{\kappa^2}(1-\cos\kappa s)\left(\kappa - \frac{2a^2}{\kappa}\sin^2\theta\right),\right.$$
$$\left.\frac{a}{\kappa^2}(1-\cos\kappa s)\sin 2\theta,\ \frac{b}{\kappa^2}(1-\cos\kappa s)\sin^2\theta\right) \tag{2.37}$$

for a suitable choice of the coordinates with respect to e_1, e_2, e_3, e_4 and e_5 described as above.

Considering the cut-locus $\mathrm{Cut}(o)$ of the point o in both the cases that $\dim(\mathrm{Im}\,\sigma)_o = 2$ and $\dim(\mathrm{Im}\,\sigma)_o = 3$, we see that M is a Blaschke surface at o.

Consequently, if the immersion x has the form (2.29), (2.34) and (2.37) or x is a standard imbedding of $S^2(1/\kappa)$ into E^3, then it is easily checked that the surface M satisfies the property $(*_1)$.

Thus we can classify surfaces in E^5 satisfying the property $(*_1)$.

THEOREM 2.10 (Classification). – Let M be a compact connected surface in E^5. Then M satisfies the property $(*_1)$ if and only if M is a standard sphere in E^3 or a Blaschke surface at a point of the form (2.34) which lies in E^4 or a Blaschke surface at a point of the form (2.37) which lies in E^5 or a Blaschke surface at a point of the form (2.29) which lies in E^5. All such Blaschke surfaces are diffeomorphic to RP^2.

REMARK. – Let M be a compact connected surface in E^m $(m \geq 5)$. Since the dimension of the first normal space at a point is at most 3, we can conclude that M satisfies the property $(*_1)$ and the geodesic are planar if and only if M lies in E^5 and M

is one of four spaces stated in Theorem 2.10 except the case of a Blaschke surface of the form (2.29).

On the other hand, the Veronese surface certainly satisfies the property $(*_1)$. So the following question naturally arises. What is the characterization of the Veronese surface in terms of the property $(*_1)$? Since the Veronese surface is fully immersed in E^5, that is, the Veronese surface cannot lie in a hyperplane of E^5, and since every geodesic in the Veronese surface is planar, we must think of the immersion which has the form (2.37).

We are going to use the theory of submanifolds of finite type introduced and mainly developed by B. Y. CHEN [Ch.B-3]. We recall some fundamental definitions and properties.

Let M be a compact orientable Riemannian manifold with Riemannian connection ∇ and Δ the Laplacian operator of M acting on $C^\infty(M)$ where

$$\Delta = -\sum_i(\nabla_{E_i}\nabla_{E_i} - \nabla_{\nabla_{E_i}E_i})$$

for an orthonormal basis $\{E_i\}$ of TM. We define an inner product $(,)$ on $C^\infty(M)$ by

$$(f, g) = \int_M fg\, dV,$$

where dV is the volume element of M. Then Δ is a self-adjoint elliptic operator with respect to $(,)$ and it has an infinite, discrete sequence of eigenvalues:

$$0 = \lambda_o < \lambda_1 < \lambda_2 < \dots < \lambda_k < \dots \uparrow +\infty.$$

Let $V_k = \{f \in C^\infty(M) \mid \Delta f = \lambda_k f\}$ be the eigenspace of Δ with eigenvalue λ_k. Then $\sum_{k=0}^{\infty} V_k$ is dense in $C^\infty(M)$ in the L^2-sense. Denote by $\widehat{\oplus}_k V_k$ the completion of $\sum_{k=0}^{\infty} V_k$. We have

$$C^\infty(M) = \widehat{\oplus}_k V_k.$$

For each function $f \in C^\infty(M)$, let f_t be the projection of f onto the subspace V_t $(t = 0, 1, 2, \dots)$. Then we have the spectral decomposition

$$f = \sum_{t=0}^{\infty} f_t \quad \text{(in tre } L^2\text{-sense)}.$$

Because V_0 is 1-dimensional, for any non-constant function $f \in C^\infty(M)$, there is a positive integer $p \geqslant 1$ such that $f_p \neq 0$ and

$$f - f_0 = \sum_{t \geqslant p} f_t,$$

where $f_0 \in V_0$ is a constant. If there are infinitely many f_t's which are nonzero, we put

$q = +\infty$. Otherwise, there is an integer q, $q \geqslant p$, such that $f_q \neq 0$ and

$$f - f_0 = \sum_{t=p}^{q} f_t .$$

If we allow q to be $+\infty$, we have the decomposition as above for any $f \in C^\infty(M)$.

For an isometric immersion x: $M \to E^m$ of a compact Riemannian manifold M into E^m, we put

$$x = (x_1, x_2, \dots, x_m),$$

where x_A is the A-th Euclidean coordinate function of M in E^m. For each x_A, we have the spectral decomposition

$$x_A - (x_A)_0 = \sum_{t=p_A}^{q_A} (x_A)_t, \qquad A = 1, 3, \dots, m.$$

For each isometric immersion x: $M \to E^m$, we put

$$p = p(x) = \inf_A \{p_A\}, \qquad q = q(x) = \sup_A \{q_A\}.$$

where A ranges among all A such that $x_A - (x_A)_0 \neq 0$. It is easy to see that p is an integer $\geqslant 1$ and q is either an integer $\geqslant p$ or ∞. Moreover, p and q are independent of the choice of the Euclidean coordinate system in E^m. Thus p and q are well-defined. Consequently, for each isometric immersion x: $M \to E^m$ of a compact Riemannian manifold, we have a pair $[p, q]$ associated with M. We call the pair $[p, q]$ the *order of the submanifold M*. If we use the spectral decomposition of the coordinate functions of the immersion x: $M \to E^m$,

$$x = x_0 + \sum_{t=p}^{q} x_t . \tag{2.38}$$

DEFINITION. – A compact submanifold M in E^m is said to be of *finite type* if q is finite. Otherwise M is of *infinite type.*

DEFINITION. – A compact submanifold M is said to be of *k-type* ($k = 1, 2, 3, \dots$) if there are exactly k nonzero x_t's ($t \geqslant 1$) in the decomposition (2.38).

We can restate *Takahashi's Theorem* in terms of 1-type:

LEMMA 2.11 (TAKAHASHI [Tk] and CHEN [Ch.B-3]). – Let x: $M \to E^m$ be an isometric immersion of a compact Riemannian manifold M into E^m. Then x is of 1-type if and only if M is a minimal submanifold of a hypersphere of E^m.

B.-Y. CHEN gave the following characterization of submanifolds of finite type.

LEMMA 2.12 (CHEN [Ch.B-2]). – Let x: $M \to E^m$ be an isometric immersion of a compact Riemannian manifold M into E^m. Then

(i) M is of finite-type if and only if there is a non-trivial polynomial Q such that

$$Q(\Delta)H = 0.$$

(ii) If M is of finite type, then there is a unique monic polynomial P of least degree such

$$P(\Delta)H = 0.$$

(iii) If M is of finite type, then M is of k-type if and only if $\deg P = k$.

Now, coming back to the problem. We shall compute the Gaussian curvature $\boldsymbol{K}$ and find a condition which gives constant Gaussian curvature. Furthermore, we shall characterize the Veronese surface by examining the surface with constant Gaussian curvature.

From (2.37) we get

$$x_*(\partial/\partial s)(s,\theta)$$

$$= \left(\cos\kappa s\cos\theta,\ \cos\kappa s\sin\theta,\ \frac{1}{\kappa}\sin\kappa s\left(\kappa - \frac{2a^2}{\kappa}\sin^2\theta\right),\ \frac{a}{\kappa}\sin\kappa s\sin 2\theta,\ \frac{b}{\kappa}\sin\kappa s\sin^2\theta\right)$$

and

$$x_*(\partial/\partial\theta)(s,\theta) = \left(-\frac{1}{\kappa}\sin\kappa\sin\theta,\ \frac{1}{\kappa}\sin\kappa s\cos\theta,\ \frac{-2a^2}{\kappa^3}(1-\cos\kappa s)\sin 2\theta,\right.$$

$$\left.\frac{2a}{\kappa^2}(1-\cos\kappa s)\cos 2\theta,\ \frac{b}{\kappa^2}(1-\cos\kappa s)\sin 2\theta\right).$$

Then the induced first fundamental form g_{ij} is derived as

$$g_{11} = \langle x_*(\partial/\partial s), x_*(\partial/\partial s)\rangle = 1, \qquad g_{12} = g_{21} = \langle x_*(\partial/\partial s), x_*(\partial/\partial\theta)\rangle = 0,$$

$$g_{22} = \langle x_*(\partial/\partial\theta), x_*(\partial/\partial\theta)\rangle = \frac{1}{\kappa^2}\sin^2\kappa s + \frac{4a^2}{\kappa^4}(1-\cos\kappa s)^2.$$

Thus the line element $d\rho^2$ of M in E^5 has the form

$$d\rho^2 = ds^2 + \left\{\frac{1}{\kappa^2}\sin^2\kappa s + \frac{4a^2}{\kappa^4}(1-\cos\kappa s)^2\right\}d\theta^2.$$

So the Gaussian curvature $\boldsymbol{K}$ is given by

$$\boldsymbol{K} = -\frac{1}{\sqrt{G}}\frac{\partial^2\sqrt{G}}{\partial s^2}, \tag{2.39}$$

where

$$G = g_{22} = \frac{1}{\kappa^2}\sin^2\kappa s + \frac{4a^2}{\kappa^4}(1-\cos\kappa s)^2 .$$

Suppose that the Gaussian curvature $\boldsymbol{K}$ is a constant. Then (2.39) is equivalent to

$$2G\frac{\partial^2\sqrt{G}}{\partial s^2} - \left(\frac{\partial G}{\partial s}\right)^2 + 4\boldsymbol{K}G^2 = 0 .$$

By a straightforward and long computation, we have

$$\begin{aligned}
&-\frac{96a^4}{\kappa^6} - \frac{3}{2\kappa^2}\left(\frac{4a^2}{\kappa^2}-1\right)^2 + 4\boldsymbol{K}\left\{\left(\frac{1}{2\kappa^2}+\frac{6a^2}{\kappa^4}\right)^2 + \frac{32a^4}{\kappa^8} - \frac{1}{8\kappa^4}\left(\frac{4a^2}{\kappa^2}-1\right)^2\right\}\\
&\quad+\left\{\frac{16a^2}{\kappa^2}\left(\frac{1}{2\kappa^2}+\frac{6a^2}{\kappa^4}\right)\left(1-\frac{4\boldsymbol{K}}{\kappa^2}\right) + \frac{4a^2}{\kappa^4}\left(\frac{4a^2}{\kappa^2}-1\right)\left(7-\frac{4\boldsymbol{K}}{\kappa^2}\right)\right\}\cos\kappa s\\
&\quad+\left\{4\left(\frac{4a^2}{\kappa^2}-1\right)\left(\frac{1}{2\kappa^2}+\frac{6a^2}{\kappa^4}\right)\left(\frac{\boldsymbol{K}}{\kappa^2}-1\right) - \frac{32a^2}{\kappa^6} + \frac{128a^2}{\kappa^8}\boldsymbol{K}\right\}\cos 2\kappa s\\
&\quad+\frac{4a^2}{\kappa^4}\left(\frac{4a^2}{\kappa^2}-1\right)\left(3-\frac{4\boldsymbol{K}}{\kappa^2}\right)\cos 3\kappa s + \frac{1}{2\kappa^2}\left(\frac{4a^2}{\kappa^2}-1\right)^2\left(\frac{\boldsymbol{K}}{\kappa^2}-1\right)\cos 4\kappa s = 0 .
\end{aligned}$$

Since 1, $\cos\kappa s$, $\cos 2\kappa s$, $\cos 3\kappa s$ and $\cos 4\kappa s$ are linearly independent, we get

$$\boldsymbol{K} = \frac{\kappa^2}{4} .$$

Thus we have

PROPOSITION 2.13. – Let M be a compact connexted surface in E^5 satisfying $(*_1)$ whose immersion is given by (2.37). Then the Gaussian curvature $\boldsymbol{K}$ is constant if and only if $\kappa = 4a^2$. In this case, the Gaussian curvature $\boldsymbol{K} = \kappa^2/4 = a^2$.

In such a case, the induced metric (g_{ij}) looks like

$$(g_{ij}) = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{1}{\kappa^2}\sin^2\kappa s + \dfrac{1}{\kappa^2}(1-\cos\kappa s)^2 \end{bmatrix} . \tag{2.40}$$

Using this induced metric, we can compute the Christoffel symbols Γ^h_{ij}:

$$\Gamma^1_{11} = 0, \quad \Gamma^2_{11} = 0, \quad \Gamma^1_{22} = -\frac{1}{2}\frac{\partial}{\partial s}(\log G) = -\frac{1}{\kappa}\sin\kappa s,$$

$$\Gamma^2_{22} = 0, \quad \Gamma^1_{12} = 0, \quad \Gamma^2_{12} = -\frac{1}{2}\frac{\partial}{\partial s}(\log G) = -\frac{1}{\kappa}\sin\kappa s,$$

where $G = (1/\kappa^2)\sin^2\kappa s + (1/\kappa^2)(1-\cos\kappa s)^2$.

LEMMA 2.14. – Let M be a compact connected surface in E^5 satisfying the property $(*_1)$ whose immersion has the form (2.37). If the Gaussian curvature is constant, then the Laplacian operator Δ is given by

$$\Delta = -\left(\frac{\partial^2}{\partial s^2} + \frac{1}{G}\frac{\partial^2}{\partial \theta^2}\right) - \frac{1}{2}\frac{\partial}{\partial s}(\log G)\frac{\partial}{\partial s}, \tag{2.41}$$

where $G = (1/\kappa^2)\sin^2\kappa s + (1/\kappa^2)(1-\cos\kappa s)^2$.

It is well-known that

$$\Delta x = -2H,$$

where H is the mean curvature vector field of M. Using this equation and computing H where ΔH by means of (2.41), we obtain the following

$$\Delta H - \frac{3}{2}\kappa^2 H = 0.$$

According to Lemma 2.12, M is of 1-type and hence M is a minimal submanifold of a hypersphere of E^5.

On the other hand, we can easily check that the Gaussian curvature $\boldsymbol{K}$ cannot be constant if the immersion has the forms (2.29) or (2.34) by computing $\boldsymbol{K} = -(1/\sqrt{G})((\partial^2\sqrt{G})/\partial s^2)$.

So if we apply Calabi's Theorem [C], we conclude

THEOREM 2.15 (Characterization of a Veronese surface). – Let M be a compact connected surface in E^5. Then M is a Veronese surface if and only if M has a constant Gaussian curvature and there is a point p which is not umbilical such that every geodesic through p is a helix of the same curvature.

In this case, x is the first standard imbedding RP^2 into E^5 and the second standard immersion of 2-sphere into S^4.

3. – Characterization of surfaces of revolution in a 3-dimensional Euclidean space.

Let M be a surface in E^3. We now define $(*_2)$.

($*_2$) There is a point p in M such that every geodesic through p is a normal section of M at p.

Suppose M has the property ($*_2$). Let γ be a geodesic parametrized by the arc length s and let $\gamma(0)=p$. Then we have

$$\gamma'(s)=T,$$

$$\gamma''(s)=\sigma(T,T),$$

$$\gamma'''(s)=-A_{\sigma(T,T)}T+(\bar{\nabla}_T\sigma)(T,T).$$

Since γ is a normal section at p in the direction $t=T(0)$, $A_{\sigma(t,t)}t\wedge t=0$, that is,

$$\langle\sigma(t,t),\sigma(t,t^{\perp})\rangle=0.$$

Since this is true for any orthonormal vectors t and $t^{\perp}$ tangent to M at p, M is isotropic at p and p is indeed an umbilical point since the ambient manifold is E^3. Since γ is a plane curve, $\gamma'(s)\wedge\gamma''(s)\wedge\gamma'''(s)=0$ for all $s\in\operatorname{Dom}\gamma$. So we can obtain

$$T\wedge A_{\sigma(T,T)}T\wedge\sigma(T,T)=0,$$

which implies

$$\langle\sigma(T,T),\sigma(T,T^{\perp})\rangle=0 \tag{3.1}$$

along γ.

Without loss of generality, we may assume p as the origin o of E^3. Since every geodesic through o is planar, we can express the immersion $x\colon M\to E^3$ locally on a neighborhood U of o in terms of geodesic polar coordinates (s,θ) as

$$x(s,\theta)=(h(s,\theta)\cos\theta,\ h(s,\theta)\sin\theta,\ k(s,\theta)) \tag{3.2}$$

for a suitable choice of Euclidean coordinates of E^3 where h and k are differentiable functions satisfying $h(0,\theta)=k(0,\theta)=0$.

Differentiating (3.2) with respect to s and θ, we obtain two orthogonal vector fields tangent to M on U

$$x_*\left(\frac{\partial}{\partial s}\right)=\left(\frac{\partial h}{\partial s}\cos\theta,\ \frac{\partial h}{\partial s}\sin\theta,\ \frac{\partial k}{\partial s}\right), \tag{3.3}$$

$$x_*\left(\frac{\partial}{\partial\theta}\right)=\left(\frac{\partial h}{\partial\theta}\cos\theta-h\sin\theta,\ \frac{\partial h}{\partial\theta}\sin\theta+h\cos\theta,\ \frac{\partial k}{\partial\theta}\right), \tag{3.4}$$

where $x_*(\partial/\partial s)(0,\theta)=(\cos\theta,\sin\theta,0)$.

For a fixed θ, $x(s,\theta)$ is a geodesic and we thus have

$$\left\langle x_*\left(\frac{\partial}{\partial s}\right),\ x_*\left(\frac{\partial}{\partial s}\right)\right\rangle=\left(\frac{\partial h}{\partial s}\right)^2+\left(\frac{\partial k}{\partial s}\right)^2=1.$$

We may put

$$\frac{\partial h}{\partial s} = \cos f(s, \theta), \qquad \frac{\partial k}{\partial s} = \sin f(s, \theta), \tag{3.5}$$

for a smooth function $f(s, \theta)$ defined on U satisfying $\cos f(0, \theta) = 1$ and $\sin f(0, \theta) = 0$.

LEMMA 3.1. – $(\partial/\partial\theta)(\kappa(s, \theta))^2 = 0$ on the neighborhood U, where $\kappa(s, \theta)$ is the Frenet curvature of the geodesic $x(s, \theta)$ for a fixed θ.

PROOF. – Let γ be a geodesic such that $\gamma(s) = x(s, \theta)$ for some θ. Then we have

$$(\kappa(s, \theta))^2 = \langle \sigma(T, T), \sigma(T, T)\rangle,$$

where $T(s) = x_*(\partial/\partial s)(s, \theta)$. We now compute $(\partial/\partial s)(\kappa(s, \theta))^2$:

$$\frac{1}{2}\frac{\partial}{\partial\theta}\langle \sigma(T, T), \sigma(T, T)\rangle = \langle D_{\partial/\partial\theta}\sigma(\partial/\partial s, \partial/\partial s), \sigma(\partial/\partial s, \partial/\partial s)\rangle$$

$$= \langle (\overline{\nabla}_{\partial/\partial\theta}\sigma)(\partial/\partial s, \partial/\partial s), \sigma(\partial/\partial s, \partial/\partial s)\rangle + 2\langle \sigma(\nabla_{\partial/\partial\theta}\partial/\partial s, \partial/\partial s), \sigma(\partial/\partial s, \partial/\partial s)\rangle$$

$$= \langle \sigma(\nabla_{\partial/\partial s}\partial/\partial\theta, \partial/\partial s), \sigma(\partial/\partial s, \partial/\partial s)\rangle \quad \text{(because of the Codazzi equation and (3.1))}$$

$$= \langle D_{\partial/\partial s}\sigma(\partial/\partial\theta, \partial/\partial s), \sigma(\partial/\partial s, \partial/\partial s)\rangle - \langle \sigma(\nabla_{\partial/\partial s}\partial/\partial\theta, \partial/\partial s), \sigma(\partial/\partial s, \partial/\partial s)\rangle$$

$$= \frac{\partial}{\partial s}\langle \sigma(\partial/\partial\theta, \partial/\partial s), \sigma(\partial/\partial s, \partial/\partial s)\rangle - \langle \sigma(\partial/\partial\theta, \partial/\partial s), (\overline{\nabla}_{\partial/\partial s}\sigma)(\partial/\partial s, \partial/\partial s)\rangle$$
$$- \langle \sigma(\nabla_{\partial/\partial\theta}\partial/\partial s, \partial/\partial s), \sigma(\partial/\partial s, \partial/\partial s)\rangle$$

$$= -\langle \sigma(\partial/\partial\theta, \partial/\partial s), (\overline{\nabla}_{\partial/\partial s}\sigma)(\partial/\partial s, \partial/\partial s)\rangle \quad \text{(because of (3.1))}.$$

Let $\sigma(\partial/\partial\theta, \partial/\partial s) = f_1(s)N$ and let $\sigma(\partial/\partial s, \partial/\partial s) = g_1(s)N$, where N is the unit vector field normal to M along γ and f_1 and g_1 are some smooth functions defined along γ

$$(\overline{\nabla}_{\partial/\partial s}\sigma)(\partial/\partial s, \partial/\partial s) = g_1'(s)N + g_1(s)\frac{\partial N}{\partial s}.$$

(3.1) leads to $f_1(s)g_1(s) = 0$ for all s.

If $g_1(0) \neq 0$, then there exists an interval I contained in $\operatorname{Dom}\gamma$ such that $g_1(s) \neq 0$ for $s \in I$. So $f_1(s) = 0$ on I. Thus we have

$$\langle \sigma(\partial/\partial\theta, \partial/\partial s), (\overline{\nabla}_{\partial/\partial s}\sigma)(\partial/\partial s, \partial/\partial s)\rangle = f_1(s)g_1'(s) = 0 \quad \text{on } I.$$

Suppose that $g_1(0) = 0$. Let $s_0 = \inf\{s \mid g_1(s) \neq 0\}$. If $s_0 = 0$, then $g_1(s) \neq 0$ for $s > 0$ and thus $f_1(s) = 0$ for $s > 0$. So, $f_1(s)g_1'(s) = 0$ for $s > 0$. By continuity, $f_1(s)g_1'(s) = 0$ for $s \geqslant 0$. If $s_0 > 0$, then $g_1(s) = 0$ for $s < s_0$. Thus $f_1(s)g_1'(s) = 0$ for $s < s_0$.

If there is some $s \in \operatorname{Dom}\gamma$ such that $g_1(s) = 0$, then we keep doing this argument and thus we get $(\partial/\partial\theta)(\kappa(s, \theta))^2 = 0$ for $s \in \operatorname{Dom}\gamma$. Since this is true for every θ, we have

$$\frac{\partial}{\partial\theta}\langle\sigma(\partial/\partial s, \partial/\partial s), \sigma(\partial/\partial s, \partial/\partial s)\rangle = 0$$

on U. In other words, the curvature $\kappa(s, \theta)$ is independent of the choice of θ.

LEMMA 3.2. – The functions h and k are independent of the choice of θ.

PROOF. – Differentiating (3.3) with respect to s, we get

$$\frac{\partial^2 x}{\partial s^2} = \left(\frac{\partial^2 h}{\partial s^2}\cos\theta, \frac{\partial^2 h}{\partial s^2}\sin\theta, \frac{\partial^2 k}{\partial s^2}\right). \tag{3.6}$$

Thus the curvature $\kappa(s, \theta)$ satisfies

$$(\kappa(s, \theta))^2 = \left(\frac{\partial f}{\partial s}\right)^2. \tag{3.7}$$

On the other hand, (3.1) gives

$$\left\langle \frac{\partial^2 x}{\partial s^2}, \frac{\partial}{\partial\theta}\left(\frac{\partial x}{\partial s}\right)\right\rangle = 0, \tag{3.8}$$

which implies

$$\frac{\partial f}{\partial\theta}\frac{\partial f}{\partial s} = 0. \tag{3.9}$$

By Lemma 3.1, the curvatures do not depend on θ and so we choose a geodesic $x(s, \theta)$ for some θ and examine its curvature.

Suppose that $\kappa(0, \theta) = 0$. Let $s_0 = \inf\{s \mid \kappa(s, \theta) \neq 0\}$. If $s_1 = 0$, then $\kappa(s, \theta) \neq 0$ for $s > 0$. (3.7) and (3.9) imply $\partial f/\partial\theta = 0$ for $s > 0$. By continuity, $\partial f/\partial\theta = 0$ for $s \geqslant 0$. If $s_1 > 0$ (possibly $+\infty$), then $\kappa(s, \theta) = 0$ for $0 \leqslant s < s_1$. Then the inside of the geodesic circle S_1 centered at o with radius s_1 lies in E^2. In this case, h and k are clearly independent of the choice of θ.

Suppose that $\kappa(0, \theta) \neq 0$. Then we can choose a sufficiently small neighborhood of o where $\kappa(s, \theta) \neq 0$. Evidently $\partial f/\partial s \neq 0$ and thus $\partial f/\partial\theta = 0$ on this neighborhood. Developing this argument continuously if $\kappa(s, \theta) = 0$ for some $s \neq 0$, we see that h and k are independent of the choice of θ in the neighborhood U because $h(0,\theta) = k(0,\theta) = 0$.

Since the functions h and k only depend on the arc length s, (3.2) defines a surface of revolution around the point o.

Conversely, a meridian of a surface of revolution is always a geodesic and all the normal sections at the point o are geodesics through o if M is locally a surface of revolution with axis of symmetry passing through o. Thus we have

THEOREM 3.3 (Local characterization). – Let M be a surface in E^3. Then M is locally a surface of revolution with vertex p (around a neighborhood of p) if and only if every geodesic through p is a normal section of M at p.

THEOREM 3.4 (Global characterization). – Let M be a complete connected surface in E^3. Then M is a surface of revolution if and only if there is a point p in M such that every geodesic through p is a normal section of M at p.

4. – Surfaces of a Euclidean space with planar geodesic through a point.

Let M be a surface in E^m $(m \geq 3)$. We define the property $(*_3)$.

$(*_3)$ There is a point p in M such that every geodesic through p is planar.

LEMMA 4.1. – Let M be a surface in E^m and let γ be a geodesic in M through p. If γ is a planar curve, then γ is a normal section of M at p.

PROOF. – Let us assume that γ is parametrized by the arc length s and let $\gamma(0) = p$. Then we have

$$\gamma'(s) = T,$$

$$\gamma''(s) = \sigma(T, T),$$

$$\gamma'''(s) = -A_{\sigma(T,T)}T + (\overline{\nabla}_T \sigma)(T, T).$$

Since γ is a plane curve, $\gamma'(s) \wedge \gamma''(s) \wedge \gamma'''(s) = 0$ along γ. Thus we get

$$T \wedge \sigma(T, T) \wedge (-A_{\sigma(T,T)}T + (\overline{\nabla}_T \sigma)(T, T)) = 0,$$

which implies

$$T \wedge A_{\sigma(T,T)}T = 0 \tag{4.1}$$

and

$$\sigma(T, T) \wedge (\overline{\nabla}_T \sigma)(T, T) = 0. \tag{4.2}$$

Suppose first that $\sigma(t, t) \neq 0$, where $t = T(0)$. We can choose a neighborhood U of p such that $\sigma(u, u) \neq 0$ for any nonzero vector $u \in T_qM$, $q \in U$. So γ lies in $p + \mathrm{Span}\{t, \theta(t, t), (\overline{\nabla}_t \sigma)(t, t)\}$ and hence γ is a normal section at p.

Suppose $\sigma(t, t) = 0$. It is enough to consider $\sigma(t, t) = 0$ and $\sigma(T, T) \neq 0$ for $s > 0$. Let N be a normal vector field to M which is parallel to $\sigma(T, T)$ along γ for $s > 0$. Then we can choose a vector field $T^\perp$ which is tangent to M along γ and perpendicular to the plane Π spanned by $\{T(s), N(s)\}$ $(s > 0)$. Extend $T^\perp(s)$ up to the point p, which will be denoted by the same notation $T^\perp$. Then $\{t, T^\perp(0)\}$ is an orthonormal basis for T_pM and $T^\perp(0)$ is perpendicular to the plane Π. $N(0)$ is thus a normal vector to M at

p. Consequently, γ lies in $p + \mathrm{Span}\{t, N(0)\}$ and hence γ is a normal section at p in the direction t. (Q.E.D.)

Making use of this lemma, we see that the property $(*_2)$ is equivalent to the property $(*_3)$ if the ambient manifold is a 3-dimensional Euclidean space E^3.

Thus we have

THEOREM 4.2. – Let M be a surface in E^3. Then, M satisfies the property $(*_3)$ if and only if M is locally a surface of revolution.

COROLLARY 4.3. – Let M be a complete connected surface in E^3. Then M satisfies the property $(*_3)$ if and only if M is a surface of revolution.

We now suppose that a surface M in E^m satisfies the property $(*_3)$. By virtue of (4.1), we get

$$\langle \sigma(T, T), \sigma(T, T^{\perp}) \rangle = 0, \tag{4.3}$$

where $\gamma'(s) = T$, γ being a geodesic through p. In particular, $\langle \sigma(t, t), \sigma(t, t^{\perp})\rangle = 0$, $t = T(0)$. It is true for any unit vector t in T_pM and thus M is isotropic at p. So we may use Lemma 1.2 later.

Let (s, θ) be the geodesic polar coordinate system about p. We may assume that p is the origin o of E^m. Let $\exp_o(s\, e(\theta)) = x(s, \theta)$, where $e(\theta) = \cos\theta\, e_1 + \sin\theta\, e_2$ for some orthonormal basis $\{e_1, e_2\}$ for T_oM which is associated with the geodesic polar coordinates (s, θ).

LEMMA 4.4. – Let M be a surface in E^m with the property $(*_3)$. Then

$$\frac{\partial}{\partial\theta}(\kappa(s, \theta))^2 = 0, \tag{4.4}$$

where $\kappa(s, \theta)$ is the Frenet curvature of $x(s, \theta)$. In other words, the curvature of each geodesic through o does not depend on θ.

This lemma can be proved similarly to Lemma 3.1.

Since every geodesic through o is a plane curve and it is a normal section at o, we may represent the immersion x: $M \to E^m$ locally as

$$x(s, \theta) = h(s, \theta)\cos\theta\, e_1 + h(s, \theta)\sin\theta\, e_2 + k(s, \theta)\, N(\theta), \tag{4.5}$$

where $N(\theta)$ is a unit vector normal to M at o which may depend on θ and h and k are some smooth functions satisfying $h(0, \theta) = k(0, \theta) = 0$ for all θ.

LEMMA 4.5. – The functions h and k described as above do not depend on θ.

PROOF. – Since (s, θ) is the geodesic polar coordinate system, we have the following orthogonal vector fields tangent to M about o:

$$x_*\left(\frac{\partial}{\partial s}\right) = \frac{\partial h}{\partial s}\cos\theta\, e_1 + \frac{\partial h}{\partial s}\sin\theta\, e_2\, \frac{\partial k}{\partial s} N(\theta), \tag{4.6}$$

$$x_*\left(\frac{\partial}{\partial \theta}\right) = \left(\frac{\partial h}{\partial \theta}\cos\theta - h\sin\theta\right) e_1 + \left(\frac{\partial h}{\partial \theta}\sin\theta + h\cos\theta\right) e_2 + \frac{\partial k}{\partial \theta} N(\theta) + kN'(\theta). \tag{4.7}$$

Since $\langle x_*(\partial/\partial s), x_*(\partial/\partial s)\rangle = 1$, we get

$$\left(\frac{\partial h}{\partial s}\right)^2 + \left(\frac{\partial k}{\partial s}\right)^2 = 1,$$

from which, we may put

$$\frac{\partial h}{\partial s} = \cos f(s, \theta), \qquad \frac{\partial k}{\partial s} = \sin f(s, \theta), \tag{4.8}$$

where f is a smooth function defined on a neighborhood of o. Since $x_*(\partial/\partial s)(0, \theta) = = \cos\theta\, e_1 + \sin\theta\, e_2$, $\cos f(0, \theta) = 1$ and $\sin f(0, \theta) = 0$. Using (4.7) and (4.8), the curvature κ is represented as

$$(\kappa(s, \theta))^2 = \left(\frac{\partial f}{\partial s}\right)^2. \tag{4.9}$$

On the other hand, (4.3) implies

$$\left\langle \frac{\partial^2 x}{\partial s^2}, \frac{\partial}{\partial \theta}\left(\frac{\partial x}{\partial s}\right)\right\rangle = 0,$$

which yields

$$\frac{\partial f}{\partial s}\frac{\partial f}{\partial \theta} = 0. \tag{4.10}$$

The rest of the proof is exactly same as that of Lemma 3.2. (Q.E.D.)

We assume that a surface M lies in E^4 satisfying the property $(*_3)$ where the base point p in the property $(*_3)$ is not an isolated flat point. An isolated flat point p means a point such that the curvature of every geodesic through p vanishes only at p in some neighborhood of p. The curvature tensor R obviously vanishes at flat points. We also assume that the point p as the origin o of E^4. Let $\{e_1, e_2\}$ be an orthonormal basis of T_oM.

Suppose first that $\dim(\operatorname{Im}\sigma)_o \leqslant 1$. In this case, by considering Lemma 1.2, we see that o is an umbilical point of M. If $\dim(\operatorname{Im}\sigma)_o = 1$, then by choosing an appropriate Euclidean coordinate system of E^4 the immersion $x\colon M \to E^4$ can be locally expressed in terms of the geodesic polar coordinate system as

$$x(s, \theta) = (h(s)\cos\theta, h(s)\sin\theta, k(s), 0)$$

for some smooth functions h and k of the arc length s due to Lemma 4.4, where (s, θ) is the system of geodesic polar coordinates related to $\{e_1, e_2\}$. Thus M is locally a surface of revolution about o with axis of symmetry in the direction of the mean curvature vector at o.

Suppose that $\dim(\operatorname{Im}\sigma)_o = 0$. Since o is not an isolated flat point, there exists a neighborhood of o which is contained in a plane E^2 and this is a special case of above surface of revolution.

We now suppose $\dim(\operatorname{Im}\sigma)_o = 2$. As we showed in Lemma 2.4, the mean curvature vector H vanishes at o and $\|\sigma(t, t^{\perp})\|$ does not depend on the choice of orthonormal vectors t and $t^{\perp}$ tangent to M at o. Choose two unit vectors N_1 and N_2 normal to M at o such that

$$N_1 = \frac{\sigma(e_1 e_1)}{\kappa(0)} \quad \text{and} \quad N_2 = \frac{\sigma(e_1, e_2)}{\kappa(0)}, \tag{4.11}$$

where $\kappa(0)$ is the Frenet curvature at o. Since the functions h and k in (4.5) are independent of the choice of θ, (4.5) can be reduced to

$$x(s, \theta) = h(s)\cos\theta\, e_1 + h(s)\sin\theta\, e_2 + k(s)N(\theta). \tag{4.12}$$

As we did before, computing the length of $x_*(\partial/\partial s)$ by using (4.12), we may put

$$h'(s) = \cos f(s), \qquad k'(s) = \sin f(s) \tag{4.13}$$

for some smooth function f satisfying $f(0) = 0$.

On the other hand, we obtain from (4.12)

$$\begin{aligned} k''(0) &= \sigma(\partial/\partial s, \partial/\partial s)(0, \theta) \\ &= \sigma(\cos\theta\, e_1 + \sin\theta\, e_2, \cos\theta\, e_1 + \sin\theta\, e_2) = \kappa(0)\cos 2\theta N_1 + \kappa(0)\sin 2\theta N_2. \end{aligned}$$

Since $k''(0) = (\cos f(0))f'(0) = f'(0) = \pm\kappa(0)$,

$$N(\theta) = \pm(\cos 2\theta N_1 + \sin 2\theta N_2). \tag{4.14}$$

Thus, for a suitable choice of Euclidean coordinates of E^4 associated with e_1, e_2, N_1 and N_2, the immersion x is locally determined by

$$x(s, \theta) = \left(\cos\theta\int_0^s \cos f(t)\,dt,\ \sin\theta\int_0^s \cos f(t)\,dt,\ \pm\cos 2\theta\int_0^s \sin f(t)\,dt,\ \pm\sin 2\theta\int_0^s \sin f(t)\,dt\right), \tag{4.15}$$

where $f(s) = \pm\int_0^s \kappa(t)\,dt$ and κ is the Frenet curvature of geodesics through o.

If a surface M has the form (4.15), it is easily checked that M satisfies $(*_3)$. Thus we conclude

THEOREM 4.6. – Let M be a surface in E^4 without isolated flat points. Then M satisfies the property $(*_3)$ if and only if M is locally a surface of revolution which lies in E^3 or a surface that locally has the form (4.15).

COROLLARY 4.7. – Let M be a complete connected surface without isolated flat point in E^4. Then M satisfies the property $(*_3)$ if and only if M is a surface of revolution which lies in E^3 or a surface is globally of the form (4.15).

We now consider a surface M which lies in E^5 satisfying the property $(*_3)$. Again, we assume the base point in the property $(*_3)$ is the origin of E^5, where o is not an isolated flat point.

If $\dim(\operatorname{Im}\sigma)_o \leq 2$, then M is locally a surface of revolution in E^3 or M is a surface with local representation about o of the form (4.15) which lies in E^4 by the exact same argument.

Suppose $\dim(\operatorname{Im}\sigma)_o = 3$. Then

$$\sigma(e_1, e_1) \wedge \sigma(e_1, e_2) \wedge \sigma(e_2, e_2) \neq 0 .$$

Choose three orthonormal normal vectors to M at o:

$$N_1 = \frac{\sigma(e_1, e_1)}{\kappa(0)}, \qquad N_2 = \frac{\sigma(e_1, e_2)}{a}, \qquad N_3 = \frac{\tilde{N}_3}{\|\tilde{N}_3\|}, \tag{4.16}$$

where $a = \|\sigma(e_1, e_2)\|$ and $\tilde{N}_3 = \sigma(e_2, e_2) - \langle\sigma(e_1, e_1), N_1 > N_1$. If we compute the second fundamental form at o as we did to derive (4.14), then we obtain

$$\sigma(\partial/\partial s, \partial/\partial s)(0, \theta) = \left(\kappa(0)\cos^2\theta - \frac{\kappa(0)^2 - 2b^2}{\kappa(0)}\right) N_1 + a \sin 2\theta N_2 + b \sin^2\theta N_3 ,$$

where $b = \|\tilde{N}_3\|$. Using this equation and (4.5), we can find $N(\theta)$:

$$N(\theta) = \pm\left(\cos^2\theta - \frac{\kappa(0)^2 - 2b^2}{\kappa(0)^2}\right) N_1 \pm \frac{a}{\kappa(0)} \sin 2\theta N_2 \pm \frac{b}{\kappa(0)} \sin^2\theta N_3 .$$

Thus locally the immersion x: $M \to E^5$ may be written in terms of a suitable choice of Euclidean coordinates of E^5 as

$$\begin{aligned} x(s, \theta) = \Bigg(&\cos\theta \int_0^s \cos f(t)\,dt,\ \sin\theta \int_0^s \cos f(t)\,dt,\ \pm\left(\cos^2\theta - \frac{\kappa(0)^2 - 2b^2}{\kappa(0)^2}\right) \int_0^s \sin f(t)\,dt, \\ &\pm \frac{a}{\kappa(0)} \sin 2\theta \int_0^s \sin f(t)\,dt,\ \pm \frac{b}{\kappa(0)} \sin^2\theta \int_0^s \sin f(t)\,dt\Bigg), \end{aligned} \tag{4.17}$$

where $f(s) = \pm \int_0^s \kappa(t)\,dt$ and κ is the Frenet curvature of geodesic through o.

Conversely, if a surface M has the form (4.17), then we can easily see that M satisfies the property $(*_3)$.

THEOREM 4.8. – Let M be a surface in E^5 without isolated flat points. Then M satisfies the property $(*_3)$ if and only if M is locally a surface of revolution in E^3 or a surface in E^4 which has a local representation of the form (4.15) or a surface of the form (4.17) which fully lies in E^5.

Let M be a surface in E^m. Since the dimension of the first normal space at o is at most three, we obtain the following theorem.

THEOREM 4.9. – Let M be a surface in E^m $(m \geqslant 3)$ without isolated flat points. Then M satisfies the property $(*_3)$ if and only if M lies locally in E^5 and M is one of the three model spaces described in Theorem 4.8.

From now on we study a surface M in E^m satisfying the property $(*_3)$ whose base point o, say the origin of E^m, is an isolated flat point.

We now assume that M is an analytic surface in E^m.

We first define the degree of an isolated flat point.

Let p be an isolated flat point of a Riemannian manifold. Then for every geodesic γ parametrized by the arc length s through $p = \gamma(0)$, its curvature $\kappa_\gamma(s)$ satisfies $\kappa_\gamma(0) = 0$ and $\kappa_\gamma(s) \neq 0$ for sufficiently small s. Let $d(p) = \inf\{n \in \boldsymbol{Z}_+ \mid \kappa_\gamma^{(n)}(0) \neq 0\}$. Then $d(p)$ is well-defined integer $\geqslant 1$.

DEFINITION. – $d(p)$ is called the *degree* of the isolate flat point p.

Suppose that the analytic surface M in E^m satisfies the property $(*_3)$ and that the base point in the property $(*_3)$ is an isolated flat point. We also assume that the base point is the origin o of E^m.

According to (4.5) and Lemma 4.5, the immersion $x\colon M \to E^m$ is locally represented in terms of geodesic polar coordinates (s, θ) about o:

$$x(s, \theta) = h(s)\, e(\theta) + k(s)\, N(\theta), \tag{4.18}$$

where h and k are analytic functions of s such that $h(0) = k(0) = 0$, $h'(s) = \cos f(s)$, $k'(s) = \sin f(s)$, $f(s) = \pm \int_0^s \kappa(t)\, dt$, $e(\theta) = \cos\theta\, e_1 + \sin\theta\, e_2$ and $N(\theta)$ is a unit vector normal to M at o depending on θ.

For $r \geqslant 2$, the r-th derivatives of h and k are:

$$h^{(r)}(s) = -(\sin f(s))\, f^{(r-1)}(s) + O_1(f', f'', \ldots, f^{(r-2)})$$

and

$$k^{(r)}(s) = (\cos f(s))\, f^{(r-1)}(s) + O_2(f', f'', \ldots, f^{(r-2)}),$$

where $O_i(f', f'', \ldots, f^{(r-2)})$ $(i = 1, 2)$ are certain polynomials with respect to $f', f'', \ldots, f^{(r-2)}$. Since o is an isolated flat point and since the curvature of each geodesic through o is independent of the choice of θ, there is an integer p (> 1) such that $\kappa(0) = \kappa'(0) = \ldots = \kappa^{(p-2)}(0) = 0$ and $\kappa^{(p-1)}(0) \neq 0$, that is, the degree of o is $p - 1$. Since $\kappa(s) = \pm f'(s)$, we see that

$$h^{(r)}(0) = 0 \quad \text{for any } r \geqslant 2,$$

and

$$k^{(r)}(0) = 0 \quad (0 \leqslant r \leqslant p - 1), \qquad k^{(p)}(0) \neq 0.$$

In other words,

$$\frac{\partial^r x}{\partial s^r}(0, \theta) = 0 \quad (2 \leqslant r \leqslant p - 1) \tag{4.19}$$

and

$$\frac{\partial^p x}{\partial s^p}(0, \theta) = k^{(p)}(0)\, N(\theta) = \kappa^{(p-1)}(0)\, N(\theta) \neq 0. \tag{4.20}$$

We now define the r-th $(r \geqslant 1)$ covariant derivative of σ by

$$(\overline{\nabla}^r \sigma)(X_1, X_2, \ldots, X_{r+2})$$

$$= D_{X_1}((\overline{\nabla}^{r-1}\sigma)(X_2, \ldots, X_{r+2})) - \sum_{i=2}^{r+2} (\overline{\nabla}^{r-1}\sigma)(X_2, \ldots, \nabla_{X_1} X_i, \ldots, X_{r+2}).$$

Then $\overline{\nabla}^r \sigma$ is a normal bundle valued tensor of type $(0, r+2)$. Moreover, it can be proved that

$$(\overline{\nabla}^r \sigma)(X_1, X_2, X_3, \ldots, X_{r+2}) - (\overline{\nabla}^r \sigma)(X_2, X_1, X_3 \ldots, X_{r+2}) \tag{4.21}$$

$$= R^N(X_1, X_2)((\overline{\nabla}^{r-2}\sigma)(X_3, \ldots, X_{r+2})) + \sum_{i=3}^{r+2} (\overline{\nabla}^{r-2}\sigma)(X_3, \ldots, R(X_1, X_2) X_i, \ldots, X_{r+2})$$

for $r \geqslant 2$, where $X_1, X_2, X_3, \ldots, X_{r+2}$ are vector fields tnagent to M, R^N the normal curvature tensor, R the curvature tensor of M and $\overline{\nabla}^0 \sigma = \sigma$.

On the other hand, for $r \in \boldsymbol{Z}_+$,

$$\frac{\partial^r x}{\partial s^r}(0, \theta) = (\overline{\nabla}^r \sigma)(t^{r+2}),$$

where $t = e(\theta) = \cos\theta\, e_1 + \sin\theta\, e_2$ and $(\overline{\nabla}^r \sigma)(t^{r+2}) = (\overline{\nabla}^r \sigma)(t, t, t, \ldots, t)$. Then we can easily prove

LEMMA 4.10. – If $(\overline{\nabla}^r\sigma)(t^{r+2}) = 0$ $(0 \leqslant r \leqslant p-1)$ and $(\overline{\nabla}^p\sigma)(t^{p+2}) \neq 0$ for all t in T_oM then $\overline{\nabla}^p\sigma$ is symmetric and $\overline{\nabla}^r\sigma = 0$ at the point o for $0 \leqslant r \leqslant p-1$.

Thus if the point o is an isolated flat point of degree $p-1$, then we have

$$\overline{\nabla}^r\sigma = 0 \qquad (0 \leqslant r \leqslant p-1)$$

and

$$\overline{\nabla}^r\sigma \text{ is symmetric}$$

at the point o.

Denote by

$$(\overline{\nabla}^r\sigma)(e_1^i, e_2^j) = (\overline{\nabla}^r\sigma)(e_{11}, e_{12}, \ldots, e_{1i}, e_{21}, e_{22}, \ldots e_{2j}),$$

where $i+j = r+2$ and $e_{1h} = e_1$, $e_{2k} = e_2$ for all $h = 1, 2, \ldots, i$ and $k = 1, 2, 3, \ldots, j$. Since the curvature κ is independent of the choice of the geodesic through o, we get

$$\|(\overline{\nabla}^r\sigma)(e_1^{p+2})\| = \|(\overline{\nabla}^r\sigma)(e(\theta)_1^{p+2})\| \qquad \text{for all } \theta.$$

So we obtain the following equation

$$-\sum_{r=1}^{p+1}\binom{p+2}{r}\cos^{2(p+2-r)}\theta\,\sin^{2r}\theta\|(\overline{\nabla}^p\sigma)(e_1^{p+2})\|^2$$

$$-\sum_{r=1}^{p+1}\binom{p+2}{r}\cos^{2(p+2-r)}\theta\,\sin^{2r}\theta\|(\overline{\nabla}^p\sigma)(e_1^{p+2-r}, e_2^r)\|^2$$

$$+2\sum_{r<s}\binom{p+2}{r}\binom{p+2}{s}\cos^{2(p+2)-r-s}\theta\,\sin^{r+s}\theta\langle(\overline{\nabla}^p\sigma)(e_1^{p+2-r}, e_2^r), (\overline{\nabla}^p\sigma)(e_1^{p+2-s}, e_2^s)\rangle = 0,$$

which yields

$$\binom{p+2}{q}\|(\overline{\nabla}^p\sigma)(e_1^{p+2})\|^2 = \binom{p+2}{q}\|(\overline{\nabla}^p\sigma)(e_1^{p+2-q}, e_2^q)\|^2 + 2\sum_{r<s\leqslant 2q}\binom{p+2}{r}\binom{p+2}{s}\langle(\overline{\nabla}^p\sigma)(e_1^{p+2-r}, e_2^r), (\overline{\nabla}^p\sigma)(e_1^{p+2-s}, e_2^s)\rangle \tag{4.22}$$

for $r = 2q$ $(q \geqslant 1)$ and

$$\sum_{r<s\leqslant 2q-1}\binom{p+2}{r}\binom{p+2}{s}\langle(\overline{\nabla}^p\sigma)(e_1^{p+2-r}, e_2^r), (\overline{\nabla}^p\sigma)(e_1^{p+2-s}, e_2^s)\rangle = 0 \tag{4.23}$$

for $r = 2q-1$ $(q \geqslant 1)$.

On the other hand, the maximal dimension of $\{(\overline{\nabla}^p\sigma)(e_{i_1}, e_{i_2}, \ldots, e_{i_{p+2}}) \mid e_{i_j} = e_1 \text{ or } e_2\}$ is $p+3$.

From (4.20), we see that

$$(4.24)\qquad N(\theta) = \frac{1}{\|(\overline{\nabla}^p\sigma)(e_1^{p+2})\|^2}(\overline{\nabla}^p\sigma)(e(\theta)_1^{p+2})$$

$$= \frac{1}{\|(\overline{\nabla}^p\sigma)(e_1^{p+2})\|^2}\sum_{r=0}^{p+2}\binom{p+2}{r}\cos^{p+2-r}\theta\,\sin^r\theta\cdot(\overline{\nabla}^p\sigma)(e_1^{p+2-r}, e_2^r).$$

Thus, if o is an isolated flat point of degree $p-1$, then M is locally contained in at most $(p+5)$-dimensional Euclidean space p^{p+5} and in this case the second fundamental form σ satisfies (4.22) and (4.23) and the immersion $x\colon M \to E^m$ becomes

$$x(s,\theta) = h(s)\cos\theta\, e_1 + h(s)\sin\theta\, e_2 + k(s)N(\theta),$$

where h and k are analytic functions such that $h^{(r)}(0) = 0$ for all $r \geqslant 2$, $k^{(r)}(0) = 0$ $(0 \leqslant r \leqslant p-1)$, $k^{(p)}(0) \neq 0$ and $N(\theta)$ is given by (4.24).

Thus we have

THEOREM 4.11. – If a surface M in E^m satisfies the property $(*_3)$ whose base point, say the origin o of E^m, is an isolated flat point of degree $p-1$, then M locally lies in at most $(p+5)$-dimensional Euclidean space E^{p+5} about o and is of the form:

$$(4.25)\qquad x(s,\theta) = \left(\int_0^s \cos f(t)\,dt\right)(\cos\theta\, e_1 + \sin\theta\, e_2) + \left(\int_0^s \sin f(t)\,dt\right)N(\theta),$$

where $N(\theta)$ is of the form (4.24), $f(s) = \pm\int_0^s \kappa(t)\,dt$ and $\kappa(s)$ is the Frenet curvature of geodesics through o.

Combining Theorem 4.9 and Theorem 4.11, we can classify analytic surfaces in E^m satisfying the property $(*_3)$.

THEOREM 4.12 (Classification). – Let M be an analytic surface in E^m. If M satisfies the property $(*_3)$, then M is one of the following:

(1) M is locally a surface of revolution about o which lies in E^3,

(2) M is a surface of the form (4.15) about o which fully lies in E^4,

(3) M is a surface of the form (4.17) about o which fully lies in E^4,

(4) M is a surface of the form (4.25) about o which lies in E^{p+5}, where the degree of the isolate flat point o is $p-1$.

REMARK. – According to K. SAKAMOTO [S-1], and J. A. LITTLE [L], a surface in a Euclidean space E^m with planar geodesics must be an open portion of a plane E^2, a standard sphere S^2 or a real projective space RP^2. So M must lie in a 5-dimensional

Euclidean space E^5. However, a surface M in E^m ($m \geqslant 3$) satisfying the property ($*_3$) may lie fully in a higher dimensional Euclidean space depending on the degree of the isolate flat point if the base point in the property ($*_3$) is an isolated flat point.

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