Annali di Matematica pura ed applicata
(IV), Vol. CLVIII (1991), pp. 231-305

A New Approach to the Morse-Conley Theory and Some Applications (*).

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Summary. – *We present a new approach to the Morse theory which is based on a generalization of the Conley index to non locally compact spaces. The variant of the Morse theory which we obtain seems suitable for the applications to nonlinear functionals analysis. Some applications are given here; they mainly concern the study of periodic solutions of second order Hamiltonian systems. Other applications are in some quoted papers.*

Introduction.

In the first four sections of this paper we present a new approach to the Conley-index theory to non locally-compact spaces (cf. also [B1] and [B2]). The definitions and the theorems are carried out in a fairly large generality as far as this generality does not complicate too much the theory.

We have worked in a metric space and we have not imposed any compactness assumption to the flow even if in the applications a reacher structure needs to be added.

We have chosen this approach because we think that this level of generality helps to understand the underlying structure and also allows to compare this theory with other versions of the Conley index theory in infinite dimensional spaces (cf. the comparison with Rybakowsky [RY] in section 3).

Moreover it is possible that the study of this structure without any compactness will help the analysis of the critical points at infinity in the sense of Bahri (cf. e.g. [BA] and its references).

In the following sections we are interested to the study of critical points of a C^1-functional defined on a Banach space (or more in general on a Finsler manifold).

In this framework we are able to write the analogous of the Morse inequalities for C^1-functionals whose critical points may not be isolated.

To do this we introduce the concept of ε-Morse covering (cf. Def. 5.11, Th. 5.12 and Th. 5.14).

(*) Entrato in Redazione il 26 settembre 1988.
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In section 6 we consider the study of C^2-functionals on a Hilbert manifold equipped with a Riemannian structure.

At this level the theory becomes comparable with the classical Morse theory. Thus if the functional is a Morse functional (i.e. all the critical points of f are nondegenerate) we obtain the same results of the Morse theory. If the functional has only isolated, but possible degenerate critical points then we get the results of the Gromoll Meyer theory ([GM], cf. also [CH1]). However our theory does not suppose «a priori» that the critical points are isolated, since also in this case it is possible to use the concept of ε-Morse covering. This fact will be very usefull in the applications of the three following sections.

In Section 7 we present some existence theorem for C^1-functionals. Some of these theorems can be obtained via minimax methods (cf. e.g. [R1]). However the use of the Morse theory gives an extra information which may be essential in some cases. For example we refer to [BF1], [BF3] and [BG] where Theorem 7.5 and Corollary 7.9 are the main tools in solving the problems studied there.

In Section 7 we consider also an elliptic equation for which an existence result is obtained in a relatively way (Th. 7.14). We consider this equation as an example for which the Morse theory works better than the minimax theory (of corse we do not mean that the Morse theroy works always better than the minimax theory: it depends on the problem!).

In Section 8 we apply this theory to equivariant functionals. In this case, the most natural thing to do would be to adapt the equivariant Morse theory (cf. *e.g.* [BO1] and [PAC] for the nonvariational case) to our theory. This fact would not present particular difficulty.

Instead we exploited the equivariance of the functional in a simper way using only the theory developed in the previous sections. We did this for two reasons: the first is to keep the paper to a simpler level and to avoid technicalities when it is possible. The second is that this level is sufficient to the applications we considered in the following.

The last four sections are devoted to the applications of the Morse theory to the study of periodic solutions of second order conservative systems.

In sections 7 and 8 we define the Maslov index and the twisting number in such a may that it can be easily related to the Morse index of our theory. The results of this sections are the «easiest translation» in our theory of ideas already existing (cf. e.g. [BO2], [COZ1], [COZ2], [EK], [EKH] and their references). The existence of periodic solutions and their relation to the twisting number is investigated in the last two sections (cf. also [B4]).

1. – The generalized Conley index.

Let M be a metric space on which a flow η is defined *i.e.* a (continuous) map

$$\eta: \boldsymbol{R} \times M \to M$$

such that $\eta(0, x) = x$ and $\eta(t_1, \eta(t_2, x)) = \eta(t_1 + t_2, x)$; $(t_1, t_2 \in \boldsymbol{R}, x \in M)$. When no ambiguity is possible we will write $x \cdot t)$ instead of $\eta(t, x)$. If X is any subset of M and T a positive constant we set

$$(1.1) \qquad G^T(X) = G^T(X, \eta) = \{x \in M | X \cdot [-T, T] \subset \overline{X}\} = \bigcap_{t \in [-T,T]} \eta(t, \overline{X})$$

where $\overline{X}$ denotes the closure of X.

Also we set

$$\Sigma = \Sigma(\eta) = \{X \in M | X \text{ is open and } \exists T > 0 \text{ s.t. } G^T(X, \eta) \subset X\}$$

DEF. 1.1. – A pair of closed subset of X, (N, N_0) with $N_0 \subset N$ is called *index pair* if

(i) $\exists T \in R^+$: $G^T(N - N_0) \subset int(N - N_0)$;

(ii) N_0 is positively invariant with respect to N (i.e. $x \in N_0$ and $x \cdot [0, t] \subset N \Rightarrow x \cdot [0, t] \subset N_0$);

(iii) N_0 is an *exit* set for N (i.e. $x \in N$ and $x \cdot [0, t] \not\subset N \Rightarrow \exists t^* \in [0, t]$ such that $x \cdot t^* \in N_0$).

We say that (N, N_0) is an index pair for $X \in \Sigma$ if

(iv) $\overline{N - N_0} \subset \overline{X}$ and there exists $T > 0$ such that $G^T(X) \subset \overline{N - N_0}$.

Now it is necessary to recall some concepts from the homotopy theory.

If X is a topological space and A is a closed subset then X/A denotes the spaces obtained by X identifing all the points of A.

Two spaces X/A and Y/B are called homotopic equivalent if there are maps φ: $X/A \to Y/B$ and ψ: $Y/B \to X/A$ such that $\varphi([A]) = [B]$; $\psi([B]) = [A]$ and such that $\varphi \circ \psi$ and $\psi \circ \varphi$ are homotopic to the identity by homotopies which leave the points $[A]$ and $[B]$ fixed respectively.

The class of all spaces homotopically equivalent to X/A is called homotopy type of X/A and denote by $[X/A]$.

The homotopy type of X/X is denoted by **0**; if X is a contractible space, the homotopy type of X/Φ is denoted by **1**. Moreover, by convenction, we set $\Phi/\Phi = \mathbf{0}$.

DEF. 1.2. – For $X \in \Sigma$, the homotopy index of X is the homotopy type of an index pair (N, N_0) relative to X; in formula we write

$$h(X) = h(X, \eta) = [N/N_0].$$

We shall call $h(X)$ the (generalised) Conley index of X.

The definition 1.2 makes sense if we prove that

$$(1.2) \qquad \begin{cases} (a) \ \forall \ X \in \Sigma \text{ there exists an index pair } (N, N_0) \text{ for } X \\ (b) \text{ if } (N, N_0) \text{ and } (\tilde{N}, \tilde{N}_0) \text{ are two index pairs relative to } X, \\ \quad \text{then } [N/N_0] = [N/N_0]. \end{cases}$$

In order to prove (1.2) some work is necessary. First, we need an other notation; for $T>0$ we set

$$\Gamma^T(X)=\Gamma^T(X,\eta)=\{X\in G^T(X,\eta)|x\cdot[0,T],\ \partial X\neq\Phi\}. \tag{1.3}$$

We need now a technical lemma:

LEMMA 1.3. – Suppose that $X, Y\in\Sigma$; then

(i) $X\subset Y\rightarrow G^T(X)\subset G^T(Y)$ for every $T>0$;

(ii) $T_1>T_2>0\Rightarrow G^{T_1}(X)\subset G^{T_2}(X)$;

(iii) $G^{T_1+T_2}(X)=G^{T_1}(G^{T_2}(X))$; (iv) if $G^T(X)\subset\mathring{X}$ then $G^{2T}(X)\subset\operatorname{int}[G^T(X)]$;

(v) $G^T(X)$ is closed;

(vi) if $X\in\Sigma$ then $G^T(X)$ and $\eta(t,X)\in\Sigma$;

(vii) $\Gamma^T(X)\subset\partial G^T(X)$.

PROOF. – (i), (ii) and (iii) follow easily from the definition of $G^T(X)$.

(iv) In order to prove (iv) we argue indirectly and we suppose that there exists $y\in G^{2T}(X)\cap\partial G^T(X)$. Then there exists a sequence $y_n\rightarrow y$ such that $y_n\cdot[-T,T]\not\subset\overline{X}$. This implies that there exist times $t_n\in[-T,T]$ such that $y_n\cdot t_n\notin\overline{X}$; we can extract a sequence t_n' such that $t_n'\rightarrow\bar{t}$, so we have that $y_n\cdot t_n\rightarrow y\cdot\bar{t}\in\partial X$. Since $y\in G^{2T}(X)$, $y\cdot[-2T,2T]\subset X$ and so $y\cdot\bar{t}\in G^T(\overline{X})$ (since $|\bar{t}|\leqq T$).

And this contradicts our assumption that $G^T(X)\cap\partial X\neq\Phi$.

(v) $G^T(X)$ is closed because by (1.1) it is the intersection of a family of closed sets.

(vi) $G^T(X)\in\Sigma$ by (iv) and (v). $\eta(t,X)\in\Sigma$ by the continuity of η.

(vii) Let $\{x_n\}\subset\Gamma^T(X)$ with $x_n\rightarrow\bar{x}$. Then there exists $t_n\in[0,T]$ such that $x_n\cdot t_n\in\partial X$. Let t_n' be a subsequence of t_n converging to some $\bar{t}\in[0,T]$; then $x_n\cdot t_n'\rightarrow\bar{x}\cdot\bar{t}\in\partial X$. Therefore $\bar{x}\in\Gamma^T(X)$.

(viii) Let $x\in\Gamma^T(X)$; then $\exists t\in[0,T]$ such that $x\cdot t\in\partial X$; thus there exists a sequence $y_n\in\overline{X}^c$ (X^c denotes the complement of X in M) converging to $x\cdot t$. This implies that $y_n(-t)\rightarrow x$. But $y_n(-t)\notin G^T(\overline{X})$, therefore $x\in\partial G^T(\overline{X})$. ■

Now we can prove (1.1) (*a*).

THEOREM 1.4. – (Existence of index pairs). Let $X\in\Sigma$ and let T be a large enough that $G^T(X)\subset int(X)$. Then

$$(G^T(X),\Gamma^T(X))$$

is an index pair for X.

PROOF. – By Lemma 1.3 (vi), (vii), $G^T(X)$ and $\Gamma^T(X)$ are closed. We have to check points (i), (ii) and (iii) of Def. 1.2.

(i) By Lemma 1.2 (viii), $int\left(\overline{G^T(X) - \Gamma^T(X)}\right) = G^T(X)$; so the conclusion follows by Lemma 1.3 (iv).

(ii) Let $x \in \Gamma^T(X)$ and suppose that

$$x \cdot [0, t] \subset G^T(X) \tag{1.4}$$

we want to prove that $x \cdot [0, t] \subset \Gamma^T(X)$. Suppose that this fact is not true; then there exists $\bar{t} \in [0, t]$ such that $x \cdot \bar{t} \notin \Gamma^T(X)$.

Now set

$$t^* = \inf\{\tau \in [0, t] | x \cdot \tau \notin \Gamma^T(X)\}.$$

Clearly $t^* \in [0, t)$ and

$$\begin{aligned} &(a)\ \ x \cdot t^* \in \Gamma^T(X) \quad \text{since } \Gamma^T(X) \text{ is closed by Lemma 1.3 (vii);} \\ &(b)\ \ x(t^* + \varepsilon_n) \notin \Gamma^T(X) \quad (\text{with } \varepsilon_n > 0 \text{ and } \varepsilon_n \to 0). \end{aligned} \tag{1.5}$$

If we set $y = x \cdot t^*$, by (1.5) and the definition of $\Gamma^T(X)$, we have

$$y \cdot [0, T] \cap \partial X \neq \Phi, \qquad y \cdot [\varepsilon_n, T] \cap \partial X = \Phi.$$

From the above formulas we have that

$$y \in \partial X. \tag{1.6}$$

On the other hand, by (1.4), $y \in G^T(X)$ and since $G^T(X) \subset X$, $y \in X$; this fact contradicts (1.6) since X is open.

(iii) It is trivial. ∎

THEOREM 1.5. – (Equivalence of index pairs). Let (N, N_0) and $(\tilde{N}, \tilde{N}_0)$ be two index pairs such that exists $T > 0$ such that

$$G^T(N - N_0) \subset \overline{\tilde{N} - \tilde{N}_0} \qquad \text{and} \qquad G^T(\tilde{N} - \tilde{N}_0) \subset \overline{N - N_0}.$$

Then $[N/N_0] = [\tilde{N}/\tilde{N}_0]$.

REMARK. – The Proof of Theorem 1.5 is essentially contained in Salamon [S].

He gave a short and elegant Proof of Conley's theorem of equivalence of index pairs (in the compact case). Salamon's Proof can be adapted to our case.

SKETCH OF THE PROOF OF TH. 1.5. – We can suppose that $G^T(N - N_0) \subset \text{int}(\tilde{N} - \tilde{N}_0)$ and that $G^T(\tilde{N} - \tilde{N}_0) \subset \text{int}(N - N_0)$; if not it is enough to replace T by $2T$ and use

Lemma 1.2 (iv). Now let $f\colon N_1/N_0 \to \tilde{N}/\tilde{N}_0$ be defined as follows

$$f([x]) = \begin{cases} [x\cdot 3T] & \text{if } x\cdot[0,2T] \subset N_1 - N_0 \text{ and } x\cdot[T,3T] \subset \tilde{N}_1 - \tilde{N}_0 \\ [N_0] & \text{otherwise} \end{cases}$$

the function f is continuous (for details of the proof see [S] Lemma 4.7). In an analogous way we can define a map $\tilde{f}\colon \tilde{N}/\tilde{N}_0 \times [T, +\infty) \to N/N_0$.

We have to prove that $\tilde{f}\circ f$ and $f\circ\tilde{f}$ are homotopic to the identity in N/N_0 and $\tilde{N}/\tilde{N}_0$ respectively.

For $t \in [0,T]$ define the map $h\colon [0.T] \times N/N_0 \to N/N_0$ as follows

$$h(t,[x]) = \begin{cases} [x\cdot 6t] & \text{if } x\cdot[0,6t] \subset N_1 - N_0 \\ [N_0] & \text{otherwise.} \end{cases}$$

It is easy to show that h is continuous and that

$$h(T,[x]) = \tilde{f}\circ f \qquad \text{and} \quad h(0,[x]) = Id_{N/N_0}.$$

In the some way it is possible to construct a homotopy $h\colon [0,T] \times \times \tilde{N}/\tilde{N}_0 \to \tilde{N}/\tilde{N}_0$. ∎

COROLLARY 1.6. – If (N, N_0) and $(\tilde{N}, \tilde{N}_0)$ are two index pairs for X, then $[\tilde{N}, \tilde{N}_0] = [N, N_0]$. In particular (1.1) (*b*) holds.

PROOF. – If (N, N_0) and $(\tilde{N}, \tilde{N}_0)$ are two index pairs for X, we have

$$G^T(N - N_0) \subset G^T(X) \subset \tilde{N} - \tilde{N}_0$$

by definition 1.1 and

$$G^T(\tilde{N} - \tilde{N}_0) \subset G^T(X) \subset N - N_0.$$

The conclusion follows from theorem 1.5. ∎

So at this point $h(X)$ is well defined. An other consequence of theorem 1.5 is the following Corollary:

COROLLARY 1.7. – Let $X, Y \in \Sigma$ and suppose that $\exists T \geqq 0$ such that

(1.7) $$G^T(X) \subset \overline{Y} \qquad \text{and} \qquad G^T(Y) \subset \overline{X}.$$

Then $h(X) = h(Y)$.

PROOF. – Let (N, N_0) and $(\tilde{N}, \tilde{N}_0)$ be two index pairs for X and Y respectively. Then

(1.9) $$G^T(N - N_0) \subset G^T(X) \subset Y \qquad \text{by Definition 1.1 and (1.7).}$$

Since $(\tilde{N}, \tilde{N}_0)$ is an index pair for $Y, \exists T_1 > 0$ such that

$$G^{T_1}(Y) \subset \operatorname{int}(\tilde{N}_1 - \tilde{N}_0).$$

Therefore by the above formula, (1.9) and Lemma 1.2 (iii), we have that

$$G^{T+T_1}(N - N_0) \subset \tilde{N} - \tilde{N}_0.$$

For the some reason there exists $T_2 > 0$ such that

$$G^{T_1+T_2}(\tilde{N} - \tilde{N}_0) \subset N - N_0.$$

Thus by theorem 1.5 (replacing T with $T + \max(T_1, T_2)$) the conclusion follows. ■

COROLLARY 1.8. – For every $T > 0$ $h(G^T(X)) = h(X)$.

PROOF. – Trivial. ■

COROLLARY 1.9. – If ther is $T > 0$ such that $G^T(X) = \Phi$, then $h(X) = \mathbf{0}$.

Notice that Corollary 1.9 cannot be inverted as the following example shows.

EXAMPLE 1.10. – Take

$$M = \mathbf{R}; \qquad \eta(t, x) = x - t; \qquad X = [0, +\infty)$$

Then $h(X) = 0$ but $G^T(X) \neq \Phi$ for every $T > 0$.

However there is a good test to see if the index of a set is $\mathbf{0}$.

THEOREM 1.11. – Suppose that $X \in \Sigma$ and that

(1.10) for every $x \in X$, there is $t > 0$ such that $x \cdot t \notin X$. Then $h(X) = 0$

We need some lemmas to prove Theorem 1.11.

LEMMA 1.12. – Suppose that (N, N_0) is an index pair and that τ is a positive constant such that

$$x \cdot [0, \tau] \subset N - N_0. \tag{1.11}$$

Then there exists an open neighborhood V of x such that for every $y \in V \cap N$,

$$y \cdot [0, \tau] \subset N - N_0.$$

PROOF. – We argue indirectly and suppose that the conclusion of the lemma is not true. Then exists a sequence $x_n \to x (x_n \in N - N_0)$ and a sequence $t_n \in [0, \tau]$ such that

$$x_n \cdot t_n \notin N - N_0.$$

We set

$$\tilde{t}_n = \sup\{t \in [0, t_n] \text{ such that } x_n \cdot [0, t] \subset N\}$$

$\tilde{t}_n$ is a bounded sequence; so we can suppose that it is convergent to some $\bar{t} \in [0, \tau]$. By our construction, $x_n \cdot \tilde{t}_n \in N_0$; so $x \cdot \bar{t} \in N_0$ since N_0 is closed. This last statement contradicts (1.11); so the lemma is proved. ∎

LEMMA 1.13. – Let $(N, N_0) = (G^T(X); \Gamma^T(X))$ be an index pair for X (cf. Th. 1.4). We set

$$U = \{x \in N | \exists t \in [0, 2T] \text{ such that } x \cdot t \in N^c\}$$

where N^c denotes $M - N$.

Then U satisfies the following properties:

(i) U is relatively open in N;

(ii) given two positive constant $t_1 < t_2$ such that

$$x \cdot t_i \in U \quad \text{and} \quad x \cdot [0, t_i] \subset N \quad (i = 1, 2)$$

then for every $t \in [t_1, t_2]$, $x \cdot t \in U$;

(iii) $N_0 \subset U$;

(iv) $(\overline{U}, N_0)$ is an index pair and $[\overline{U}/N_0] = \mathbf{0}$.

PROOF. – (i) and (ii) are easy to check.

In order to prove (iii) we argue indirectly and suppose that there is $x \in N_0$ such that $x \cdot [0, 2T] \subset N$. Since N_0 is positively invariant with respect to N, $x \cdot [0, 2T] \subset N_0$. Then if we set $y = x \cdot T$, it follows that $y \in N_0$ and $y \in G^T(N)$. Since $G^T(N) \subset \mathring{N}$ by Lemma 1.3 (iii) and (iv) and $N_0 \subset \partial N$, by Lemma 1.3 (vii), we have obtained a contradiction.

Now let us prove (iv). First observe that $N_0 \subset \overline{U}$ by (iii). (i) of Def. 1.1 is satisfied since $\overline{\overline{U} - N_0} = \overline{U}$ and $G^{2T}(\overline{U}) = \Phi \subset \operatorname{int}(\overline{U})$.

To check (ii), it is enough to observe that $\overline{U} \subset N$. (iii) follows directly by the definition of U. So $(\overline{U}, N_0)$ is an index pair.

$[\overline{U}/N_0] = h(U) = \mathbf{0}$ by Corollary (1.9). ∎

PROOF OF TH. 1.11. – Let N, N_0 and U as in Lemma 1.13. For every $x \in N$, we choose a $t(x) > 0$ such that

$$x \cdot [0, t(x)] \subset N \quad \text{and} \quad x \cdot t(x) \in U.$$

This is possible by (1.10) and Lemma 1.13 (iii). If $x \in U$ we choose $t(x) = 0$. Also if $x \notin U$, we can choose $t(x)$ such that $t(x) \notin N_0$.

Now for $x \in N - N_0$, let V_x be an open neighborhood of N (open in the topology of

N) such that

$$\text{for every } y \in V_x, y \cdot [0, t(x)] \subset N \quad \text{and} \quad y \cdot t(x) \in U. \tag{1.12}$$

This is possible by our choice of $t(x)$, Lemma 1.12 and Lemma 1.13 (i).

For $x \in N_0$, set $V_x = U$. Thus $\{V_x\}_{x \in N}$ is an open cover of N (open in the relative topology of N).

Let $\{V_i\}_{i \in I}$ be a locally finte rifiniment of $\{V_x\}_{x \in N}$ which exists since N is a metric space.

Observe that, by our construction, for every $i \in I$, there exists $t_i \geqq 0$ such that

$$\eta(t_i, V_i) \subset U \quad \text{and} \quad \eta([0, t_i], V_i) \subset N. \tag{1.13}$$

Now let $\{\beta_i(x)\}_{i \in I}$ be a partition of the unity relative to $\{V_i\}_{i \in I}$ i.e. a set of function $\beta_i : N \to \boldsymbol{R}$ whose support is $\overline{V}_i$ and $\sum_{i \in I} \beta_i(x) = 1$ for every $x \in N$. Such partition exists since N is a metric space.

Now set

$$\tau(x) = \sum_{i \in I} \beta_i(x) t_i.$$

Clearly $\tau(x)$ is a continuous function. We claim that

$$x \cdot \tau(x) \in U. \tag{1.14}$$

In order to see this, fix $\overline{x} \in N$ and set

$$t_1(\overline{x}) = \min \{t_i | \overline{x} \cdot t_i \in V_i\}; \qquad t_2(\overline{x}) = \max \{t_i | \overline{x} \cdot t_i \in V_i\}.$$

By (1.13), $\eta(t_i, \overline{x}) \in U$ $(i = 1, 2)$ and $\eta([0, t_1], \overline{x}) \subset N$.

Therefore (1.14) follows from Lemma 1.13 (ii).

Moreover observe that by our construction

$$\tau(x) = 0 \quad \text{for every } x \in N_0. \tag{1.15}$$

Now consider the map h: $[0, 1] \times N \to U$ defined by

$$h(s, x) = \eta(s \cdot t(x), x)$$

h is an homotopy equivalence between N and $\overline{U}$, and by (1.15) it is also an homotopy equivalence between N/N_0 and $\overline{U}/N_0$.

Therefore, by Lemma 1.13 (iv)

$$h(X) = [N/N_0] = [\overline{U}/N_0] = 0. \qquad \blacksquare$$

REMARK 1.14. – Now, few words to compare the Conley index with our generalization.

A closed set X is called by Conley [C] an isolating neighborhood if $I(X) \subset \mathring{X}$ where $I(X) = \{x \in X : x \cdot \boldsymbol{R} \subset X\}$ or, using our notation, $I(X) = \bigcap_{t \geqq 0} G^T(X)$.

Let $\tilde{\Sigma}$ be the family of isolating neighborhoods in M; then if M is compact $\Sigma = \tilde{\Sigma}$. If M is not compact, in general, $\Sigma \subsetneqq \tilde{\Sigma}$. So, in our approach it was necessary to restrict the class of sets X for which to define index pairs (and introduce the operator $G^T(\cdot)$).

Now, observe that the relationship (1.7) gives an equivalence relation on Σ (which we will denote by $\approx$).

Corollary 2.4 states that the index is constant on each equivalence class of $\approx$. If M is compact, then $X \approx Y$ if and only if $I(X) = I(Y)$ (the easy proof of this left to the reader).

So, when M is compact, h depends only on the maximal invariant set $I(X)$ contained in X; therefore it is an *index of isolated invariant sets*. Example 1.10 shows that this is not the case when the compactness is not assumed (in fact $h(X) = \mathbf{1}$ but $I(X) = \Phi$). Concluding, the Conley index is an index of isolated invariant sets; our generalization is an index of a class of open sets Σ which has been chosen in order that the main properties of the Conley theory can be preserved.

EXAMPLE 1.15. – Let $M = E$ be an Hilbert space and let L be a bounded normal invertible operator whose spectrum is far from the imaginary axis.

We consider the flow η defined by the differential equation

$$\dot{x} = Lx. \tag{1.16}$$

We want to compute $h(X, \eta)$ where X is a bounded open neighborhood of 0. By our assumption E can be splitted as follows

$$E = E^+ \oplus E^- \tag{1.17}$$

where E^+ and E^- are two mutually ortogonal subspaces such that exists a constant $\alpha > 0$

$$\begin{cases} \langle Lx, x\rangle \geqq \alpha\|x\|^2 & \forall x \in E^+ \\ \langle Lx, x\rangle \leqq -\alpha\|x\|^2 & \forall x \in E^-. \end{cases} \tag{1.18}$$

According to the splitting (1.17), (1.16) can be written as follows

$$\dot{x}^+ = L^+x^+, \qquad \dot{x}^- = L^-x^-$$

where $x = x^+ + x^-$ with $x \in E$ and $L^\pm = L|_{E^\pm}$.

Now, if Y is any other bounded open neighborhood of 0, by (1.18), it is easy to check that $X, Y \in \Sigma(\eta)$ and that (1.7) is satisfied. Then $h(X) = h(Y)$. In particular we can take

$$Y = (B_R \cap E^+) \times (B_R \cap E^-)$$

where B_R is the open ball of radious R.

It is easy to check that

$$(\overline{Y}, \overline{(B_R \cap E^+) \times \partial(B_R \cap E^-)})$$

is an index pair and that it is homotopically equivalent to

$$(\overline{B_R \cap E^-}, \partial(B_R \cap E^-)).$$

Also we have

$$\left[\frac{\overline{B_R \cap E^-}}{\partial(B_R \cap E^-)}\right] = \begin{cases} [S^N, *] & \text{if } \dim E^- = N \\ [S^\infty, *] = \mathbf{0} & \text{if } E^- \text{ is infinite dimensional.} \end{cases}$$

So concluding, we have

$$h(X) = h(Y) = [S^N, *]$$

where N is $\dim E^-$ and remembering that $[S^\infty, *] = [*, *] = \mathbf{0}$.

2. – Stability and homotopy invariance of the generalised Conley index.

The stability and homotopy invariance of the Conley index is a very important tool in the applications. However, in this paper, we consider only applications relative to variational systems (cf. section 5) where a much richer structure is exploited. In the case of variational systems the stability of the index is much easier to prove and independent proof of it will be given in Th. 5.16. Nevertheless we think that the stability of the index is so important that a general proof is worth to be presented at the level of generality of section 1. In the compact case the homotopy invariance of the index (or «continuation property») has been investigated by Conley himself and others ([CO], [CHU], [SA]).

We start with some notation:

$$\mathcal{T}(M) = \{X \subset M | X \text{ is open}\}$$

and

$$d_H(X, Y) = \sup_{x \in X} d(x, Y) + \sup_{y \in Y} d(y, X).$$

If M is bounded, then d_H is the Housdorff metric between $\overline{X}$ and $\overline{Y}$; in general, however, the function d_H can take the value $+\infty$.

We need also the following notation

$$X \overset{\delta}{\sqsubset} Y \Leftrightarrow N_\delta(X) \subset Y, \qquad X \sqsubset Y \Leftrightarrow \exists \delta > 0 \quad \text{s.t. } X \overset{\delta}{\sqsubset} Y.$$

We set

$\Sigma_0 = \Sigma_0(\eta) = \{X \in \mathcal{F}(M) | \exists T, \delta > 0 \text{ s.t. } G^T(N_\delta(X)) \sqsubset X, \text{ and}$

$\eta(\cdot, T)|_{N_\delta(X)}$ is uniformly continuous$\}$.

Clearly $\Sigma_0 \subset \Sigma$.

THEOREM 2.1. – Let $X \in \Sigma_0(\eta)$ and let $\tilde{\eta}$ be a flow such that

$$d(\eta(t,x), \tilde{\eta}(t,x)) \leqq \varepsilon \qquad \forall t \in [-T, T], \ \forall x \in X$$

where ε and t are suitable positive constants, which depend on X and η. Then

(i) $X \in \Sigma_0(\tilde{\eta})$

(ii) $h(X, \tilde{\eta}) = h(X, \eta)$.

Before proving Theorem 2.1 we will see two important consequences of this theorem.

COROLLARY 2.2. – Let $X, \eta, \tilde{\eta}$ be as in Theorem 2.1 and let $\tilde{X}$ be an open set such that

$$d_H(X, \tilde{X}) < \varepsilon_1 \tag{2.1}$$

where ε_1 is a positive distance depending on $X, \eta, \tilde{\eta}$ but not on $\tilde{X}$.
Then

(i) $\tilde{X} \in \Sigma_0(\tilde{\eta})$

(ii) $h(\tilde{X}, \tilde{\eta}) = h(X, \eta)$.

PROOF. – By Th. 2.1 (i), there exist $T, \delta_1, \delta_2 > 0$ such that

$$G^T(N_{\delta_1}(X), \tilde{\eta}) \overset{\delta_2}{\sqsubset} X.$$

Now choose $\varepsilon_1 > 0$ smaller then $\min(\delta_1/2, \delta_2/2)$. Then by (2.1)

$$G^T(N_{\delta_1}(X), \tilde{\eta}) \overset{\varepsilon_1}{\sqsubset} \tilde{X}X. \tag{2.2}$$

Moreover, by the choice of ε_1, we have

$$\tilde{X} \subset N_{\varepsilon_1}(X) \subset N_{\delta_1/2}(X) \qquad \text{and } N_{\varepsilon_1}(\tilde{X}) \subset N_{\delta_1}(X).$$

By the above formula and (2.2) we get,

$$G^T(N_{\varepsilon_1}(\tilde{X}), \tilde{\eta}) \subset G^T(N_{\delta_1}(X), \tilde{\eta}) \overset{\varepsilon_1}{\sqsubset} X \cap \tilde{X}. \tag{2.3}$$

The above formula proves (i).

Moreover by (2.2) and (2.3), we have that

$$G^T(X, \tilde{\eta}) \subset \tilde{X} \qquad \text{and } G^T(\tilde{X}, \tilde{\eta}) \subset X.$$

Then by Corollary 1.7 we have

$$h(X, \tilde{\eta}) = h(\tilde{X}, \tilde{\eta}).$$

The conclusion follows by (ii) of Theorem 2.1. ∎

COROLLARY 2.3. – Let η_λ, $\lambda \in [0, 1]$, be a family of flows depending continuously on λ with respect to the topology of the uniform convergence on $X \times [-T, T]$ for every $T > 0$ where $X \subset M$.

Suppose that X_λ is a family of sets contained in X and depending uniformly on λ with respect to the Hausdorff topology.

Finally suppose that $X_\lambda \in \Sigma_0(\eta_\lambda)$ for every $\lambda \in [0, 1]$. Then $h(X_\lambda, \eta_\lambda)$ does not depend on λ.

PROOF. – By Corollary 2.2, for every $\bar{\lambda} \in [0, 1]$, there exists a neighborhood of λ, $I_{\bar{\lambda}}$ such that

$$h(X_\lambda, \eta_\lambda) \qquad \text{is constant for } \lambda \in I_{\bar{\lambda}}.$$

Then the conclusion follows strightforward. ∎

The proof of Theorem 2.1 is involved and relies on several lemmas.

LEMMA 2.4. – Take $X \in \Sigma$ and T large enough such that

$$G^{T/2}(X) \subset X. \tag{2.4}$$

Set $\varphi_1 : \operatorname{int}(G^{T/2}(X)) \to G^T(X)/\Gamma^T(X)$ be defined as follows

$$\varphi_1(x) = \begin{cases} [x \cdot T] & \text{if } x \cdot T \in G^T(X) \\ [\Gamma^T(X)] & \text{if } x \cdot T \notin G^T(X). \end{cases}$$

Then φ_1 is continuous.

PROOF. – It is obvious that $\varphi_1(x)$ is continuous if $x \cdot T \in \operatorname{int}(G^T(x))$ or $x \cdot T \notin G^T(X)$. So we have to consider only the case $x \cdot T \in \partial G^T(X)$. First notice that

$$x \in \operatorname{int}(G^{T/2}(X)) \Rightarrow x \cdot [0, T/2] \subset X. \tag{2.5}$$

Moreover

$$x \cdot T \in G^T(X) \Rightarrow x \cdot \left[\frac{1}{2}T, \frac{3}{2}T\right] \subset G^{T/2}(X).$$

Thus by (2.4) and the above formula $x\cdot[(1/2)T,(3/2)T]\subset X$ and by (2.5) it follows that

$$x\cdot\left[0,\frac{3}{2}T\right]\subset X. \tag{2.6}$$

We claim that

$$x\cdot T\in\partial G^T(X)\Rightarrow x\cdot T\in\Gamma^T(X). \tag{2.7}$$

In fact if $x\cdot T\in\partial G^T(X)$ there exists $t\in[0,2T]$ such that $x\cdot t\in\partial X$.

By (2.6) we have that $t\geqq(1/2)T\geqq T$. Then by the definition of $\Gamma^T(X)$, (2.7) follows. So we have that

$$x\cdot T\in\partial G^T(X)\Rightarrow\varphi_1(x)=[\Gamma^T(X)]$$

and by the above formula the continuity of φ_1 at x follows easily. ∎

LEMMA 2.5. – The function $\varphi_2\colon G^T(X)/\Gamma^T(X)\to G^T(X)/\Gamma(X)$ defined as follows

$$\varphi_2([x])=\begin{cases}[x\cdot T] & \text{if } x\cdot T\in G^T(X)\\ [\Gamma^T(X)] & \text{otherwise}\end{cases}$$

is continuous.

PROOF. – The proof of thie lemma is contained in the proof of Th. 1.5 when it is shown that $h(t,[x])$ is continuous. ∎

LEMMA 2.6. – Let $X\in\Sigma_0(\eta)$ and let $T>0$ be large enough that

$$G^{T/2}(X)\overset{\delta}{\sqsubset}X\qquad\text{for some }\delta>0. \tag{2.8}$$

Then there exists $\delta_1=\delta_1(\eta,X)$ such that

$$x\in N_{\delta_1}(\Gamma^T(X))\Rightarrow x\cdot\left[0,\frac{3}{2}T\right]\cap\partial X\neq\varphi.$$

PROOF. – Choose δ_1 small enough that

$$d(x_1,x_2)<\delta_1\Rightarrow d(x_1\cdot T,x_2\cdot T)\leqq\delta/2\qquad\forall x_1,x_2\in X, \tag{2.9}$$

This is possible by the uniform continuity of $\eta(T,\cdot)$.

So we have

$$
\begin{aligned}
& x \in N_{\delta_1}(\Gamma^T(X)) \Rightarrow && \\
& \exists \bar{x} \in \Gamma^T(X)\colon d(x, \bar{x}) \leqq \delta_1 \Rightarrow && \text{[by (2.9)]} \\
& d(x \cdot T, \bar{x} \cdot T) < \delta/2 \Rightarrow && [\text{since } \bar{x} \cdot T \in \partial X] \\
& d(x \cdot T, \partial X) < \delta/2 \Rightarrow && \text{[by (2.8)]} \\
& x \cdot T \notin G^{T/2}(x) \Rightarrow && [\text{by the definition of } G^{T/2}(X)] \\
& x \cdot \left[\frac{1}{2}T, \frac{3}{2}T\right] \cap \partial X \neq \varphi. \quad \blacksquare &&
\end{aligned}
$$

In the following lemmas we shall write $\eta_t(x)$ instead of $\eta(t, x)$ to simplify the notation.

LEMMA 2.7. – Take $X \in \Sigma_0(\eta)$ and choose T large enough that

$$G^T(X) \overset{\delta}{\sqsubset} G^{T/2}(X) \overset{\delta}{\sqsubset} X. \tag{2.10}$$

Let $\tilde{\eta}$ be a flow such that

$$d(\tilde{\eta}_t(x), \eta_t(x)) \leqq \frac{1}{2} \qquad \forall x \in X \ \forall t \in [-T, T] \tag{2.11}$$

where $\delta_1 = \delta_1(\eta, X) < \delta$ is defined in Lemma 2.6.

Let $h\colon [0, 1] \times G^T(X)/\Gamma^T(X) \to G^T(X)/\Gamma^T(X)$ be defined as follows

$$h(\lambda, [x]) = \begin{cases} [\eta_{2T} \circ \eta_{-\lambda T} \circ \eta_{\lambda T}(x)] & \text{if } \eta_{[0, 2T]}(x) \subset G^T(X) \\ [\Gamma^T(X)] & \text{otherwise}. \end{cases}$$

Then h is continuous.

PROOF. – By (2.11) taking $t = -\lambda T$ and replacing x with $\eta_{-\lambda T}(x)$ we get

$$d(\tilde{\eta}_{-\lambda T} \circ (x), x) \leqq \delta_1/2)(\leqq \delta/2) \qquad \forall \lambda \in [0, 1], \forall x \in G^T(X). \tag{2.12}$$

Then by (2.10), the function

$$x \to \eta_{-\lambda T} \circ \eta_{\lambda T}(x)$$

maps $G^T(X)$ into $(G^{T/2}(X))$ for every $\lambda \in [0, 1]$.

Now consider the function $g\colon [0, 1] \times G^T(X) \to G^T(X)/\Gamma^T(X)$ defined as follows

$$g(\lambda, x) = \begin{cases} [\eta_{2T} \circ \eta_{-\lambda T} \circ \eta_{\lambda T}(x)] & \text{if } \eta_{[0, 2T]} \subset G^T(X) \\ [\Gamma^T(X)] & \text{otherwise} \end{cases}$$

Then we have $g(\lambda, x) = \phi_2 \circ \phi_1 \circ (\tilde{\eta}_{-\lambda T} \circ \eta_{\lambda T})$ where

$$\tilde{\eta}_{-\lambda T} \circ \eta_{\lambda T}\colon [0,1] \times G^T(X) \to G^{T/4}(X)$$

$$\varphi_1\colon G^{T/4}(X) \to G^T(X)/\Gamma^T(X) \qquad \text{is defined by Lemma 2.4.}$$

$$\varphi_2\colon G^T(X)/\Gamma^T(X) \to G^T(X)/\Gamma^T(X) \quad \text{is defined by Lemma 2.5.}$$

Since all the above maps are continuous also g is continuous.

It remains to prove that

$$h(\lambda, [x]) = g(\lambda, x).$$

So we have to prove that if $x \in \Gamma^T(X)$ then $g(t, x)$ is constant, so that the above equality makes sense.

By (2.12) we have

$$x \in \Gamma^T(X) \Rightarrow \tilde{\eta}_{-\lambda T} \circ \eta_{\lambda T}(x) \in N_{\delta_1}(\Gamma^T(X)).$$

By the above formula and Lemma 2.6 we have that

$$x \in \Gamma^T(X) \Rightarrow \eta_{[0,2T]} \circ \eta_{\lambda T} \circ \eta_{-\lambda T}(x) \cap \partial X \neq \varphi.$$

Therefore $g(\lambda, x) = [\Gamma^T(X)]$ $\forall x \in \Gamma^T(X)$. ■

LEMMA 2.8. – Take T large enough that

$$G^T(X) \overset{\delta}{\sqsubset} G^{T/2}(X) \overset{\delta}{\sqsubset} X$$

and take $\delta_1 = \delta_1(X, \eta) < \delta/4$.

Now take $\tilde{\eta}$ close enough to η such that

$$(2.13) \quad \begin{cases} \text{(i)} & d(\tilde{\eta}_t, (x), \eta_t(x)) \leqq \delta_1 \quad \text{for every } x \in X \text{ and } t \in [-2T, 2T] \\ \text{(ii)} & N_{\delta_1}(G^T(X)) \subset \tilde{G}^{T/2}(X) \\ \text{(iii)} & \tilde{G}^T(X) \overset{2\delta_1}{\sqsubset} X. \end{cases}$$

Then the function $f\colon G^T(X)/\Gamma^T(X) \to \tilde{G}^T(X)/\tilde{\Gamma}^T(X)$ defined as follows

$$f([x]) = \begin{cases} [\tilde{\eta}_{2T} \circ \eta_{-T}(x)] & \text{if } \tilde{\eta}_{2T} \circ \eta_T(x) \in \tilde{G}^T(x) \\ [\tilde{\Gamma}^T(x)] & \text{otherwise} \end{cases}$$

is continuous (we have used the notation $\tilde{G}^T(X) = G^T(X, \tilde{\eta})$ and $\tilde{\Gamma}^T(X) = \Gamma^T(X, \tilde{\eta})$).

PROOF. – By (2.13) (i) we set

$$d(\tilde{\eta}_{-t} \circ \eta_t(x), x) \leqq \delta_1 \qquad \forall x \in G^T(X) \quad \forall t \in [0, T]. \tag{2.14}$$

Then, by (2.13) (ii), the function $\tilde{\eta}_T \circ \eta_T$ maps $G^T(X)$ into $\operatorname{int} G^{T/2}(X)$.

Now define $g\colon G^T(X) \to \tilde{G}^T(X)/\tilde{\Gamma}^T(X)$ as follows

$$g(x) = \begin{cases} \tilde{\eta}_{2T} \circ \eta_{-T}(x) & \text{if } \tilde{\eta}_{2T} \circ \eta_{-T}(x) \in \tilde{G}^T(X) \\ [\tilde{\Gamma}^T(x)] & \text{otherwise}. \end{cases}$$

Notice that

$$g(x) = \tilde{\phi}_1 \circ (\tilde{\eta}_T \circ \eta_{-T})$$

where $\tilde{\varphi}_1\colon \operatorname{int} \tilde{G}^{T/2}(X) \to \tilde{G}^T(X)/\tilde{\Gamma}^T(X)$ is the map of Lemma 2.4 with $G^T(X), \Gamma^T(X)$ and $\tilde{\eta}_t$ replaced by $\tilde{G}^T(X), \tilde{\Gamma}^T(X)$ and η_t respectively.

Therefore g is continuous. It remains to prove that

$$f([x]) = g(x).$$

So we have to prove that

$$x \in \Gamma^T(X) \Rightarrow g(x) \quad \text{is constant}$$

or more exactly $g(X) = [\tilde{\Gamma}(X)]$.

Use (2.13) (i) with $t = 2T$ and x replaced by $\eta_{-T}(x)$ with $x \in \Gamma^T(X)$; then we have

$$d(\tilde{\eta}_{2T} \circ \eta_{-T}(x), \eta_{2T} \circ \eta_{-T}(x)) \leqq \delta_1$$

or

$$d(\tilde{\eta}_{2T} \circ \eta_{-T}(x), \eta_T(x)) \leqq \delta_1 .$$

Since $x \in \Gamma^T(X)$, we have that $\eta_T(X) \in \partial X$, and by the above formula

$$d(\tilde{\eta}_{2T} \circ \eta_{-T}(x), \partial X) \leqq \delta_1 .$$

Thus $\tilde{\eta}_{2T} \circ \eta_{-T}(x) \in G^T(X)$. So we have proved that

$$x \in \Gamma^T(X) \Rightarrow g(x) = \tilde{\Gamma}^T(X)$$

and this completes the proof of the lemma. ∎

PROOF OF THEOREM 2.1. – Take T and ε such that (2.11) and (2.13) are satisfied with $\delta_1 < 2\varepsilon$.

Moreover, if ε is small enough, we have also

$$(2.14)\quad \begin{cases} (a) & N_{\delta_1}(\tilde{G}^T(X)) \subset G^{T/2}(X) \\ (b) & \tilde{G}^T(X) \overset{2\delta_1}{\sqsubset} \overline{X}. \end{cases}$$

Now let $f\colon G^T(X)/\Gamma^t(X) \to \tilde{G}^T(X)/\tilde{\Gamma}^T(X)$ be the function defined in Lemma 2.8. We have to prove that f is an homotopy equivalence.

We clain that $\tilde{f}\colon \tilde{G}^T(X)/\tilde{\Gamma}^T(X) \to G^T(X)/\Gamma^T(X)$ is the homotopy inverse of f ($\tilde{f}$ is defined as f replacing $G^T(X)$ with $\tilde{G}^T(X)$, etc...).

f and $\tilde{f}$ are continuous by virtue of Lemma 2.8 and (2.14).

Moreover $\tilde{f} \circ f = h(1, \cdot)$ where h is defined in Lemma 2.7.

Lemma 2.7 shows that $\tilde{f} \circ f \sim h(0, \cdot)$ (where «$\sim$» means homotopy equivalence).

Moreover it is strightforward to show that $(h, \cdot) \sim Id$. Thus $\tilde{f} \circ f \sim Id$.

Analogously we can show that $f \circ \tilde{f} \sim Id$ and this proves Theorem 2.1.

EXAMPLE 2.9. – Let η be the flow defined on M by the defferential equation

$$\dot{x} = F(x).$$

We suppose that M is an Hilbert space E (or an Hilbert manifold). Let $\overline{x}$ a nondegenerate critical point for F i.e. $F(\overline{x}) = 0$ and $F'(\overline{x})\colon T_{\overline{x}}M \to T_{\overline{x}}M$ (where $T_{X_0}M$ denotes the tangent space at X_0) is defined (as Frechèt derivative) and it is an invertible normal operator.

Since $F'(x)$ is a normal operator, we have (cf. Ex. 1.15)

$$T_{\overline{x}}M = E^+ \oplus E^-$$

where E^+ is the stable manifold of η and E^- the unstable manifold.

Now let η_0 be the flow defined by the following equation

$$\dot{x} = \overline{x} + F'(x_0) \cdot x.$$

By Theorem 2.1 it follows that

$$h(U, \eta) = h(U, \eta_0)$$

where U is a neighborood of $\overline{x}$ sufficiently small.

Therefore by the Example 1.15, if follows that

$$(2.15)\qquad h(U, \eta) = (S^{m(x)}, *)$$

where $m(x) = \dim E^-$.

3. – The generalized Conley index and compactness..

For $X \in \Sigma(\eta)$ we set

$$I(X) = \bigcap_{T>0} G^T(X) = \{x \in X | \eta(t, x) \in X \text{ for every } t \in \boldsymbol{R}\}.$$

The following compactness assumption is very important for our theory:

DEF. 3.1. – Let $X \in \Sigma$. We say that X satisfies the property (C) if for every neighborhood U of $I(X)$ there exists $T > 0$ such that

$$G^T(X) \subset U.$$

PROP. 3.2. – Suppose that $X, Y \in \Sigma$ and that satisfy the property (C). Then

$$I(X) = I(Y) \Rightarrow h(X) = h(Y).$$

PROOF. – Let $S = I(X) = I(Y)$. $U = X \cap Y$ is a neighborhood of S. Then, since X and Y satisfy the property (C) there exists $T > 0$ such that

$$G^T(X) \subset U \subset Y \quad \text{and} \quad G^T(Y) \subset U \subset X.$$

The conclusion follows by Corollary 1.7. ■

DEF. 3.3. – We say that $S \subset X$ is a (C)-invariant set if

(i) S is an invariant set

(ii) S has a neighborhood U which satisfies the property (C) and such that $I(U) = S$.

By the remarks before Prop. 3.3 and by the Prop. 3.3, it follows that any neighborhood sufficiently small of S has the same homotopy index.

Therefore it is natural to define the index of a (C)-invariant set S as follows:

(3.1) $\quad h(S) = h(U) \quad$ where $U \in \Sigma$ neighborhood of S sufficiently small.

The following proposition gives a critierium to check if a set $U \in \Sigma$ satisfies the property (C).

PROP. 3.4. – Let $U \in \Sigma$ and suppose that

(3.2) given a sequence $x_n \in U$ and a sequence $t_n \to +\infty$ such that $x_n \cdot [0, t_n] \subset U$, then the sequence $x_n \cdot t_n$ has a limit point.

Then U satisfies the property (C).

PROOF. – We argue indirectly and suppose that there exists a neighborhood V of

$I(U)$ such that for every $T > 0$

$$G^T(U) \not\subset V.$$

Then there exists a sequence $y_n \in U$ and a sequence $t_n \to +\infty$ such that

$$y_n \in G^{t_n}(U) - V.$$

If we set $x_n = y_n(-t_n)$, then $x_n \cdot [0, t_n] \subset U$. Then by (3.2) $x_n \cdot t_n$ is convergent to some $\bar{y}$ (may be considering a subsequence). By its construction $\bar{y} \cdot \boldsymbol{R} \subset U$, therefore $\bar{y} \in S$.

However, since $\bar{y} = \lim_{n \to +\infty} y_n$, we have that $\bar{y} \notin \mathring{V}$. And this is a contradiction, since V is a neighborhood of S. ■

COROLLARY 3.5. – Let M be a locally compact space. Then any compact invariant isolated set $S \subset M$ is a (C)-invariant set.
Therefore, the index (3.1) is defined for such S.

PROOF. – Clearly every compact neighborhood of S satisfies (3.2). ■

REMARK 3.6. – When M is locally compact we get the «classical» Conley theory (of Remark 1.14).

The property (3.2) (which was introduced by Rybakowsky [R]) can replace the local compactness of M in such a way that the main properties of the «original» Conley index are preserved (in particular it is possible to define the index of an isolated invariant set).

Our theory has been developed without any request of compactness, replacing the index of an invariant set with the index of a set $X \in \Sigma$.

A compactness property, as the property (C), is required only to define the index of an invariant set as in the original Conley theory.

PROP. 3.7. – Let U satisfy the property (C) and suppose that $I(U)$ is compact. Then $U \in \Sigma_0$.

PROOF. – Let

$$\varepsilon = d(\partial U, I(U)).$$

Since $I(U)$ is compact then $\varepsilon > 0$. Then setting $V = N_{\varepsilon/2}(I(U))$, we have that $V \in \Sigma$ and that, for T large enough

$$G^T(U) \subset V \quad \text{(since } U \text{ satisfies the property (C))}.$$

Thus $V \overset{\varepsilon/2}{\sqsubset} U$ as we wanted to prove. ■

EXAMPLE 3.8. – Let $\bar{x}$ be as in Example 2.9. Then $\bar{x}$ is a (C)-invariant set and

$$h(\bar{x}) = (S^{m(\bar{x})}, *).$$

4. – The generalized Morse index.

Let $\bar{H}^*(\cdot,\cdot)$ denote the Alexander-Spanier cohomology with coefficients in some field F (cf. [Sp]).

We recall that the Alexander-Spanier cohomology satisfies the following property which is not shared by the singular cohomology theory.

TH. 4.1. – Let (X, A) and (Y, B) two pairs of topological spaces. We suppose that X and Y are paracompact Hausdorff spaces and that A and B are closed in X and Y respectively. Moreover suppose that $X - A$ and $Y - B$ are homeomorphic. Then

$$H^*(X, A) \simeq H^*(Y, B).$$

PROOF. – See [Sp], Th. 5, pag. 318. ■

Now for every pairs of closed spaces (X, A) we set

$$p(X, A) = p_t(X, A) = \sum_{q=0}^{\infty} [\dim \bar{H}^q(X, A)]\, t^q$$

$p(X, A)$ is a formal series whose coefficients are cardinal numbers; these numbers are known as Betti numbers.

If X is compact manifold with boundary A, then $p(X, A)$ reduces to a polinomial, called Poincaré or Betti polynomial.

$p(X, A)$ is a topological invariant which carries part of the information contained in the cohomology algebra $\bar{H}^*(X, A)$.

When $A = \emptyset$ we shall write $p(X)$ instead of $p(X, \emptyset)$. Wil shall denote by S the set of formal series with cardinal coefficients. The following properties of $p(X, A)$ will be used to study the generalized Morse index.

LEMMA 4.2. – Let (X, A) and (Y, B) be couples of closed subspaces of a metric space. Then

(i) $p(X, A) = p(X/A, [A])$

(ii) if $X \cap Y = \Phi$ then $p(X \cup Y, A \cup B) = p(X, A) + p(Y, B)$

(iii) $p(X, A) \times (Y, B) = p(X, A) \cdot p(Y, B)$
where $(X, A) \times (Y, B) = (X \times Y, X \times B \cup Y \times A)$

(iv) if $B \subset A \subset X$ then there exists $Q(t) \in S$ s.t.
$p_t(X, A) + p_t(A, B) = p_t(X, B) + (1 + t)\, Q(t)$.

PROOF. – (i) Let $\pi\colon X \to X/A$ be the projection map. Then $\pi_{|X-A}$ is a homeomorphism between $X - A$ and $X/A - [A]$. Thus the conclusion follows from Th. 4.1.

(ii) trivial.

(iii) Since (X, A) and (Y, B) are closed pairs, there is an exact Mayer-Vietoris sequence for the $\overline{H}^*$ cohomology (cf. [Sp] pag. 291).

But every closed pairs of Hausdorff-paracompact spaces is a tout pair for the Alexander-Spanier cohomology (cf. [Sp] pag. 315).

Therefore $\overline{H}^* = H^*$ on such pairs. Therefore the Künneth formula can be applied to such pairs (cf. [Sp] pag. 249) and get

$$H^*((X,A)\times(Y,B)) = H^*(X,A) \oplus H^*(Y,B).$$

From the above formula the conclusion follows.

(iv) Let us consider the exact sequence relative to the triple $B \subset A \subset X$:

$$\dots \xrightarrow{\delta^*_{q-1}} H^q(X,A) \xrightarrow{i^*_q} H^q(X,B) \xrightarrow{j^*_q} H^q(A,B) \xrightarrow{\delta^*_q} \dots \tag{4.1}$$

and set

$$a_q = \dim(\ker i^*_q)$$

$$b_q = \dim(\ker j^*_q)$$

$$c_q = \dim(\ker \delta^*_q)$$

By the exactness of (4.1) we get

$$\dim H^q(X,A) = c_{q-1} + a_q \qquad \text{(with the convenction that } c_{-1} = 0)$$

$$\dim H^q(X,B) = a_q + b_q$$

$$\dim H^q(A,B) = b_q + c_q.$$

Then we have

$$p(X,A) = \sum_{q=0}^{\infty} (c_{q-1} + a_q)\, t^q$$

$$p(X,B) = \sum_{q=0}^{\infty} (a_q + b_q)\, t^q$$

$$p(A,B) = \sum_{q=0}^{\infty} (b_q + c_q)\, t^q$$

Then

$$p(X,A) + p(A,B) = p(X,B) + \sum_{q=0}^{\infty} (c_{q-1} + c_q)\, t^q = p(X,B) + (1+t) \sum_{q=0}^{\infty} c_q t^q .$$

The conclusion follows setting $Q(t) = \sum_{q=0}^{\infty} c_q t^q$.

Notice that the formula (iv) holds even if some of the coefficients are infinite cardinal numbers. ∎

We can now define the generalized Morse index:

DEF. 4.3. – The generalized Morse index is a map

$$i\colon \Sigma(\eta) \to S$$

defined by

$$i_t(X, \eta) = p_t(N, N_0)$$

where (N, N_0) is an index pair for X.

When no ambiguity is possible we shall write $i(X)$ instead of $i_t(X, \eta)$.

Using Th. 1.4 we could define the GIM in the following (formally) simpler way

$$i_t(X, \eta) = \lim_{T\to+\infty} p_t(G^T(X), \Gamma^T(X)).$$

EXAMPLE 4.3. – Let $\eta, \bar{x}, U$ be as in the Example 2.9. Then

$$i(U) = \sum_{q=0}^{\infty} \dim H^q(S^{m(\bar{x})}, *)\, t^q = t^{m(\bar{x})} \qquad \text{[by 3.9]}$$

since we have

$$H^q(S^k, *) = \begin{cases} 0 & \text{if } q \neq k \\ F & \text{if } q = k. \end{cases}$$

REMARK 4.4. – By Lemma 4.2 (i), $p(N, N_0) = p(N/N_0, [N_0])$; so the generalized Morse index depends only on $h(X)$; thus it is well defined by (1.1) (a) and (b). The above remark implies that the generalized Morse index carries less information than the Conley index. Nevertheless it is more usefull since it is much easier to deal with. The following theorem illustrates the first properties of the generalized Morse index:

THEOREM 4.5. – The generalized Morse index satisfies the following properties

(i) if $X \in \Sigma$ and for every $x \in X$, there is $t > 0$ such that $x \cdot t \notin X$, then $i(X) = 0$; in particular if $G^T(X) = \emptyset$ for some $T > 0$, then $i(X) = 0$;

(ii) if $X \in \Sigma$ is contractible and positively invariant, then $i(X) = 1$;

(iii) if $X, Y \in \Sigma$ and $X \cap Y = \emptyset$ then $i(X \cup Y) = i(X) + i(Y)$;

(iv) if η_i is a semiflow on M_i $(i = 1, 2)$, then a semiflow $\eta_1 \times \eta_2$ is defined on $M_1 \times M_2$ as follows

$$(\eta_1 \times \eta_2)(t, (x_1, x_2)) = (\eta_1(t, x_1), \eta_2(t, x_2));$$

then if $X_i \in \Sigma(\eta_i)$ $(i = 1, 2)$, we have that $X_1 \times X_2 \subset \Sigma(M_1 \times M_2, \eta_1 \times \eta_2)$ and $i(X_1 \times \times X_2, \eta_1 \times \eta_2) = i(X_1, \eta_1) \cdot i(X_2, \eta_2)$.

PROOF. – (i) follows from Theorem 1.11; (ii) follows by the fact that

$$H^q(X) = 1 \qquad \text{if and only if } q = 0 .$$

(iii) and (iv) follow by Lemma 4.3 (ii) and (iii) respectively. ■

Next we are going to prove a property of the GIM which is a generalization of the classical Morse inequalities.

DEF. 4.6. – Take $X_1, X_2 \in \Sigma$ with $X_1 \cap X_2 = \emptyset$. We say that X_2 is over X_1 if there exists $T > 0$ such that $\overline{X}_1 \cap G^T(X_1 \cup X_2)$ is positively invariant with respect to $G^T(X_1 \cup X_2)$.

If X_2 is over X_1 and X_1 is over X_2 then we say that X_1 and X_2 are η-disconnected. Otherwise we say that are η-connected.

EXAMPLE 4.7.

(I): if $\overline{X}_1 \cap \overline{X}_2 = \emptyset$, then X_1 and X_2 are η-disconnected.

(II): Let f be a Liapunov function for (M, η) and let c be a constant which is a regular value for f(i.e. $f(x) = c \Rightarrow f'(x) \neq 0$). We set

$$X_1 = \{x \in M | f(x) < c\}; \qquad X_2 = \{x \in M | f(x) > c\} .$$

Then $X_1, X_2 \in \Sigma$ and X_2 is over X_1.

DEF. 4.8. – Let $X \in \Sigma$. A family of sets $\{X_k\}_{k \leq N}$ is called a *Morse decomposition* of X if

(i) $\overline{X} = \bigcup_{k=1}^{N} \overline{X}_k$

(ii) $X_k \in \Sigma$ for $k = 1, \dots, N$

(iii) $X_k \cap X_h = \emptyset$ for $k \neq h$

(iv) X_{h+1} is over int $\bigcup_{k=1}^{h} \overline{X}$ for $h = 1, \dots, N-1$.

EXAMPLE 4.9. – Let f be a Liapunov function for (M, η) and let $c_1 < c_2 < \dots < c_{N-1}$ be a sequence of regular values for f. Let $c_0 = -\infty$ and $c_N = +\infty$ and

$$X_k = \{x \in X | \, c_{k-1} < f(x) < c_k\}$$

then $\{X_k\}$ is a Morse decomposition of X.

The next theorem states one for the most important properties of the index (as far as the applications are concerned).

THEOREM 4.10. – If $\{X_k\}_{k\leqq N}$ is a Morse decomposition of X, then there exists $Q \in S$ such that

$$\sum_{h=1}^{N} i(X_k) = i(X) + (1+t)\,Q(t) \qquad Q \in S.$$

In order to prove Theorem 4.9 some lemmas are necessary.

LEMMA 4.10′. – Let $X = \operatorname{int}(\overline{X}_1 \cup \overline{X}_2)$ and suppose that X_2 is over X_1. Then there exist closed spaces $N_0 \subset N_1 \subset N_2$ such that (N_2, N_0), (N_2, N_1), (N_1, N_2) are index pairs for X, X_2 and X_1 respectively.

PROOF. – Take T big enough in order that

$$(4.4)\qquad \begin{cases} (a)\ \overline{X}_1 \cap G^T(X) & \text{is positively invariant with respect to } G^T(X) \\ (b)\ (G^T(X), \Gamma^T(X)) & \text{is an index pair for } X \\ (c)\ G^T(X_1) \subset X_1 . \end{cases}$$

We set

$$N_0 = \Gamma^T(X)$$
$$N_1 = (X_1 \cap G^T(X)) \cup \Gamma^T(X)$$
$$N_2 = G^T(X).$$

We want to prove that N_0, N_1, N_2 satisfy the required properties. We now prove that (N_1, N_0) is an index pair for X. Let us check (i) of Def. 1.1. Since $\overline{N_1 - N_0} = \overline{X}_1 \cap G^T(X)$

$$(4.5)\qquad G^T(N_1 - N_0) = G^T(X_1 \cap G^T(X)) \subset G^T(X_1) \subset X_1 \qquad \text{by (4.4) } (c).$$

Also by Lemma 1.3 (i), (iii) and (iv)

$$(4.6)\qquad G^T(N_1 - N_0) = G^T(G^T(X)) = G^{2T}(X) \subset \operatorname{int}[G^T(X)].$$

Then by (4.5) and (4.6)

$$G^T(N_1 - N_0) \subset \operatorname{int}(N_1 - N_0).$$

(iii) of Def. 4.8 holds since $(X_1 \cap G^T(X))$ is positively invariant in $G^T(X)$ by definition and $\Gamma^T(X)$ is positively invariant in $G^T(X)$ by Th. 1.4. Now let us check (iii) of Def. 1.1. If $x \in N_1$ and it leaves N_1 at some times, it has to leave $G^T(X)$ also, since N, is positively invariant in $G^T(X)$. Thus there exists t^* such that $x \cdot t^* \in \Gamma^T(X)$ since $\Gamma^T(X)$ is an exit set for $G^T(X)$. Finally since $G^T(X_1) \subset N_1 - N_0$, (iv) of Def. 1.1 holds. Let us check that (N_2, N_1) is an index pair for X_2.

$$\overline{N_2 - N_1} = \overline{G^T(X) - X_1} = G^T(X) \cap \overline{X}_2 .$$

Then arguing as we have done for $G^T(X) \cap X_1$, it follows that $\operatorname{int}(N_2 - N_1) \in \Sigma$.

(ii) of Def. 1.1 holds since N_1 is positively invariant in N_2 and (iii) holds since $N_1 \supset \Gamma^T(X)$ and $\Gamma^T(X)$ is an exit set for N_2.

(iv) follows by the fact that $G^T(X_2) \subset \overline{N_2 - N_1}$. ∎

COROLLARY 4.11. – If $\overline{X} = \overline{X}_2 \cup \overline{X}_1$ and X_2 is over X_1, then there exists $Q \in \mathcal{S}$ such that

$$i(X_1) + i(X_2) = i(X) + (1+t)\,Q(t).$$

PROOF. – By Lemma 4.2 (iv) applied to the triple N_0, N_1, N_2 defined in Lemma 4.10 we have

$$p(N_2, N_1) + p(N_1, N_0) = p(N_2, N_0) + (1+t)\,Q(t).$$

The conclusion follows by Lemma 4.10 and the definition of the cohomological index. ∎

REMARK 4.12. – It is easy to check that if X_1 and X_2 are η-disconnected, then, for T large enough

$$G^T(X_1 \cup X_2) = G^T(X_1) + G^T(X_2) \qquad \text{and } G^T(X_1) \cap G^T(X_2) = \emptyset.$$

Then

$$\begin{aligned} i(X) &= i(G^T(X_1 \cup X_2) && \text{by Corollary 1.8} \\ &= i(G^T(X_1)) + i(G^T(X_2)) && \text{by Th. 4.5 (iii)} \\ &= i(X_1) + i(X_2). \end{aligned}$$

Comparing this result with Corollary 4.11 we deduce that $Q(t) \neq 0$ implies that X_1 and X_2 are η-connected.

PROOF OF TH. 4.9. – We argue by induction. For $N = 2$ it is true since it is nothing else but Corollary 2.11.

We can suppose that it is true for $N-1$; so there exists $Q_1 \in \mathcal{S}$ such that

$$\sum_{k=1}^{N-1} i(X_k) = i\left(\text{int}\left(\bigcup_{k=1}^{N-1} \overline{x}_k\right)\right) + (1+t)\,Q_1(t).$$

Now, since $\overline{X}_N$ is over $\text{int}\left(\bigcup_{k=1}^{N-1} \overline{x}_k\right)$, applying Corollary 4.11 an other time, we get

$$i(X_N) + i\left(\text{int}\left(\bigcup_{k=1}^{N-1} x_k\right)\right) = i(X) + (1+t)\,Q_2(t) \qquad \text{with } Q_2(t) \in \mathcal{S}.$$

Then the conclusion follows with $Q(t) = Q_1(t) + Q_2(t)$. ∎

If we have enough compactness we can define the Morse index of an isolated invariant set as follows (cf. also (3.1)).

DEF. 4.13. – Let S be a (C)-invariant set (cf. Def. 3.1), then we set $i(S) = i(U)$ where $U \in \Sigma$ is a sufficiently small open neighborhood of S. From the above definition and Theorem 4.10 we get

COROLLARY 4.14. – Let X and $\{X_k\}_{k \leq N}$ be as in Theorem 4.10.

Moreover suppose that X_k satisfy the property (C) ($k = 1, \ldots, N$) and set $S_k = I(X_k)$. Then we have

$$\sum_{k=1}^{N} i(S_k) = i(X) + (1 + t)\, Q(t) \qquad Q \in \mathcal{S}.$$

Observe that in Corollary 4.14 the property (C) for X is not required.

EXAMPLE 4.15. – Let η be a flow as in Example 2.9. Suppose that X and the X_k's satisfy the assumptions of Lemma 4.14. Moreover suppose that each X_k contains only one nondegenerate critical point x_k.

Therefore, by the Example 4.4 and Corollary 4.14, we get

$$\sum_{k=1}^{N} t^{m(x_k)} = i(X) + (1 + t)\, Q(t) \qquad Q \in \mathcal{S}. \tag{4.7}$$

More in particular, if $F(X) = Df(x)$, then $m(x)$ reduces to the classical Morse index and (4.7) reduces to the classical Morse inequalities.

5. – Variational systems.

The generalized Conley theory concerns general flows, however it can be applied to the study of the critical points of a function f defined on a manifold, constructing a flow whose stationary points correspond to the critical points of f and to obtain a theory comparable with the classical Morse theory. This will be done in the next sections.

Let M be a Finsler manifold, i.e., a C^1-manifold modelled on a Banach space with a continuous assignment of a norm to each tangent space which is compatible with its Banach structure locally uniformly (actually every Banach manifold admits a Finsler structure which determines a metric compatible with its structure cf. e.g. [Pa 3]). If M has a Riemannian structure on a possible infinite dimensional Hilbert space, we will simply say that M is a Hilbert manifold.

We will write $f \in C^1(M)$ if f is a function which is Frechét differentiable on M and such that its derivative is continuous for every $x \in M$, $f'(x) \in T_x^* M$ will denote the differential of F, and $\langle \cdot, \cdot \rangle$ the paring between T^*M and TM; C^2 will denote the class of twice differentiable functions on M. Also we shall use the following notation:

$$f_a^b := \{x \in M | a < f(x) < b\}$$
$$f^b := \{x \in M | f(x) < b\}$$
$$f_a := \{x \in M | f(x) > a\}$$
$$K(A) = K(A, f) := \{x \in A | f'(x) = 0\}$$

If η is a flow on M of class C^1 we set

$$Df(x) \overset{\text{def}}{=} \frac{d}{dt} f(\eta(t)) \bigg|_{t=0} = \langle f'(x), \dot{\eta} \rangle .$$

When M is not compact and in particular when M is not even locally compact, there is a compactness assumption which plays a crucial role in the study of the critical points.

DEFINITION 5.1. – (i) Let $f \in C^1(M)$. We say that f satisfies (P.S.) (i.e. the Palais-Smale assumption) if any sequence $\{x_n\}$ such that $f(x_n)$ is bounded and $f'(x_n) \to 0$ has a converging subsequence.

(ii) If $S \subset \boldsymbol{R}$ is an open set, we say that f satisfy P.S. in S if any sequence $\{x_n\}$ such that $f(x_n) \to c \in S$ and $f'(x_n) \to 0$ has a converging subsequence.

(iii) If $A \subset M$, we say that f satisfies P.S. in A if (i) holds only for sequences $\{x_n\} \subset A$.

The assumption (P.S.) has been introduced by Palais and Smale [PAS] in a pioneering paper on the infinite dimensional Morse theory. Since then it has always been used in the infinite dimensional critical point theory. Also many generalizations have been done [cf. e.g. CE, BBF, B4] for particular problems. Theese generalizations could be adapted to our theory; however we will not discuss them to avoid extra technicalities.

DEFINITION 5.2. – A variational system relative to f is a couple $\{\eta, \Gamma(\eta)\}$ where η is a C^1-flow which satisfies the following properties

(i) (*a*) $Df(x) \leqq \alpha(\|f'(x)\|) \cdot \alpha(\text{dist}(x, K(M))$ where α is a strictly increasing function with $\alpha(0) = 0$.

(*b*) $\|\dot{\eta}\| \leqq 1$

and $\Gamma = \Gamma(\eta)$ is the family of sets A which satisfy the following assumptions:

(ii) (*a*) $A \in \Sigma(\eta)$ (i.e. $\Gamma(\eta) \subset \Sigma(\eta)$)

(*b*) $f|_A$ is bounded

(*c*) f satisfies P.S. in A.

Notice that if $B \in \Sigma(\eta)$ and $B \subset A \in \Gamma(\eta)$ then $B \in \Gamma(\eta)$.

As we shall see in Theorem 5.5, given any $f \in C^1(M)$ there exists always a flow η which satisfies (i). Therefore a variational system can be defined for every $f \in C^1(M)$. However, because of the assumption (ii) (*c*), the family $\Gamma(\eta)$ could be very small and this theory may not give good information. Nevertheless, if f satisfies P.S., then the family Γ is reasonably large.

The next examples will clarify what we mean:

EXAMPLE 5.3. – Let M be a compact manifold and $f \in C^2(M)$. Let η be the flow relative to equation

$$\dot{x} = -k\,\nabla f(x) \qquad \text{with } k = \left[\max_{x\in M} \|\nabla f(x)\|\right].$$

It is not difficult to check that $\{\eta, \Sigma(\eta)\}$ is a variational system relative to f.

EXAMPLE 5.4. – Set $f \in C^2(M)$ where M is a Hilbert manifold and let η be the flow relative to the differential equation

$$\dot{x} = -F(x) \qquad \text{where } F(x) = \frac{\nabla f(x)}{1 + \|\nabla f(x)\|}$$

Then if F satisfies (P.S), we get a variational system $\{\eta, \Gamma\}$ with

$$\Gamma = \{A \in \Sigma(\eta) | f|_A \text{ is bounded}\}.$$

In particular, if a and b are regular values of $f, f_a^b \in \Gamma$.

In many applications to P.D.E.'s usually f is only of class $C^1(M)$ and not $C^2(M)$, and sometimes M is a Banach space or a Finsler manifold and not a Hilbert manifold. In this case the costruction of a variational system relative to f is more involved as the following theorem will show.

THEOREM 5.5. – Let $f \in C^1(M)$ where M is a Finsler manifold. Then there exists a flow η which satisfies the assumptions (i) of Def. 5.1.

PROOF. – Let Φ be a pseudogradient vector field, i.e. a function

$$\Phi \colon \tilde{M} \to M$$

(where $\tilde{M} = M - K$ and $K = K(M) = \{x \in M | f'(x) = 0\}$) which satisfies the following

(i) $\|\Phi(x)\| \leqq 2\|f'(x)\|$

(ii) $\langle f'(x), \Phi(x)\rangle \geqq \|f'(x)\|$

The concept of pseudogradient vector field has been introduced by Palais [P2]. He has also proved its existence for every $f \in C^1(M)$. (See also [R1]. Appendix A).

Now let φ be a function defined as follows

$$\varphi(x) = \frac{\operatorname{dist}(x, K)}{1 + \operatorname{dist}(x, K)}$$

and

$$F(x) = \begin{cases} 0 & \text{if } x \in K \\ \varphi(x) \dfrac{\Phi(x)}{1 + \|\Phi(x)\|} & \text{if } x \in \tilde{M}. \end{cases}$$

Now consider the following differential equation:

$$\dot{x} = -F(x). \tag{5.1}$$

We claim that (5.1) has a unique solution defined for all $t \in \boldsymbol{R}$ for every $x \in M$. Notice that $F(x)$ is not Lipschitz continuous for $x \in K(M)$. Thus the existence and uniqueness needs to be proved directly.

EXISTENCE. – If the initial data $x_0 \in K(M)$ then we have the solution $x(t) \equiv x_0$.

If $x_0 \in \tilde{M}$, let (t^-, t^+) be the maximum interval for which the solution is defined. By standard theorems on O.D.E.'s it is sufficient to show that $x(t)$ does not go to infinity or to $K(M)$ for $t \to t^{\pm}$ unless $t^{\mp} = \pm\infty$ (so that it can be extended).

$x(t)$ cannot go to ∞ since $\|F(x)\|$ is bounded.

In order to show that $x(t)$ does not reach K in a positive time we have to prove that $\varphi(x_0) > 0$ implies that $\varphi(x(t)) > 0$ for every $t \in (t^-, t^+)$.

The function $t \to \varphi(x(t))$ is in $H^{1,\infty}(t^-, t^+)$ since φ is Lipschitz continuous. So it is differentiable for a.e. $t \in (t^-, t^+)$ and we have

$$\frac{d}{dt}\varphi(x(t)) \geqslant -\|\dot{x}\| = -\left\|\varphi(x(t)) \cdot \frac{\Phi(x(t))}{1 + \Phi(x(t))}\right\| \geqq -\varphi(x(t)).$$

The above inequality implies that

$$\varphi(x(t)) \geqq \varphi(x_0) \exp[-|t|] \quad \text{for } t \in (t^-, t^+). \tag{5.2}$$

Thus the solution can be extended to all $\boldsymbol{R}$; i.e. $t^+ = +\infty$. In the same way we can prove that $t^- = -\infty$.

UNIQUENESS. – It is an immediate consequence of (5.2).

So a flow η is defined and it is easy to check that the function

$$x \to \eta(t, x)$$

is continuous for every $x \in M$.

Clearly η satisfies the assumption (i) and (ii) of the Def. 5.1. ∎

COROLLARY 5.6. – If $f \in C^1(M)$ (where M is a Finsler manifold) and it satisfies P.S. then we have a variational system $\{\eta, \Gamma)$ where η is constructed in Theorem 5.5 and

$$\Gamma = \{A \in \Sigma(\eta): f|_A \text{ is bounded}\}.$$

REMARK 5.7. – If f satisfies (P.S.) only in an open set $S \subset \mathbf{R}$, then the characterization of Γ is more complicated; however if $A \in \Sigma(\eta)$, $f|_A$ is bounded and $f(A) \subset S$, then $A \in \Gamma$.

Before beginning our study of variational systems an other notation is necessary

$$\mathcal{K}_0 = \{K(A) | A \in \Gamma; K(A) \text{ is connected}\}.$$

Notice that, by virtue of Def. 5.1 (ii) (*c*), the sets in $\mathcal{K}_0$ are compact.

THEOREM 5.8.

(i) if f satisfies (P.S.) in S and K is a connected set of critical points such that $f(K) \subset S$ and $\operatorname{dist}(K, K(M) - K) > 0$, then $K \in \mathcal{K}_0$

In particular all the isolated critical points of a function which satisfies (P.S.) belong to $\mathcal{K}_0$.

(ii) if $K \in \mathcal{K}_0$, then it is a (C)-invariant set (cf. Def. 3.3); in particular $i(K)$ is well defined

(iii) if $K \in \mathcal{K}_0$ and $\{\tilde{\eta}, \tilde{\Gamma}\}$ is an other variational system relative to f, then

$$i(K, \eta) = i(K, \tilde{\eta}).$$

This means that $i(K)$ depends only on f and not on the particular variational system we have chosen.

(iv) if $x_0 \in \mathcal{K}_0$ is an isolated local minimum point, then $i(x_0) = 1$.

(v) if $U \in \Gamma$ and $i(U) \neq 0$, then $K(U) \neq \emptyset$.

In order to prove Theorem 5.8 we need some work.

LEMMA 5.9. – Let f satisfies P.S. in an open set $A \subset M$ and be bounded in A and let η be a flow which satisfies (i) (*a*) of Def. 5.2, then

$$\forall \varepsilon > 0 \; \exists \delta > 0 \qquad \text{such that } Df(x) \leqq -\delta \; \forall x \in A - N_\varepsilon(K(A)).$$

In particular the above formula holds for any $A \in \Gamma$.

PROOF. – Suppose that the Lemma is false. Then there exists a sequence $\{x_n\} \subset A - N_\varepsilon(K(A))$ such that $Df(x_n) \to 0$. Then by Def. 5.2 (i) (*a*),

$$\alpha(\|f'(x_n)\|) \cdot \alpha(\operatorname{dist}(x, K(A)) \to \mathbf{0}.$$

This implies that at least one of the two factors converges to 0.

If $\alpha(\|f'(x_n)\|) \to 0$, then $f'(x_n) \to 0$, and since f satisfies P.S., x_n has a converging subsequence $x_n' \to \bar{x}$. By the continuity of f', we have that $f'(\bar{x}) = 0$; so $\bar{x} \in K$ and this is a contradiction since $\operatorname{dist}(x_n, K) > 0$. If $\alpha(\operatorname{dist} x, K(A)) \to 0$, then $\operatorname{dist}(x, K(A)) \to 0$ and also this is a contradiction since $d(x_n, K) > \varepsilon$.

LEMMA 5.10. – Let f and A be as in Lemma 5.9 and let η satisfies the assumptions of Def. (5.2) (i).

Moreover suppose f has only one critical value c corresponding to a critical point in A and that $K = K(\bar{A}) \subset A$.

Then A satisfies the property (C) (cf. Def. 3.1) and K is a (C)-invariant set (cf. Def. 3.3).

PROOF. – Since f satisfies P.S., K is compact. Then by our assumptions, there exists $\varepsilon > 0$ such that $N_{3\varepsilon}(K) \subset A$.

We have to prove that there exists $T > 0$ such that

$$G^T(A) \subset N_{2\varepsilon}(A). \tag{5.3}$$

Let

$$\delta = \inf\{\|Df(x)\|: x \in A - N_\varepsilon(A)\}.$$

By Lemma 5.9, $\delta > 0$. Now suppose that $x \in A - N_{2\varepsilon}(A)$ and that $x \cdot t \subset N_\varepsilon(A)$. Then by Def. 5.2 (i, b),

$$\varepsilon < \|x \cdot \bar{t} - x\| = \left\| \int_0^{\bar{t}} \dot{\eta}(x, t)\, dt \right\| \leqq \int_0^{\bar{t}} \|\dot{\eta}(x, t)\| dt \leqq |\bar{t}|.$$

Then $|\bar{t}| > \varepsilon$. By (5.3), we get that

$$|f(x \cdot \bar{t}) - f(x)| = \left| \int_0^{\bar{t}} Df(\eta(x, t)) dt \right| \geqq \left| \int_0^{\bar{t}} \delta\, dt \right| \geqq \delta|\bar{t}| > \delta\varepsilon. \tag{5.4}$$

Now we set $\varepsilon_1 = \delta\varepsilon$ and

$$\delta_1 = \inf\{\|Df(x)\|: x \in A - f_{c-\varepsilon_1}^{c+\varepsilon_1}\}. \tag{5.5}$$

Since c is the only critical value of in A, then, by Lemma 5.9, $\delta_1 > 0$. Now we set

$$T = \frac{b-a}{\delta} + \frac{b-a}{\delta_1} \qquad \text{where } b = \sup_{x \in A} f(x) \text{ and } a = \inf_{x \in A} f(x).$$

With this choice of T we can prove (5.3). If we take $x \in N_{2\varepsilon}(A)$, we have to show that there exists $t \in [-T, T]$ such that $x \cdot t \in A$.

Now we suppose that $f(x) \leqq c$; then it is sufficient to prove that $x \cdot T \in A$. (If $f(x) \geqq c$, using the same argument we prove that $x \cdot (-T) \in A$).

We now distinguish two cases

(i) $\forall t \in \left[0, \dfrac{b-a}{\delta}\right], \quad x \cdot t \notin N_\varepsilon(A)$

(ii) $\exists \bar{t} \in \left[0, \dfrac{b-a}{\delta}\right] \quad$ s. t. $x \cdot \bar{t} \in N_\varepsilon(A)$.

In the first case we have that

$$f(x \cdot T) \leqq f\left(x \cdot \frac{b-a}{\delta}\right) = f(x) + \int_0^{(b-a)/\delta} Df(x \cdot t)\, dt \leqslant c - \int_0^{(b-a)/\delta} \delta\, dt$$

$$= c - b + a < c \qquad \text{(by (5.3), supposing that } x \cdot [0, T] \subset A).$$

Therefore $x \cdot T \in A$.

In case (ii), we have:

$$f(x \cdot T) \leqq f(\tau) = f(x \cdot \bar{t}) + \int_{t}^{\tau} Df(x \cdot t)\, dt \qquad \text{with } \tau = \bar{t} + \frac{\delta - a}{\delta_1}.$$

Using (5.4), we have that $f(x \cdot \bar{t}) \leqq c - \delta\varepsilon = c - \varepsilon_1$, then supposing that $x \cdot [0, T] \subset A$, by (5.5), we get

$$f(x \cdot) \leqq c - \int_{\bar{t}}^{\tau} \delta_1\, dt = c - (\tau - \bar{t})\, \delta_1 = c - \delta_1 \tau - (b - a) < c - (b + a).$$

Then also in this case $x \cdot T \in A$.

PROOF. OF TH. 5.8. – (i) Since f satisfies P.S. in $S \supset f(K)$, then K is compact. Moreover since K is connected, $f(K)$ consists only of a point, i.e. a critical value c. Then the conclusion follows from the Lemma 5.10.

(ii) If $K \in \mathcal{K}_0$, then K has a neighborhood A which satisfies the assumptions of Lemma 5.10. Then the conclusion follows from this lemma.

(iii) Let η and $\tilde{\eta}$ be the flows relative to the following equations

$$\dot{x} = -F(x) \qquad \dot{x} = -\tilde{F}(x).$$

Now let η_λ be the flow relative to the following equation

$$\dot{x} = -(1 - \lambda) F(x) - \lambda \tilde{F}(x) \qquad \lambda \in [0, 1].$$

Clearly for every $\lambda \in [0, 1]$, $\{M, \eta_\lambda\}$ is a variational system relative to F and K is a (C)-invariant set for η_λ by the part (ii) of this theorem.

Take $\bar{\lambda} \in [0, 1]$ and let $U_{\bar{\lambda}}$ be a neighborhood of K which satisfies the property (C); it exists by (ii).

By Proposition 3.7, $U_{\bar\lambda} \in \Sigma_0$.

Then by Theorem 2.1, $i(U_{\bar\lambda}, \eta_\lambda)$ is constant for $\lambda \in I_{\bar\lambda}$ where $I_{\bar\lambda}$ is a suitable neighborhood of $\bar\lambda$.

This implies that $i(K, \eta_\lambda)$ is constant for $\lambda \in I_{\bar\lambda}$ for every $\bar\lambda \in [0, 1]$. Thus it follows that

$$i(\eta, K) = i(\eta_0, K) = i(\eta_1, K) = i(\bar\eta, K).$$

(iv) Let $f(x_0) = c$ and let U_ε be the connected component of $\{x | c \leqq f(x) < c + \varepsilon\}$ containing x_0.

If ε is small enough, then U_ε is contractible. Then the conclusion follows from Th. 4.5 (ii).

(v) It is a trivial consequence of the fact that f satisfies P.S. in U. ∎

Now we can state the «Morse relations» for variational systems as defined above.

DEF. 5.11. – Let $X \in \Sigma(\eta)$ and $K = K(X)$.

A family of sets $\{U_j\}_{j \leqq I}$ is called an ε-Morse covering of K if

(i) $\overline{U}_j$ is connected for $j \in I$.

(ii) $K \subset \bigcup_{j \in I} U_j \subset N_\varepsilon(K)$.

(iii) $U_j \in \Gamma$ and $\sum_{j \in I} i(U_j) = i(x) + (1 + t)\, Q(t)\ \ Q \in \mathcal{S}$.

The above definition is justified by the following theorem.

THEOREM 5.12. – If $X \in \Gamma$, then for every $\varepsilon > 0$ there exists a finite ε-Morse covering of $K(X)$.

PROOF. – For every $c \in \overline{f(X)}$ there exists $\delta(c) > 0$ such that $f^{c+\delta(c)}_{c-\delta(c)} \in \Gamma$ and that

$$G^{T(c)}(f^{c+\delta(c)}_{c-\delta(c)}) \subset N_{\varepsilon/2}(K). \tag{5.6}$$

Since $\overline{f(X)}$ is compact there exists a finite covering $\{(c_i - \delta(c_i), c_i + \delta(c_i))\}_{i \leqq N}$ of $\overline{f(X)}$.

Now let $b_0, \ldots, b_n$ an increasing sequence of regular values of f such that $b_0 = \inf_X f$; $b_n = \sup_X f$ and for every $l = 1, \ldots, n-1$

$$c_i + \delta(c_i) \leqq b_{l-1} < b_l \leqq c_i + \delta(c_i) \qquad \text{for some } i = 1, \ldots, N. \tag{5.7}$$

Now we set

$$A_l = X \cap f^{b_l}_{b_{l-1}} \qquad l = 1, \ldots, n. \tag{5.8}$$

By our construction we have that A_l is a Morse decomposition of Γ (cf. Example 4.9) and by (5.6), (5.7) and (5.8) we get

$$G^T(A_l) \subset N_{\varepsilon/2}(K) \quad \text{for } T \text{ large enough.} \tag{5.9}$$

By Theorem 4.10, we have

$$\sum_{l=1}^{n} i(A_l) = i(X) + (1+t)\,Q(t) \qquad Q \in \mathcal{S}. \tag{5.10}$$

Setting

$$\sigma = \{l \mid A_l \cap K \neq \emptyset)$$

by (5.10) and Th. 4.5 (i) we have

$$\sum_{l\in 0} i(A_l) = i(X) + (1+t)\,Q(t) \qquad Q \in \mathcal{S}. \tag{5.11}$$

Now set

$$U_l = A_l \cap N_\varepsilon(K). \tag{5.12}$$

Then by (5.9) and the fact that $A_l \in \Gamma$ we have that

$$G^T(A_l) \subset U_l \qquad \text{and } G^T(U_l) \subset A_l.$$

Then, by Corollary 1.7, we have

$$i(A_l) = i(U_l).$$

Using the above formula and (5.11) we get

$$\sum_{l\in\sigma} i(U_l) = i(X) + (1+t)\,Q(t) \qquad Q \in \mathcal{S}. \tag{5.13}$$

Now for $l \in \sigma$, let $\{\overline{U}_{l,k}\}_{k \leqq n_l}$ be the family of connected components of $\overline{U}_l$. We claim that $\{U_{l,k}\}_{l\in\sigma; k\leqq n_l}$ is a ε-Morse covering of $K(X)$. (i) and (ii) of the Def. 5.8 are trivially satisfied.

$$U_{l,k} \in \Gamma \qquad \text{since } U_l \in \Gamma. \text{ Moreover, since}$$
$$\overline{U}_{l,k} \cap \overline{U}'_{l,h} = \Phi \qquad \text{for } k \neq l.$$

By Theorem 4.5 (iii), we have

$$\sum_{k=1}^{n_l} i(U_{k,l}) = i(U_l).$$

By the above formula and (5.13) we get

$$\sum_{\substack{l\in\sigma\\ k\leqq n}} i(U_{k,l}) = i(X) + (1+t)\,Q(t).$$

So we have proved the theorem. ■

COROLLARY 5.13. – Suppose that the assumptions of Theorem 5.9 are satisfied. Moreover suppose that $K(X)$ consists of a finite number of connected components $K_1, \ldots, K_n$. Then

$$\sum_{l=1}^{n} i(K_l) = i(X) + (1+t)\,Q(t) \qquad Q \in \mathcal{S}.$$

PROOF. – It follows from Th. 5.12 and Th. 5.8 (i, ii). ■

The next theorem generalises the Morse relations to a set where f is not bounded above.

THEOREM 5.14. – Let f be a function which satisfies P.S. and let $K = K(f_c)$. Then, for every $\varepsilon > 0$ there exists an ε-Morse covering of K.

Notice that, in Theorem 1.16, the series $\sum_{j\in I} i(U_j)$ and $Q(t)$ (which appears in (iii) of Def. 5.11) may have some coefficients equal to $+\infty$.

The following lemma will simplify the proof of Th. 5.14.

LEMMA 5.15. – If a and b are regular values of f, then

$$i(f_a^b) = \sum_{n=0}^{\infty} \dim\,[H_n(f^b, f^a)]\,t^n$$

where H_* denotes the singular homology with coefficients in $\boldsymbol{Q}$.

PROOF. – Since a and b are regular values of f, f_a^b is a ANR; then the Alexander-Spanier cohomology coincides with the singular cohomology. Moreover, since our coefficients are in $\boldsymbol{Q}$,

$$\dim H^n(f^b, f^a) = \dim H_n(f^b, f^a). \quad ■$$

PROOF OF TH. 5.14. – Let $c_n > c$ be increasing sequence of regular values of f diverging to $+\infty$. By Theorem 4.10 we have, for every $n \in N$,

$$i(f_c^{c_n}) + i(f_{c_n}^{\infty}) = i(f_c) + (1+t)\,Q_n^{(1)}(t) \qquad Q_n^{(1)} \in \mathcal{S}. \tag{5.14}$$

By Theorem 5.12 (with $X = f_c^{c_n}$) we have

$$\sum_{j=1}^{k_n} i(U_j) = i(f_c^{c_n}) + (1+t)\,Q_n^{(2)}(t) \qquad Q_n^{(2)} \in \mathcal{S}.$$

Comparing the above formula with (5.14) we gwet

$$\sum_{j=1}^{k_n} i(U_j) + i(f_{c_n}) = i(f_c) + (1+t)\,Q_n \qquad Q_n = Q_n^{(1)} + Q_n^{(2)}. \tag{5.15}$$

Now if $p = \sum_{n=0}^{\infty} a_l t^l \in \mathcal{S}$, we set $\{p\}_l = a_l$.

Then (5.15) reads

$$\left\{\sum_{j=1}^{k_n} i(U_j)\right\}_l + \{i(f_{c_n})\}_l = \{i(f_c)\}_l + \{(1+t)\,Q_n^{(t)}\}_l. \tag{5.16}$$

The theorem is proved if we can take the limit in (5.16) for every $l \in N$.

We consider two cases

(*a*) $\{i(f_{c_n})\}_l = 0$ for n large enough

(*b*) $\{i(f_{c_n})\}_l \neq 0$ for *a* subsequence $c'_n \to +\infty$.

If (*a*) holds we have done, since we can take the limit in (5.16) (notice that the sequence $\left\{\sum_{j=1}^{k_n} i(U_j)\right\}_l$ is monotonically increasing as $n \to +\infty$).

If (*b*) holds, then, by Lemma 5.15, we have that

$$H_l(M, f^{c'_n}) \neq 0 \qquad \text{for the subsequence } c'_n.$$

Let Δ denote the support of a representative nontrivial homology class $\alpha \in H_l(M, f^{c'})$ and let $c'_m > \max_{x\in\Delta} f(x)$.

Consider the exact homology sequence:

$$\ldots \to H_l(f^{c'_m}, f^{c'_n}) \xrightarrow{i_l} H_l(M, f^{c'_n}) \xrightarrow{j_l} H_l(M, f^{c'_m}) \xrightarrow{\partial_l} \ldots$$

By our choice of c'_m, $j_l(\alpha) = 0$; then by exactness of the sequence $\exists \beta \in H_l(f^{c'_m}, f^{c'_n})$ s.t. $j_l(\beta) = \alpha$.

This fact shows that

$$\{i(f_{c'_n}^{c'_m})\}_l \neq 0$$

and by Theorem 5.12 there exists $U_m \subset f_{c'_n}^{c'_m}$ such that $U \in \Gamma$ and

$$\{i(U_m)\}_l \neq 0.$$

Since this true for all the terms of the subsequence c'_n defined by (*b*), it follows that taking the limit in (5.16)

$$\left\{\sum_{j=1}^{k_n} i(U_i)\right\}_l$$

diverges to $+\infty$.

Thus the equality (iii) of Def. 5.11 is satisfied also in this case. ∎

Next we shall prove a perturbation theorem for variational system whose a proof is simpler than the proof of Th. 2.1 and whose result is more general.

THEOREM 5.16. – (Perturbation theorem for variational systems).

Let $\{\eta, \Gamma(\eta)\}$ be a variational system for f, and let $A \in \Gamma(\eta)$. Then there exists $\varepsilon_0 > 0$ such that: if $\tilde{f} \in C^1(M)$ and

$$\|f - \tilde{f}\|_{C^1(M)} < \varepsilon_0,$$

there exists a variational system $\{\tilde{\eta}, \tilde{\Gamma}\}$ (relative to $\tilde{f}$) such that

(a) $A \in \Sigma(\tilde{\eta})$

(b) $i(A, \tilde{\eta}) = i(A, \eta)$.

Moreover if $\tilde{f}|_A$ satisfies P.S., $A \in \tilde{\Gamma}(A)$.

PROOF. – Let T be large enough such that

$$(G^T(A, \eta), \Gamma^T(A, \eta))$$

be an index pair for A.

Now let

$$\delta = \inf\{-Df(x) | x \in A - G^T(A)\}.$$

By Lemma 5.9, $\delta > 0$.

We now set

$$\tilde{f}(x) = f(x) + \frac{\delta}{2} g(x)$$

where $\|g(x)\|_{C^1(A)} \leqq 1$. We have to prove (a) and (b) and we will get the first part of the theorem with $\varepsilon_0 = \delta/2$. If $x \in A - G^T(A, \eta)$ we have

$$(5.17) \qquad \frac{d}{dt}\tilde{f}(\eta(t))\bigg|_{t=0} = \frac{d}{dt}\left[f((t)) + \frac{\delta}{2} g(\eta(t))\right]_{t=0} =$$

$$= Df(x) + \frac{\delta}{2}\langle g'(x), \dot{\eta}\{t\}\rangle \leqq -\delta + \frac{\delta}{2}\|g'(x)\| \cdot \|\dot{\eta}(x)\| \leqq -\frac{\delta}{2}.$$

Now let $\{\hat{\eta}, \Gamma(\hat{\eta})\}$ be a variational system relative to $\tilde{f}$ and let $\hat{F}(x)$ be the vector field which defines $\hat{\eta}$ (i.e. $\hat{F}(x) = (d/dt)\,\hat{\eta}(t, x)|_{t=0}$). Now let $\varphi(x)$ be a function which is 1 in a neighborhood of $M - G^T(A, \eta)$ and 0 for $x \in B_\varepsilon$ where $B_\varepsilon : \overset{\text{det}}{=} M - N_\varepsilon(M - G^T(A))$. Now consider the flow $\tilde{\eta}$ relative to the differential equation

$$\dot{x} = \varphi(x) F(x) + (1 - \varphi(x)) \hat{F}(x)$$

where $F(x)$ is the vector field relative to η.

We claim that $\{\tilde{\eta}, \Gamma(\tilde{\eta})\}$ is a variational system relative to $\tilde{f}$ provided that ε is small enough (i.e. we claim that $\tilde{\eta}$ satisfies (i, a, b) of Def. 5.2 where f is replaced by $\tilde{f}$).

In fact (i, a, b) are satisfied in $M - G^T(A)$ since $\tilde{\eta} = \dot{\eta}$ and in B_ε since $\tilde{\eta} = \dot{\hat{\eta}}$ In $G^T(A) - B_\varepsilon$ (i, a, b) holds too if we take ε small enough.

Since in a neighborhood of $\partial G^T(A)$ we have that $\dot{\eta}(x) = \tilde{\eta}(x)$, it follows that

$(G^T(A), \Gamma^T(A))$ is an index pair for $\tilde{\eta}$ and hence $A \in \Sigma(\tilde{\eta})$ and $i(A, \eta) = i(A, \tilde{\eta})$. The last part of the theorem follows by the Def. 5.2 (ii) and the fact that $g|_A$ is bounded. ■

An interesting consequence of Theorem 5.16 is the following Corollary.

COROLLARY 5.17. – Let $f \in C^1(M, R)$ (M is an Hilbert Manifold) and let $A \in \Gamma$. Suppose that f is the limit in $C^1(A)$ of a sequence of C^2-function f_n which satisfies P.S. Then $i(A)$ is finite (i.e. $i_1(A) < +\infty$).

PROOF. – It is a trivial conseuqence of Th. 5.16 and Corollary 6.6 which will be proved in the next section. ■

COROLLARY 5.18. – Let $\{f_\lambda\}_{\lambda \in [0,1]}$ be a family of C^1-functions depending continuously on the parameter λ and let $\{\eta_\lambda, \Gamma_\lambda\}$ be a family of variational systems relative to f_λ with η_λ depending continuously on λ.

Let A be a set such that $A \in \Gamma_\lambda$ for every $\lambda \in [0, 1]$. Then $i(A, \eta_\lambda)$ is independent of λ.

REMARK 5.19. – Given a family $\{f_\lambda\}$ as above, a family η_λ with the above property can be constructed easily.

6. – Variational systems and differentiability..

In this section we are going to relate the Generalized Morse Index with the differential structure of (M, f). Suppose that M is a Hilbert manifold, x is a critical point of f such that

$$f''(x)\colon T_x M \to T_x M$$

is defined.

For the rest of this section we shall suppose that the nonpositive part of the spectrum of $f''(x)$ consists of isolated eigenvalues of finite multiplicity. Now we set

$m(x)$ = dimension of the space spanned by the eigenvectors of $f''(x)$ corresponding to negative eigenvalues

$n(x) = \dim[\ker f''(x)]$

$m^*(x) = m(x) + n(x)$.

We shall call $m(x)$ the (numerical) Morse index of x.

We recall that a critical point x is called nondegenerate, if $f''(x)$ exists and it is invertible. In this case we have $m(x) = m^*(x)$. If $f|_X$ has only nondegenerate critical points then it is called a Morse function (on X).

We recall a theorem of Marino and Prodi [MP] «translated» in our language.

THEOREM 6.1. – If $\{f, \Gamma\}$ is a variational system, then for every $X \in \Gamma$ and for every $\varepsilon \in (0, \bar{\varepsilon}]$ (where $\bar{\varepsilon} = \bar{\varepsilon}(X)$) there exists a Morse function on X such that $|f - f_\varepsilon|_{C^2(X)} < \varepsilon$ and f_ε satisfies P.S. in X.

The following theorem characterizes the index of nondegenerate critical points.

THEOREM 6.2. – If x_0 is a nondegenerate critical point of f, then $\{x_0\} \in \mathcal{K}_0$ and

$$i(x_0) = t^{m(x_0)}.$$

Moreover if $U \in \Gamma$ and $K(U) = \{x_0\}$ we have that

$$i(U) = t^{m(x_0)}. \tag{6.1}$$

REMARK. – Observe that in Theorem 6.2 we do not assume that $f''(x)$ is defined in a neighborhood of x_0; it is sufficient that it is defined in x_0. A similar result has been obtained in the contest of the classical Morse theory by Mercuri and Palmieri [Me.P].

PROOF. – Since x_0 is nondegenerate, it is isolated; thus $\{x_0\} \in \mathcal{K}_0$. Now let $\bar{\eta}$ be the flow relative to the differential equation

$$\dot{x} = -f''(x_0) \cdot (x - x_0).$$

If U is a small enough neighborood of x_0, $U \in \Gamma(\bar{\eta})$. Then by Theorem 5.8 (iii) $i(x_0, \eta) = i(x_0, \bar{\eta})$. But we know by Example 4.4 that $i(x_0, \bar{\eta}) = t^{m(x_0)}$.

The second part of the theorem follows from Lemma 5.10. ■

Theorem 6.2 suggests the following definition:

DEF. 6.3. – A critical point is called topologically nondegenerate if $\{x\} \in \mathcal{K}_0$ and $i_1(x) = 1$ (i.e. if $i(x) = t^m$ for some $m \in \boldsymbol{N}$).

As consequence of Theorem 6.2 we can write the «classical Morse» relations.

THEOREM 6.4. – Suppose that $X \in \Gamma$ contains only topologically nondegenerate critical points of f. Let a_m denote the number of critical points having Morse index m. Then

$$\sum_{n=0}^{N} a_m t^m = i(X) + (1 + t) Q(t) \qquad Q \in \mathcal{S}.$$

PROOF. – First observe that the number of critical points is finite. Infact, since they are nondegenerate they are isolated and by Def. 5.2 (ii) (*c*) it follows that they are a finite number. The conclusion follows from Theorem 5.12 and 6.2. ■

Notice that theorem 6.4 reduces to the classical Morse relations when $f \in C^2$. If f satisfies P.S. we have the following variant of the above Theorem.

COROLLARY 6.5. – Let $f \in C^2(M)$ be a function which satisfies P.S. and let c be a regular value of f.

Suppose that all the critical points of $K(f_c)$ are topologically nondegenerate. Let a_m denote the number of critical points having Morse index m. Then

$$\sum_{m=0}^{\infty} a_m t^m = i(f_c) + (1+t)\,Q(t) \qquad Q \in \mathcal{S}.$$

PROOF. – It follows from Th. 5.14 and Def. 6.3. ■

An other consequence of Th. 6.2 is the following one

COROLLARY 6.6 – Suppose that f satisfies P.S. and that $X \in \Gamma$. Then $i(X)$ is finite, i.e. $i_1(X) < +\infty$.

PROOF. – By Theorem 6.1 and Theorem 2.1 we can find a Morse function f_ε such that $i(X, f_\varepsilon) = i(X, f)$. Then the conclusion follows from Corollary 6.4. ■

From this corollary, the proof of Corollary 5.17 follows strighforward.

Theorem 6.2 suggest the following definition.

DEF. 6.7. – If x is a critical point of f, the number $i_1(x)$ will be called the multiplicity of f.

Notice that the definition 6.7 (as well as definition 6.3) can be extended also to critical sets $K \in \mathcal{K}_0$. Using this definition we have:

COROLLARY 6.8. – If $X \in \Gamma$, then $f|_X$ has at least $i_1(X)$ critical points if counted with their multiplicity.

PROOF. – Obvious.

Notice that Corollary 6.8 does not need the function f to be of class C^2. Now we shall consider the degenerate situation. If K is a set of critical points of f we set

$$\text{(6.2)} \qquad \begin{cases} m(K) = \inf\limits_{n \in K} m(X) \\ m^*(K) = \sup\limits_{n \in K} m^*(X). \end{cases}$$

THEOREM 6.9. – Suppose that $U \in \Gamma$, and that $f \in C^2(U)$. Then

$$i(U) = \sum_{l=m(K)}^{m^*(K)} a_l t^l$$

where $K = K(U)$.

PROOF. – Let $\lambda_1(x) \leqq \lambda_2(x) \leqq \ldots \leqq \lambda_k(x) \leqq \ldots$ be the eigenvalues of $f''(x)$. They are continuous functions of x in U since $f \in C^2(U)$.

Now let $s = m(K)$ and $r = m^*(K) + 1$.

By the definition of $m(K)$ and $m^*(K)$ we have that

$$\lambda_s(x) < 0 < \lambda_r(x) \quad \text{for every } x \in K. \tag{6.3}$$

Now take ε_1 small enough in order that (6.3) holds for every $x \in N_{\varepsilon_1}(K)$. This is possible since the $\lambda_k(x)$ are continuous in x and K is compact. Now let

$$\lambda_1^\varepsilon(x) \leqq \lambda_2^\varepsilon(x) \leqq \ldots \leqq \lambda_k^\varepsilon(x) \leqq \ldots \quad \text{be the eigenvalues of}$$

the operator $f''_\varepsilon(x)$ where f_ε is a function as in Theorem 6.1.

Now chose $\varepsilon < \varepsilon_1$ small enough that

$$\lambda_s^\varepsilon(x) < 0 < \lambda_r^\varepsilon(x) \quad \forall x \in N_{\varepsilon_1}(K). \tag{6.4}$$

Thus we have that all the critical points $x_1, \ldots, x_N$ of f_ε are nondegenerate, contained in N_{ε_1}, and by (6.4)

$$s \leqq m(x_k) \leqq r - 1 \qquad k = 1, \ldots, N \tag{6.5}$$

where $m(x_k)$ is the Morse index of x_k for f_ε.

Now, if ε has been chosen small enough, by Theorem 5.16, there exists a flow η_ε relative to f_ε such that

$$i(U, \eta) = i(U, \eta_\varepsilon). \tag{6.6}$$

By Corollary 6.4

$$\sum_{k=1}^{N} t^{m(x_k)} = i(U, \eta_\varepsilon) + (1 + t)\, Q(t) \qquad Q \in \mathcal{S} \tag{6.7}$$

where the x_ks are the critical values of f_ε in U.

By (6.5) we have

$$\sum_{k=1}^{N} t^{m(x_k)} = \sum_{l=s}^{r-1} a_l t^l.$$

By (6.6). (6.7) and the above formula we have

$$\sum_{l=s}^{r-1} a_l t^l = i(U, \eta) + (1 + t)\, Q(t).$$

From the definition of s and r the conclusion follows. ∎

THEOREM 6.10. – If $i(U) = t^n$ + other possible terms, $(U \in \Gamma)$, then there exists $\bar{x} \in K(U)$ such that

$$m(\bar{x}) \leqq n \leqq m^*(\bar{x}).$$

PROOF. – Set

$$K_1 = \{x \in K(U) \mid m(x) \leqq n\}$$

and

$$K_2 = \{x \in K(U) | m^*(x) \geqq n\}$$

Since $f''(x)$ is continuous, it follows that K_1 is closed; in fact if $x_n \to x_0$, $\lim_{n\to+\infty} m(x_n) \geqq m(x_0)$. Also K_2 is closed since $x_n \to x_0 \Rightarrow \lim_{n\to+\infty} m^*(x_n) \geqq m(x_0)$. We want to prove that $K_1 \cap K_2 \neq \varphi$. We argue indirectly and suppose that $K_1 \cap K_2 = \varphi$. Since K_1 and K_2 are compact, there exists $\varepsilon > 0$ such that

$$d(K_1, K_2) \geqq 3\varepsilon .$$

Now let $\{U_j\}$ be an ε-Morse covering of U.

By our construction each U_j contains points of K_1 or points of K_2, but does not contains points of both.

If $K(U_j) \subset K_1$, then for every $x \in K(U_j)$, we have that $m^*(x) \leqq n-1$ since $x \in K_2$.

Then $i(U_j) = \sum_{m=m_1}^{n-1} a_m t^m$ where m_1 is a suitable number $\leqq n-1$.

Arguing in the same way, if $K(U_j) \subset K_2$, we have that $i(U_j) = \sum_{m=n+1}^{m_2} a_m t^m$. Therefore, none of the $i(U_j)$'s contains the term t^n and this is a contradiction since, by the Morse relations, we have

$$\sum_j i(U_j) = i(U) + (1+t)\,Q(t) = t^n + \text{other possible terms}. \qquad \blacksquare$$

Now we are going to see some consequences of the Gromoll-Meyer theory on our theory. The basic tool is the following theorem:

THEOREM 6.11. – *Suppose that $U \in \Gamma$ and that x_0 is the only critical point of f in U. Then there exists a neighborhood $N_\varepsilon(x_0)$, a homomorphism Φ: $N_\varepsilon(x_0) \to N_\varepsilon(x_0)$ with $\Phi(x_0) = x_0$ and a C^1-mapping*

$$h\colon N_\varepsilon(x_0) \cap (x_0 + \ker f''(x_0)) \to x_0 + [\ker f''(x_0)]^\perp$$

such that

$$f \circ \Phi(z+y) = \frac{1}{2} f''(x_0)[z, z] + f(h(y) + y)$$

where $y = Px; z = P^\perp x$ and P is the orthogonal projector on $\ker f''(x_0)$.

Theorem 6.11 is essentially due to Gromoll and Meyer. Here we have presented an improved version of the theorem due to Chang, we refer to [CH] for the proof.

From the above theorem we obtain the following results which is a «translation» in our theory of the «shifting theorem» of [GM].

THEOREM 6.12. – Let x_0 be an isolated critical point of a C^2-variational system. Then

$$i(x_0) = t^{m(x)} \cdot i(x_0, \tilde{f})$$

where $\tilde{f} = f|_V$ and $V = \{(h(y) + y) | y \in N_\varepsilon(x_0) \cap (x_0 + \ker f''(x_0))\}$ (h and ε are as in theorem 6.11).

PROOF. – Let

$$N_\varepsilon^1 = N_\varepsilon(x_0) \quad (x_0 + \ker f''(x_0))$$

$$N_\varepsilon^2 = N_\varepsilon(x_0) \quad (x_0 + [\ker f''(x_0)])$$

where ε is small enough that Theorem 6.11 holds.

The by theorem 5.8 (i)

$$i(x_0) = i(\Phi^{-1}(N_\varepsilon^1 \oplus N_\varepsilon^2)).$$

Since the index is a topological invariant, we have:

$$i(x_0) = i(N_\varepsilon^1 \oplus N_\varepsilon^2, f \circ \Phi).$$

Now, using theorem 4.5 (iv) and theorem 6.11, we get

$$i(N_\varepsilon^1 \oplus N_\varepsilon^2, f \circ \Phi) = i\left(N_\varepsilon^1, f'(y + h(y)) \cdot i\left(N_\varepsilon^2, \frac{1}{2}(Az, z)\right)\right). \tag{6.8}$$

Since the index is the topological invariant, using Th. 5.8, we get

$$i(N_\varepsilon^1, f(y + h(y))) = i(V, \tilde{f}) = i(x_0, \tilde{f}). \tag{6.9}$$

By theorem 6.2, we have that

$$i\left(N_\varepsilon^2, \frac{1}{2}(Ax, x)\right) = t^{m(x_0)}. \tag{6.10}$$

The conclusion follows from (6.8), (6.9) and (6.10). ■

THEORREM 6.13. – Let x_0 be a degenerate isolated critical point of a C^2-variational system with $i(x_0) \neq 0$; then

$$\dim \ker [f''(x_0)] \geqq 2 \qquad \text{and } i(x_0) = \sum_{q=m(x)+1}^{m^*(x)-1} a_q t^q.$$

Before proving the above theorem, we shall prove the following lemma:

LEMMA 6.14. – If x_0 is an isolated maximum point of a variational system defined on a n-dimensional manifold M_n. Then $i(x_0) = t^n$.

PROOF. – Let $U = \{x \in M_n : c - \varepsilon < f \leqq c\}$ with ε small enough such that $\overline{U}_\varepsilon$ be a contractible set. By duality we have that

$$H^*(\overline{U}_\varepsilon, \partial U_\varepsilon) = H^{h-*}(\overline{U}_\varepsilon)$$

and since $\overline{U}_\varepsilon$ is contractible, $H^n(\overline{U}_\varepsilon) \cong \boldsymbol{R}$ if $n = 0$ and 0 otherwise. Since $(\overline{U}_\varepsilon, \partial U_\varepsilon)$ is an index pair for U_ε, it follows that $i(U_\varepsilon) = t^n$ and hence $i(x_0) = t^n$. ∎

PROOF OF THE 6.13. – Let V and $\tilde{f}$ be as in Corollary 6.12.

We claim that x_0 is not a maximum point of $f|_V$. To prove this we argue indirectly.

If x_0 is a minimum point, then by corollary 6.12 and theorem 5.8 (iv), we get

$$i(x_0, f) = t^{m(x)} \cdot i(x_0, \tilde{f}) = t^{m(x)} \cdot 1 = t^{m(x)},$$

then x_0 is topologically non degenerate against the assumption.

If x_0 is a maximum point, then by corollary 6.12 and lemma 6.14, we have

$$i(x_0, f) = t^{m(x)} \cdot i(x_0, V) = t^{m(x)} \cdot t^{\dim V} = t^{m(x)} \cdot t^{n(x)} = t^{m^*(x)};$$

thus, also in this case x_0 is topologically non degenerate against our assumption.

Now, since x_0 is not a maximum or a minimum point, we have that

$$i(x_0, \tilde{f}) = \sum_{q=1}^{n(x)-1} b_q t^q$$

and hence using again corollary 6.12, we have that

$$i(x, f) = \sum_{q=m(x)+1}^{m^*(x)-1} a_q t^q \qquad \text{with } a_{q-m(x)} = b_q .$$

Moreover since we have supposed that $i(x_0) \neq 0$, some a_q needs to be different from 0. Then we have $m^*(x) - m(x) = \dim[\ker f''(x)] \geqq 2$. ∎

REMARK 6.15. – Theorem 6.14 is the translation in our theory of some results of Dancer [DA] cf. also [TI].

7. – Some existence theorem.

As first application of our index theory we are going to prove a well known theorem of Ambrosetti and Rabinowitz (see e.g. [AR] or [R1]) with an additional information on the Morse index of the critical points; a proof of this theorem and the following one found also in [B3].

THEOREM 7.1. – (Mountain Pass Theorem). Suppose that $f \in C^1(M)$ where M is a connected Finsler manifold, and that there is a set S in M which splits

M in two conncected components. Moreover suppose that there exist $\alpha, \beta \in \mathbf{R}(\alpha < \beta)$ such that

(i) f satisfies P.S. in (α, β)

(ii) $\inf_{x \in S} f(x) > \alpha$

(iii) $f(x_i) < \alpha\ i = 1, 2$ where x_1 and x_2 are two points belonging to two different connected components of $M - S$

(iv) $\sup_{x \in Q} f(x) < \beta$ where Q is a curve joining x_1 and x_2.

Then $K = K(f_\beta^\alpha) \neq \emptyset$ and if $f \in C^2(N_\varepsilon(K))$ there exists a point $\bar{x}$ such that

$$m(\bar{x}) \leqq 1 \leqq m^*(\bar{x}).$$

PROOF. – We suppose that there exist two constants

$$a \in (\alpha, \inf_{x \in S} f(x))$$

and

$$b \in (\sup_{x \in Q} f(x), \beta)$$

such that a and b are not critical values of f, thus $f_a^b \in \Gamma$. Otherwise we can use a perturbation argument (i.e. take $f_n(x) = f(x) + g_n(x)$ and then let $\|g_n\|_{c^2} \to 0$).

Since f^a has at least two connected components $H_0(f^a)$ has at least two generators $[x_1]$ and $[x_2]$.

Now consider the map $i_0 \colon H_0(f^a) \to H_0(f^b)$ induced by the natural embedding. Since x_1 and x_2 belong to the same connected component of f^b, then $i_0([x_2] - [x_1]) = 0$. Then, by the exactness of the sequence,

$$\to H_1(f^b, f^a) \xrightarrow{j_1} H_0(f^a) \xrightarrow{i_0} H_0(f^b).$$

It follows that $[x_1] - [x_2] \in \operatorname{Im} j_1$. Therefore $H_1(f^b, f^a) \neq 0$. Then, by lemma 5.15

$$i(f_a^b) = t + \text{other possible terms}.$$

Thus the first statement of the theorem follow by theorem 5.8 (ν). In order to obtain the second part of the theorem use corollary 6.10 with $U = f_a^b$. ∎

REMARK 7.2. – The interest of the above theorem is in the fact that we give some information about the generalized Morse index of the critical points. A similar result for C^2-functionals has been obtained by Hofer [Ho] and Solimini [So].

Now we are going to generalize the Mountain Pass Theorem to a more general situation i.e. to the so called Linking Theorem:

DEF. 7.3. – Let Q and S be two disjoint subsets of a manifold M and suppose that

(i) Q is diffeomorphic to B^{n+1}

(ii) ∂Q is the support of a nontrivial homology class in $H_n(M-S)$.

In this case we say that S and ∂Q link homologically.

Roughly speaking we can say that S and ∂Q link homologically if any n-dimensional manifold sharing the same boundary with ∂Q intersects S. However this picture is not alwais correct; in fact the homological linking depends on the coefficient of our homology theory etc. In this paper we have chosen to use homology with real coefficients since it is sufficient for our applications.

The next example will describe a fairly general situation of homological linking.

EXAMPLE 7.4. – Let S be any connected manifold of codimension $n+1$ ($n \geqq 1$) in a Banach space E and let Q be homeotopic to the $(n+1)$-dimensional ball.

Suppose that S has a tubolar neighborhood N such that $\partial N \cap Q$ is homeomorphic to ∂Q in $E-S$. Then S and ∂Q link homologically.

PROOF. – N has the structure of fiber bundle on S with fiber (B^{n+1}, S^n). Then by Thom isomorphism theorem

$$H_{q+n+1}(N, \partial N) \cong H_q(S)$$

and in particular $H_{n+1}(N, \partial N) = H_0(S) = \boldsymbol{R}$ with generator $[Q \cap N/\partial N]$.

Also we have that $H_{n+1}(N, \partial N) \cong H_{n+1}(N, N-S)$ and by excision we have that $H_{n+1}(N, N-S) \cong H_{n+1}(E, E-S)$. By the exactness of the sequence

$$\ldots \to H_{n+1}(E) \to H_{n+1}(E, E-S) \xrightarrow{\partial} H_n(E-S) \to H_n(E) \to \ldots$$

it follows that $H_n(N-S) \cong \boldsymbol{R}$ with generator $[\partial N \cap Q] \simeq [\partial Q]$. ∎

THEOREM 7.5. – (Linking theorem). Suppose that $f \in C^1(M)$ where M is a Finsler manifold and let S and Q two subsets of M such that S and ∂Q link homologically. Moreover suppose that exist $\alpha, \beta \in \boldsymbol{R}$ such that

(i) f satisfies F.S. in (α, β);

(ii) $\inf_{x \in S} f(x) > \alpha$;

(iii) $\sup_{x \in \partial Q} f(x) < \alpha$;

(iv) $\sup_{x \in Q} f(x) < \beta$.

Then $K = K(f_a^b) \neq \emptyset$ and if $f \in C^2(N_\varepsilon(K))$ there exists a point $\bar{x} \in K$ such that

$$m(\bar{x}) \leqq n+1 \leqq m^*(\bar{x}).$$

Notice that the Mountain Pass Theorem is a particular case of Linking Theorem when $n=0$.

PROOF. As in the proof of lemma 7.1, we can suppose that there exists

$$a \in \left(\alpha, \inf_{x \in S} f(x)\right) \tag{7.1}$$

and

$$b \in \left(\sup_{x \in Q} f(x), \beta\right) \tag{7.2}$$

such that a and b are not critical values of f.

Let $i_1: \partial Q \to E - S$, $i_2: \partial Q \to f^a$, and $i_3: f^a \to E - S$ be the embedding maps. i_2 and i_3 are well defined by (ii), and (7.1) respectively.

Using these maps we get the following diagram

$$\begin{array}{ccc} H_n(\partial Q) & \xrightarrow{i_2} & H_n(f^a) \\ & {}_{i_1}\searrow \quad \swarrow_{i_3} & \\ & H_n(E - S) & \end{array}$$

Since $i_1 \neq 0$ by the definition of homological linking, we have that $i_2 \neq 0$ and hence $H_n(f^a) \neq 0$.

Now consider the embedding $i: f^a \to f^b$. Since ∂Q is contractible in f^b by (7.2) it follows that $i_n([\partial Q]) = 0$.

Now consider the exact sequence

$$\ldots \to H_{n+1}(f^b, f^a) \xrightarrow{\partial_{n+1}} H_n(f^a) \xrightarrow{i_n} H_n(f^b) \to \ldots$$

Since $[\partial Q] \in \ker i_n$, it follows that $[\partial Q] \in \operatorname{Im}(\partial_{n+1})$ and in particular we have that $H_{n+1}(f^b, f^a) \neq \{0\}$.

So by lemma 5.15, we have that

$$i(f_a^b) = t^{n+1} + \text{other possible terms}.$$

Now the conclusion follows from theorem 5.8 (v) and corollary 6.10 with $U = f_a^b$. ■

REMARK 7.6. – As in the case of Th. 7.1 the interest of Th. 7.5 does not rely on the existence result which can be obtained in an easier way with minimax methods (see e.g. [R1]). The interest lies in the information about the Morse index of the critical points which is relevant in some class of problems. A somewhat weaker version of this theorem can be four in [B3]. An other slight variant of the theorem is now appearing [LS].

COROLLARY 7.7. – (Saddle point theorem). Suppose that $f \in C^1(E)$ and E_{n+1} be a $(n+1)$-dimensional space. Moreover suppose that $\exists \alpha, \beta \in \boldsymbol{R}$ such that

(i) f satisfies P.S. in (α, β);

(ii) $\inf\limits_{x \in E_{n+1}} f(x) > \alpha$;

(iii) $\exists R > 0$: $\sup\limits_{x \in \partial Q_R} f(x) < \alpha$ where $\partial Q_R \stackrel{\text{def}}{=} E_{n+1} \cap \partial B_R$,

(iv) $\sup\limits_{x \in Q_R} f(x) < \beta$ where $Q_R \stackrel{\text{def}}{=} E_{n+1} \cap B_R$.

Then the same conclusion of Theorem 7.5 holds.

PROOF. – In order to apply Theorem 7.5 it sufficient to prove that S and ∂Q link homologically with $S = E_{n+1}$ and $Q = Q_R$.

To see this we can apply the result of the example 7.4 or it can be seen directly, since $E - S$ is homotopically equivalent to $E_{n+1} - \{0\}$ which is homotopically equivalent to S^n. ■

REMARK 7.8. – Rabinowitz [cf. e.g. R1] gave a proof of the Saddle Point Theorem without any information on the Morse index using the Minimax method.

COROLLARY 7.9. – Let $f \in C^1(E)$ and let E_n be an n-dimensional subspace of E. Suppose that there exist constants $\alpha, \beta, \rho, R_1, R_2$ (with $R_1 > \rho > 0$ and $R_2 > 0$) and $z \in E_n$ (with $\|z\| = 1$) such that

(i) f satisfies P.S. in (α, β);

(ii) $\inf\limits_{E_n \in \partial B_\rho} f(x) > \alpha$;

(iii) $\sup\limits_{x \in \partial Q_z} f(x) < \alpha$;

(iv) $\sup\limits_{x \in Q_z} f(x) < \beta$ where $Q_z = \{y + tz | y \in E_n; \|y\| \leqq R_2; t \in [0, R_1]\}$.

Then the same conclusion of Theorem 7.5 holds.

PROOF. – Take $S = E_n \cap \partial B_\rho$ and $Q = Q_R$. By the Example 7.4 it follows that S and ∂Q link homologically. ■

REMARK 7.10. – There are several version of Corollary 7.9 without any statement about the Morse index (cf. [R1, 3], [BER] and [BBE]). However the estimate of the Morse index of the critical value of f is very relevant for some problems. We refer to [BF3], [BF1] and [B4] for such applications. In [BF1] there is also a simpler proof of a variant of Corollary 7.9.

Now we consider a case in which f has a special form.

We suppose that $f \in C^1(E)$ where E is a Hilbert space and that

(f_1) $f(x) = 1/2 \langle Lx, x \rangle - \psi(x)$ where L is a bounded strictly positive selfadjoint operator and ψ' is compact.

(f_2) $\forall x \in E - \{0\}$, $\lim\limits_{\rho \to +\infty} f(\rho x) = -\infty$.

(f_3) $\exists p > 0$ and $M \geqq 0$ such that $M + \langle \psi'(x), x \rangle \geqq p\psi(x) \geqq 0$ $\forall x \in E$.

LEMMA 7.10. – Let f satisfy (f_1), (f_2) and (f_3) and let $\{\eta, \Gamma\}$ be a variational system relative to f. Then if $c < \min(f(0), -M/2)$, $f_c \in \Sigma(\eta)$ and $i(f_c) = 0$.

PROOF. – Let $\tilde{f}(x) = -1/2\|x\|^2$ and let $\{\tilde{\eta}, \tilde{\Gamma}\}$ be a variational system relative to $\tilde{f}$ with $\tilde{\eta}(t, x) = xe^t$. We want to prove that $(\bar{f}_c, \partial f_c)$ is an index pair relative for $\tilde{\eta}$. (i) and (iii) of Definition 1.1 are trivially satisfied. Let us prove (ii). We have

$$\frac{d}{dt} f(\tilde{\eta}(t, x))|_{t=0} = \langle Lx, x\rangle - \langle \psi'(x), x\rangle \leqq \quad \text{[by } (f_3)] \tag{7.3}$$

$$\leqq \langle Lx, x\rangle - p\psi(x) - M \leqq \quad \text{[since } p > 2]$$

$$\leqq 2f(c) - M.$$

Thus if $c \in \partial f_c$, we have that $(d/dt) f(\eta(t, x))|_{t=0} < 0$ and this implies that $(\bar{f}_c, \partial f_c)$ is an index pair for $\tilde{\eta}$ and that $i(F_c, \tilde{\eta}) = p(\bar{f}_c, \partial f_c)$.

Now take ρ small enough in order that $B_\rho \subset f_c$.

The family $\{B_\rho, f_c - \bar{B}_\rho\}$ is a Morse decomposition (cf. Example 4.7), then we have, by Theorem 4.9

$$i(B_\rho, \tilde{\eta}) + i(f_c - B_\rho, \tilde{\eta}) = i(f_c, \tilde{\eta}) + (1 + t) Q(t). \tag{7.4}$$

Also we have

$$i(f_c - \bar{B}_\rho) = 0 \qquad \text{by Theorem 4.5 (i) and } (f_2). \tag{7.5}$$

Since $(\bar{B}_\rho, \partial B_\rho)$ is an index pair for $\tilde{\eta}$ we have that

$$i(B_\rho) = \sum_{q=0}^{\infty} [\dim \bar{H}^q(\bar{B}_\rho, \partial B_\rho)]\, t^q = 0 \tag{7.6}$$

since ∂B_ρ is contractible.

By (7.4), (7.5) and (7.6) it follows that $i(f_c, \tilde{\eta}) = 0$ and hence $p(\bar{f}_c, \partial f_c) = 0$. But $(\bar{f}_c, \partial f_c)$ is an index pair for η too since, by virtue of (7.3), c is not a critical value of f. Therefore

$$i(f_c, \eta) = p(\bar{f}_c, \partial f_c) = 0. \qquad \blacksquare$$

LEMMA 7.11. – If (f_1) and (f_2) hold then f satisfies P.S.

PROOF. – Let $\{x_k\}$ be a sequence such that

$$Lx_k - \psi'(x_k) = v_k \qquad \text{with } v_k \to 0 \text{ as } k \to +\infty \tag{7.7}$$

and

$$\frac{1}{2}\langle Lx_k, x_k\rangle - \psi(x_k) \to c \qquad \text{as } k \to +\infty.$$

By the above formula, for k large enough, we have that

$$\frac{1}{2}\langle Lx_k, x_k\rangle \leqq \psi(x_k) + c + 1 \leqq \qquad \text{[by } (f_3)]$$

$$\leqq \frac{1}{p}\langle \psi'(x_k), x_k\rangle + M_1 \qquad [\text{where } M_1 = c + 1 + M]$$

Multiplying (7.7) by x_k and using the above formula we get

$$\frac{1}{2}\langle Lx_k, x_k\rangle \geqq \frac{1}{p}\langle Lx_k, x_k\rangle - \langle v_k, x_k\rangle - M_1 .$$

Then, since L is positive,

$$\left(\frac{1}{2} - \frac{1}{p}\right)\langle Lx_k, x_k\rangle \leqq \|v_k\|\|x_k\| - M_1 .$$

Since $\langle Lx_k, x_k\rangle \geqq \nu\|x_k\|^2$, by the above formula it follows that $\|x_k\|$ is bounded. Then $\psi'(x_k)$ is convergent (may be considering a subsequence). By (7.7) we get that

$$x_k = L^{-1}\psi'(x_k) + L^{-1}v_k$$

is convergent. ■

THEOREM 7.12. – Suppose that f satisfies (f_1), (f_2) and (f_3). Moreover suppose that O is a topologically nondegenerate critical point of f with index t^m. Then f has at least an other critical point $\bar{x}$ and if this point is nondegenerate has Morse index t^{m+1} or t^{m-1}. Moreover if $f \in C^2$, we have

$$m(\bar{x}) - 1 \leqq m \leqq m^*(\bar{x}) + 1 .$$

PROOF. – By Theorem 5.14 we have that

$$t^m + \sum_{j\in I} i(U_j) = i(f_c) + (1+t)\,Q(t)$$

where $\{U_j\}_{j\in I}$ is an ε-Morse covering of $K(f_c) - \{0\}$.

By Lemma 1.10, $i(f_c) = 0$, then there exists a U_j such that

$$i(U_j) = t^{m+1} + \text{other possible terms}$$

or

$$i(U_j) = t^{m-1} + \text{other possible terms}.$$

Then U_j contains at least one critical point and if it is nondegenerate it has index t^{m+1} or t^{m-1}. The last statement follows from Corollary 6.10. ■

We now apply Theorem 7.12 to obtain an existence result which seems not easy to be obtained with minimax methods (cf. [R1]).

EXAMPLE 7.13. – Consider the following boundary value problem

$$(7.8)\qquad \begin{cases} u \in H_0^1(\Omega) & \Omega \subset \mathbf{R}^N \text{ smooth and bounded, } n \geqslant 3 \\ \Delta u + g(u) = 0 & \text{where } g \in C^2(\mathbf{R}). \end{cases}$$

Suppose that

$$(7.9)\qquad \begin{cases} (a) & g(0) = 0; \\ (b) & g'(0) \text{ is not an eigenvalue of } -\Delta; \\ (c) & \exists p > 2,\ \exists R > 0 \text{ s.t. } g(t)\cdot t \geqq p\int_0^t g(\tau)\,d\tau > 0 \text{ if } |t| > R; \\ (d) & \exists K_1, K_2 \text{ such that } |g(t)| \leqq K_1 + K_2|t|^\alpha \text{ where } \alpha < \dfrac{N+2}{N-2}. \end{cases}$$

THEOREM 7.14. – If g satisfies (7.9) then the problem (7.8) has at least a nontrivial solution.

To prove Theorem 7.14 we can use theorem 7.12 setting

$$E = H_0^1(\Omega) \qquad \text{with norm } \|u\|_E^2 = \int_\Omega |\nabla u|^2\,dx$$

$$L = \text{Identity}$$

$$\psi = a + \int_\Omega G(u(x))\,dx \quad \text{where } G(t) = \int_0^t g(t)\,dt$$

and a is a positive constant which makes $\psi \geqq 0$ for every $u \in H_0^1(\Omega)$.

By virtue of well known results and the asumption (7.9) (d) the functional $f(x) = 1/2\|x\|^2 - \psi(x)$ satisfies (f_1).

Moreover, by (7.9) (c), we have that

$$c(t) > C_1 + C_2|t|^p, \qquad (C_1 \in \mathbf{R}, C_2 > 0).$$

Then, since $p > 2$, it is not difficult to prove f_2.

In order to prove (f_3), more work is necessary:

LEMMA 7.15. – By virtue of (7.9) (c), ψ satisfies (f_3).

PROOF. – Let $\Omega_1 = \{x \mid |u(x)| > R\}$ and $\Omega_2 = \{x \mid |u(x)| \leqq R\}$.

Then we have

$$\langle \psi'(x), x\rangle = \int_{\Omega_1} g(u)\,u + \int_{\Omega_2} g(u)\,u \geqq \qquad \text{[by (7.9)(c)]}$$

$$\geqq p\int_{\Omega_1} G(u)\,dx - M_1 \geqq \quad \left[\text{where } M_1 = \sup_{\|u\|_{L^\infty} \leqq R} \left|\int_\Omega g(u)\,u\,dx\right|\right]$$

$$\geqq p\int_\Omega G(u)\,dx - M_1 - M_2 = \left[\text{where } M_2 = \sup_{\|u\|_{L^\infty}\leqq R}\left|p\int_\Omega G(u)\,dx\right|\right]$$

$$= p\psi(u) - M_1 - M_2 - a\,.$$

Thus (f_3) is proved with $M = M_1 + M_2 + a$. ∎

PROOF OF THEOREM 7.14. – With our notation f satisfies (f_1), (f_2), and by lemma (7.15), f satisfies (f_3).

Moreover, by (7.9) (b), O is nondegenerate critical point of f.
Then the conclusion follows from Theorem 7.12. ∎

8. – Some existence theorems for invariant functionals.

Now we consider how to use the index in a symmetric situation. We suppose that a compact Lie group G acts on E, i.e. that there exists a map

$$\varphi\colon G\times E\to E$$

such that $\varphi(g_1,\varphi(g_2,x)) = \varphi(g_1\cdot g_2, x)$. As usual we shall write gx instead of $\varphi(g,x)$. We recall some definition. If $x\in E$, the subgroup of G defined by

$$G_x = \{g\in E|gx = x\}$$

is called the isotropy group of x.

We say that G acts freely on $A\subset E$ if $G_x = Id$ for every $x\in A$. A point x is called a fix point if $G_x = G$. The set of all fix points of G will be denoted Fix(G). The set $O_x = \{g|\exists g\in G\colon g = gx\}$ is called the orbit of G passing through x. A set $A\subset E$ such that $gx\in A$ for every $g\in G$ and every $x\in A$ is called G-invariant. A functional $f\colon E\to R$ is called G-invariant if

$$f(gx) = f(x)\qquad \forall g\in G\,.$$

If the function $f\in C^1(E)$, then it is possible to construct a variational system

$$\{\eta,\Gamma\}\qquad \text{where } \eta \text{ is a } G\text{-invariant flow, i.e.:}$$

$$\eta(t,gx) = g\eta(t,x)\qquad \forall t\in R \text{ and } \forall g\in G\,.$$

We shall call the triple $\{\eta,\Gamma,G\}$ an equivariant variational system (notice that we do not require that the sets in Γ are G-invariant).

If x is a critical point for a G-invariant function f, then all the points of the orbit are critical points. Such an orbit is called «critical orbit». We have the following proposition:

PROPOSITION 8.1. – Let $\{\eta,\Gamma,G\}$ be an equivariant variational system relative to $f\in C^1(E)$ where G is a group of finite order, and let O_x be an isolated critical orbit of f.

Then there exist a polynomial $P(t) \in \mathcal{S}$ such that

$$(8.1) \qquad i(t, O_x) = \gamma \cdot P(t) \quad \text{where } \gamma = \frac{\operatorname{Ord} G}{\operatorname{Ord} G_x}$$

(notice that γ is the number of points of the critical orbit).

PROOF. – Let $x = x_1, \ldots, x_\gamma$ be the points of the orbit O_x. Then by theorem 4.5 (iii) we have

$$(8.2) \qquad i(t, O_x) = i(t, N_\varepsilon(O_x)) \sum_{l=1}^{\gamma} i(t, N_\varepsilon(x_l))$$

where ε is small enough that $N(x_{l_1}) \cap N(x_{l_2}) = \emptyset$ for $l_1 \neq l_2$. Now since the index is a local property, $i(N_\varepsilon(x_i)) = i(N_\varepsilon(x_j))$. Then the conclusion follows immediately by (8.2). ■

PROPOSITION 8.2. – Let $\{\eta, \Gamma, G\}$ be an equivariant system relative to f where G is a group of finite order p. Take $A \in \Gamma$ such that

(i) A is G-invariant

(ii) G acts freely on $K(A)$

(iii) f is of class C^2 is a neighborhood of $K(A)$.

Then there exists $m \in N$ such that

$$i(p-1, A) = p \cdot m.$$

PROOF. – Since the set of points on which G acts freely is an open set, it is possible to choose $\varepsilon > 0$ such that G acts freely on $N_\varepsilon(K)$ where $K{:} = K(A)$. Also we can take ε small enough that $f \in C^2(N_\varepsilon(K))$. Now let $\{U_j\}$ be an ε-Morse covering of K (cf. Th. 5.12).

Now by (iii) of Def. 5.11 we get

$$(8.3) \qquad i(N_\varepsilon(K)) = i(A) + (1+t)\, Q_1, \qquad Q_1 \in \mathcal{S}.$$

By (iii) and the fact that G acts freely on $N_\varepsilon(K)$ it is possible to choose a equivariant Morse function f_ε arbitrarily close to f (apply the Theorem 6.1 at the function $f \circ \pi^{-1}$ where π: $N_\varepsilon \to N_\varepsilon/G$ is the natural projection).
Then by Theorem 1.5 we have

$$(8.4) \qquad i(t, U_j, f_\varepsilon) = i(t, U_j, f)$$

and by Theorem 6.4 we have

$$(8.5) \qquad \sum_{m=0}^{N} a_m t^m = i(t, U_j, f) + (1+t)\, Q_2 \qquad Q_2 \in \mathcal{S}.$$

Notice that all the a_m's are multiple of p since the action is free. Then by (8.3), (8.4) and (8.5) we have

$$\sum_{m=0}^{N} a_m t^m = i(A) + (1+t)(Q_1 + Q_2).$$

Since all the a_m's are multiple of p, the conclusion follows taking $t = p - 1$. ■

Now let us apply the theory developed to some existence theorems:

THEOREM 8.3. – Suppose that on $S = \{x \in E \,|\, \|x\| = 1\}$ a group G of finite order acts. Suppose that $f \in C^1(S, \boldsymbol{R})$ is a G-invariant function bounded from below which satisfy P.S. in $[m_0, m_\infty)$ where $m_0 = \min_S f$ and $m_\infty = \sup_S f$ (m is allowed to be $+\infty$). Moreover, suppose that

(8.6) there exists $\gamma \geqq 2$ such that every critical orbit has a cardinality multiple of γ.

Then f has infinitely many critical orbits.

PROOF. – Since S is contractible, $i(S) = 1$. We argue indirectly and suppose that f has only a finite number of critical orbits $O_1, \ldots, O_k$. Then by Proposition 8.1 and Corollary 5.13 we get

$$p \cdot \sum_{l=1}^{k} P_l(t) = 1 + (1+t)\, Q(t).$$

If you take $t = p - 1$, we get

$$p \cdot (\text{number}) = 1 + pQ(p-1)$$

and this is a contradiciton. ■

THEOREM 8.4. – Suppose that $f \in C^1(E, R)$ satisfies (f_1), (f_2), (f_3) of section 7, and it is ivariant for the action of a finite group G. Moreover, suppose that

$$(8.7)\qquad \begin{cases} (a) & O \text{ is the only fixed point of } G \\ (b) & O \text{ is a critical point of } f \text{ and } i(O) = t^\alpha \text{ for some } \alpha \in N \\ (c) & (8.6) \text{ holds for any critical point different from } O. \end{cases}$$

Then f has infinitely many critical orbits.

PROOF. – Remember that by Lemma 7.11, f satisfies P.S. Now take c small enough in order that, by Lemma 7.10, we have

$$(8.8)\qquad i(f_c) = 0.$$

Now we argue indirectly and suppose that f has only a finite number of critical orbits

$O_1, \ldots, O_k$ in f_c. Then by Corollary 5.13 we have

$$i(0) + \sum_{l=1}^{k} i(O_l) = i(f_c) + (1+t)\,Q(t).$$

By (8.7) (*b*), (8.8) and Proposition 8.1 (with the assumption (8.6)), we get

$$t^{\alpha} + \gamma \sum_{l=1}^{k} m_l p_l(t) = (1+t)\,Q(t).$$

Taking $t = \gamma - 1$ we get

$$(\gamma - 1)^{\alpha} = \gamma \cdot m \qquad \text{where } m = Q(\gamma - 1) - \sum_{l=1}^{k} m_l p_l(\gamma - 1)$$

and this is a contradiction. ■

Next we want to consider an example where the group G is continuous. We consider the group $S^1 = \{z \in \boldsymbol{C} \mid |z| = 1\}$ with the multiplicative structure.

PROPOSITION 8.5. – Let $\{\eta, \Gamma, S^1\}$ be an equivariant system relative to f. Take $A \in \Gamma$ such that

(i) A is S^1 invariant

(ii) $K(A) \cap \mathrm{Fix}(S^1) = \emptyset$

(iii) f is of class C^2 in a neighborhood of $K(A)$.

The there exists a polynomial $P(t)$ with coefficients in $\boldsymbol{Z}$ such that

$$i(t, A) = (1+t)\,P(t).$$

PROOF. – We claim that $\exists \bar{p}$ such that $\boldsymbol{Z}_p \subset S^1$ acts freely on $K(A)$ for every $p \geqq \bar{p}$. (Here $\boldsymbol{Z}_p = \exp[2\pi k i/p] \mid k = 0, \ldots, p-1\}$). To prove this we argue indirectly and suppose not. Then there exists a sequence $p(k) \to +\infty$ and points $x_k \in K(A)$ such that

$$(g_{p(k)})^l \cdot x_k = x_k \qquad l = 0, \ldots, p(k) - 1 \tag{8.9}$$

where

$$g_{p(k)} = \exp[2\pi i/p(k)] \in \boldsymbol{Z}_{p(k)}.$$

Since $K(A)$ is compact, we can suppose that x_k converges to some $\bar{x} \in K(A)$. Moreover, for every $g \in S^1$, there exists a sequence of $l(k)$'s such that

$$(g_{p(k)})^{l(k)} \to g \qquad \text{for } k \to +\infty.$$

Then taking the limit in (8.9) with $l = l(k)$ we get

$$g\bar{x} = \bar{x}. \tag{8.10}$$

Since g has been chosen arbitrarily, (8.10) implies that $\bar{x} \in \mathrm{Fix}(S^1) \subset K(A)$ against our assumptions. So the claim is proved.

Then, by Proposition 8.2, for every p sufficiently large, there exists $m(p) \in \boldsymbol{N}$ such that

$$i(p-1, A) = p \cdot m(p). \tag{8.11}$$

By Corollary 6.6, $i(t, A)$ is a polynomial. Then there exists a polynomial P and an integer number a_0 such that

$$i(t, A) = (1+t)P(t) + a_0. \tag{8.12}$$

Then, by (8.10) and the above formula, we get

$$\begin{aligned} a_0 &= i(t, A) - (1+t)P(t) && [\text{for every } t \in \boldsymbol{R}], \\ &= i(p-1, A) - pP(p-1) && [\text{for every } p \geqq \bar{p}], \\ &= p[m(p) - P(p-1)] && [\text{by } 8.11]. \end{aligned}$$

Since $m(p) - P(p-1)$ is an integer number, the above formula implies that a_0 must be O (otherwise $|a_0| = +\infty$). ■

PROPOSITION 8.6. – Suppose that the same assumption of Proposition 8.5 are satisfied. Moreover suppose that $i(t, A) = (1+t)P(t)$ where

$$P(t) = \sum_{m=m_1}^{m_2} b_m t^m. \tag{8.13}$$

Then if $b_n \neq 0$ there exists a critical point of $f, \bar{x} \in A$ such that

$$m(\bar{x}) \leqq n \leqq m^*(\bar{x}).$$

REMARK 8.7. – Proposition 8.6 is in the same spirit of Th. 6.10; however, by virtue of the S^1 invariance, it gives a better information. In fact, if

$$i(t, A) = 1 + t^{2m+1} = (1+t)(1 - t + t^2 + \dots + t^{2m}) \qquad (m \geqq 1)$$

by Th. 6.10, it is not possible to deduce the existence of a point $\bar{x}$ with

$$m(\bar{x}) \leqq n \leqq m^*(\bar{x})$$

if $1 < n < 2m$.

PROOF OF PROPOSITION 8.6. – We set

$$K_1 = \{x \in K(A) \colon m(x) \leqq n\}$$

$$K_2 = \{x \in K(A) \colon m^*(x) \geqq n\}.$$

We argue indirectly and we suppose that $K_1 \cap K_2 = \emptyset$.

Let $\{U_j\}$ be a Morse ε-covering. Arguing again as in Th. 6.10 we have

$$i(t, U_j) = \sum_{m=m_1}^{n-1} a_m^j t^m \qquad \text{if } U_j \cap K_2 = \emptyset$$

and

$$i(t, U_j) = \sum_{m=n+1}^{m_2} a_m^j t^m \qquad \text{if } U_j \cap K_1 = \emptyset .$$

With this notation, the Morse relations reads

$$\sum_j i(t, U_j) = i(t, A) + (1+t)\, Q(t) \tag{8.14}$$

If we set $Q(t) = \sum_m q_m t^m$, since the left hand side of this formula does not have any term of degree n, we have that

$$q_n = 0 . \tag{8.15}$$

Now, using Proposition 8.5, we have that

$$i(t, U_j) = (1+t)\, P_j(t) \qquad \forall j$$

where $P_j(t)$ has the term of degree n equal to 0. Replacing the above expression and (8.13) in (8.14), and dividing by $(1+t)$ we get

$$\sum_j P_j(t) = P(t) + Q(t) .$$

Since $b_n \neq 0$ by assumption and the left hand side of the above expression does not have any term of degree n, it follows that $q_n = -b_n \neq 0$ and this contradicts (8.15). ■

9. – The Maslov index and the twisting number.

Now we want to apply the theory developed in the last four sections to the study of the periodic solutions of second order Hamiltonian systems. Before doing this we will study the Maslov index and the twisting number. This will be done in this and the next section.

For $\sigma \in S^1 = \{z \in \boldsymbol{C} \,|\, |z| = 1\}$ we set

$$L^2_{\sigma, T} = \{x \in L^2_{\text{loc}}(\boldsymbol{R}, \boldsymbol{C}^N) | x(t+T) = \sigma \cdot x(t)\}$$

where $L^2_{\text{loc}}(R, \boldsymbol{C}^N)$ in the set of function $x\colon \boldsymbol{R} \to \boldsymbol{C}^N$ which are measurable and whose square is locally integrable. $L^2_{\sigma, T}$ is a Hilbert space if it is equipped with the following scalar product

$$(9.1)\qquad (x, y)_{L^2_{\sigma,T}} = \frac{1}{T}\int_0^T (x(t), y(t))_{C^N}\, dt.$$

Now let $A(t)$ be a family of real simmetric $N \times N$ matrices depending continuously on t and periodic of period τ and consider the following ordinary differential equation

$$(9.2)\qquad \ddot{y} + A(t)\, y = -\lambda y \qquad y \in \boldsymbol{C}^N,\ \lambda \in \boldsymbol{R}$$

with the condition

$$(9.3)\qquad y(t+T) = \sigma \cdot y(t) \qquad \sigma \in S^1,\ T = k\tau,\ k \in N.$$

Now let $W^2_{\text{loc}}(\boldsymbol{R}, \boldsymbol{C}^N)$ denote the space of functions having two square locally integrable derivative.

If $\mathcal{L}_{\sigma,T}$ is the extension to $W^2_{\text{loc}}(\boldsymbol{R}, \boldsymbol{C}^N) \cap L^2_{\sigma,T}$ of the operator

$$(9.3')\qquad -\ddot{y} - A(t)\, y$$

then it is well known that $\mathcal{L}_{\sigma,T}$ is a selfadjoint unbounded operator on $L^2_{\sigma,T}$. Then the eigenvalue problem (9.2), (9.3) becomes

$$(9.4)\qquad \mathcal{L}_{\sigma,T}\, y = \lambda y \qquad y \in D(\mathcal{L}_{\sigma,\tau}) = L^2_{\sigma,T} \cap W^2_{\text{loc}}(\boldsymbol{R}, \boldsymbol{C}^N).$$

It is easy to check from elementary facts of spectral theory that $\mathcal{L}_{\sigma,T}$ has discrete spectrum with only a finite number of negative eigenvalues.

This fact allows us to define a function

$$j(T, \cdot)\colon S^1 \to N$$

as follows.

$j(T, \sigma) = \{$number of negative eigenvalues of $\mathcal{L}_{\sigma,T}$ counted with their multeplicity$\}$.

We shall call the function $j(T, 1)$ the Maslov index relative to the equation $\ddot{y} + A(t)\, y = 0$ in the interval $[0, T]$.

Now let $W(t)$ be the Wronskian matrix relative to equation

$$\ddot{y} + A(t)\, y = 0$$

i.e. the matrix which sends the initial data $\begin{bmatrix} x_0 \\ v_0 \end{bmatrix}$ to $\begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}$.

The map $W(T)\colon \boldsymbol{C}^{2n} \to \boldsymbol{C}^{2n}$ (where T is the period of $A(t)$) is called the Poincaré map or the monodromy map.

The eigenvalues of $W(T)$ are usually called Floquet multipliers (relative to the interval $(0, T)$).

PROPOSITION 9.1. – The function $j(T, \sigma)$ satisfies the following properties

(i) $j(T, \bar{\sigma}) = j(T, \sigma)$ where $\bar{\sigma}$ is the complex coniugate of σ

(ii) if $j(1, \sigma)$ is discontinuous at the point σ^* then σ^* is a Floquet multiplier

(iii) $|j(T, \sigma_2) - j(T, \sigma_1)| \leqq l \ \forall \sigma_1, \sigma_2 \in S^1 - \{+1, -1\}$ where $2l$ is the number of non-real Floquet multiplies on S^1 counted with their multeplicity. Thus, in particular

$$|j(T, \sigma_2) - j(T, \sigma_1)| \leqq N \qquad \text{for every } \sigma_1, \sigma_2 \in S^1 - \{+1, -1\}$$

(iv) $j(kT, \theta) = \sum_{j=0}^{k-1} j(T, \sigma_j)$ where $\sigma_0, \dots, \sigma_{k-1}$ are the k values of $\sqrt[k]{\theta}$.

PROOF. – (i) If $y(t)$ is an eigenfunction of $\mathcal{L}_{\sigma,T}$, the complex coniugate function $\overline{y(t)}$ is an eigenfunction of $\mathcal{L}_{\bar{\sigma},T}$ corresponding to the same eigenvalue. Therefore $\mathcal{L}_{\sigma,T}$ and $\mathcal{L}_{\bar{\sigma},T}$ have the same number of negative eigenvalues.

(ii) The eigenvalues of $\mathcal{L}_{\sigma,T}$ depend continuously on σ. Since $\mathcal{L}_{\sigma,T}$ is selfadjoint, they are all real. Therefore the number of the nonpositive eigenvalues $j(T, \sigma)$ can change only for those σ^* such that O is an eigenvalue of $\mathcal{L}_{\sigma^*,T}$. This means that if σ^* is a discontinuity of $j(\sigma, T)$, the following problem

$$\ddot{y} + A(t)\, y = 0 \tag{9.5}$$

$$y(t + T) = \sigma^* y(t) \tag{9.6}$$

has a nontrivial solution $y(t)$. If $W(t)$ is the Wronskian matrix relative to this problem, then

$$\begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} = W(t) \begin{bmatrix} x \\ v \end{bmatrix} \qquad \text{for some x, } v \in \mathbf{C}^N .$$

Then the condition (2.9) for $t = T$ reads

$$W(T) \begin{bmatrix} x \\ v \end{bmatrix} = \sigma^* \begin{bmatrix} x \\ v \end{bmatrix}.$$

Therefore σ^* is an eigenvalue of $W(T)$.

(iii) $W(T)$ is a sympletic matrix; then if λ is an eigenvalue of $W(T)$, also $\bar{\lambda}, \lambda^{-1}, \bar{\lambda}^{-1}$ are eigenvalues of $W(T)$. Therefore

(*a*) the number of eigenvalues of $W(T)$ different from ± 1 is even;

(*b*) the sum of the multiplicity of the eigenvalues $+1$ and -1 is even (if ± 1 are not eigenvalues then their multiplicity has to be assumed 0 in order to make sense of the above statement).

In particular the eigenvalues of $W(T)$ on $S^1 - \{+1, -1\}$ is an even number

$2l$. We can assume that all the eigenvalues are simple (otherwhise use a perturbation argument).

Therefore $j(T,\sigma)$ has at most $2l$ points of discontinuity and at each of them the jump of $j(T,\sigma)$ is ± 1 since we have assumed that all the eigenvalues are simple.

Now

$$|j(T,\sigma_1)-j(T,\sigma_2)| = |j(T,\exp[i\omega_1])-j(T,\exp[i\omega_2])| \text{ with } \omega\in(-\pi,\pi)-\{0\}, j=1,2.$$

By (i) we have that $j(T,\exp[i\omega]=j(T,\exp[-i\omega])$. Then we can assume that ω_1 and $\omega_2\in(0,\pi)$. But the function $\omega\mapsto j(T,\exp[i\omega])$ has at most l jumps in $(0,\pi)$ and this proves the statement.

(iv) A direct computation shows that

$$L_{kT,\theta}=\bigoplus_{j=0}^{k-1} L^2_{T,\sigma_j}; \qquad \text{where } \sigma_0,\dots,\sigma_{k-1} \text{ are the } k \text{ values of } \sqrt[k]{\theta}.$$

Since T is a multiplier of τ, $A(t)$ is T-periodic; then the operator $\mathcal{L}_{kT,\theta}$ leaves the spaces L^2_{T,σ_j} $(j=0,\dots,k-1)$ invariant.

By our definitions, we have that

$$\mathcal{L}_{kT,\theta}|_{L^2_{T,\sigma_j}}=\mathcal{L}_{T,\sigma_j} \qquad j=0,\dots,k-1.$$

Therefore we have the following decomposition of $\mathcal{L}_{kT,\theta}$

$$\mathcal{L}_{kT,\theta}=\bigoplus_{T=0}^{k-1}\mathcal{L}_{T,\sigma_j} \qquad j=0,\dots,k-1.$$

From the above formula the conclusion follows. ∎

Now we can define the twisting number as follows:

$$\rho=\frac{1}{2\pi\tau}\int_{S^1} j(\tau,\sigma)\,d\sigma=\frac{1}{2\pi\tau}\int_0^{2\pi} j(\tau,\exp[i\omega])\,d\omega$$

PROPOSITION 9.2. – The twisting number satisfies the following properties:

(i) $\rho=\lim\limits_{\substack{T\to+\infty\\ T=k\tau}}\frac{1}{T}j(T,1)$;

(ii) $\rho=\frac{1}{2\pi T}\int_{S^1} j(T,\sigma)\,d\sigma \qquad T=k\tau$;

(iii) $|T\rho-j(T,\sigma)|\leqq l$ for every $\sigma\in S^1-\{+1,-1\}$, $T=k\tau$, $2l$ is the number of Floquet multiplier on $S^1-\{+1,-1\}$;

(iv) for every $\sigma\in S^1$ we have $\lim\limits_{\substack{T\to+\infty\\ T=k\tau}}\frac{1}{T}j(T,\sigma)=\rho$.

PROOF. – (i) By Proposition (9.1) (iv) we have

$$\lim_{k\to+\infty}\frac{1}{k\tau}j(k\tau,1)=\lim_{k\to+\infty}\frac{1}{k\tau}\sum_{l=0}^{k-1}j(\tau,\exp[2\pi il/k])\,. \tag{9.7}$$

By the definition of the Cauchy integral, we have

$$\lim_{k\to+\infty}\frac{2\pi}{k}\sum_{l=0}^{k-1}j(\tau,\exp[2\pi il/k])=\int_0^{2\pi}j(\tau,\exp[i\omega])\,d\omega=2\pi\tau\rho\,. \tag{9.8}$$

Then, by (9.7) and (9.8) we have

$$\lim_{\substack{T\to+\infty\\ T=k\tau}}\frac{1}{T}j(k\tau,1)=\frac{1}{2\pi\tau}\lim_{k\to+\infty}\frac{2\pi}{k}\sum_{l=0}^{k-1}j(\tau,\exp[2\pi l/k])=\rho\,.$$

(ii) From (i) it follows that ρ is indipendent on $T=k\tau$.

(iii) $$|T\rho-j(T,\sigma)|=\frac{1}{2\pi}\left|\int_{S^1}j(T,\theta)\,d\theta-\int_{S^1}j(T,\sigma)\,d\theta\right|\leqq$$

$$\leqq\frac{1}{2\pi}\int_{S^1-\{+1,-1\}}|j(T,\theta)-j(T,\sigma)|d\theta\leqq l \text{ by Proposition 9.1 (iv).}$$

(iv) It follows from (i) and (iv). ∎

EXAMPLE. – Consider the equation

$$\ddot{y}+Ay=0$$

where A is a time indipendent real simmetric matrix with l positive eigenvalues $\omega_1^2,\dots,\omega_l^2$ and $N-l$ negative eigenvalues.

Then the negative eigenvalues of $-\ddot{y}-Ay$ on $L^2_{1,T}$ are

$$\lambda_{n,j}=\left(\frac{2\pi}{T}\right)^2n^2-\omega_j^2\qquad\text{with } n\in N,\ \ l=0,\dots,k-1 \text{ and } n<\frac{\omega_jT}{2\pi}\,.$$

Notice that for $n\geqq1$ they have double multiplicity. Therefore

$$j(T,1)=l+2\,\#\left\{(n,j)\,\middle|\,n\leqq\frac{\omega_jT}{2\pi}\right\}=l+2\cdot\sum_{j=1}^{l}\left[\frac{\omega_jT}{2\pi}\right].$$

Then by Proposition 9.2 (i) we have

$$\rho=\lim_{T\to+\infty}\frac{1}{T}j(T,1)=\lim_{T\to+\infty}\frac{l}{T}+\frac{2}{T}\sum_{j=1}^{l}\left[\frac{\omega_jT}{2\pi}\right]=\frac{1}{\pi}\sum_{j=1}^{l}\omega_j\,.$$

10. – The generalized Morse index for periodic solutions of second order Hamiltonian systems.

In this section we consider the following system of ordinary differential equations

$$\ddot{x} + V_x(t, x) = 0 \qquad x \in \mathbf{R}^N \tag{10.1}$$

with $V \in C^2(\mathbf{R} \times R^N)$. We suppose that $V(t, \cdot)$ is τ-periodic. We set

$$W^T = \{x \in W^2_{\text{loc}}(\mathbf{R}, \mathbf{R}^N) | x \text{ is } T\text{-periodic}\}$$

W^T is an Hilbert space if it is equipped with the following scalar product:

$$(x, y)_{W^T} = \frac{1}{T}\int_0^T (\dot{x} \cdot \dot{y} + x \cdot y)\, dt$$

where «·» denotes the scalar product un $\mathbf{R}^N$.

The equations (10.1) are the Euler-Lagrange equations corresponding to the functional

$$f(x) = \frac{1}{T}\int_0^T \left\{\frac{1}{2}|\dot{x}|^2 + V(t, x)\right\} dt \qquad x \in W^T. \tag{10.2}$$

It is well known that $f(x)$ is a functional class C^2 on W^T.

Therefore, any T-periodic solution of (10.1) can be interpreted as a critical point of the functional (10.2).

If we apply the theory of section 6, we can define a Morse index for every T-periodic solution $\bar{x}$ of (10.1) which we shall denote by $m(\bar{x}, T)$ to enphatise the fact that the Morse index is computed in the space W^T.

Of course we can also define the nullity $n(\bar{x}, T)$ and the number $m^*(\bar{x}, T) = m(\bar{x}, T) + n(\bar{x}, T)$ as in section 6. Now let us consider the linearization of the equation (10.1) at x:

$$\ddot{y} + V_{xx}(t, \bar{x}(t))\, y = 0. \tag{10.3}$$

It is easy to check that $m(\bar{x}, T)$ is the number of negative eigenvalues of the selfadjoint operator

$$y \to -\ddot{y} - V''_{xx}(t, \bar{x}(t))\, y \qquad \text{in } L^2((0, T), \mathbf{R}^n) \tag{10.3'}$$

$n(\bar{x}, T)$ is the multiplicity of the eigenvalue 0 of (10.3′) and hence it is the number of indipendent solutions of equation (10.3).

A T-periodic solution $\bar{x}$ of (10.1) is called nondegenerate if it is nondegenerate as critical point of the functional (10.2) i.e. if $n(x, T) = 0$. Clearly $\bar{x}$ is nondegenerate if and only if the linear system (10.3) does not have any nontrivial T-periodic solution, or, if you like, if 1 is not a Floquet multiplier of the equation (10.3) relative to the interval $(0, T)$. We recall that a number $\alpha \in C$ is called a Floquet exponent if e^{α} is a Floquet multiplier.

DEFINITION 10.1. – Let $\bar{x}$ be a T-periodic solution of the equation (10.1) and let $(2\pi i \omega_j$ $(j = 1, \ldots, N)$ be the purely immaginary Floquet exponent of the linearised equation (10.3). Then if $\omega_j \notin Q$ for $j = 1, \ldots, l$ we say that $\bar{x}$ is nonresonant.

It is easy to check that if $\bar{x}$ is a nonresonant τ-periodic solution, then $\bar{x}$ is T-nondegenerate for every $T = k\tau$, $k \in N$.

If $\bar{x}$ is a T-degenerate solution of (10.1) then the definition (6.7) can be applied to define the multiplicity of $\bar{x}$.

We can associate to the equation (10.3) a Maslov index $j(T, \sigma)$ as in section 9 where $A(t) = V''(t, \bar{x}(t))$ and consequently a twisting number $\rho(\bar{x})$.

PROPOSITION 10.2. – If $\bar{x}$ is a T-periodic solution of (10.1) $(T = k\tau, k \in N)$ then

(i) $m(\bar{x}, T) = j(T, 1)$.

Moreover if $\bar{x}$ is nondegenerate

(ii) $T \cdot \rho(\bar{x}) - N \leqq m(\bar{x}, T) \leqq T \cdot \rho(\bar{x}) + N$.

PROOF. – (i) is a trivial consequence of the definitions.

(ii) Since 1 is not a Floquet multiplier, then for σ_1 very close to 1 $(\sigma \in S^1)$, σ_1 is not a Floquet multiplier and

$$m(T, \bar{x}) = j_{\bar{x}}(T, \sigma) \qquad \text{by Prop. 9.1 (ii)}.$$

Then the conclusion follows from Proposition 9.2. (iii). ∎

Now let Γ^T be the family of subsets of W^T defined in Def. 5.2.

Now we want to examine the relationship between the index of a set $U (U \in \Gamma^T)$ and the twisting number of the solution of (10.1) contained in U.

PROPOSITION 10.3. – Let $\bar{x} \in W^T$ be a possibly degenerate critical point of f. Then

$$m^*(\bar{x}, T) - N \leqq \rho(\bar{x})\, T \leqq m(\bar{x}, T) + N.$$

Proof. – By our assumption $\bar{x}$ satisfies the following equation

$$\ddot{\bar{x}} + V_x(t, \bar{x}) = 0 \tag{10.4}$$

and the following operator

$$y \mapsto -\ddot{y} - V_{xx}(t, \bar{x})\, y \tag{10.5}$$

has $m_1 := m(\bar{x}, T)$ negative eigenvalues and $m_2 := m^*(\bar{x}, T)$ nonpositive eigenvalues.

Now, for every $\varepsilon \neq 0$, consider the following functional on W^T

$$f_\varepsilon(x) = \int\left\{\frac{1}{2}\dot{x}^2 - V(t, x) - \frac{\varepsilon}{2}|x - \overline{x}|^2\right\} dt.$$

Obviously $\overline{x}$ is a critical point of f_ε since by (10.4) it satisfies the equation

$$\ddot{x} + V_x(t, x) + \varepsilon(x - \overline{x}) = 0.$$

Moreover, if $\varepsilon \neq 0$ is small enough, ε is not an eigenvalue of (10.5); this implies that $\overline{x}$ is a nondegenerate critical point of f_ε and its Morse index is m_1 if $\varepsilon < 0$ and m_2 if $\varepsilon > 0$. If we denote by $\rho_\varepsilon(\overline{x})$ the twisting number of $\overline{x}$ with respect to the function f_ε, using Proposition 3.2 we get

$$m_1 - N \leqq \rho_\varepsilon(\overline{x}) \cdot T \leqq m_1 + N \quad \text{ if } \varepsilon < 0$$

$$m_2 - N \leqq \rho_\varepsilon(\overline{x}) \cdot T \leqq m_2 + N \quad \text{ if } \varepsilon > 0.$$

It is not difficult to check that ρ_ε depends continuously on ε. Therefore letting $\varepsilon \mapsto 0$ in the above formula we get

$$m_2 - N \leqq \rho(\overline{x}) \cdot T \leqq m_1 + N. \quad ■$$

COROLLARY 10.4. – (i) Let $U \in \Gamma^T$ and let

$$i(U) = \sum_{l=m_1}^{m_2} a_l t^l \quad \text{ with } a_m \neq 0$$

Then there exists $\overline{x} \in U$, solution of (10.1), such that

$$\frac{m - N}{T} \leqq \rho(\overline{x}) \leqq \frac{m + N}{T}.$$

(ii) Suppose now that the equation (10.1) is autonomous (i.e. $\partial V/\partial t = 0$) and let $U \in \Gamma^T$ be a set which does not contain constant solutions. If

$$i(U) = (1 + t) \sum_{l=m_1}^{m_2} b_l t^l \quad \text{ with } b_m \neq 0.$$

Then the same conclusion of (i) holds.

PROOF. – (i) Apply Theorem 6.10 and Proposition 10.3.

(ii) Apply Proposition 8.6 and Proposition 10.3. ■

11. – Some applications to nonautonomous systems.

In this section we try to get some information on the structure of the periodic solutions of the equation (10.1). We suppose that $V(t, x)$ satisfies the following asymptotic conditions

(11.1) there exists $R > 0$ and $p > 2$ such that

$$0 < V(t, x) \leqq \frac{1}{p} V_x(t, x) \cdot x \quad \forall t \in \mathbf{R} \ \forall x \text{ with } |x| > R.$$

Condition (11.1) implies that $V(t, x)$ grows more than $|x|^2$ as $|x| \to +\infty$. Moreover this condition implies the following facts:

LEMMA 11.1. – Suppose that V satisfies (11.1). Then the functional (10.2) satisfies f_1, f_2 and f_3 of section 7.

PROOF. – (f_1) is verified if we define L and ψ by the following equations:

$$\langle Lx, v \rangle = \int_0^T (\dot{x} \cdot \dot{v} + x \cdot v)\, dt \qquad \forall v \in W^T$$

$$\psi(x) = \int_0^T \left\{ V(t, x) + \frac{1}{2} |x|^2 \right\} dt.$$

(f_2) is a consequence of the superlinear growth of V.
(f_3) can be verified reasoning as in Lemma 7.11. ■

COROLLARY 11.2. – The functional (3.2) satisfies (P.S.). Moreover there exists c_0 such that $i(f_c) = 0$ for every $c \leqq c_0$.

PROOF. – It follows from Lemma (11.1) Lemma (7.11) and Lemma (7.10). ■

THEOREM 11.3. – Suppose that V satisfies (11.1) and let x_0 be a nonresonant τ-periodic solution of 10.1.

Then, for every $\varepsilon > 0$ there exists a T-periodic solution $x \neq x_0$ (with $T = k\tau$, $T < \tau + (2N+1)/\varepsilon$), such that

$$|\rho(x) - \rho(x_0)| \leqq \varepsilon.$$

PROOF. – Take $T = k\tau$ with $(2N+1)/\varepsilon \leqq T \leqq \tau + ((2N+1)/\varepsilon)$. Since x_0 is nonresonant, there is a neighborhood $N_\delta(x_0)$ in W which does not contain periodic solutions of (10.1). Now take a δ-Morse covering $\{U_l\}$ of f_c (where f_c is as in Lemma 11.2, $c \leqq c_0$). Then, by Theorem 5.14

$$i(x_0) + \sum_{l \in I} i(U_l) = (1+t)\, Q(t).$$

By the above formula there exists $l \in I$ such that either

(11.2) $$i(U_l) = t^{m+1}$$

or

$$i(U_l) = t^{m-1}$$

where m is the Morse index of x_0 (i.e. $i(x_0) = t^m$).

We consider the first possibility (if the second one holds we argue in the same way).

By Proposition 10.2 (ii) we have

$$i(x_0) = t^m \qquad \text{with } \rho(x_0)\,T - N \leqq m \leqq \rho(x_0)\cdot T + N. \tag{11.3}$$

By Corollary 10.4 (i) and (11.2), there exists $x \in U_l$ such that

$$\frac{1}{T}(m+1-N) \leqq \rho(x) \leqq \frac{1}{T}(m+1+N). \tag{11.4}$$

Comparing (11.3) and (11.4) we get

$$|\rho(x) - \rho(x_0)| \leqq \frac{1}{T}(2N+1) \leqq \varepsilon. \qquad \blacksquare$$

The next theorem we are going to prove has stronger assumptions and gives a better information about the T-periodic solution of equation (11.1).

THEOREM 11.4. – Suppose that V satisfies (11.1). Let $T = p\tau$ with p prime number, and suppose that all the T-periodic solutions of (11.1) are isolated (as points in W^T). Let $x_1, x_2, \ldots, x_n, \ldots$ be the τ-periodic solutions of equation (10.1). We suppose that they are T-nondegenerate and ordered by increasing twisting number:

$$\rho(x_1) \leqq \rho(x_2) \ldots \leqq \rho(x_n) \leqq \ldots$$

Then for every number $\rho \in [\rho(x_{2n-1}), \rho(x_{2n})]$, $(2n < p)$ there is a T-periodic solution $\bar{x}$ such that

$$|\rho(\bar{x}) - \rho| \leqq \frac{N+1}{T}.$$

PROOF. – By the Theorem 5.14 relative to the space W^T we have

$$\sum_{j\in J} i(x_j) + \sum_{j\in I} i(U_j) = (1+t)\,Q(t) \qquad \text{with } Q(t) = \sum_l q_l t^l \tag{11.5}$$

where $\{U_j\}_{j\in I}$ is an ε-Morse covering of the T-periodic solutions of (11.1) which are not τ-periodic and $\{x_j\}_{j\in J}$ is the set of τ-periodic solutions. Now fix $\rho \in [\rho(x_{2n-1}) + T\cdot(N + 1), \rho(x_{2n}) - T\cdot(N+1)]$ and take $m = \{\text{integer part of } \rho\cdot T\}$.

Consider only the terms of (11.5) of order less or equal to m:

$$\sum_{l=1}^{m} a_l t^l + \sum_{l=0}^{m} b_l t^l = (1+t)\sum_{l=0}^{m-1} q_l t^l + q_m t^m \tag{11.6}$$

where

$$\sum_{l=1}^{m} a_l t^l = \sum_{j=1}^{2n-1} i(x_j) \tag{11.7}$$

and the term $\sum_{l=0}^{m} b_l t^l$ comes from the ε-Morse covering relative to the solutions which are not τ-periodic.

Since we have supposed that these solutions are isolated, by Proposition 8.1 we have that

$$b_l = p\beta_l \qquad \text{for some } \beta_l \in \boldsymbol{N}.$$

Then rewriting (11.6) for $t = -1$, we get

$$\sum_{l=1}^{m} (-1)^l a_l + p \sum_{l=0}^{m} (-1)^l \beta_l = (-1)^m q_m . \tag{11.8}$$

By (11.7), the first term of (11.8) is an odd number less or equal to $2n-1$, and by our assumption less than p.

Thus the sum of the two terms of the left hand side of (11.8) is different from 0. Thus $q_m \neq 0$. Then, by (4.5), there exists U_j such that

$$i(U_j) = t^m + \text{possible other terms}.$$

Corollary 10.4 (i) implies that there exists $\bar{x} \in U_j$ such that

$$\frac{1}{T}(m-N) \leqq \rho(\bar{x}) \leqq \frac{1}{T}(m+N)$$

and by the definition of m we have that

$$\rho - \frac{N+1}{T} \leqq \rho(\bar{x}) \leqq \rho + \frac{N+1}{T}.$$

Thus the theorem is proved for $\rho \in [\rho(x_{2n-1}) + T(N+1), \rho(x_{2n}) - T(N+1)]$.

Considering also the solutions x_{2n-1} and x_{2n} the theorem is proved for every $\rho \in [\rho(x_{2n-1}), \rho(x_{2n})]$. ■

We conclude this section with a theorem which is the analogous of Theorem 11.3 in the asymptotically quadratic case.

We say that $V(t, x)$ is asymptotically quadratic if there exists a matrix $A_\infty(t)$ such that

$$V_x(t, x) = A_\infty(t)\, x + O(|x|) \qquad \text{as } |x| \to +\infty. \tag{11.9}$$

If V is asymptotically quadratic we can consider the linearised system at ∞

$$\ddot{y} + A_\infty(t)\, y = 0 \tag{11.10}$$

and associate to (11.10) a twisting number ρ_∞.

Then we have the following result:

THEOREM 11.5. – Suppose that V satisfies (11.9) and suppose that (11.10) has not T-periodic solution different from 0.

Let x_0 be a nondegenerate T-periodic solution of (10.1) with twisting number $\rho(x_0)$ such that

$$|\rho(x_0) - \rho_\infty| > \frac{2N}{T}. \tag{11.11}$$

Then the system (10.1) has a T-periodic solution $\bar{x}$ such that

$$|\rho(\bar{x}) - \rho(x_0)| < \frac{2N+1}{T}.$$

SKETCH OF THE PROOF. – If we take a ball in W^T of sufficiently large radious R, arguing as in [B3], we have that

$$B_R \in \Gamma^T \qquad \text{and } i(B_R) = t^{m(\infty)}.$$

It is easy to check that

$$T \cdot \rho_\infty - N \leqq m(\infty) \leqq T \cdot \rho_\infty + N. \tag{11.12}$$

Then the Morse relation take the form

$$i(x_0) + \sum_{l \in I} i(U_l) = t^{m(\infty)} + (1+t)\, Q(t).$$

Let $i(x_0) = t^m$.

Then, by (11.11) and (11.12)

$$|m - m(\infty)| \neq 0.$$

Therefore we have that $Q(t) \neq 0$.

From now on we can argue as at the end of the Theorem 4.3. ■

It is well known that if $V(t, x)$ is even in x and satisfies (11.1), then the equation (10.1) has infinitely many τ-periodic solution.

Now we want to show that a more general symmetry assumption on V will give the same result.

We assume that

(11.13)
- (a) $V(t, \cdot)$ is G-invariant, where $G \subset O(N)$ is a finite group which satisfies the following:
  - (i) O is the only fix point of G
  - (ii) there exists $\gamma \geqq 2$ such that every orbit O_x passing through $x \in \boldsymbol{R}^N - \{O\}$ has a cardinality multiple of γ
- (b) O is a nondegenerate T-solution of (10.1).

THEOREM 11.6. – Let V satisfy (11.1) and (11.3). Then the equation (10.1) has infinitely many τ-periodic solutions.

PROOF. – It is an immediate consequence of Lemma 11.1 and Theorem 8.4. ■

REMARK 11.7. – It has been conjectured that the statement of Theorem 11.6 is true without the assumption 11.3. However until now this result is known only for $n = 1$ (cf. [JA]) or if V has the following form

$$V(t, x) = U(x) - f(t) \cdot x$$

(Cf. Bahri-Beresticky [BAB]).

12. – One application to autonomous systems..

Now we consider the following autonomous equation

$$\ddot{x} + V_x(x) = 0 \qquad x \in \boldsymbol{R}^n. \tag{12.1}$$

We restrict ourselves to the superlinear case i.e. we still assume that V satisfies (11.1).

In this case the Theorems 11.3 and 11.4 do not apply since every non constant solution of equation (11.4) is degenerate.

In fact if x is a T-periodic solution of (12.1), $y = \dot{x}$ is a T-periodic solution of the linearised equation

$$\ddot{y} + V_{xx}(x(t))\, y = 0.$$

In this section we shall prove the following theorem:

THEOREM 12.1. – For every $\rho \geqq \rho_0$ there is a T-periodic solution $\bar{x}$ such that

$$|\rho - \rho(\bar{x})| \leqq \frac{N+1}{T}$$

where $\rho_0 = \max\{\rho(x) |\, x$ is a constant solution of (12.1)$\}$.

REMARK 12.2. – The existence of infinitely many T-periodic solution of (12.1) for any $T > 0$ under the only assumption (11.1) has been proved some years ago (cf. [BF2] Th 4.1 or [RA2]).

So the interest of Theorem 12.1 relies not in the existence of T-periodic solutions, but in the relation between the T-periodic solutions and their twisting number.

PROOF OF THEOREM 12.1. – Let $\{U_j\}_{j \in I}$ be an ε-Morse covering of the solutions of (12.1). We divide $\{U_j\}_{j \in I}$ in two families.

$\{U_j\}_{j \in I_1}$ if U_j contains a constant solution of (12.1) and $\{U_j\}_{j \in I_2}$ otherwise.

Set m_0= integer part of $(T\rho_0 + N + 1)$. We claim that

$$\sum_{j\in I_1} i(1, U_j) = \sum_{l=0}^{m_0} a_l t^l + (1+t) P_1(t) \tag{12.2}$$

where $\sum_l a_l = \{\text{odd number}\}$ and $P_1(t)$ is a polynomial with integer coefficients.

Since the $\overline{U}_j$ are disjoined, by Theorem 4.5 (iii), we have that

$$\sum_{j\in I_1} i(t, U_j) = i(t, A)$$

where

$$A = \bigcup_{j\in I_1} U_j .$$

Now let $\tilde{V}$ be a small C^2-perturbation of V and we set

$$\tilde{f}(x) = \int_0^T \left\{ \frac{1}{2} |\dot{x}|^2 - \tilde{V}(x) \right\} dx$$

and let $\tilde{\eta}$ be the flow relative to $\tilde{f}$.

We can chose $\tilde{V}$ close enough to V in order that

$$\begin{cases} (a)\ A \in \Gamma(\tilde{\eta}) \\ (b)\ i(t, A, \tilde{\eta}) = i(t, A) \quad \text{where } i(t, A) = i(t, A, \eta). \end{cases} \tag{12.3}$$

This is possible by Theorem 5.16.

Moreover, by well known generic properties, $\tilde{V}$ can be chosen such that

$$\begin{cases} (a) \text{ the critical point } x_1, \ldots, x_s \in \boldsymbol{R}^N \text{ of } \tilde{V} \text{ are nondegenerate in } \boldsymbol{R}^N \\ (b) \text{ the positive eigenvalues of } \tilde{V}_{xx}(x_j) \text{ are not of the form } \left(\frac{2\pi}{T}\right)\cdot n^2 \\ \quad (n \in \boldsymbol{N}, j = 1, \ldots s). \end{cases} \tag{12.4}$$

The x_j's are critical points of the perturbed functional $\tilde{f}$ and if $\tilde{V}$ has been chosen close enough to V, their twisting number is less or equal to $\rho_0 + 1/T$. Moreover by virtue of (12.4) (*b*), they are not degenerate and by Prop. 10.2 their Morse index $m(x_j)$ is less or equal to m_0. Now, the critical points $x_1, \ldots, x_s$ of $\tilde{V}$ satisfy the following Morse relation (cf. Th. 6.4)

$$\sum_{j=1}^{s} t^{m_j} = i(\boldsymbol{R}^n) + (1+t) Q(t)$$

where m_j is the Morse index of x_j in $\boldsymbol{R}^N$. Since, by our assumption on the potential V,

$i(\boldsymbol{R}^n) = 1$, taking the above relation with $t = 1$ we get

$$s = 1 + 2Q(1) = \text{odd number}.$$

Now let us write the Morse relation in A for the functional $\tilde{f}$: we get

$$\sum_{j=1}^{s} t^{m(x_j)} + \sum_{k} i(t, U_k) = i(t, A) + (1 + t)\, Q(t)$$

where the x_j's are the constant critical points of $\tilde{f}$ and $\{U_k\}$ is an ε-Morse covering of the other critical points of $\tilde{f}$ in A. By Proposition 8.5, we have that

$$i(t, V_k) = (1 + t)\, P_k(t).$$

By the above formula and (12.5) we get

$$\sum_{j=1}^{s} t^{m(x_j)} + (1 + t) \sum_{k} P_k(t) = i(t, A) + (1 + t)\, Q(t).$$

Now taking $t = -1$, we get that $i(1, A)$ is an odd number and hence, by (12.3) (*b*), (12.2) is proved.

Now we write the Morse relation for f using Corollary 11.2 and we get

$$\sum_{j \in I_1} i(t, U_j) + \sum_{j \in I_2} i(t, U_j) = (1 + t)\, Q(t). \tag{12.5}$$

We use again Proposition 8.5 and we get

$$i(t, U_j) = (1 + t)\, P_j(t) \qquad j \in I_2.$$

Then taking account of (12.2) the equation (12.5) can be written as follows

$$\sum_{l=0}^{m_0} a_l t^l + (1 + t) \sum_{l} b_l t^l = (1 + t) \sum_{l} q_l t^l \tag{12.6}$$

where $\sum_{l} b_l t^l := P_1(t) + \sum_{j \in I_2} P_j(t)$ and $\sum_{l} q_l t^l := Q(t)$.

Now we can prove the statement of the theroem for any given $\rho > \rho_0 + (N + 1)/T$ (if $\rho\ [\rho_0, (N + 1)/T]$, the statement is true taking the constant solution with twisting number ρ_0).

If we set $m =$ integer part of $\rho \cdot T$, we have that $m \geqq m_0$.

The equation (12.6) up to the order m reads

$$\sum_{l=0}^{m_0} a_l t^l + (1 + t) \sum_{l=1}^{m-1} b_l t^l + b_m t^m = \sum_{l=1}^{m-1} q_l t^l + q_m t^m. \tag{12.7}$$

Now, taking $t = -1$, from the above equation we get

$$\sum_{l=0}^{m_0} a_l t^l + (-1)^m b_m = (-1)^m q_m.$$

Since $\sum_{l=0}^{n} a_l t^l$ is an odd number, it follows that b_m (or q_m) is different from zero.

In either case, from Proposition 8.6, there exists a solution $\bar{x}$ of (12.1) such that

$$m(\bar{x}) \leqq m \leqq m^*(\bar{x}).$$

Then by Proposition 10.3

$$\frac{m-N}{T} \leqq \rho(\bar{x}) \leqq \frac{m+N}{T}.$$

The conclusion follows from the definition of m. ∎

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