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Existence for the Cahn-Hilliard Phase Separation Model with a Nondifferentiable Energy (*).

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Summary. – *The Cahn-Hilliard model for phase separation in a binary alloy leads to the equations* (I) $u_t = \Delta w$, (II) $w = \psi'(u) - \gamma\Delta u$ *with an associated energy functional* $F(u) = \int_\Omega [\psi(u) + \gamma|\nabla u|^2/2]\,dx$. *In this paper we discuss the existence theory for initial boundary value problems arising from modifications to the Cahn-Hilliard model due to the addition of the nondifferentiable term* $\alpha\int|\nabla u|\,dx$ *to the energy* $F(u)$.

1. – Introduction.

The Cahn-Hilliard equation

$$\frac{\partial u}{\partial t} = \Delta w \qquad x \in \Omega, \quad t > 0, \tag{1.1a}$$

$$w = \varphi(u) - \gamma\Delta u \qquad x \in \Omega, \quad t > 0, \tag{1.1b}$$

$$\varphi(u) \equiv u^3 - \beta u, \tag{1.1c}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0 \qquad x \in \partial\Omega, \quad t > 0, \tag{1.1d}$$

holding in a bounded domain $\Omega \subset \mathbb{R}^n$ $(n \leq 3)$ for $t > 0$, with ν denoting the unit outward pointing normal to $\partial\Omega$, where γ and β are positive constants, arises in the study of phase separation in binary mixtures; see CAHN and HILLIARD [1958], CAHN [1965], HILLIARD [1970], LANGER [1971], GUNTON, SAN-MIGUEL and SAHNI [1983], NOVICK-COHEN and SEGEL [1984] and the references cited therein. Here $u(x, t)$ is a suitably scaled concentration of one of the two components of the mixture, w is a generalized

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chemical potential and $\varphi(\cdot)$ is the derivative of the homogeneous free energy

$$\psi(u) = \frac{1}{4}(u^2 - \beta)^2 . \tag{1.2}$$

The equations (1.1) are derived by consideration of the Landau-Ginsburg free-energy functional

$$F_\gamma(u) = \int_\Omega \left[\psi(u) + \frac{\gamma}{2}|\nabla u|^2\right] dx \tag{1.3}$$

for which the chemical potential w is the functional derivative. It is the purpose of this paper to study modifications of the original Cahn-Hilliard model based on the energy functionals:

$$F^\alpha_{\gamma,\varepsilon}(u) = F_\gamma(u) + \alpha \int_\Omega [\varepsilon^2 + |\nabla u|^2]^{1/2} dx, \tag{1.4a}$$

$$F^\alpha_\gamma(u) = F_\gamma(u) + \alpha \int_\Omega |\nabla u| \, dx, \tag{1.4b}$$

$$F^\alpha(u) = \int_\Omega [\psi(u) + \alpha|\nabla u|] \, dx. \tag{1.4c}$$

In particular in sections 3, 4 and 5 we prove global existence theorems for the following initial-value problems where we use the notation $W(0,T) = \{\eta \in L^2(0,T; H^1(\Omega)) : d\eta/dt \in L^2(0,T; (H^1(\Omega))')\}$.

$(P^\alpha_{\gamma,\varepsilon})$ Find $u \in L^\infty(0,T; H^1(\Omega)) \cap W(0,T)$ and $w \in L^2(0,T; H^1(\Omega))$

such that

$$\left\langle \frac{\partial u}{\partial t}, \eta \right\rangle + (\nabla w, \nabla \eta) = 0 \quad \forall \eta \in H^1(\Omega), \quad \text{a. e. } t \in (0,T), \tag{1.5a}$$

$$(w - \varphi(u), \eta) = \gamma(\nabla u, \nabla \eta) + \alpha\left(\frac{\nabla u}{\sqrt{\varepsilon^2 + |\nabla u|^2}}, \nabla \eta\right) \quad \forall \eta \in H^1(\Omega), \quad \text{a. e. } t \in (0,T), \tag{1.5b}$$

$$u(0) = u_0 . \quad \square \tag{1.5c}$$

(P^α_γ) Find $u \in L^\infty(0,T; H^1(\Omega)) \cap W(0,T)$ and $w \in L^2(0,T; H^1(\Omega))$

such that

(1.6a) $\left\langle \frac{\partial u}{\partial t}, \eta \right\rangle + (\nabla w, \nabla \eta) = 0 \quad \forall \eta \in H^1(\Omega), \text{ a.e. } t \in (0,T),$

(1.6b) $\gamma(\nabla u, \nabla \eta - \nabla u) + \alpha \int_\Omega |\nabla \eta| dx - \alpha \int_\Omega |\nabla u| dx \geqslant (w - \varphi(u), \eta - u)$

$\forall \eta \in H^1(\Omega), \text{ a.e. } t \in (0,T),$

(1.6c) $u(0) = u_0 .$ □

(P^α) Find $u \in L^\infty(0,T; BV(\Omega) \cap L^4(\Omega))$ and $w \in L^2(0,T; H^1(\Omega))$ with $du/dt \in L^2(0,T;(H^1(\Omega))')$ such that

(1.7a) $\left\langle \frac{\partial u}{\partial t}, \eta \right\rangle + (\nabla w, \nabla \eta) = 0 \quad \forall \eta \in H^1(\Omega), \text{ a.e. } t \in (0,T),$

(1.7b) $\alpha \int_\Omega |\nabla \eta| - \alpha \int_\Omega |\nabla u| - (w - \varphi(u), \eta - u) \geqslant 0 \quad \forall \eta \in BV(\Omega), \text{ a.e. } t \in (0,T).$

The motivation for studying (P_γ^α) and (P^α) comes from the papers by TILLER, POUND and HITRH [1970], CAHN and HILLIARD [1971] and HAGAN and COHEN [1985] which discuss the possibility and desirability of adding a term of the form $\alpha \int_\Omega |\nabla u|\, dx$ to the energy functional. In particular HAGAN and COHEN [1985] study the partial differential equation version of (P_γ^α)

(1.8) $$\frac{\partial u}{\partial t} = \varphi(u)_{xx} - \gamma u_{xxxx} - \mathrm{H}(u_x)_{xxx}, \qquad x \in (0,L),$$

where $\mathrm{H}(\cdot)$ denotes the Heaviside operator, using approximate analytical methods with the interpretation of (1.8) being the limit as $\varepsilon \to 0$ in (1.5). (See also COHEN and ALEXANDER [1988].)

Global existence results for the Cahn-Hilliard equation (1.1) have been obtained by ELLIOTT and ZHENG [1986]. The behaviour as $t \to \infty$ was studied by ZHENG [1987]. Results of a numerical study of (1.1) can be found in ELLIOTT & FRENCH [1987]. We note that (P_γ^α) with (1.6a) replaced by $w = f - \partial u/\partial t$ and $\varphi(\cdot) \equiv 0$ is a problem arising in non-Newtonian fluid mechanics studied by GLOWINSKI, LIONS and TREMOLIERES [1981].

Problems (P_γ^α) and (P^α) and some variants have previously been studied by A. VISINTIN, in the context of the Stefan problem with surface tension where $\varphi(u)$ is taken to be $-u$. Global existence results are obtained in VISINTIN [1984]; see also VISINTIN [1988].

2. – Preliminaires.

Throughout the paper the norm of $H^s(\Omega)$ $(s \geq 0)$ is denoted by $\|\cdot\|_s$, the semi-norm $\|D^s\eta\|_0$ is denoted by $|\eta|_s$ and the $L^2(\Omega)$ inner-product by $(\cdot,\cdot)$. For $v \in L^1(\Omega)$ we set

$$\int_\Omega |\nabla v| := \sup\left\{-\int_\Omega v\nabla\cdot\vec{\eta}\,dx : \vec{\eta} \in C_0^1(\Omega)^n, |\vec{\eta}| \leq 1 \text{ in } \Omega\right\} \tag{2.1a}$$

and the Banach space of functions with bounded total variation in Ω, $BV(\Omega)$, is defined by

$$BV(\Omega) := \left\{v \in L^1(\Omega) : \int_\Omega |\nabla v| < \infty\right\} \tag{2.1b}$$

with norm

$$\|v\|_{BV(\Omega)} = \|v\|_{L^1(\Omega)} + \int_\Omega |\nabla v|. \tag{2.1c}$$

It follows that if $\{v_j\}_{j=1}^\infty$ is a sequence in $L^1(\Omega)$ which converges in $L^1(\Omega)$ to v then

$$\int_\Omega |\nabla v| \leq \liminf_{j\to\infty} \int_\Omega |\nabla v_j|\,dx. \tag{2.2}$$

Furthermore it is known that bounded sets in $BV(\Omega)$ are compact in $L^1(\Omega)$.

Setting

$$J(v) = \alpha\int_\Omega |\nabla v|\,dx, \qquad J_\varepsilon(v) = \alpha\int_\Omega (\varepsilon^2 + |\nabla v|^2)^{1/2}\,dx \tag{2.3}$$

we have that $J(\cdot)$ is convex and continuous on $H^1(\Omega)$ and $J_\varepsilon(\cdot)$ is convex and differentiable on $H^1(\Omega)$. Furthermore, a simple calculation, yields

$$0 < J_\varepsilon(v) - J(v) \leq \varepsilon|\Omega| \qquad \forall v \in H^1(\Omega), \tag{2.4}$$

and

$$\langle J_\varepsilon'(v), \eta\rangle = \alpha\left(\frac{\nabla v}{\sqrt{\varepsilon^2 + |\nabla v|^2}}, \nabla\eta\right) \qquad \forall \eta \in H^1(\Omega). \tag{2.5}$$

Convexity also implies that the following statements are equivalent for $f \in (H^1(\Omega))'$:

$$\langle J_\varepsilon'(v), \eta\rangle = \langle f, \eta\rangle \qquad \forall \eta \in H^1(\Omega), \tag{2.6a}$$

(2.6*b*) $$J_\varepsilon(\eta) - J_\varepsilon(v) \geq \langle f, \eta - v \rangle \quad \forall \eta \in H^1(\Omega).$$

It is convenient to use the Green's operator $\mathcal{G}_N$ for the Laplacian with Neumann boundary data: given $f \in \mathcal{F} \equiv \{f \in (H^1(\Omega))' : \langle f, 1 \rangle = 0\}$ we define $\mathcal{G}_N f \in H^1(\Omega)$ to be the unique solution of

(2.7*a*) $$(\nabla \mathcal{G}_N f, \nabla \eta) = \langle f, \eta \rangle \quad \forall \eta \in H^1(\Omega),$$

(2.7*b*) $$(\mathcal{G}_N f, 1) = 0$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $(H^1(\Omega))'$ and $H^1(\Omega)$ such that

(2.8) $$\langle f, \eta \rangle \equiv (f, \eta) \quad \forall f \in L^2(\Omega).$$

For $f \in \mathcal{F}$ we define

(2.9*a*) $$\|f\|_{-1} = |\mathcal{G}_N f|_1$$

and note that if $f \in \mathcal{F} \cap L^2(\Omega)$ then

(2.9*b*) $$\|f\|_{-1} = (\mathcal{G}_N f, f)^{1/2}.$$

3. – The $(P^\alpha_{\gamma,\varepsilon})$ problem.

THEOREM 3.1. – If $u_0 \in H^1(\Omega)$ then there exists a unique solution to $(P^\alpha_{\gamma,\varepsilon})$.

PROOF. We use a Galerkin method. Let $\{z_j\}$ be the orthogonal basis for $H^1(\Omega)$ consisting of the eigenfunctions for

(3.1*a*) $$-\Delta z + z = \lambda z \quad \text{in } \Omega, \qquad \partial z / \partial \nu = 0 \quad \text{on } \partial\Omega$$

and normalised so that

(3.1*b*) $$(z_i, z_j) = \delta_{ij}.$$

Note that $\{z_j\}$ is an orthogonal basis for $L^2(\Omega)$. Denote by V_m the finite dimensional subspace of $H^1(\Omega)$ spanned by $\{z_j\}_{j=1}^m$. A Galerkin approximation to $(P^\alpha_{\gamma,\varepsilon})$ is

(3.2*a*) $$u^m(t) = \sum_{j=1}^m c_j(t) z_j, \qquad w^m(t) = \sum_{j=1}^m d_j(t) z_j,$$

(3.2*b*) $$\left(\frac{du^m}{dt}, \eta^m\right) + (\nabla w^m, \nabla \eta^m) = 0 \quad \forall \eta^m \in V_m,$$

(3.2*c*) $$(w^m - \varphi(u^m), \eta^m) = \gamma(\nabla u^m, \nabla \eta^m) + \alpha\left(\frac{\nabla u^m}{\sqrt{\varepsilon^2 + |\nabla u^m|^2}}, \nabla \eta^m\right) = 0 \quad \forall \eta^m \in V_m,$$

(3.2*d*) $$u^m(0) = \boldsymbol{P}_m(u_0)$$

where $\boldsymbol{P}_m \colon H^1(\Omega) \to V_m$ is the projection defined by:

$$\boldsymbol{P}_m v = \sum_{j=1}^{m} (v_j, z_j) z_j \qquad \forall v \in H^1(\Omega) \tag{3.3a}$$

so that

$$(\boldsymbol{P}_m v - v, \eta^m) = (\nabla \boldsymbol{P}_m v - \nabla v, \nabla \eta^m) = 0 \qquad \forall \eta^m \in V_m . \tag{3.3b}$$

Clearly it holds that

$$|\boldsymbol{P}_m v|_i \leq |v|_i \qquad i = 0, 1. \tag{3.4}$$

It follows from (3.1*b*) that (3.2) can be rewritten in the simplified form, for $i = 1, 2, \ldots, m$

$$\frac{dc_i}{dt}(t) = -(\lambda_i - 1) d_i(t), \tag{3.5a}$$

$$d_i(t) = \gamma(\lambda_i - 1) c_i(t) + (\mathscr{H}^m(\boldsymbol{c}))_i + (\boldsymbol{\Phi}^m(\boldsymbol{c}))_i , \tag{3.5b}$$

$$(\mathscr{H}^m(\boldsymbol{c}))_i = \left(\frac{\nabla u_m}{\sqrt{\varepsilon^2 + |\nabla u^m|^2}}, \nabla z_i \right), \qquad (\boldsymbol{\Phi}^m(\boldsymbol{c}))_i = (\varphi(u^m), z_i), \tag{3.5c}$$

$$c_i(0) = (u_0, z_i). \tag{3.5d}$$

Obviously $\mathscr{H}^m(\boldsymbol{c})$ and $\boldsymbol{\Phi}^m(\boldsymbol{c})$ are continuously differentiable functions of $\boldsymbol{c} = (c_1, \ldots, c_m)^T$. Therefore there exists a positive $t_m > 0$ such that the finite system of ordinary differential equations has a unique solution with $\boldsymbol{c}(t)$ and $\boldsymbol{d}(t)$ being absolutely continuous on the interval $[0, t_m]$.

Using the definition (1.4*a*) and differentiating with respect to t we obtain,

$$\frac{d}{dt} F^{\alpha}_{\gamma,\varepsilon}(u^m) = \int_{\Omega} \left[\gamma \nabla u^m \nabla \frac{\partial u^m}{\partial t} + \varphi(u^m) \frac{\partial u^m}{\partial t} + \alpha \frac{\nabla u^m \nabla \partial u^m / \partial t}{\sqrt{\varepsilon^2 + |\nabla u^m|^2}} \right] \tag{3.6}$$

and (3.2*b*), (3.2*cc*) imply

$$\frac{d}{dt} F^{\alpha}_{\gamma,\varepsilon}(u^m) = \left(w^m, \frac{\partial u^m}{\partial t} \right) = -|w^m|_1^2 . \tag{3.7}$$

Finally we obtain

$$F^{\alpha}_{\gamma,\varepsilon}(u^m(t)) + \int_0^t |w^m(\tau)|_1^2 \, d\tau = F^{\alpha}_{\gamma,\varepsilon}(\boldsymbol{P}_m(u_0)). \tag{3.8}$$

It follows from (3.4) and (1.2) that

$$\gamma|u^m(t)|_1^2 + \frac{1}{8}\|u^m(t)\|_{L^4(\Omega)}^4 + \int_0^t |w^m(\tau)|_1^2\, d\tau \leqslant C \tag{3.9}$$

where C is independent of m.

Noting (2.7) and (2.9) it follows from (3.2*b*) that

$$|w^m|_1 = \|u_t\|_{-1},$$

hence we obtain the bound

$$\|du^m/dt\|_{L^2(0,T;(H^1(\Omega))')} \leqslant C \tag{3.10}$$

where C is independent of m and T.

Taking $\eta^m = 1$ in (3.2*b*), (3.2*c*) (this is valid since $z_1 = 1/|\Omega|^{1/2}$) yields

$$(u^m(t) - \boldsymbol{P}_m u_0, 1) = (w^m(t) - \varphi(u^m(t)), 1) = 0. \tag{3.11}$$

Inequality (3.9) and equation (3.11) imply that

$$\|u^m\|_{L^\infty(0,T;H^1(\Omega))} \leqslant C, \tag{3.12a}$$

$$\|w^m\|_{L^2(0,T;H^1(\Omega))} \leqslant C(1+T) \tag{3.12b}$$

where C is independent of m and T.

Now we can conclude that there exists a subsequence of $\{u^m, w^m\}_{m=1}^\infty$ (again denoted by $\{u^m, w^m\}$) and functions $u \in W(0,T) \cap L^\infty(0,T;H^1(\Omega))$ and $w \in L^2(0,T;H^1(\Omega))$ such that

$$u^m \to u \quad \text{weakly in} \quad W(0,T), \tag{3.13a}$$

$$u^m \to u \quad \text{weak* in} \quad L^\infty(0,T;H^1(\Omega)), \tag{3.13b}$$

$$w^m \to w \quad \text{weakly in} \quad L^2(0,T;H^1(\Omega)), \tag{3.13c}$$

$$u^m \to u \quad \text{strongly in} \quad L^2(0,T;H^\beta(\Omega)) \quad 0 \leqslant \beta < 1, \tag{3.13d}$$

$$u^m \to u \quad \text{strongly in} \quad L^2(0,T;L^p(\Omega)) \quad p < 6, \tag{3.13e}$$

The strong convergence in $L^2(0,T;H^\beta(\Omega))$ is a consequence of the compactness theorem in LIONS [1969; p. 58]. Furthermore the continuous embedding $W(0,T) \hookrightarrow C([0,T];L^2(\Omega))$ together with (3.13*a*) and the strong convergence in $L^2(\Omega)$ of $\boldsymbol{P}_m u_0$ to u_0 implies that

$$u(0) = u_0 \in H^1(\Omega).$$

For any $\eta \in H^1(\Omega)$, let $\eta^m = \boldsymbol{P}_m \eta$ in (3.2*b*) and (3.2*c*). Passing to the limit in (3.2*b*) immediately yields (1.5*a*). It remains to prove that (1.5*b*) holds. Rewriting (3.2*c*), using the equivalence (2.6), yields the inequality

$$(3.14)\qquad \gamma\int_0^T \xi(t)(\nabla u^m, \nabla\eta^m - \nabla u^m)\,dt + \int_0^T \xi(t)(J_\varepsilon(\eta^m) - J_\varepsilon(u^m))\,dt \geqslant \int_0^T \xi(t)(w^m - \varphi(u^m), \eta^m - u^m)\,dt$$

for all non-negative $\xi \in C[0,T]$ and $\eta^m = \boldsymbol{P}_m\eta$. Passing to the limit in (3.14) as $m \to \infty$ one obtains

$$(3.15)\qquad \int_0^T \xi(t)[\gamma(\nabla u, \nabla\eta) + J_\varepsilon(\eta) - (w - \varphi(\eta), \eta - u)]\,dt \geqslant \liminf_{m\to\infty}\int_0^T \xi(t)[\gamma(\nabla u^m, \nabla u^m) + J_\varepsilon(u^m)]\,dt \geqslant \int_0^T \xi(t)[\gamma(\nabla u, \nabla u) + J_\varepsilon(u)]\,dt$$

where we have used the convergence properties (3.13), the fact that $\varphi(\cdot)$ is a polynomial of degree 3 and the weak lower semi-continuity of $J_\varepsilon(\cdot)$. It follows from (3.15) that

$$\gamma(\nabla u, \nabla\eta - \nabla u) + J_\varepsilon(\eta) - J_\varepsilon(u) - (w - \varphi(u), \eta - u) \geqslant 0$$

for all $\eta \in H^1(\Omega)$ and a.e. $t \in (0,T)$. Therefore (1.5*b*) holds after noting the equivalence (2.6).

Finally we prove uniqueness. Let $\{u^i, w^i\}_{i=1}^2$ be two solutions. Consideration of (1.5*a*) and (1.5*b*) in the usual way leads to, for a.e. $t \in (0,T)$,

$$(3.16)\qquad \left\langle \frac{\partial\theta^u}{\partial t}, \eta\right\rangle + (\nabla\theta^w, \nabla\eta) = 0 \qquad \forall\eta \in H^1(\Omega),$$

$$(3.17)\qquad \gamma|\theta^u|_1^2 - (\theta^w, \theta^u) \leqslant (\varphi(u^2) - \varphi(u^1), \theta^u),$$

where

$$(3.18)\qquad \theta^u = u^1 - u^2, \qquad \theta^w = w^1 - w^2.$$

Rewriting (3.16) using the Green's operator $\mathscr{G}_N$ defined by (2.7) (note that $(u^i(t), 1) = (u_0, 1)$), we have that $\theta^w = -\mathscr{G}_N\partial\theta^u\partial t + m(\theta^w)$, where $m(\cdot)$ denotes the mean value on Ω, and substitution into (3.17) yields

$$(3.19)\qquad \frac{1}{2}\frac{d}{dt}(\mathscr{G}_N\theta^u, \theta^u) + \gamma|\theta^u|_1^2 \leqslant (\varphi(u^1) - \varphi(u^2), \theta^u).$$

Recalling that $\varphi'(s) = \psi''(s) \geq -\beta \ \forall s$, we obtain from (3.19) that

$$\frac{1}{2}\frac{d}{dt}\|\theta^u\|_{-1}^2 + \gamma|\theta^u|_1^2 \leq \beta|\theta^u|_0^2 = \beta(\nabla\theta^u, \nabla\mathcal{G}_N\theta^u) \leq \beta|\theta^u|_1\|\theta^u\|_{-1}.$$

Hence it follows that

$$\frac{d}{dt}\|\theta^u\|_{-1}^2 + \gamma|\theta^u|_1^2 \leq \frac{\beta^2}{\gamma}\|\theta^u\|_{-1}^2. \tag{3.20}$$

An application of Gronwall's lemma together with the fact that $\theta^u(0) = 0$ yields that $\theta^u = 0$. Since $(w^i(t) - \varphi(u^i(t)), 1) = 0$ we have that $(\theta^w, 1) = 0$ and we can conclude from (3.16) that $\theta^w = 0$. □

THEOREM 3.2. – If $\Omega \in C^3$ then a solution $\{u_\varepsilon, w_\varepsilon\}$ for $(P^\alpha_{\gamma,\varepsilon})$ satisfies $u_\varepsilon \in L^2((0, T: W^1_p(\Omega)), \ \forall p \in [2, +\infty)$ and

$$\|u_\varepsilon\|_{L^2(0,T;W^1_p(\Omega))} \leq C \tag{3.21}$$

where the constant C does not depend on ε.

PROOF. We write equation (1.5*b*) in the form

$$\begin{cases} -\gamma\Delta u_\varepsilon = w_\varepsilon - \varphi(u_\varepsilon) + \alpha \operatorname{div}\dfrac{\nabla u_\varepsilon}{\sqrt{\varepsilon^2 + |\nabla u_\varepsilon|^2}} \equiv F^\varepsilon + \alpha \operatorname{div}\mathcal{H}^\varepsilon, \\ \dfrac{\partial u_\varepsilon}{\partial \nu}\Big|_{\partial\Omega} = 0. \end{cases} \tag{3.22}$$

Note that $|\mathcal{H}_\varepsilon| < 1$ and $F^\varepsilon \in L^2(Q_T)$, $n \leq 3$. Let us study the following two auxilliary problems

$$-\gamma\,\Delta v_\varepsilon = F^\varepsilon; \ \frac{\partial v_\varepsilon}{\partial \nu}\Big|_{\partial\Omega} = 0$$

and

$$-\gamma\,\Delta g_\varepsilon = \alpha \operatorname{div}\mathcal{H}^\varepsilon; \ \frac{\partial g_\varepsilon}{\partial \nu}\Big|_{\partial\Omega} = 0.$$

It follows from NECAS [1967; p. 320] and (3.12*a*) that

$$\|v_\varepsilon\|_{L^2(0,T;L^\infty(\Omega))} \leq C\{\|F^\varepsilon\|_{L^2(Q_T)} + \|v_\varepsilon\|_{L^2(0,T;H^1(\Omega))}\} \leq C. \tag{3.23}$$

Also results from NECAS [1967; p. 318] and (3.12*a*) imply that

$$\|g_\varepsilon\|_{L^2(0,T;L^\infty(\Omega))} \leq C\{\|\mathcal{H}^\varepsilon\|_{L^2(0,T;L^\infty(\Omega))} + \|g^\varepsilon\|_{L^2(0,T;H^1(\Omega))}\} \leq C. \tag{3.24}$$

The estimates (3.23) and (3.24) imply

$$\|u_\varepsilon\|_{L^2(0,T;L^\infty(\Omega))} \leq C, \tag{3.25}$$

where C does not depend on ε.

In order to conclude (3.21) we write F^ε in the form

$$F^\varepsilon = \operatorname{div} \boldsymbol{Q}^\varepsilon, \qquad \boldsymbol{Q}^\varepsilon \cdot \nu|_{\partial\Omega} = 0.$$

It follows from (3.25) that

$$F^\varepsilon \in L^2(0,T;L^6(\Omega)) \quad \text{and} \quad \boldsymbol{Q}^\varepsilon \in L^2(0,T;(W_6^1(\Omega))^n) \cap L^2(0,T;(L^\infty(\Omega))^n).$$

Now (3.22) can be written in the form

$$\operatorname{div}\{\gamma\nabla u_\varepsilon + \boldsymbol{Q}^\varepsilon + \alpha\mathscr{H}^\varepsilon\} = 0, \qquad \gamma\frac{\partial u_\varepsilon}{\partial\nu} + (\boldsymbol{Q}^\varepsilon + \alpha\mathscr{H}^\varepsilon)\nu\Big|_{\partial\Omega} = 0 \tag{3.26}$$

with $\boldsymbol{Q}^\varepsilon + \alpha\mathscr{H}^\varepsilon \in L^2(0,T;L^\infty(\Omega)^n)$.

It follows from ANTONCEV, KAZIKHOV and MONAKHOV [1983; p. 232-236], that the solution for (3.26) is from $L^2(0,T;W_p^1(\Omega))$, $\forall p \in [2,\infty)$ and that

$$\|u_\varepsilon\|_{L^2(0,T;W_p^1(\Omega))} \leq C\|\boldsymbol{Q}^\varepsilon + \alpha\mathscr{H}^\varepsilon\|_{L^2(0,T;L^\infty(\Omega))}, \qquad \forall p \in [2,\infty).$$

Now it is easy to deduce (3.21). □

4. – The (P_γ^α) problem.

Denoting by $\{u^\varepsilon, w^\varepsilon\}$ the unique solution of $(P_{\gamma,\varepsilon}^\alpha)$ we shall construct a solution of (P_γ^α) by taking the limit as $\varepsilon \to 0$ of $\{u^\varepsilon, w^\varepsilon\}$.

THEOREM 4.1. – If $u_0 \in H^1(\Omega)$ then there exists a unique solution to (P_γ^α).

PROOF. – It follows from (3.8) and (3.10) that the estimates

$$\|u_\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} \leq C, \tag{4.1a}$$

$$\|\partial u_\varepsilon/\partial t\|_{L^2(0,T;(H^1(\Omega))')} \leq C, \tag{4.1b}$$

$$\|w_\varepsilon\|_{L^2(0,T;H^1(\Omega))} \leq C(1+T^{1/2}), \tag{4.1c}$$

hold with C independent of ε and T. Furthermore taking $\eta = 1$ in (1.5a), (1.5b) yields

$$(u_\varepsilon(t), 1) = (u_0, 1), \tag{4.2a}$$

$$(w_\varepsilon(t), 1) = (\varphi(u_\varepsilon(t)), 1). \tag{4.2b}$$

Therefore one can again conclude that there exists a subsequence of $\{u_\varepsilon, w_\varepsilon\}$ (again denoted by $\{u_\varepsilon, w_\varepsilon\}$) and functions $u \in L^\infty(0,T;H^1(\Omega)) \cap W(0,T)$ and $w \in L^2(0,T;H^1(\Omega))$ such that

$$u_\varepsilon \to u \quad \text{weakly in} \quad W(0,T), \tag{4.3a}$$

$$u_\varepsilon \to u \quad \text{weak* in} \quad L^\infty(0,T;H^1(\Omega)), \tag{4.3b}$$

(4.3*c*) $w_\varepsilon \to w$ weakly in $L^2(0,T;H^1(\Omega))$,

(4.3*d*) $u_\varepsilon \to u$ strongly in $L^2(0,T;H^\beta(\Omega)) \quad 0 \leq \beta < 1$,

(4.3*e*) $u_\varepsilon \to u$ strongly in $L^2(0,T;L^p(\Omega)) \quad p \in (1,6)$.

It is easy to obtain the equations

$$\frac{\partial u}{\partial t} = \nabla w \quad \text{in } L^2(0,T;(H^1(\Omega))'),$$

$$u(0) = u_0 \quad \text{in } H^1(\Omega).$$

In order to complete the proof of existence it remains to obtain (1.6*b*). Rewriting (1.5*b*) in the equivalent form (3.15) we obtain

$$\int_0^T \xi(t)[\gamma(\nabla u_\varepsilon, \nabla\eta - \nabla u_\varepsilon) - (w_\varepsilon - \varphi(u_\varepsilon), \eta - u_\varepsilon) + J_\varepsilon(\eta) - J_\varepsilon(u_\varepsilon)]\,dt \geq 0 \tag{4.4}$$

for all non-negative $\xi \in C[0,T]$ and $\eta \in H^1(\Omega)$.

In passing to the limit $\varepsilon = 0$ in (4.4) we see that the only new terms in comparison with (3.14) are $J_\varepsilon(\eta)$ and $J_\varepsilon(u_\varepsilon)$. Clearly

$$\lim_{\varepsilon\to 0}\int_0^T J_\varepsilon(\eta)\xi(t)\,dt = \int_0^T J(\eta)\xi(t)\,dt$$

and noting (2.4),

$$\liminf_{\varepsilon\to 0}\int_0^T \xi(t)J_\varepsilon(u_\varepsilon)\,dt \geq \liminf_{\varepsilon\to 0}\int_0^T \xi(t)J(t_\varepsilon)\,dt \geq \int_0^T \xi(t)J(u)\,dt$$

by the weak lower semi-continuity of $J(\cdot)$ in $H^1(\Omega)$. Therefore we obtain

$$\int_0^T \xi(t)[\gamma(\nabla u, \nabla\eta - \nabla u) - (w - \varphi(u), \eta - u) + J(\eta) - J(u)]\,dt \geq 0 \tag{4.5}$$

for all non-negative $\xi \in C[0,T]$ and $\eta \in H^1(\Omega)$. This concludes the proof of existence. Uniqueness follows by an argument identical to that of Theorem 3.1. □

THEOREM 4.2. – Let u_ε and u be the solutions to $(P^\alpha_{\gamma,\varepsilon})$ and (P^α_γ). Then the following estimates hold

$$\|u - u_\varepsilon\|_{L^\infty(0,T;(H^1(\Omega))')} \leq C_T\,\varepsilon^{1/2}, \tag{4.6a}$$

$$\|u - u_\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} \leq C_T\,\varepsilon^{1/2}. \tag{4.6b}$$

PROOF. – Take $\eta = u$ and u_ε in (1.5*b*) and (1.6*b*) respectively and adding the resulting inequalities yields

$$\gamma|\theta^u|_1^2 - (\theta^u, \theta^w) \leqslant J_\varepsilon(u) - J(u) + J(u_\varepsilon) - J_\varepsilon(u_\varepsilon) + (\varphi(u) - \varphi(u_\varepsilon), \theta^u) \tag{4.7}$$

where $\theta^u = u - u_\varepsilon$, $\theta^w = w - w_\varepsilon$. It follows from (2.4) that

$$\gamma|\theta^u|_1^2 - (\theta^u, \theta^w) \leqslant \alpha\varepsilon|\Omega| + (\varphi(u) - \varphi(u_\varepsilon), \theta^u).$$

and employing the arguments used in the proof of uniqueness we obtain

$$\frac{d}{dt}\|\theta^u\|_{-1}^2 + \gamma|\theta^u|_1^2 \leqslant \frac{\beta^2}{\gamma}\|\theta^u\|_{-1}^2 + 2\alpha\varepsilon|\Omega|. \tag{4.8}$$

Applying Gronwall's inequality to (4.8) immediately yields (4.6). □

COROLLARY 4.1. – The variational inequality (1.6*b*) is equivalent to the existence of $\boldsymbol{\lambda} \in \Lambda = \{\boldsymbol{\eta} \in L^2(\Omega) \times L^2(\Omega) \colon |\boldsymbol{\eta}(x)| \leqslant 1 \text{ a.e. } \Omega\}$ such that for a.e. $t \in (0, T)$

$$\gamma(\nabla u, \nabla \eta) + \alpha(\boldsymbol{\lambda}, \nabla\eta) = (w - \varphi(u), \eta) \qquad \forall \eta \in H^1(\Omega), \tag{4.9a}$$

$$\boldsymbol{\lambda} \cdot \nabla u = |\nabla u| \qquad \text{a.e. } x \in \Omega. \tag{4.9b}$$

PROOF. – Following GLOWINSKI, LIONS and TREMOLIERES [1981; Ch.5] (4.9) is an easy consequence of the equations

$$\gamma(\nabla u_\varepsilon, \nabla\eta) + \alpha(\boldsymbol{\lambda}_\varepsilon, \nabla\eta) = (w_\varepsilon - \varphi(u_\varepsilon), \eta) \qquad \forall \eta \in H^1(\Omega) \qquad \text{a.e. } t \in (0, T), \tag{4.10a}$$

$$\boldsymbol{\lambda}_\varepsilon = \frac{\nabla u_\varepsilon}{(\varepsilon^2 + |\nabla u_\varepsilon|^2)^{1/2}} \tag{4.10b}$$

and the convergence properties of $\{u_\varepsilon, w_\varepsilon\}$. □

THEOREM 4.3. – If $\Omega \in C^3$ then a solution $\{u, w\}$ for

$$(P_\gamma^\alpha) \quad \text{satisfies} \quad u \in L^2(0, T; W_p^1(\Omega)) \cap L^2(0, T; C^{0,\alpha}(\overline{\Omega}))$$

for every $p \in [2, +\infty)$ and for every $\alpha \in (0, 1)$. Furthermore if $\partial\Omega$ is smooth then $u \in L^2(0, T; H^2(\Omega))$.

PROOF. – The first statement is an easy consequence of *Theorem* 3.2.

Replacing $(w - \varphi(u))$ with f in $(1 = 6b)$ yields a variational inequality of a form studied by BREZIS [1971] in the case of Dirichlet boundary conditions. His proof of $H^2(\Omega)$ can be generalized to the case of zero Neumann conditions, QUITTNER [1988]. This implies the second statement. □

REMARK. – Noting (2.7) equation (1.6*a*) implies that

$$w = m(w) - \mathcal{G}_N \partial u/\partial t$$

and (1.6*a*), (1.6*b*) are equivalent to the variational inequality

$$(4.11)\qquad \left(\mathcal{G}_N \frac{\partial u}{\partial t}, \eta - u\right) + \gamma(\nabla u, \nabla\eta - \nabla u) + \alpha \int_\Omega |\nabla \eta|\, dx - \alpha \int_\Omega |\nabla u|\, dx \geqslant$$

$$\geqslant (m(w) - \varphi(u), \eta - u) \qquad \forall \eta \in H^1(\Omega) \qquad \text{a.e. } t \in (0, T).$$

Furthermore taking $\eta = u \pm c$ where c is any constant we find that $m(w) = m(\varphi(u))$ and (P_γ^α) is equivalent to

$$(4.12a)\qquad \left(\mathcal{G}_N \frac{\partial u}{\partial t}, \eta - u\right) + \gamma(\nabla u, \nabla\eta - \nabla u) + \alpha \int_\Omega |\nabla \eta|\, dx - \alpha \int_\Omega |\nabla u|\, dx \geqslant$$

$$\geqslant (m(\varphi(w)) - \varphi(u), \eta - u) \qquad \forall \eta \in H^1(\Omega) \qquad \text{a.e. } t \in (0, T),$$

$$(4.12b)\qquad u(0) = u_0 .$$

Another equivalent formulation is: find $u \in L^\infty(0, T; H^1(\Omega)) \cap W(0, T)$ such that

$$(4.13a)\qquad \left(\mathcal{G}_N \frac{\partial u}{\partial t}, \eta - u\right) + \gamma(\nabla u, \nabla\eta - \nabla u) + \alpha \int_\Omega |\nabla \eta|\, dx - \alpha \int_\Omega |\nabla u|\, dx \geqslant$$

$$\geqslant (-\varphi(u), \eta - u) \qquad \forall \eta \in H^1(\Omega), \qquad m(\eta) = m(m_0), \qquad \text{a.e. } t \in (0, T),$$

$$(4.13b)\qquad u(0) = u_0 \quad \text{and} \quad m(u(t)) = m(u_0).$$

5. – The (P^α) problem.

Denoting by u_γ the unique solution to (P_γ^α) constructed in § 4, we have the following a priori estimates for $\gamma \in (0, 1]$

$$(5.1)\qquad \gamma |u_\gamma(t)|_1^2 + \frac{1}{8} \|u_\gamma(t)\|_{L^4(\Omega)}^4 + \alpha \int_\Omega |\nabla u_\gamma(t)|\, dx + \int_0^t \left\| \frac{\partial u_\gamma}{\partial t} \right\|_{-1}^2 + \int_0^t |w_\gamma(\tau)|_1^2\, d\tau \leqslant C$$

where C does not depend on γ and t. Furthermore the equations

$$(5.2a)\qquad (u_\gamma(t), 1) = (u_0, 1),$$

$$(5.2b)\qquad (w_\gamma(t), 1) = (\varphi(u_\gamma(t)), 1),$$

hold. Thus the following estimates are uniform in γ:

$$(5.3a)\qquad \|u_\gamma(t)\|_{BV(\Omega) \cap L^4(\Omega)} \leqslant C,$$

(5.3*b*) $$\|\nabla w_\gamma\|_{L^2(\Omega\times(0,T))} \leqslant C,$$

(5.3*c*) $$\|w_\gamma\|_{L^2(0,T;H^1(\Omega))} \leqslant C(1+T),$$

(5.3*d*), $$\left\|\frac{\partial u}{\partial t}\gamma\right\|_{L^2(0,T;(H^1(\Omega))')} \leqslant C$$

where C is independent of T.

Applying the compactness theorem of SIMON [1987], (Corollary 4, p. 85 with $X = BV(\Omega)$, $B = L^q(\Omega)$ $(q < n/(n-1))$ and $Y = (H^1(\Omega))')$ (note that $BV(\Omega) \hookrightarrow L^q(\Omega)$, $q < (n/n-1)$ is compact embedding see GILBARG and TRUDINGER [1977]) one obtains the existence of

$$u \in L^\infty(0,T; BV(\Omega)\cap L^4(\Omega)) \quad \text{and} \quad w \in L^2(0,T;H^1(\Omega))$$

such that: there exists a subsequence $\{u_\gamma, w_\gamma\}$ still denoted by $\{u_\gamma, w_\gamma\}$ with the convergence properties

(5.4*a*) $$u_\gamma \to u \quad \text{weak* in} \quad L^\infty((0,T;L^4(\Omega)),$$

(5.4*b*) $$u_\gamma \to u \quad \text{weakly in} \quad L^2(0,T;L^4(\Omega)),$$

(5.4*c*) $$u_\gamma \to w \quad \text{strongly in} \quad C[0,T;L^q(\Omega)], \qquad q < n/(n-1),$$

(5.4*d*) $$u_\gamma \to u \quad \text{strongly in} \quad L^2(0,T;L^q(\Omega)), \qquad q < n/(n-1),$$

(5.4*e*) $$w_\gamma \to w \quad \text{weakly in} \quad L^2(0,T;H^1(\Omega)).$$

It also follows from (5.1) that

(5.5) $$\gamma^{1/2}u_\gamma \to 0 \quad \text{weakly in} \quad L^2(0,T;H^1(\Omega)).$$

Let us write (1.6*b*) in the form

(5.6) $$\int_0^T \xi(t)\left[\gamma(\nabla u_\gamma, \nabla\eta - \nabla u_\gamma) + \alpha\int_\Omega |\nabla\eta|\,dx - \alpha\int|\nabla u_\gamma|\,dx - (w_\gamma - \varphi(u_\gamma), \eta - u_\gamma)\right]dt \geqslant 0$$

for all non-negative $\xi \in C[0,T]$ and $\eta \in H^1(\Omega)$. It follows immediately from the convergence properties (5.4) and (5.5) that

(5.7*a*) $$\lim_{\gamma\to\infty}\int_0^T \xi(t)(\gamma^{1/2}\nabla u_\gamma, \gamma^{1/2}\nabla\eta)\,dt = 0,$$

(5.7*b*) $$\lim_{\gamma\to\infty}\int_0^T \xi(t)(w_\gamma, \eta)\,dt = 0.$$

Also since w_γ is bounded in $L^2(0,T;L^6(\Omega))$ and u_γ converges strongly to u in

$L^2(0, T; L^{4/3}(\Omega))$ we have

$$\lim_{\gamma\to\infty} \int_0^T \xi(t)(w_\gamma, u_\gamma)\,dt = \int_0^T \xi(t)(w, u)\,dt. \tag{5.7c}$$

Furthermore it holds that, by (2.2),

$$\lim_{\gamma\to 0} \int_0^T \xi(t) \int_\Omega |\nabla u_\gamma|\,dx\,dt \geqslant \int_0^T \xi(t) \int_\Omega |\nabla u|. \tag{5.7d}$$

Pssing to the limit $\gamma = 0$ in (5.6) using (5.7) yields

$$\int_0^T \xi(t)\left[\alpha \int_\Omega |\nabla \eta|\,dx - \alpha \int_\Omega |\nabla u| - (w, \eta - u)\right] dt \geqslant \lim_{\gamma\to 0} \int_0^T \xi(t)(\varphi(u_\gamma), u_\gamma - \eta)\,dt. \tag{5.8}$$

Introducing $A: L^4(\Omega) \to L^{4/3}(\Omega)$ defined by

$$A(v) = \varphi(v) + \beta v \tag{5.9}$$

we have, by the inequality $\varphi'(s) \geqslant -\beta\ \forall s$, that A is a continuous monotone operator. Since (5.4*b*) and (5.4*d*) with $q = 4/3$ hold it follows that the right-hand side of (5.8) is equal to

$$-\beta \int_0^T \xi(t)(u, u - \eta)\,dt + \lim_{\gamma\to 0} \int_0^T \xi(t)(A(u_\gamma), u_\gamma - \eta)\,dt. \tag{5.10}$$

Choosing $\eta = u$ implies that

$$\limsup_{\gamma\to 0} \int_0^T \xi(t)(A(u_\gamma), u_\gamma - u)\,dt \leqslant 0.$$

Hence the usual method of monotonicity LIONS [1969; p. 172] yields

$$\lim_{\gamma\to 0} \int_0^T \xi(t)(A(u_\gamma), u_\gamma - \eta)\,dt = \int_0^T \xi(t)(A(u), u - \eta)\,dt$$

so that (5.10), (5.8) finally yield

$$\int_0^T \xi(t)\left[\alpha \int_\Omega |\nabla \eta|\,dx - \alpha \int_\Omega |\nabla u| - (w - \varphi(u), \eta - u)\right] dt \geqslant 0. \tag{5.11}$$

Therefore (1.7*b*) holds. Limiting (1.6*b*) yields (1.7*a*) and (1.7*c*) follows from (5.4*c*). Hence we have proved the following theorem.

THEOREM 5.1. – If $u_0 \in H^1(\Omega)$ then there exists a solution to (P^α). □

6. – Regularity and asymptotic behaviour in one dimension.

The next question which we would like to answer concerns the continuity of the solution and some further smoothness. We are able to prove it only in one dimension. Now $\Omega = (0, L)$ and $Q_T = (0, L) \times (0, T)$. We shall state a result about smoothness in the Holder space $H^{\lambda, \lambda/2}(\overline{Q}_T)$. For definition of these spaces see LADYZENSKAYA, SOLONNIKOV and URALCEVA [1968; p. 80].

THEOREM 6.1. – Let us suppose the following additional regularity and compatibility conditions on the data:

$$u_0 \in H^3(0, L), \qquad \frac{du_0}{dx} \geq 0 \quad \text{on } (0, L), \tag{6.1}$$

$$\frac{du_0}{dx}(0) = \frac{du_0}{dx}(L) = 0 \tag{6.2}$$

and let $\{u, w\}$ be a solution for (P^α_γ). Then

$$u \in L^\infty(0, T; H^2(0, L)) \cap H^{1/2, 1/4}(\overline{Q}_T), \tag{6.3}$$

$$w \in L^\infty(0, T; H^1(0, L)). \tag{6.4}$$

PROOF. – Instead of $[\varepsilon^2 + |du/dx|^2]^{1/2}$ we choose a regularization from GLOWINSKI, LIONS and TREMOLIERES [1981; Ch. 5].

$$Z_\varepsilon(z) = \begin{cases} \dfrac{2\varepsilon}{\pi}\left(1 - \cos\dfrac{\pi z}{2\varepsilon}\right), & |z| \leq \varepsilon, \\[2ex] |z| - \varepsilon\left(1 - \dfrac{2}{\pi}\right), & |z| > \varepsilon. \end{cases} \tag{6.5}$$

Note that $Z_\varepsilon \in W^3_{\infty, \mathrm{loc}}(\mathbb{R})$; $Z''_\varepsilon = Z'''_\varepsilon = 0$ for $|z| > \varepsilon$.

Let us define (a new) ε-problem:

Find $u_\varepsilon \in L^\infty(0, T; H^1(\Omega)) \cap W(0, T)$ and $w_\varepsilon \in L^2(0, T; H^1(\Omega))$

$$\left\langle \frac{\partial u_\varepsilon}{\partial t}, \eta \right\rangle + \left(\frac{\partial w_\varepsilon}{\partial x}, \frac{\partial \eta}{\partial x} \right) = 0, \qquad \eta \in H^1(\Omega), \text{ a.e. } t \in (0, T), \tag{6.6a}$$

$$(6.6b)\qquad (w_\varepsilon - \varphi(u_\varepsilon), \eta) = \gamma\left(\frac{\partial u_\varepsilon}{\partial x}, \frac{\partial \eta}{\partial x}\right) + \alpha\left(Z'_\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x}\right), \frac{\partial \eta}{\partial x}\right) - $$

$$-(2\gamma\varepsilon + \alpha)(\eta(L) - \eta(0)), \qquad \forall \eta \in H^1(\Omega), \qquad \text{a.e. } t \in (0, T),$$

$$(6.6c)\qquad u_\varepsilon(0) = u_0(x) + 2\varepsilon x = u_{0\varepsilon}(x).$$

Let $\{v_j\}$ be the orthogonal basis for $H^1(\Omega)$ consisting of the eigenfunctions for

$$(6.7a)\qquad -\frac{d^2 z}{dx^2} = \lambda z; \qquad \frac{dz}{dx}(0) = \frac{dz}{dx}(L) = 0, \qquad \int_0^L z\,dx = 0$$

and normalized so that

$$(6.7b)\qquad (v_i, v_j) = \delta_{ij}, \qquad \left(\frac{dv_i}{dx}, \frac{dv_j}{dx}\right) = \lambda_i \delta_{ij}.$$

Note that $\{v_j\}$ is an orthonormal basis for $L^2(0, L)/\mathbb{R}$. Denote by V_m the finite dimensional subspace of $H^1(0, L)$ spanned by $\{v_j\}_{j=1}^m$ and let $\boldsymbol{P}_m$ be given by (3.3*a*).

A Galerkin approximation to (6.6*a*)-(6.6*c*) is:

$$(6.8a)\qquad \begin{cases} u^m(t) = \sum\limits_{j=1}^m c_j(t) v_j + \dfrac{1}{L}\displaystyle\int_0^L u_0\,dx + 2\varepsilon x, \\ w^m(t) = \sum\limits_{j=1}^m d_j(t) v_j + \dfrac{1}{L}\displaystyle\int_0^L \varphi(u^m(t))\,dx, \end{cases}$$

$$(6.8b)\qquad \left(\frac{\partial u^m}{\partial t}, v_j\right) + \left(\frac{\partial w^m}{\partial x}, \frac{\partial v_i}{\partial x}\right) = 0 \qquad j = 1, \ldots, m,$$

$$(6.8c)\qquad (w^m - \varphi(u^m), v_j) = \gamma\left(\frac{\partial u^m}{\partial x}, \frac{\partial v_j}{\partial x}\right) + \alpha\left(Z'_\varepsilon\left(\frac{\partial u^m}{\partial x}\right), \frac{\partial v_j}{\partial x}\right) - $$

$$-(\alpha + 2\varepsilon\gamma)(v_j(L) - v_j(0)) \qquad j = 1, \ldots, m,$$

$$(6.8d)\qquad u^m(0) = \boldsymbol{P}_m\left(u_0 - \frac{1}{L}\int_0^L u_0\right) + \frac{1}{L}\int_0^L u_0 + 2\varepsilon x.$$

One can easily obtain results similar to those in § 3, for the system (6.8*a*)-(6.8*d*). We are interested in obtaining some new estimates. After differentiating in t the equation (6.8c) one obtains

$$(6.9)\qquad \left(\frac{\partial w^m}{\partial t} - \varphi'(u^m)\frac{\partial u^m}{\partial t}, v_j\right) = \gamma\left(\frac{\partial}{\partial x}\frac{\partial u^m}{\partial t}, \frac{\partial v_j}{\partial x}\right) + \alpha\left(Z_\varepsilon''\left(\frac{\partial u^m}{\partial x}\right)\frac{\partial}{\partial x}\frac{\partial u^m}{\partial t}, \frac{\partial v_j}{\partial x}\right).$$

After multiplying (6.9) by dc_i/dt and summing, we have

$$(6.10)\qquad \int_0^L\left[\gamma + \alpha Z_\varepsilon''\left(\frac{\partial u^m}{\partial x}\right)\right]\left|\frac{\partial^2 u^m}{\partial x\partial t}\right|^2 dx = -\int_0^L \varphi'(u^m)\left|\frac{\partial u^m}{\partial t}\right|^2 +$$

$$+\int_0^L \frac{\partial w^m}{\partial t}\frac{\partial u^m}{\partial t} = -\int_0^L \varphi'(u^m)\left|\frac{\partial u^m}{\partial t}\right|^2 - \frac{1}{2}\frac{\partial}{\partial t}\int_0^L\left|\frac{\partial w^m}{\partial x}(t)\right|^2.$$

By using the convexity of Z_ε and (2.8b) we are able to derive the inequalities

$$(6.11a)\qquad \left\|\frac{\partial w^m}{\partial x}\right\|_{L^\infty(0,T;L^2(0,L))} \leq C(T)\left\|\frac{\partial w^m}{\partial x}(0)\right\|_{L^2(0,L)},$$

$$(6.11b)\qquad \left\|\frac{\partial u^m}{\partial t}\right\|_{L^2(0,T;H^1(0,L))} \leq C(T)\left\|\frac{\partial w^m}{\partial x}(0)\right\|_{L^2(0,L)},$$

$$(6.11c)\qquad \left\|\int_0^L \varphi(u^m)\right\|_{L^\infty(0,T)} \leq C(T).$$

Now it remains to estimate $\|w_m'(0)\|_{L^2(0,L)}$. We use the equations (6.8c) and (6.7a):

$$\int_0^L w^m\left(-\frac{1}{\lambda_i}v_i''\right) = \gamma\int_0^L (u^m)'\left(-\frac{1}{\lambda_i}v_i'''\right) + \alpha\int_0^L Z_\varepsilon'((u^m)')\left(-\frac{1}{\lambda_i}v_i'''\right) +$$

$$+\int_0^L \varphi(u^m)\left(-\frac{1}{\lambda_i}v_i''\right) - [2\gamma\varepsilon + \alpha]\left(-\frac{1}{\lambda_i}v_i''(L) + \frac{1}{\lambda_i}v_i''(0)\right).$$

After partial integration we have

$$(6.12)\qquad -\int_0^L \frac{\partial w^m}{\partial x}\frac{\partial v_i}{\partial x} + \int_0^L \varphi'(u^m)\frac{\partial u^m}{\partial x}\frac{\partial v_i}{\partial x} = -\gamma\int_0^L \frac{\partial^2 u^m}{\partial x^2}\frac{\partial^2 v_i}{\partial x^2} -$$

$$-\alpha\int_0^L Z_\varepsilon''\left(\frac{\partial u^m}{\partial x}\right)\frac{\partial^2 u^m}{\partial x^2}\,\frac{\partial^2 v_i}{\partial x^2} = \gamma\int_0^L \frac{\partial^3 u^m}{\partial x^3}\,\frac{\partial v_i}{\partial x} +$$

$$+\alpha\int_0^L Z_\varepsilon'''\left(\frac{\partial u^m}{\partial x}\right)\left|\frac{\partial^2 u^m}{\partial x^2}\right|^2\cdot\frac{\partial v_i}{\partial x} + \alpha\int_0^L Z_\varepsilon''\left(\frac{\partial u^m}{\partial x}\right)\frac{\partial^3 u^m}{\partial x^3}\,\frac{\partial v_i}{\partial x}.$$

Let us multiply (6.12) by $d_i(t)$ and make summation. One obtains

$$(6.13)\qquad -\int_0^L\left|\frac{\partial w^m}{\partial x}(t)\right|^2 + \int_0^L \varphi'(u^m)\frac{\partial u^m}{\partial x}\,\frac{\partial w^m}{\partial x} = \int_0^L \frac{\partial^3 u^m}{\partial x^3}\,\frac{\partial w^m}{\partial x} +$$

$$+\alpha\int_0^L Z_\varepsilon'''\left(\frac{\partial u^m}{\partial x}\right)\cdot\left|\frac{\partial^2 u^m}{\partial^2}\right|^2\frac{\partial w^m}{\partial x} + \alpha\int_0^L Z_\varepsilon''\left(\frac{\partial u^m}{\partial x}\right)\frac{\partial^3 u^m}{\partial x^3}\,\frac{\partial w^m}{\partial x}.$$

We have particular interest in the case $t=0$. Note that $(\partial u^m/\partial x)(x,0)\geqslant\varepsilon\cdot 3/2$ for every $x\in(0,L)$ and sufficiently large m. Therefore $Z_\varepsilon'''((\partial u^m/\partial x)(0)) = Z_\varepsilon''((\partial u^m/\partial x)(0)) = 0$. Also note that (6.1) and (6.2) imply $P_m\left(u_0 - (1/L)\int_0^L u_0\right)\to u_0 - (1/L)\int_0^L u_0$ in $H^3(0,L)$. Hence one has

$$\left\|\frac{\partial w^m}{\partial x}(0)\right\|^2_{L^2(0,L)} = \gamma\int_0^L \frac{\partial w^m}{\partial x}(0)\cdot\frac{\partial^3}{\partial x^3}u^m(0) - \int_0^L \varphi'(u^m)\frac{\partial u^m}{\partial x}(0)\,\frac{\partial w^m}{\partial x}(0)$$

and consequently

$$(6.14)\qquad \left\|\frac{\partial w^m}{\partial x}(0)\right\|_{L^2(0,L)}\leqslant C$$

where the constant C does not depend on ε and m. Note that the uniform estimate for $\partial w^m/\partial x$ in $L^\infty(0,T;L^2(0,L))$ and equation (6.12) give a uniform estimate for u in $L^\infty(0,T;H^2(0,L))$.

Now in the limit $m\to+\infty$ and $\varepsilon\to+0$ one obtains

$$(6.15)\qquad \begin{cases} w\in L^\infty(0,T;H^1(0,L)); \qquad u\in L^\infty(0,T;H^2(0,L)) \\ \text{and} \\ \dfrac{\partial u}{\partial t}\in L^2(0,T;H^1(0,L)), \end{cases}$$

Finally (6.15) implies $u\in W_2^{2,1}(Q_T) = \{z|(\partial^r/\partial t^r)(\partial^s/\partial x^s)z\in L^2(Q_T); 2r+s\leqslant 2\}$. Now an embedding theorem from LADYZENSKAYA, SOLONNIKOV and URAL'CEVA [1986; p. 80] gives $u\in H^{1/2,1/4}(\overline{Q}_T)$.

Let us now study the asymptotic behaviour of solutions for problem (P_γ^α). We restrict ourselves to the one dimensional case. Our first goal is to obtain estimates of the form

$$\|u\|_{L^\infty(0,+\infty;H^2(0,L))} \leqslant C, \qquad \|w\|_{L^\infty(0,+\infty;H^1(0,L))} \leqslant C. \tag{6.16}$$

LEMMA 6.1. – Let all assumptions of *Theorem* 6.1 be valid. Then inequalities (6.16) hold true.

PROOF. – It is enough to prove that the constant in inequality (6.11a) does not depend on T. In order to prove it we study equality (6.10). An immediate consequence of the equality (6.10) is

$$\frac{1}{2}\frac{\partial}{\partial t}\int_0^L \left|\frac{\partial w^m}{\partial x}(t)\right|^2 dx + \gamma\int_0^L \left|\frac{\partial^3 w^3}{\partial x^3}(t)\right|^2 dx \leqslant \tag{6.17}$$

$$\leqslant C\int_0^L \left|\frac{\partial^2 w^m}{\partial x^2}(t)\right|^2 dx \leqslant C\left\|\frac{\partial^3 w^m}{\partial x^3}(t)\right\|_{L^2(0,L)}\left\|\frac{\partial w^m}{\partial x}\right\|_{L^2(0,L)}.$$

It follows from (6.17) that

$$\frac{\partial}{\partial t}\int_0^L \left|\frac{\partial w^m}{\partial x}(t)\right|^2 dx \leqslant C\int_0^L \left|\frac{\partial w^m}{\partial x}(t)\right|^2 dx$$

or, after integration and using (3.9)

$$\left\|\frac{\partial w^m}{\partial x}(t)\right\|^2_{L^2(0,L)} \leqslant C\left\|\frac{\partial w^m}{\partial x}(0)\right\|^2_{L^2(0,L)}. \qquad \square \tag{6.18}$$

Now we will construct a continuous nonlinear semigroup connected with our evolution problem.

Let us construct a metric space X in the following way: we pick up a bounded set of functions u_0 from $H^3(0,L)$ which satisfy conditions (6.1) and (6.2). Solving a parabolic problem (P^α_γ) for such initial data one produces a new function for every $t \in (0,+\infty)$. Because of inequalities (6.16) we make a weak closure in $H^2(0,L)$ of that whole range. Now X consists of these functions. We equip X with a metric induced by the $(H(0,L))'$ norm. Obviously we have

$$\sup_{u\in X}\|u\|_{H^2(0,L)} \leqslant C \tag{6.19}$$

X is a metric space, but not a complete metric space (and not a linear space).

Let us define a family of maps $\{T(t): X \to X, t \geqslant 0\}$ by setting

$$T(t)u_0 = u(x,t) \qquad \forall u_0 \in X. \tag{6.20}$$

LEMMA 6.2. – The family of maps $\{T(t): X \to X, t \geq 0\}$ is a continuous nonlinear semigroup.

PROOF. – We check conditions from HENRY [1981; Chapter 4, Def. 4.1.1.].

First we have to prove that for fixed $t > 0$ the map $T(t): X \to X$ is continuous.

Let u_{01} and u_{02} be two different initial data. Then $u_1(t)$ and $u_2(t)$ are corresponding solutions for time t. We have

$$\gamma \int_0^L \frac{\partial u_1}{\partial x} \cdot \frac{\partial}{\partial x}(u_2 - u_1) + \alpha \int_0^L \left| \frac{\partial u_2}{\partial x} \right| - \alpha \int_0^L \left| \frac{\partial u_1}{\partial x} \right| \geq \int_0^L [w_1 - \varphi(u_1)](u_2 - u_1), \tag{6.20a}$$

$$\gamma \int_0^L \frac{\partial u_2}{\partial x} \cdot \frac{\partial}{\partial x}(u_1 - u_2) + \alpha \int_0^L \left| \frac{\partial u_1}{\partial x} \right| - \alpha \int_0^L \left| \frac{\partial u_2}{\partial x} \right| \geq \int_0^L [w_2 - \varphi(u_2)](u_1 - u_2). \tag{6.20b}$$

After adding (6.20*a*) and (6.20*b*) one obtains

$$\frac{d}{dt} \|(u_1 - u_2)(t)\|^2_{(H^1(0,L))'} + \gamma \int_0^L \left| \frac{\partial}{\partial x}(u_1 - u_2) \right|^2 \leq \beta \int_0^L |(u_1 - u_2)(t)|^2$$

and, consequently

$$\|u_1(t) - u_2(t)\|_{(H^1(0,L))'} \leq C(T) \|u_{01} - u_{02}\|.$$

Therefore $T(t) \in \mathcal{L}(X, X)$ is continuous.

Next we prove that for fixed $u_0 \in X$, the mapping $t \to T(t)u_0$ is continuous. Really $T(t)u_0 \in L^\infty(0, T; H^1(0, L))$ and $(d/dt)(T(t)u_0) \in L^2(0, T; H^1(0, L)')$ imply $T(t)u_0 \in C([0, T]; X)$.

The last two properties are easy to check. Obviously $T(0)$ is an identity on X and $T(t)(T(\tau)u_0) = T(t + \tau)u_0$ $\forall u_0 \in X$ and $t, \tau \geq 0$. Q.E.D.

We define the ω-limit set for u_0 by

$$\omega(u_0) = \{z \in X : \exists t_n \to +\infty \quad \text{such that} \quad T(t_n)u_0 \to z\} \tag{6.21}$$

PROPOSITION 6.1. – For every $u_0 \in X$, $\omega(u_0)$ is a compact connected and non-empty subset of X. The functional F given by

$$F(z) = \frac{\gamma}{2} \int_0^L \left| \frac{\partial z}{\partial x} \right|^2 + \alpha \int_0^L \left| \frac{\partial z}{\partial x} \right| + \int_0^L \psi(z) \tag{6.22}$$

is a Liapunov functional for T.

PROOF. – The first assertion is a direct consequence of Proposition 2.1 from

DAFERMOS [1977]. From equality (3.8) one easily concludes that F is a Liapunov functional for T in the sense of Definition 2.1 from DAFERMOS [1977].

THEOREM 6.2. – For a given $u_0 \in X$ there exists a sequence $\{t_n\}$, $t_n \to +\infty$, a function $\bar{u} \in H^2(0, L)$ and constant M such that

$$(6.23)\qquad \begin{cases} u(t_n) \to \bar{u} & \text{weakly in} \quad H^2(0,L), \\ w(t_n) \to M & \text{weakly in} \quad H^1(0,L) \\ \dfrac{\partial u}{\partial t}(t_n) \to 0 & \text{weakly in} \quad (H^1(0,L))' . \end{cases}$$

The function $\bar{u} \in \omega(u_0)$ satisfies

$$(6.24)\qquad \gamma \int_0^L \frac{\partial \bar{u}}{\partial x} \frac{\partial}{\partial x}(v - \bar{u}) + \alpha \int_0^L \left| \frac{\partial v}{\partial x} \right| - \alpha \int_0^L \left| \frac{\partial \bar{u}}{\partial x} \right| + \int_0^L \varphi(\bar{u})(v - \bar{u}) \geq M \int_0^L (v - \bar{u}) \qquad \forall v \in H^1(0,L),$$

$$(6.25)\qquad \int_0^L u_0 = \int_0^L \bar{u} \quad \text{and} \quad M = \int_0^L \varphi(\bar{u}) \cdot \frac{1}{L} .$$

PROOF. – In order to use the theory developed in DAFERMOS [1977] let us prove that F is continuous on X. Let $u_{0,k} \to u_0$ in X, i.e. strongly in $(H^1(0,L))'$. Then

$$(6.26)\qquad \|u_{0,k} - u_0\|_{H^1(0,L)} \leq C \|u_{0k} - u_0\|^{2/3}_{(H^1(0,L))'} \|u_{0k} - u_0\|^{1/3}_{H^2(0,L)} \leq C \|u_{0k} - u_0\|^{2/3}_{(H^1(0,L))} .$$

Therefore $u_{0k} \to u_0$ in $H^1(0,L)$ and $F(u_{0k}) \to F(u_0)$. We have concluded that F is continuous on X.

Now Proposition 2.2 from DAFERMOS [1977] holds true and F is constant on $\omega(u_0)$ for $u_0 \in X$. The Proof of (6.23), (6.24) and (6.25) is now obvious.

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