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Spreads and Classes of Maximal Subgroups of $GL_n(q)$, $SL_n(q)$, $PGL_n(q)$ and $PSL_n(q)$ (*).

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Summary. – *If r divides n then the points of $PG(n-1, q)$ can be partitioned by the $(r-1)$-subspaces of a classical spread $\mathcal{S}_r$. The underlying finite geometry of this configuration, in particular the orbits of lines, is used to prove that if r is a proper prime divisor of n then the stabilizers of $\mathcal{S}_r$ in $PGL_n(q)$ and $PSL_n(q)$ are maximal subgroups of $PGL_n(q)$ and $PSL_n(q)$ respectively. Special attention is needed for the case of $PSL_n(q)$ when $n/r = 2$ and r divides $q-1$. An explicit description is found for the stablizers.*

1. – Introduction.

$PG(n-1, q)$, projective space of dimension $n-1$ over the field $GF(q)$ of q elements, can be partitioned by a set of its $(r-1)$-subspaces whenever r divides n. Such a partition is called a *spread.* We shall be concerned with the « classical » such spread $\mathcal{S}_r$, whose construction is briefly recalled in Section 2.1. If the points of $PG(n-1, q)$ are regarded as the 1-subspaces of a vector n-space V over $GF(q)$, then $\mathcal{S}_r$ corresponds to a set $\mathcal{K}_r$ of r-subspaces of V, where each non-zero vector is in exactly one member of $\mathcal{K}_r$. Clearly, if $r = 1$ or $r = n$ then the stabilizers of $\mathcal{S}_r$ in $PGL_n(q)$ and of $\mathcal{K}_r$ in $GL_n(q)$ are $PGL_n(q)$ and $GL_n(q)$ respectively. So we avoid these uninteresting cases and take $1 < r < n$.

Suppose that t divides r with $1 < t < r$. We show in Section 3.1 that there is a classical spread $\mathcal{S}_t$ whose stabilizer in $PSL_n(q)$ strictly contains the stabilizer of $\mathcal{S}_r$ in $PSL_n(q)$. Hence if r is not a prime than the stabilizer of $\mathcal{S}_r$ in $PSL_n(q)$ $[PGL_n(q)]$ is not a maximal subgroup of $PSL_n(q)$ $[PGL_n(q)]$: analogous statements hold for $\mathcal{K}_r$ and $SL_n(q)$ and $GL_n(q)$. The major result (Theorem 4) of this paper is the converse result: if r is a proper prime divisor of n then the stabilizers of $\mathcal{S}_r$ in $PSL_n(q)$ and $PGL_n(q)$ are maximal subgroups of $PSL_n(q)$ and $PGL_n(q)$ respectively, and the stabilizers of $\mathcal{K}_r$ in $SL_n(q)$ and $GL_n(q)$ are maximal subgroups of $SL_n(q)$ and $GL_n(q)$ respectively.

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To complete the picture we identify the various stabilizers. Write $N = n/r$. To construct $\mathcal{K}_r$ (see Section 2.1) we set up a bijection between V and a vector N-space W over $GF(q^r)$. Choose a coordinate system for W. Let σ be the semi-linear bijection of W that is given by applying to each coordinate the automorphism $\lambda \to \lambda^q$ of $GF(q^r)$: σ has order r. As a group $GL_N(q^r)\langle\sigma\rangle$ is independent of the coordinate system of W: via the bijection from W to V it acts naturally on V. In fact (Theorem 1) the stabilizer of $\mathcal{K}_r$ in $GL_n(q)$ is the semi-direct product $GL_N(q^r)\langle\sigma\rangle$. Let

$$Z_N = \left\{\lambda I_N\colon \lambda \in GF(q)\setminus\{0\}\right\} < GL_N(q^r)\,. \tag{1}$$

Considered as acting on V this Z_N is the group of all scalar maps of $GL_n(q)$. Moreover σ centralizes Z_N: the action induced by σ on $PG(n-1, q)$ or $PG(N-1, q^r)$ may, without confusion, also be denoted by σ. Then (Theorem 2), the stabilizer of $\mathcal{S}_r$ in $PGL_n(q)$ is the semi-direct product $(GL_N(q^r)/Z_N)\langle\sigma\rangle \cong GL_N(q^r)\langle\sigma\rangle/Z_N$. To describe the stabilizers in $SL_n(q)$ and $PSL_n(q)$ requires a little more notation. The multiplicative group of $GF(q^r)$ is cyclic of order $q^r - 1$. Let Q be its unique subgroup of order $(q^r-1)/(q-1)$. Let

$$GL_N^*(q^r) = \{A\colon A \in GL_N(q^r) \text{ and } \det A \in Q\}\,. \tag{2}$$

Let

$$Z_N^* = \{\lambda I_N\colon \lambda \in GF(q) \text{ and } \lambda^n = 1\}\,. \tag{3}$$

We show (Theorems 1, 2) that the stabilizer of $\mathcal{K}_r$ in $SL_n(q)$ and the stabilizer of $\mathcal{S}_r$ in $PSL_n(q)$ are the respective semi-direct products $GL_N^*(q^r)\langle\sigma\rangle$ and $(GL_N^*(q^r)/Z_N^*)\langle\sigma\rangle \cong$ $\cong GL_N^*(q^r)\langle\sigma\rangle/Z_N^*$, except for the case when q is odd, r is even and $N = n/r$ is odd. In this exceptional case $\sigma \notin SL_n(q)$. However, there are members of $GF(q^r)$ with (multiplicative) order $2(q^r-1)/(q-1)$. Let α be such an element. In the exceptional case (Theorems 1, 2) the stabilizer of $\mathcal{K}_r$ in $SL_n(q)$ is $GL_N^*(q^r)\langle\alpha\sigma\rangle$, and the stabilizer of $\mathcal{S}_r$ in $PSL_n(q)$ is the semi-direct product $(GL_N^*(q^r)/Z_N^*)\langle\alpha\sigma\rangle$. Considered as acting on $PG(n-1, q)$ $\langle\alpha\sigma\rangle$ has order r. Considered as acting on V $\langle\alpha\sigma\rangle$ has order $2r$, and there is no simple way of writing $GL_N^*(q^r)\langle\alpha\sigma\rangle$ as a semi-direct product. It is worth pointing out that these identifications of the stabilizers do not require the condition that r is a prime.

The details of the proofs involve a number of strands of arguments, so it will be helpful to give an over view. The bijection from W to V shows that the stabilizer of $\mathcal{K}_r$ in $GL_n(q)$ contains $GL_N(q^r)\langle\sigma\rangle$. To establish equality we deal first with the case $N = 2$. It is known [5, pp. 176-182] that in this case there are symplectic polarities having each member of $\mathcal{K}_r$ for a totally isotropic subspace. The number of such polarities is known, and so is the stabilizer of $\mathcal{K}_r$ in the symplectic group of one of these polarities. A consideration of the orbits of these polarities under the action of the stabilizer of $\mathcal{K}_r$ in $GL_n(q)$ shows that this stabilizer has order at most $|GL_2(q^r)|r = |GL_N(q^r)|r$. The earlier containment then delivers the conclusion. For the cases $N \geqslant 3$ we take N members of $\mathcal{K}_r$ whose direct sum is V, and deduce that the stabilizer of $\mathcal{K}_r$ in $GL_n(q)$ is $GL_N(q^r)\langle\sigma\rangle$ from the fact that the subgroup of the stabilizer of $\mathcal{K}_r$ in $GL_n(q)$ that fixes each of the set of N members of $\mathcal{K}_r$ is a subgroup of $GL_N(q^r)\langle\sigma\rangle$. This last result is established by order considerations, which in turn depend on consequences of the known case $N = 2$.

The maximality proofs also use combinatorial geometric arguments. We consider the orbits of lines of $PG(n-1, q)$ under a stabilizer of $\mathcal{S}_r$ and the way in which these must fuse under the action of an over-group. Let G be either $PGL_n(q)$ or $PSL_n(q)$, and let H be the stabilizer of $\mathcal{S}_r$ in G. Suppose that

$$H < J \leqslant G .$$

Let l be a line that is contained in some member, say π, of $\mathcal{S}_r$: We show (Proposition 1) that there is a set of lines $l = l_1, l_2, \ldots, l_{r-1}$ that span π, that are all in the same orbit under H, and are such that l_i meets l_{i+1} for $i < r-1$. This is one point where we need critically the condition that r is a prime. We deduce (Proposition 2) that if r is a prime, every orbit of lines under J contains lines not in any member of $\mathcal{S}_r$. We also show (Proposition 3) that the lines not in members of $\mathcal{S}_r$ form a single orbit under H except when $N = 2$, r divides $q-1$ and $G = PSL_n(q)$. Apart from this exceptional case we immediately conclude that J acts transitively on the lines of $PG(n-1, q)$. This same conclusion holds in the exceptional case, but its justification requires a much longer and intricate investigation of the action of J on the various orbits under H of lines not in members of $\mathcal{S}_r$. In all cases it is easy to deduce that J acts 2-transitively on the points of $PG(n-1, q)$. Then a theorem of Cameron and Kantor [3, p. 384] gives $J \leqslant PSL_n(q)$, from which the conclusion $J = G$ readily follows.

This description shows that the details of the argument are very different from those used in maximality proofs for the stabilizers of spreads in symplectic and orthogonal groups [6], [7], [8], [9]. As in [7], [8] and [9] the proof is geometric: Cameron and Kantor proved their theorem by geometric arguments and made no use of group-theoretic classification theorems. Our results are complementary to the important paper [1]. There Aschbacher lists [1, pp. 472, 473] 8 classes of « obvious » candidates for maximal subgroups of the finite simple classical groups, and obtains a very significant hold on any other maximal subgroups: for $PSL_n(q)$ the stabiliser of $\mathcal{S}_r$ is in Aschbacher's class C_3. Although he does not prove the maximality, or otherwise, of his classes in [1] (see [1, p. 469]) he suggests in [2, p. 40] that [1] can be used with the known full list of finite simple groups to settle the matter. Since the completion of the present work Martin Liebeck has informed me that, using the development [11], he and P. Kleidman have successfully carried out Aschbacher's suggestion, making heavy use of the classification theorem. The reason for, and interest of, the present proof of Theorem 4 is that it is geometric and elementary.

2. – The spreads and their stabilizers.

2.1. – We recall the construction of classical spreads in a coordinate form that is convenient for our discussion of orbits of lines. It is analogous to the approach for even N given in [5].

Let ω be a primitive element of $L = GF(q^r)$. Then $1, \omega, \omega^2, \ldots, \omega^{r-1}$ form a base for L considered as a vector space over $K = GF(q)$. If $\lambda \in L$ then we may write

$$\lambda = \sum_{i=0}^{r-1} \lambda_i \omega^i = \boldsymbol{\lambda}' \hat{\omega}\,, \tag{4}$$

where $\boldsymbol{\lambda}' = (\lambda_0, \lambda_1, \ldots, \lambda_{r-1})$ has its entries in K and is uniquely determined by λ, and $\hat{\omega} = (1, \omega, \omega^2, \ldots, \omega^{r-1})'$.

Suppose that $n = Nr$. Using (4) we may define the map $x \mapsto \boldsymbol{x}$ from $W = L^N$ to $V = K^n$ by the rule

$$\boldsymbol{x}' = (\boldsymbol{x}_1', \boldsymbol{x}_2', \ldots, \boldsymbol{x}_N') \quad \text{if } x' = (x_1, x_2, \ldots, x_N)\,. \tag{5}$$

Clarly this map is a bijection from W to V. From (4) and (5) it follows that if $y \in W$ and $a \in K$ then to $x + y$ and ax in W correspond respectively $\boldsymbol{x} + \boldsymbol{y}$ and $a\boldsymbol{x}$ in V. Thus to a K-linear combination of members of W corresponds the same K-linear combination of their images in V. In particular *K-linear independent members of W produce linearly independent members of V.*

The vectors of the 1-subspace $\langle x \rangle$ of L^N spanned by $x \neq 0$ are the various K-linear combinations of $x, \omega x, \ldots, \omega^{r-1}x$. These latter vectors are K-linearly independent. Thus the image of $\langle x \rangle$ in V is an r-subspace: call this k_x. Write

$$\mathcal{K}_r = \{k_x \colon x \in W, x \neq 0\}\,. \tag{6}$$

Each non-zero vector of W is in exactly one 1-subspace of W. Consequently *each zero vector of V is in exactly one member of $\mathcal{K}_r$*. Let $PG(n-1, q)$ have for its points the 1-suspaces of V. Then to k_x corresponds a projective $(r-1)$-subspace of $PG(n-1, q)$: call this s_x: Write

$$\mathcal{S}_r = \{s_x \colon x \in W, x \neq 0\}\,. \tag{7}$$

The remarks after (6) show that $\mathcal{S}_r$ *is a partition*, i.e. a spread, of $PG(n-1, q)$.

2.2. – Let $A \in GL_N(q^r)$. On expanding each entry of A as a K-linear combination of $1, \omega, \ldots, \omega^{r-1}$, we see, from (4) and (5), that to the map $x \to Ax$ of W corresponds a linear map, $\boldsymbol{x} \to \boldsymbol{A}\boldsymbol{x}$ say, of V: the entries of $\boldsymbol{A}$ depend on those of A

and the coefficients that occur when $\omega^r, \omega^{r+1}, \ldots$ are expressed as K-linear combinations of $1, \omega, \ldots, \omega^{r-1}$. Since $x \to Ax$ is a bijection of W, so is $\boldsymbol{x} \to A\boldsymbol{x}$ a bijection of V. Thus $A \in GL_n(q)$. Since A permutes the 1-subspaces of W we see that A fixes $\mathcal{K}_r$.

By (4),

$$\lambda^q = \sum_{i=0}^{r-1} \lambda_i^q \omega^{iq} = \sum_{i=0}^{r-1} \lambda_i \omega^{iq} = \boldsymbol{\lambda}' M' \hat{\omega} \,, \tag{8}$$

where M is the $r \times r$ K-matrix whose i-th column is the coordinate vector of $\omega^{(i-1)q}$ with respect to $1, \omega, \ldots, \omega^{r-1}$. Let σ be the semi-linear bijection of W given by

$$\sigma(x) = (x_1^q, x_2^q, \ldots, x_N^q)' \,. \tag{9}$$

By (8), σ, considered as acting on V via the correspondence $x \to \boldsymbol{x}$, is a linear map whose matrix is block diagonal with N diagonal blocks all equal to M. Since σ is a bijection of W we see that, considered as acting on V, $\sigma \in GL_n(q)$. Moreover, since $\lambda^{q^r} = \lambda$ for all λ in $GF(q^r)$ and this is true for no lower power of λ, we see that σ has order r. Now σ permutes the 1-subspaces of W. Hence $GL_N(q^r)\langle\sigma\rangle$ stabilizes $\mathcal{K}_r$. In fact we have

THEOREM 1. – *Suppose that r is a proper divisor of n and $N = n/r$. Define $GL_N^*(q^r)$ as in (2), $\mathcal{K}_r$ as in (6), and σ as in (9). Then:*

(i) *the stabilizer of $\mathcal{K}_r$ in $GL_n(q)$ is the semi-direct product $GL_N(q^r)\langle\sigma\rangle$;*

(ii) *except when N is odd, r is even and q is odd the stabilizer of $\mathcal{K}_r$ in $SL_n(q)$ is the semi-direct product $GL_N^*(q^r)\langle\sigma\rangle$;*

(iii) *if N is odd, r is even and q is odd then the stabilizer of $\mathcal{K}_r$ in $SL_n(q)$ is $GL_N^*(q^r)\langle\alpha\sigma\rangle$, where α is an element of $GF(q^r)$ of multiplicative order $2(q^r - 1)/(q - 1)$.*

PROOF. – (i) Notice that the result is trivially, and uninterestingly, true if $r = 1$. It fails if $r = n$.

Consider, first, the case $N = 2$. We know from [5, pp. 176-182] that there is a nonsingular bilinear alternating form $B(\boldsymbol{x}, \boldsymbol{y})$ with respect to which each member of $\mathcal{K}_r$ is totally isotropic, and such that the stabilizer of $\mathcal{K}_r$ in the symplectic group of $B(\boldsymbol{x}, \boldsymbol{y})$ is $Sp_2(q^r)\langle\sigma\rangle = SL_2(q^r)\langle\sigma\rangle$: the notation of [5] is a little different from present usage; our σ is the ϱ of [5], and our N is, when it is even, the $2N$ of [5]. Let H_1 be the stabilizer of $\mathcal{K}_r$ in $GL_{2r}(q)$, and let $\mathcal{B}$ be the orbit of $B(\boldsymbol{x}, \boldsymbol{y})$ under H_1. Then, the stabilizer of $B(\boldsymbol{x}, \boldsymbol{y})$ in H_1 is $SL_2(q^r)\langle\sigma\rangle$ so that

$$|H_1| = |\mathcal{B}||SL_2(q^r)|r \,. \tag{10}$$

Each member of $\mathcal{B}$ is a nonsingular alternating bilinear form with respect to which every member of $\mathcal{K}_r$ is totally isotropic. We know [5, p. 182] that there are q^r-1 nonsingular alternating bilinear forms satisfying this last condition. Hence $|\mathcal{B}|\leqslant q^r-1$. Since $SL_2(q^r)$ has index q^r-1 in $GL_2(q^r)$ we see, from (10), that

$$|H_1|\leqslant(q^r-1)|SL_2(q^r)|r=|GL_2(q^r)|r=|GL_2(q^r)\langle\sigma\rangle|\,.$$

As we already know that $GL_2(q^r)\langle\sigma\rangle\leqslant H_1$ we deduce that $H_1=GL_2(q^r)\langle\sigma\rangle$, as required.

As is customary denote the multiplicative group of L by $L^\times$. Let $e_1=(1,0)'$ and $e_2=(0,1)'$. Since, by (9), σ fixes both e_1 and e_2 the subgroup of H_1 that fixes both $\langle e_1\rangle$ and $\langle e_2\rangle$ is

$$H_2=\left\{\begin{pmatrix}a_1 & 0\\ 0 & a_2\end{pmatrix}\sigma^i\colon a_1,a_2\in L^\times \text{ and } i=0,1,\dots,r-1\right\}. \tag{11}$$

The restriction of $\begin{pmatrix}a_1 & 0\\ 0 & a_2\end{pmatrix}\sigma^i$ to $\langle e_1\rangle$ is the map $\lambda e_1\to a_1\lambda^{q^i}e_1$. There are q^r-1 choices for a_1 and r for i. Different pairs a_1, i give different maps of $\langle e_1\rangle$. Hence the restriction of H_2 to $\langle e_1\rangle$ consists of $r(q^r-1)$ distinct actions. If the action is known then i and a_1 are prescribed, and there are then q^r-1 possibilities for a_2. Hence, if $A\in H_2$ then there are q^r-1 elements of H_2 having the same action on $\langle e_1\rangle$ as A, and these elements provide q^r-1 different actions on $\langle e_2\rangle$.

Pass to V. Then H_2 is the subgroup of H_1 fixing both $k_1\equiv k_{e_1}$ and $k_2\equiv k_{e_2}$: The action of $\begin{pmatrix}a_1 & 0\\ 0 & a_2\end{pmatrix}\sigma^i$ on k_j $(j=1,2)$ is completely specified by its action in W on $\langle e_j\rangle$. Since, by (5), k_1 is given by $\boldsymbol{x}_2=\boldsymbol{0}$ and k_2 by $\boldsymbol{x}_1=\boldsymbol{0}$ each element of H_2, acting on V, has matrix form $\begin{pmatrix}A_1 & O_r\\ O_r & A_2\end{pmatrix}$, where A_1, A_2 are $r\times r$ K matrices. The action of this element on k_j is specified by A_j. Hence, by the last paragraph, we see that H_2 consists of matrices of the form $\begin{pmatrix}A_1 & O_r\\ O_r & A_2\end{pmatrix}$, where there are $r(q^r-1)$ possibilities for A_1, and for each possibility for A_1 there are q^r-1 possibilities for A_2.

Now consider $N>2$. Let

$$f_1=(1,0,\dots,0)',\quad f_2=(0,1,0,\dots,0)',\dots,f_N=(0,0,\dots,0,1)'$$

in W. Let H_3 be the subgroup of the stabilizer of $\mathcal{K}_r$ in $GL_n(q)$ that fixes each of $m_1\equiv k_{f_1}$, $m_2\equiv k_{f_2},\dots,m_N\equiv k_{f_N}$. By (5), m_j is given by $\boldsymbol{x}_1=\boldsymbol{x}_2=\dots=\boldsymbol{x}_{j-1}=\boldsymbol{x}_{j+1}=\dots=\boldsymbol{x}_N=\boldsymbol{0}$ for $j=1,2,\dots,N$. Hence, if $\boldsymbol{C}\in H_3$ then $\boldsymbol{C}$ must have block diagonal form $\boldsymbol{C}=\operatorname{diag}(C_1,C_2,\dots,C_N)$, where each C_j is an $r\times r$ K-matrix. Take $j>1$. The definition of the correspondence $x\to\boldsymbol{x}$, given in (4) and (5), shows that its restriction to $\langle f_1,f_j\rangle$ is just the standard correspondence from $\langle f_1,f_j\rangle$ to $\langle m_1,m_j\rangle$ that we would take in case $N=2$. Hence to the 1-subspaces of $\langle f_1,f_j\rangle$

there correspond, via $x \to \boldsymbol{x}$, just the members of the standard $\mathcal{K}_r$ for $N = 2$. Hence $\boldsymbol{C}$ must fix this standard $\mathcal{K}_r$ for $N = 2$. The restriction of $\boldsymbol{C}$ to $\langle m_1, m_j\rangle$ is $\begin{pmatrix} C_1 & O_r \\ O_r & C_j \end{pmatrix}$. If we take, as we obviously may, e_1, e_2 of the last paragraph to be respectively f_1, f_j then $m_1 = k_1$ and $m_j = k_2$ and the conclusion of the last paragraph shows that there are at most $r(q^r - 1)$ possibilities for C_1 and, for each of these possibilities, at most $(q^r - 1)$ possibilities for C_j. Taking $j = 2, 3, \ldots, N$ in turn we deduce that

$$|H_3| \leqslant r(q^r - 1)^N . \tag{12}$$

Arguing as for the case $N = 2$ we see that the subgroup of $GL_N(q^r)\langle\sigma\rangle$ fixing each of $m_1, m_2, \ldots, m_N$ is

$$\{\operatorname{diag}(a_1, a_2, \ldots, a_N)\sigma^i \colon a_j \in L^\times \text{ for } j = 1, \ldots, N \text{ and } i = 0, 1, \ldots, r-1\} .$$

Clearly this has order $r(q^r - 1)^N$. Since $GL_N(q^r)\langle\sigma\rangle$ fixes $\mathcal{K}_r$ we deduce from (12) that

$$H_3 \leqslant GL_N(q^r)\langle\sigma\rangle . \tag{13}$$

Suppose that $\boldsymbol{D}$ is a member of the stabilizer of $\mathcal{K}_r$ in $GL_n(q)$. Since V is the direct sum of $m_1, m_2, \ldots, m_N$ it is the direct sum of $\boldsymbol{D}m_1, \boldsymbol{D}m_2, \ldots, \boldsymbol{D}m_N$. Now $\boldsymbol{D}m_j \in \mathcal{K}_r$: suppose it corresponds to $\langle u_j\rangle$ in W. If $\boldsymbol{x} \in V$ then $\boldsymbol{x}$ is the sum of N vectors, one from each of $\boldsymbol{D}m_1, \boldsymbol{D}m_2, \ldots, \boldsymbol{D}m_N$. Hence, by (4), each vector x in W is the sum of N vectors, one from each of $\langle u_1\rangle, \langle u_2\rangle, \ldots, \langle u_N\rangle$. As W is an N-space this shows that $u_1, \ldots, u_N$ is a base of W. Hence there is a member A of $GL_N(q^r)$ such that $Au_j = f_j$, for $j = 1, 2, \ldots, N$. Then $A\langle u_j\rangle = \langle f_j\rangle$ so that $\boldsymbol{A}\boldsymbol{D}m_j = m_j$ for $j = 1, 2, \ldots, N$. Thus $\boldsymbol{A}\boldsymbol{D} \in H_3$. It follows, from (13), that $\boldsymbol{D} \in GL_N(q^r)\langle\sigma\rangle$. Consequently the stabilizer of $\mathcal{K}_r$ in $GL_n(q)$ is $GL_N(q^r)\langle\sigma\rangle$. That this is a semi-direct product is obvious.

(ii) and (iii). Let $E = \operatorname{diag}(\omega, 1, 1, \ldots, 1)$ in $GL_N(q^r)$. Then $\det(E^i) = \omega^i$. As ω is a primitive element of L we see that $GL_N(q^r)$ is the semi-direct product $SL_N(q^r)\langle E\rangle$. Moreover, $Q = \langle\omega^{q-1}\rangle$. Let $A \in GL_N(q^r)$. Then $A = BE^i$ for some element B of $SL_N(q^r)$ and some i in $\{0, 1, \ldots, q^r - 2\}$. Since $\det A = \omega^i$ we deduce, from (2), that $A \in GL_N^*(q^r)$ if and only if $E^i \in \langle E^{q-1}\rangle$. In particular, $GL_N^*(q^r) = SL_N(q^r)\langle E^{q-1}\rangle$.

$SL_N(q^r)$ is generated by its transvections [4, pp. 37, 38]: one such transvection is the block matrix $T = \begin{pmatrix} C & 0 \\ 0 & I_{N-2} \end{pmatrix}$, where $C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. It is immediate from (4) and (5) that $\boldsymbol{T} = \begin{pmatrix} \hat{C} & 0 \\ 0 & I_{(N-2)r} \end{pmatrix}$, where $\hat{C} = \begin{pmatrix} I_r & I_r \\ O_r & I_r \end{pmatrix}$. Thus $\det \boldsymbol{T} = 1$. Any two transvections of $SL_N(q^r)$ are conjugate under $GL_N(q^r)$ [4, p. 4]. Hence, considered as acting on V, all transvections of $SL_N(q^r)$ are in $SL_n(q)$. Their generating property yields $SL_N(q^r) \leqslant SL_n(q)$. Let A, E, i be as in the previous paragraph. It follows that $A \in SL_n(q)$ if and only if $E^i \in SL_n(q)$. The minimum polynomial $f(t)$ of ω over K has degree r: suppose that

$$f(t) = t^r - p_{r-1}t^{r-1} - p_{r-2}t^{r-2} - \dots - p_1 t - p_0\,, \tag{14}$$

where $p_i \in K$ and $p_0 \neq 0$. Let

$$P = \begin{pmatrix} 0 & 0\cdots 0 & p_0 \\ 1 & 0\cdots 0 & p_1 \\ 0 & 1\cdots 0 & p_2 \\ \cdot & \cdots\cdots & \cdot \\ \cdot & \cdots\cdots & \cdot \\ 0 & 0\cdots 0 & p_{r-2} \\ 0 & 0\cdots 1 & p_{r-1} \end{pmatrix}. \tag{15}$$

An inspection of (4) and (14) shows that in block form $\boldsymbol{E} = \operatorname{diag}(P, I_r, I_r, \dots, I_r)$. Hence $\boldsymbol{E}^i \in SL_n(q)$ if and only if $(\det P)^i = 1$. From (15), $\det P = (-1)^{r+1}p_0$. A consideration of the effect of the field automorphism $\lambda \to \lambda^q$ shows that $f(t)$ has r distinct roots $\omega, \omega^q, \omega^{q^2}, \dots, \omega^{q^{r-1}}$. Thus, by (14), $(-1)^{r+1}p_0 = \omega\omega^q\omega^{q^2}\dots\omega^{q^{r-1}} = \omega^{|Q|}$ since $|Q| = (q^r-1)/(q-1)$. Hence $\det P$ is a primitive element of K, so that $(\det P)^i = 1$ if and only if i is a multiple of $q-1$. Thus, since E has order q^r-1, $\boldsymbol{A} \in SL_n(q)$ if and only if $\boldsymbol{E}^i \in \langle \boldsymbol{E}^{q-1}\rangle$. The concluding remarks of the last paragraph now show that $GL_N^*(q^r) = GL_N(q^r) \cap SL_n(q)$.

By section 2.2, as a K-matrix $\sigma = \operatorname{diag}(M, M, \dots, M)$. Since σ has order r so does M; i.e. M satisfies $t^r - I_r = O_r$. Now $I_r, M, M^2, \dots, M^{r-1}$ are K-linearly independent. For suppose otherwise. Then, for each λ in L, $\boldsymbol{\lambda}, M\boldsymbol{\lambda}, \dots, M^{r-1}\boldsymbol{\lambda}$ are K-linearly dependent. We deduce from Section 2.1 that $\lambda, \lambda^q, \lambda^{q^2}, \dots, \lambda^{q^{r-1}}$ are K-linearly dependent. Thus the r distinct automorphisms $\lambda \to \lambda^{q^i}$, $i = 0, 1, \dots, r-1$, of L are linearly dependent. This contradicts Dedekind's Theorem [10, p. 25]. Thus the minimum polynomial, and hence the characteristic polynomial, of M is $t^r - 1$. Hence $\det M = (-1)^{r+1}$ so that $\det \sigma = (-1)^{(r+1)N}$. Hence $\sigma \in SL_n(q)$ except when $(-1)^{(r+1)N} = -1 \neq 1$, i.e. when q is odd, N is odd and r is even. Dedekind's rule and the result of the last paragraph now yields (ii).

In the exceptional case (iii) $\alpha \in \langle \omega^{(q-1)/2}\rangle$ and has the same order $2(q^r-1)/(q-1)$ as $\omega^{(q-1)/2}$. Hence $\alpha = \omega^{a(q-1)/2}$ where the integer a is relatively prime to $2(q^r-1)/(q-1)$ and so, in particular, is odd. Considered as acting on V, αI_N has, by (14) and (15), for its matrix $\operatorname{diag}(P^{a(q-1)/2}, P^{a(q-1)/2}, \dots, P^{a(q-1)/2})$ and thus has determinant $(\det P)^{(q-1)/2]aN}$. Since $\det P$ is a primitive element of K, $(\det P)^{(q-1)/2} = -1$. Thus, since a and N are odd, αI_N, as an element of $GL_n(q)$, has determinant -1. So does σ. Hence $\alpha\sigma \in SL_n(q)$. Obviously $GL_N(q^r)\langle\sigma\rangle = GL_N(q^r)\langle\alpha\sigma\rangle$. Then Dedekind's rule yields (iii).

2.3. – Take the situation of Theorem 1 (iii). A brief calculation shows that $(\alpha\sigma)^u = \alpha^v\sigma^u$ where $v = (q^u - 1)/(q-1)$. Since σ has order r, $(\alpha\sigma)^u$ is a linear map of W if and only if u is a multiple or r. Since α has order $2(q^r-1)/(q-1)$ we have $(\alpha\sigma)^r = -I_N$. Hence, as claimed in Section 1, $\alpha\sigma$ has order $2r$. Further $\det(-I_N) = (-1)^N = -1 = \omega^{(q^r-1)/2} \in \langle\omega^{(q-1)/2}\rangle = Q$ so that $-I_N \in GL_N^*(q^r)$ and $GL_N^*(q^r)\langle\alpha\sigma\rangle$ is not semi-direct.

2.4. – The detail of the proof of Theorem 1 (ii) provides the element $\boldsymbol{E}$ of $GL_N(q^r)$ whose determinant is $\det P$, which is a primitive element of K. We obtain

COROLLARY 1. – *$GL_N(q^r)\langle\sigma\rangle$ contains elements of each coset of $SL_n(q)$ in $GL_n(q)$.*

2.5. – If $\lambda \in K^\times$ then, by (4) and (5), λI_N of $GL_N(q^r)$ acts on V as λI_n of $GL_n(q)$. Hence Z_N, given by (2), is the group of scalar maps of $GL_n(q)$. Since, by Section 2.2, $GL_N(q^r) \cap SL_n(q) = GL_N^*(q^r)$ the group of scalar maps of $SL_n(q)$ is the Z_N^* of (3), and $Z_N^* < GL_N^*(q^r)$. For the situation of Theorem 1 (iii) the discussion of Section 2.3 shows that the image of $\alpha\sigma$ under the homomorphism $GL_N^*(q^r)\langle\alpha\sigma\rangle \to GL_N^*(q^r)\langle\alpha\sigma\rangle/Z_N^*$ has order r and has no positive power before its r-th in $GL_N^*(q^r)/Z_N^*$. This image is, of course, the image of $\alpha\sigma$ in $PSL_n(q)$ acting as a projectivity on $PG(n-1, q)$. From Theorem 1 we deduce

THEOREM 2. – *Suppose that r is a proper divisor of n and $N = n/r$. Define $GL_N^*(q^r)$, $\mathcal{S}_r$, σ, Z_N and Z_N^* as in (2), (7), (9), (1) and (3), and denote the image of σ under the natural homomorphism $GL_N(q^r)\langle\sigma\rangle \to GL_N(q^r)\langle\sigma\rangle/Z_N$ also by σ. Then:*

(i) *the stabilizer of $\mathcal{S}_r$ in $PGL_n(q)$ is the semi-direct product $(GL_N(q^r)/Z_N)\langle\sigma\rangle$;*

(ii) *except when N is odd, r is even and q is odd the stabilizer of $\mathcal{S}_r$ in $PSL_n(q)$ is the semi-direct product $(GL_N^*(q^r)/Z_N^*)\langle\sigma\rangle$;*

(iii) *if N is odd, r is even and q is odd then the stabilizer of $\mathcal{S}_r$ in $PSL_n(q)$ is the semi-direct product $(GL_N^*(q^r)/Z_N^*)\langle\alpha\sigma\rangle$, where α is an element of $GF(q^r)$ of multiplicative order $2(q^r-1)/(q-1)$.*

It is worth pointing out from Theorems 1, 2 that whatever N, q, r, the stabilizer of $\mathcal{K}_r$ in $SL_n(q)$ has order $|GL_N^*(q^r)|r$, and the stabilizer of $\mathcal{S}_r$ in $PSL_n(q)$ has order $|GL_N^*(q^r)|r/|Z_N^*|$.

3. – The maximality proofs.

3.1. – Suppose that $n = Nr$ and that t divides r with $1 < t < r$. Then L has a unique subfield $F \cong GF(q^t)$ and $K < F < L$. Let

$$\mathcal{F} = \{Fx \colon 0 \neq x \in W\} . \tag{16}$$

Clearly, if $x, y \neq 0$ then $Fx = Fy$ if and only if y is an F-multiple of x, and each non-zero vector of W is in exactly one member of $\mathcal{F}$. Denote the image of Fx under the bijection $x \to \boldsymbol{x}$ by f_x and let

$$\mathcal{K}_t = \{f_x \colon 0 \neq x \in W\} .$$

Let g_x be the image of f_x in $PG(n-1, q)$ and let

$$\mathcal{S}_t = \{g_x \colon 0 \neq x \in W\} .$$

Clearly, $f_x \subseteq k_x$ and $g_x \subseteq s_x$.

Let $a_1, a_2, \ldots, a_t$ be a base of F over K. The members of Fx are the various K-linear combinations of $a_1 x, a_2 x, \ldots, a_t x$, and these t-vectors are K-linearly independent. Hence by Section 2.1, so are their images in V. Thus f_x is a t-subspace of V and g_x a $(t-1)$-subspace of $PG(n-1, q)$. By the remark after (16), each non-zero member of V is in exactly one member of $\mathcal{K}_t$. Thus $\mathcal{S}_t$ is a spread of $PG(n-1, q)$.

If $f \in F$ then $f^q \in F$. Hence, from (9) if $A \in GL_N(q^r)$ then

$$A(Fx) = F(Ax) \quad \text{ and } \sigma(Fx) = F(\sigma x) .$$

Thus $GL_N(q^r)\langle\sigma\rangle$ fixes $\mathcal{K}_t$. Define the bijection ϱ of W by

$$\varrho \colon (x_1, x_2, \ldots, x_N)' \to (x_1^{q^t}, x_2, \ldots, x_N)' .$$

Clearly $\varrho \notin GL_N(q^r)\langle\sigma\rangle$—it is not even a semi-linear map—so its action on V, obtained via the map $x \to \boldsymbol{x}$, does not fix $\mathcal{K}_r$. On the other hand, if $f \in F$ then $f^{q^t} = f$ so that $\varrho(Fx) = F(\varrho x)$. Thus ϱ fixes $\mathcal{K}_t$. By (8), ϱ acts on V as the linear map with matrix diag $(M^t, I_r, I_r, \ldots, I_r)$. Let A be an element of $GL_N(q^r)$ such that det $A =$ $=$ det (M^t): such an A is guaranteed by Corollary 1. Then ϱA^{-1} fixes $\mathcal{K}_t$, does not fix $\mathcal{K}_r$, and is in $SL_n(q)$. Hence the stabilizer of $\mathcal{K}_r$ in $SL_n(q)$ $[GL_n(q)]$ is strictly contained in the stabilizer of $\mathcal{K}_t$ in $SL_n(q)$ $[GL_n(q)]$. Passing to $PG(n-1, q)$ we see that the stabilizer of $\mathcal{S}_r$ in $PSL_n(q)$ $[PGL_n(q)]$ is strictly contained in that of $\mathcal{S}_t$. Since $\mathcal{S}_t$ is a proper non-trivial spread we deduce

THEOREM 3. – *Suppose that r is a proper composite divisor of n. Define $\mathcal{K}_r$ and $\mathcal{S}_r$ as in* (6) *and* (7). *Then*:

(i) *the stabilizers of $\mathcal{K}_r$ in $SL_n(q)$ and $GL_n(q)$ are not maximal subgroups of $SL_n(q)$ and $GL_n(q)$, respectively*;

(ii) *the stabilizers of $\mathcal{S}_r$ in $PSL_n(q)$ and $PGL_n(q)$ are not maximal subgroups of $PSL_n(q)$ and $PGL_n(q)$, respectively.*

To see that $\mathcal{S}_t$ is a classical spread—a fact that we do not need—observe that W is a vector N^*-space over F where $N^* = Nr/t$, and that if W is so regarded then Fx is a 1-subspace. It is easy to check that the map $x \to \boldsymbol{x}$ then gives the standard construction of a classical «t-spread» in $PG(n-1, q)$ from F^{N^*}: it is referred to a base of L over F less specific than the one for L over K used in Section 2.1.

3.2. – We are now in a position to deal with the main

THEOREM 4. – *Suppose that r is a proper prime divisor of n. Define $\mathcal{K}_r$ and $\mathcal{S}_r$ as in* (6) *and* (7). *Then*:

(i) *the stabilizers of $\mathcal{K}_r$ in $SL_n(q)$ and $GL_n(q)$ are maximal subgroups of $SL_n(q)$ and $GL_n(q)$ respectively*;

(ii) *the stabilizers of $\mathcal{S}_r$ in $PSL_n(q)$ and $PGL_n(q)$ are maximal subgroups of $PSL_n(q)$ and $PGL_n(q)$ respectively.*

PROOF. – By Section 2.5 the stabilizers of $\mathcal{K}_r$ in $SL_n(q)$ and $GL_n(q)$ contain, respectively, the full groups of scalar maps of $SL_n(q)$ and $GL_n(q)$. Hence, by standard group homomorphism theorems, (i) and (ii) are equivalent. We shall prove (ii). This allows us to use the more graphic geometric language of $PG(n-1, q)$; although, of course, we pass to the vector space for matrix computations.

Let G be one of $PGL_n(q)$ or $PSL_n(q)$, and let H be the stabilizer of $\mathcal{S}_r$ in G. Suppose that (see Section 1)

$$H < J \leqslant G\,. \tag{17}$$

$SL_N(q^r)$ is transitive on the non-zero vectors and doubly transitive on the 1-subspaces of W. Since $SL_N(q^r) \leqslant GL_N^*(q^r)$ we deduce, from Theorem 2, that H, and thus J, is transitive on the points of $PG(n-1, q)$, and that H is doubly transitive on the members of $\mathcal{S}_r$.

We label as Propositions the main steps of the proof. Let $\mathcal{L}$ be the set of those lines of $PG(n-1, q)$ that lie in the members of $\mathcal{S}_r$, and let $\mathcal{M}$ be the set of the other lines in $PG(n-1, q)$. Since $\mathcal{S}_r$ is a spread, if $l \in \mathcal{L}$ then l lies in a unique member of $\mathcal{S}_r$.

PROPOSITION 1. – *Suppose that $l \in \mathfrak{L}$ and is in the member π of $\mathcal{S}_r$. Then there are lines $l = l_1, l_2, \ldots, l_{r-1}$ that span π, that are in the orbit of l under H, and are such that l_i meets l_{i+1} for $i < r-1$.*

PROOF. – The result is trivial if $r = 2$! To l corresponds a 2-subspace, say $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$, of V. Then $\pi = s_x$, so that $y = \lambda x$ for some λ in $L^{\times}$. As $\boldsymbol{x}$ and $\boldsymbol{y}$ are K-linearly independent, $\lambda \in L \setminus K$, by Section 2.1. Hence, since r is prime, $L = K(\lambda)$ and the minimum polynomial of λ over K has degree r. So, $1, \lambda, \lambda^2, \ldots, \lambda^{r-1}$ are K-linearly independent, and hence so are $x, \lambda x, \lambda^2 x, \ldots, \lambda^{r-1} x$. Write $z = \lambda^2 x$, $w = \lambda^3 x, \ldots, u = \lambda^{r-2} x$, $v = \lambda^{r-1} x$. Then, by Section 2.1, $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w}, \ldots, \boldsymbol{u}, \boldsymbol{v}$ are r linearly independent vectors in k_x, and so span the r-subspace k_x. Thus, if $l = l_1, l_2, l_3, \ldots, l_{r-1}$ are the lines of s_x corresponding to $\langle \boldsymbol{x}, \boldsymbol{y} \rangle, \langle \boldsymbol{y}, \boldsymbol{z} \rangle, \langle \boldsymbol{z}, \boldsymbol{w} \rangle, \ldots, \langle \boldsymbol{u}, \boldsymbol{v} \rangle$, respectively, then $l_1, l_2, \ldots, l_{r-1}$ span s_x and l_i meets l_{i+1}. Further, there is an element A of $SL_N(q^r)$ such that $Ax = y = \lambda x$. Then $Ay = z$, $Az = w, \ldots, Au = v$. Thus $\boldsymbol{Ax} = \boldsymbol{y}$, $\boldsymbol{Ay} = \boldsymbol{z}$, $\boldsymbol{Az} = \boldsymbol{w}, \ldots, \boldsymbol{Au} = \boldsymbol{v}$, so that, by Theorem 2, H has an element taking l_i to l_{i+1}. The Proposition follows.

Notice that the condition that r is prime is crucial.

PROPOSITION 2. – *If $l^* \in \mathfrak{L}$ then the orbit of l^* under J contains members of $\mathcal{M}$.*

PROOF. – Suppose otherwise. Let $\pi \in \mathcal{S}_r$ and $j \in J$. By the transitivity of H on $\mathcal{S}_r$ there is a line l of π in the same orbit as l^* under H. Take $l = l_1, l_2, \ldots, l_{r-1}$ as in Proposition 1. Then $l^*, l_1, l_2, \ldots, l_{r-1}$ are in the same orbit order J. By our supposition $jl \in \mathfrak{L}$, so that jl is in one member, say π', of $\mathcal{S}_r$. Also, $jl_2 \in \mathfrak{L}$ and jl_2 meets jl. Thus the member of $\mathcal{S}_r$ containing jl_2 must be π'. Also, $jl_3 \in \mathfrak{L}$ and meets jl_2. Thus jl_3, and similarly $jl_4, \ldots, jl_{r-1}$, are lines of the projective $(r-1)$-subspace π'. Since $l_1, \ldots, l_{r-1}$ span π we see that $jl_1, \ldots, jl_{r-1}$ span an $(r-1)$-subspace. Hence $j\pi = \pi'$. Thus j fixes $\mathcal{S}_r$, for all j in J. This contradicts (17): the Proposition follows.

PROPOSITION 3. – *H acts transitively on $\mathcal{M}$ except when $N = 2$, r divides $q-1$ and $G = PSL_n(q)$.*

PROOF. – If $m \in \mathcal{M}$ then it corresponds to a 2-subspace of V, say $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$, where $k_x \neq k_y$. Thus $\langle x \rangle \neq \langle y \rangle$, and x, y are linearly independent in W. Now $GL_N(q^r)$ is transitive an ordered pairs of linearly independent vectors of W_j; so is $SL_N(q^r)$, and thus $GL_N^*(q^r)$, if $N > 2$. The Proposition now follows from Theorem 2 except for the case $N = 2$ and $G = PSL_n(q)$.

Assume, for the rest of this proof, that $N = 2$ and $G = PSL_n(q)$.

As in the proof of Theorem 1, take $e_1 = (1, 0)'$ and $e_2 = (0, 1)'$. A consideration of the transitivity of $GL_2(q^r)$ and the action of elements of the form diag $(\lambda, 1)$, shows that there is an element A of $SL_2(q^r)$ such that $Ay = e_2$ and $Ax = \lambda e_1$ for some λ in $L^\times$. The subgroup of $GL_2(q^r)\langle\sigma\rangle$ that fixes $\langle e_1\rangle$ and $\langle e_2\rangle$ is H_2, given by (11). Since, by (9), σ fixes Ke_2, we see, from (11), that the subgroup H_4 of $GL_2^*(q^2)\langle\sigma\rangle$ that fixes $\langle e_1\rangle$ and Ke_2 consists of all the elements diag $(a_1, a_2)\sigma^i$ for which $a_2 \in K^\times$ and $a_1a_2 \in Q$. Let

(18) $$U = K^\times Q\,.$$

Then

(19) $$H_4 = \left\{\begin{pmatrix} u & 0\\ 0 & k\end{pmatrix}\sigma^i\colon k\in K^\times,\ u\in U,\ i = 0, 1, \dots, r-1\right\}.$$

$K^\times$ and Q are subgroups of the cyclic group $L^\times$ of respective orders $q-1$ and $(q^r-1)/(q-1)$. Thus U is the cyclic subgroup of $L^\times$ of order

$$|U| = (q^r-1)/d\,, \qquad \text{where } d = (q-1, |Q|)\,.$$

But

$$|Q| = q^{r-1} + q^{r-2} + \dots + q + 1 = (q^{r-1}-1) + (q^{r-2}-1) + \dots + (q-1) + r\,.$$

Hence

(21) $$d = (q-1, r)\,.$$

Since r is prime either $d = 1$ or $d = r$; the latter possibility occurs if and only if $r|(q-1)$.

So suppose that $r \nmid (q-1)$: then $d = 1$ and, by (20), $U = L^\times$. Hence, by (19), $B = \operatorname{diag}(\lambda^{-1}, 0)$ is in H_4. Thus $C = BA \in GL_2^*(q^r)\langle\sigma\rangle$. Also $Cy = e_2$ and $Cx = e_1$. Thus $\boldsymbol{C}$ takes $\langle \boldsymbol{x}, \boldsymbol{y}\rangle$ to $\langle \boldsymbol{e}_1, \boldsymbol{e}_2\rangle$. Since m is any line of $\mathcal{M}$ we have the required transitivity.

PROPOSITION 4. – *J acts transitively on the lines of $PG(n-1, q)$.*

PROOF. – Exclude, first, the case $N = 2$, $r|(q-1)$ and $G = PSL_n(q)$. By Proposition 3 there is an orbit of lines under J than contains $\mathcal{M}$. By Proposition 2 this orbit contains every line of $\mathcal{L}$, and hence must be $\mathcal{L} \cup \mathcal{M}$: we have transitivity.

Assume, for the rest of this proof, that $N = 2$, $r|(q-1)$ and $G = PSL_n(q)$. The various strands of the argument are presented as lettered sections.

(*a*) We show first that *under H_4 there are r orbits of non-zero vectors of $\langle e_1\rangle$, and that if $\lambda \in L^\times$ then the orbit of λe_1 is $U\lambda e_1$.*

PROOF. – Since $|Q| = (q^r-1)/(q-1)$ we see that $Q = \langle \omega^{q-1}\rangle$, and thus Q contains the $(q-1)$-th power of each element in L^x. Thus $\lambda^{q^i-1} \in Q \leqslant U$. Hence $\sigma^i(\lambda e_1) = \lambda^{q^i} e_1 = \lambda^{q^i-1}\lambda e_1 \in U\lambda e_1$. Thus, by (19), the orbit of λe_1 is $U\lambda e_1$. By (20) and (21) this orbit has size $|U| = (q^r - 1)/r$. Since there are $q^r - 1$ non-zero vectors in $\langle e_1 \rangle$ they must fall into r orbits under H_4.

Write k_1 for k_{e_1} and k_2 for k_{e_2}. Suppose that k_1 and k_2 correspond to the members s_1 and s_2 respectively of $\mathbb{S}_r$. Let PH_4 be the image of H_4 acting on $PG(n-1, q)$. Then PH_4 fixes s_1 and s_2, and

$$PH_4 < H . \tag{22}$$

Suppose that $K\boldsymbol{e}_2$ in V corresponds to the point Y in s_2. Since H_4 fixes Ke_2 in W which corresponds to $K\boldsymbol{e}_2$ in V, by Section 2.1, we see that PH_4 fixes Y. Since $K^\times < U$ it also follows from the previous paragraph that under PH_4 the points of s_1 fall into r orbits each of size $(q^r - 1)/r(q-1)$.

(*b*) Information about the geometric structure of these orbits can be obtained by considering the action of σ. We show that *if $r \geq 3$ then the restriction of σ to s_1 has r distinct eigen-values in K, and that the r fixed points, which correspond to $\omega^{|Q|i/r} e_1$ for $i = 0, 1, \ldots, r-1$, are the vertices of a simplex Σ and lie one in each orbit of points in s_1 under PH_4.*

PROOF.

$$\sigma(\omega^{|Q|i/r} e_1) = \omega^{|Q|iq/r} e_1 = \omega^{|Q|i(q-1)/r} \omega^{|Q|i/r} e_1 . \tag{23}$$

Since

$$|Q| = (q^r - 1)/(q-1)$$

we have $K^\times = \langle \omega^{|Q|} \rangle$. Since $r|(q-1)$ we have $\omega^{|Q|i(q-1)/r} \in K^\times$, and as i runs from 0 to $r-1$ it takes r distinct values, namely the members of $\langle \omega^{(q^r-1)/r} \rangle$. Since a projectivity of projective $(r-1)$-space that has r distinct eigen-values has r fixed points that correspond to a base of the underlying vector space, and s_1 is an $(r-1)$-space, the statement follows from (23) except for the last clause. For that, suppose that $0 \leq j < i \leq r-1$ and that $\omega^{|Q|i/r} e_1$ and $\omega^{|Q|j/r} e_1$ are in the same orbit under H_4. By (*a*), $\omega^{|Q|(i-j)/r} \in U$. Since $|U| = (q^r - 1)/r$ we have $U = \langle \omega^r \rangle$. Thus there are integers a, b such that $|Q|(i-j)/r = ar + b(q^r - 1)$. Thus r divides $|Q|(i-j)/r$. Since r is a prime the restriction on i, j implies that $r^2 \big| |Q|$. Now

$$q^h = (q - 1 + 1)^h = 1 + h(q-1) + \text{higher powers of } q - 1 .$$

Since $r|(q-1)$ these higher powers are divisible by r^2. Hence, since

$$|Q| = q^{r-1} + q^{r-2} + \ldots + q + 1 ,$$

we have that r^2 divides

$$\sum_{h=0}^{r-1} (1 + h(q-1)) = r + \frac{(q-1)r(r-1)}{2} .$$

Since r is odd r^2 divides the second term. Then we obtain the contradiction $r^2|r$. The result is established.

(*c*) We next show that *if* $\mathcal{O}$ *is an orbit of points of* s_1 *under* PH_4 *then there is a set of* r *points* $X_1, X_2, \ldots, X_r$ *of* s_1 *and an orbit* $\mathcal{O}^*$ *of points of* s_1 *under* PH_4 *distinct from* $\mathcal{O}$ *such that* $X_1, X_2, \ldots, X_r$ *span* s_1 *and each of the lines* $X_1X_2, X_1X_3, \ldots, X_1X_r$ *contains a point of* $\mathcal{O}^*$.

PROOF. – If $r = 2$ then s_1 is a line and there are, by (i), 2 orbits under PH_4 each of size $(q+1)/2 > 1$. We need merely take X_1, X_2 as any two points of $\mathcal{O}$.

So, suppose that $r \geqslant 3$. Take, in accordance with (*b*), X_1 to be the vertex of Σ that is in $\mathcal{O}$. Let $D = \operatorname{diag}(\omega^{|Q|/r}, 1)$. Then D fixes $\langle e_1\rangle$ and $\langle e_2\rangle$. Since $\omega^{|Q|} \in K^\times$ we see that $D(\omega^{|Q|i/r}e_1)$ is $\omega^{|Q|(i+1)/r}e_1$ if $0 \leqslant i < r-1$ and is in Ke_1 if $i = r-1$. Thus, by (*b*), the action D^* induced by D on $PG(n-1, q)$ permutes cyclically the vertices of Σ. Since $D(U\lambda e_1) = U(\lambda\omega^{|Q|/r}e_1)$, by (*a*), D^* permutes the orbits of points in s_1 under PH_4. As each orbit, by (*b*), contains one vertex of Σ, D^* must permute these orbits cyclically. Thus each of the orbits has the same number of points not in hyperfaces of Σ. There are $(q-1)^{r-1}$ points of s_1 not in the hyperfaces of Σ: if Σ is taken as a coordinate simplex of reference then they are the points all of whose r coordinates are non-zero. Thus $\mathcal{O}$ has $(q-1)^{r-1}/r$ points not in the hypersurfaces of Σ. Consider the lines joining these points to X_1. Not all these lines can have all their points in $\mathcal{O}$. For if they did then $(q-1)^{r-1}/r$ points would form batches of q points, so that $q|(q-1)^{r-1}/r$ and thus $q|(q-1)^{r-1}$: an absurdity. Thus there is a point X_2 in $\mathcal{O}$ and not in a hypersurface of Σ such that X_1X_2 contains a point Z of some orbit $\mathcal{O}^* \neq \mathcal{O}$.

Let $X_3 = \sigma X_2$, $X_4 = \sigma X_3, \ldots, X_r = \sigma X_{r-1}$. Since, by (19), σ fixes $\mathcal{O}$ and $\mathcal{O}^*$ and, by (*b*), fixes X_1, we see that $X_3, \ldots, X_r \in \mathcal{O}$, and that the line X_1X_i contains the point $\sigma^{i-1}Z$ in $\mathcal{O}^*$ for $i = 2, \ldots, r$. Refererred to Σ as coordinate simplex the restriction of σ to s_1 has by (*b*), a matrix that is diagonal with distinct eigen-values. The standard general theory of such projectivities shows that $X_1, \ldots, X_r$ span s_1.

(*d*) Before applying these results to the orbits of lines through Y we need another result about the action of H. *Suppose that* $\varkappa$ *is a plane of* $PG(n-1, q)$ *containing exactly one line, say* l, *of* $\mathfrak{L}$. *Then the subgroup of* H *that fixes* $\varkappa$ *and fixes* l *pointwise acts transitively on the points of* $\varkappa$ *off* l.

PROOF. – Since H acts doubly transitively $\mathcal{S}_r$, l lies in a unique member of $\mathcal{S}_r$ and $\varkappa$ must meet another member of $\mathcal{S}_r$ in a point, we may assume that $l \subseteq s_1$ and that $\varkappa$ meets s_2 is a point. Suppose this point, say P, corresponds to $p = \mu e_2$ in W. Here $\mu \neq 0$. Any point P^* of $\varkappa \setminus l$ has for one of its representatives in V a vector $\boldsymbol{p}^*$ of the form $\boldsymbol{p}^* = \boldsymbol{p} + \boldsymbol{x}$ for some $\boldsymbol{x} \in k_1$. Then, by Section 2.1, $p^* = p + x = \mu e_2 + \nu e_1$ for some $\nu \in L$. Let $E = \begin{pmatrix} 1 & \nu\mu^{-1} \\ 0 & 1 \end{pmatrix}$. Then E fixes $\langle e_1\rangle$ vectorwise and takes p to p^*. Thus $\boldsymbol{E}$ fixes k_1 vectorwise and $\boldsymbol{Ep} = \boldsymbol{p}^*$. Consequently, the image of $\boldsymbol{E}$ on $PG(n-1, q)$ fixes s_1 and, thus l, pointwise, and takes P to P^*. It obviously fixes $\varkappa = \langle l, P\rangle = \langle l, P^*\rangle$. Moreover, $E \in SL_2(q^r) \leqslant GL_2^*(q^r)\langle\sigma\rangle$. The result follows by Theorems 1, 2.

(*e*) We are now in a position *to complete the proof of the Proposition.*

By Proposition 2, some member, say m, of $\mathcal{M}$ is in the same orbit under J as a line of $\mathcal{L}$. The action of the element A considered in the proof of Proposition 3 shows that the orbit of m under H, and thus under J, contains a line XY where $X \in s_1$. Let $\mathcal{O}_1$ be the orbit of X under PH_4. Since PH_4 fixes Y we see, from (17) and (22), that the lines joining Y to the points of $\mathcal{O}_1$ are in the same orbit under J, and thus each is in the orbit under J of a line of $\mathcal{L}$. Take $\mathcal{O}_1$ as the $\mathcal{O}$ of (*c*). Then there are points $X_1, X_2, \dots, X_r$ in $\mathcal{O}_1$ and an orbit $\mathcal{O}_2 \neq \mathcal{O}_1$ of points of s_1 under PH_4 such that $X_1, X_2, \dots, X_r$ span s_1 and the lines $X_1X_2, X_1X_3, \dots, X_1X_r$ each contain a point of $\mathcal{O}_2$. There is an element j of J such that $l = j(X_1Y) \in \mathcal{L}$. Let π be the unique member of $\mathcal{S}_r$ that contains l. Not every one of $jX_2, jX_3, \dots, jX_r$ can be in π; else j would take the r-subspace $\langle Y, s_1\rangle$ spanned by $Y, X_1, \dots, X_r$ into the $(r-1)$-subspace π. Suppose that $jX_i \notin \pi$. Let Z be a point of $\mathcal{O}_2$ on X_1X_i. The plane $j(X_1X_iY)$ is not in π. Thus l is the only line of $\mathcal{L}$ in $j(X_1X_iY)$: any other line of $\mathcal{L}$ that it contained would span the plane with l, and would meet l and thus be forced to lie in π. Hence, by (*d*), since jZ is a point of $j(X_1X_i)$ distinct from jX_1 and thus not on l, there is an element h of H fixing l pointwise such that $hjX_i = jZ$. Since $jY \in l$ we have $hjY = jY$. Thus $j^{-1}hj$ fixes Y and takes X_i to Z. Since the joins of Y to the points of $\mathcal{O}_2$ are in the same orbit under PH_4, and thus under J, we conclude that the joins of Y to $\mathcal{O}_1 \cup \mathcal{O}_2$ are in the same orbit under J, and this orbit contains lines of $\mathcal{L}$.

For $r = 2$ we immediately conclude that the lines joining Y to the points of s_1 lie in one orbit under J. Suppose that $r \geqslant 3$. We saw when proving (*c*) that there is a projectivity of s_1 permuting cyclically the r orbits of points of s_1 under PH_4. Since r is prime some power, say E^*, of this projectivity permutes the orbits cyclically and has $E^*\mathcal{O}_1 = \mathcal{O}_2$. Write $\mathcal{O}_3 = E^*\mathcal{O}_2$, $\mathcal{O}_4 = E^*\mathcal{O}_3, \dots, \mathcal{O}_r = E^*\mathcal{O}_{r-1}$, so that $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_r$ are the orbits of points of s_1 under PH_4. Now $E^*X_1, E^*X_2, \dots$ $\dots, E^*X_r$ span s_1 and are in $\mathcal{O}_2$. Further, the lines joining E^*X_1 to $E^*X_2, \dots, E^*X_r$ each contain a point of $E^*\mathcal{O}_2 = \mathcal{O}_3$. The conclusion of the last paragraph shows that $\mathcal{O}_2$ has all the properties we assumed for $\mathcal{O}_1$. Arguing as in that paragraph with $\mathcal{O}_2$ in place of $\mathcal{O}_1$, and $E^*X_1, \dots, E^*X_r$ in place of $X_1, \dots, X_r$ we see that the joins of Y to the points of $\mathcal{O}_2 \cup \mathcal{O}_3$ are in the same orbit under J, and this orbit contains lines of $\mathcal{L}$. Repeat the argument with $E^*\mathcal{O}_2, E^*\mathcal{O}_3, \dots$ in turn in place of $\mathcal{O}_1$. We conclude that under J the lines joining Y to the points of $\mathcal{O}_1 \cup \mathcal{O}_2 \cup$ $\cup \dots \cup \mathcal{O}_r = s_1$ are all in one orbit under J. The comment at the beginning of (*e*) shows that the lines of $\mathcal{M}$ lie in one orbit under J. By Proposition 2, any line of $\mathcal{L}$ is in this orbit, which is thus $\mathcal{L} \cup \mathcal{M}$: transitivity of J on lines of $PG(n-1, q)$ is established.

PROPOSITION 5. – *If $m \in \mathcal{M}$ then the stabilizer of m in H acts doubly transitively on the points of m.*

PROOF. – Suppose m corresponds to the 2-subspace $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$ of V. Then $\langle x \rangle \neq \neq \langle y \rangle$. Since $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = K\boldsymbol{x} \oplus K\boldsymbol{y}$ we see, by Section 2.1, that it corresponds to the set $Kx + Ky$ in W, which is a K-vector 2-space with base x, y. Let W^* be a complement of $\langle x, y \rangle$ in W. Suppose that $\alpha, \beta, \gamma, \delta \in K$ with $\alpha\delta - \beta\gamma = 1$. Define the linear map T by:

$$Tx = \alpha x + \beta y$$

$$Ty = \gamma x + \delta y$$

$$Tw^* = w^* \quad \text{for } w \in W^* .$$

Then $T \in SL_N(q^r)$. Further, T fixes $Kx + Ky$. The set of all such T, for all possible choices of $\alpha, \beta, \gamma, \delta$, is a group isomorphic to $SL_2(q)$ and acts as such on the K-vector space $Kx + Ky$. It thus acts doubly transitively on $\{Ku\colon 0 \neq u \in Kx + Ky\}$. Since Ku corresponds, by Section 2.1, to $K\boldsymbol{u}$ in V and thence to a point of m, we have the result.

We can now *complete the proof of Theorem* 4.

Propositions 4, 5 and (17) imply that the stabilizer of any line of $PG(n-1, q)$ in J acts doubly transitively on its points. Let R_1, R_2 and R_1^*, R_2^* be two pairs of distinct points. By Proposition 4 there is an element j taking the line $R_1^* R_2^*$ to the line $R_1 R_2$. Then jR_1^* and jR_2^* are distinct points of $R_1 R_2$, so that, by the previous remark, there is an element j^* of J such that $j^* j_1 R_1^* = R_1$ and $j^* j R_2^* = R_2$. Thus J acts doubly transitively on the points of $PG(n-1, q)$.

It follows by Theorem 1 of [2, p. 384] that either a matrix « over-group » of J contains $SL_n(q)$ and thus $J \geqslant PSL_n(q)$, or that $n = 4$, $q = 2$ and J is the alternating group A_7 in $PSL_4(2) = SL_4(2) \cong GL_4(2) \cong A_8$. This latter exceptional case cannot occur for us. For if it did then $N = r = 2$ so that, by Theorem 2,

$$|H| = (4^2 - 1)(4^2 - 4)\cdot 2 = 360 .$$

Then H has index 6 in A_7 and so is a copy of A_6. This is impossible since A_6 is simple and is thus not a semi-direct product. Thus, always, $J \geqslant PSL_n(q)$. Hence if $G = PSL_n(q)$ then $J = G$ and H is maximal in G by (17). If $G = PGL_n(q)$ it follows from Corollary 1 and Theorem 2 that H contains an element of each coset of $PSL_n(q)$ in G. Hence, by (17), so does J. Hence J contains every coset of $PSL_n(q)$ in G and so is G. Again we have maximality.

3.3. – It is easy to deduce from Proposition 5 that if $N = 2$, $r|(q-1)$ and $G = PSL_n(q)$ then two lines joining the point Y of the proof of Theorem 4 to two points of s_1 are in the same orbit under H if and only they are in the same orbit under PH_4. Consequently $\mathcal{M}$ splits into r orbits under H. Any hope of simplifying the proof of Theorem 4 in this case by having simpler or fewer orbits under H than we considered is consequently dashed!

4. – Addedum.

After the above pages were complete I learned of recent work of Li [12], [13]. If $K \subset F$ are division rings with $\dim_F K = r$ and $n = Nr$, then he has determined the overgroups of $SL_N(K)$ in $GL_n(F)$. He states his results in [12], and gives the proofs in [13]; I am grateful to Li for sending me copies of [12] and [13]. Our Theorem 4 follows as corollaries of Li's results. His arguments are also elementary, not using the classification theorem for finite simple groups, but are very different from ours. Apart from some work on groups containing root groups his proofs are based on matrix techniques. He performes a very large number of ingenious matrix manipulations and computations; a veritable tour de force. It is interesting that he has a lengthy special treatment of the case $N = 2$. It is hoped that the present proof for the finite linear groups, which explits the finite geometry and the geometric action of the groups, is still of interest.

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