Annali di Matematica pura ed applicata
(IV), Vol. CLXVIII (1995), pp. 301-316

Existence and Stability Results for the Hyperbolic Stefan Problem with Relaxation (*).

NIKOLAY V. SHEMETOV (**)

Summary. – *The aim of this paper is to study a phase transition model, based on the Cattaneo-Fourier constitutive law for the heat flux and on a relaxed constitutive law for the phase variable. In turn, the model describes fast processes of melting and crystallization with supercooling and superheating effects. We give existence and stability results for the former phase transition problem. Uniqueness is deduced from the stability result.*

1. – Introduction.

In this paper we will consider the following system of equations describing the «liquid-crystal» phase transition:

$$c_\varrho \theta_t + \lambda \chi_t + \operatorname{div} \boldsymbol{q} = 0\,, \tag{1.1}$$

$$\alpha \boldsymbol{q}_t + \boldsymbol{q} = -k \operatorname{grad} \theta\,, \tag{1.2}$$

$$\chi_t + H^{-1}(\chi) \ni F(\theta, \chi)\,. \tag{1.3}$$

Equation (1.1) is the energy balance law for the internal energy $e = c_\varrho \theta + \lambda\chi$, where c_ϱ is the specific heat at a constant pressure, θ is the relative temperature (thus $\theta = 0$ will be the equilibrium temperature for a liquid-crystal mixture), λ is the latent energy of phase transition. Equation (1.2) is the Cattaneo-Fourier constitutive equation for the heat flux $\boldsymbol{q}$, where k denotes the thermal conductivity and α the thermal relaxation time. Finally, equation (1.3) is the non-equilibrium constitutive equation for the liquid concentration χ, where $F(\theta, \chi)$ is the so-called driving force of phase transition and H^{-1} denotes the inverse of the Heaviside graph, *i.e.* $H^{-1}(0) = \{\xi \leqslant 0\}$, $H^{-1}(1) = \{\xi \geqslant 0\}$, $H^{-1}(\chi) = \{0\}$ for all $\chi \in (0, 1)$. Here, all thermodynamical parameters c_ϱ, λ, α, k are positive constants.

System (1.1)-(1.3) was suggested by Visintin (1985) for describing phase transi-

(*) Entrata in Redazione il 15 luglio 1993.

Indirizzo dell'A.: CMAF and Universidade de Lisboa, Av. Prof. Gama Pinto 2, 1699 Lisboa Codex, Portugal.

(**) On leave from: Lavrent'ev Institute of Hydrodynamics, 630090 Novosibirsk, Academy of Sciences of Russia, Russia. Current address: Universidade de Beira Interior, R. Marquês d'Àvila e Bolama, 6200 Covilhã, Portugal.

tion processes with supercooling and superheating effects. The usual phase transition model (Stefan problem) consists in coupling the balance energy law (1.1) with the Fourier's conduction law

$$q = -k \operatorname{grad} \theta \tag{1.4}$$

and the equilibrium constitutive equation for χ

$$\chi \in H(\theta). \tag{1.5}$$

System (1.1), (1.4), (1.5) is not completely satisfactory, especially for extremely rapid cooling and heating processes, since it corresponds to a parabolic equation having an infinite speed of propagation of the thermal signal. Replacing (1.4) by Cattaneo law (1.2), one gets a hyperbolic system which yields to a finite speed of propagation of the thermal signal. Moreover, in order to describe dynamical supercooling and superheating effects, (1.5) can be replaced by (1.3). Supercooling and superheating effects correspond to the cases $\chi = 1$, $\theta < 0$ and $\chi = 0$, $\theta > 0$ (respect.); these conditions are compatible with (1.3).

Under suitable assumptions on the data existence of a solution of system (1.1)-(1.3), but not uniqueness, is known only for the case F is linear (VISINTIN, 1985). A numerical approximation for this system has been given by MAZZULO et al. (1989); NOCHETTO and VERDI (1989); VERDI and VISINTIN (1992).

Others hyperbolic models for phase transitions were considered by FRIEDMAN and HU BEI (1987); GREENBERG (1987); SHOWALTER and WALKINGTON (1987); SHEMETOV (1991); COLLI and GRASSELLI (1992).

The aim of this paper is to prove existence and uniqueness of the solution of system (1.1)-(1.3), even for a nonlinear driving force F, in the one-dimensional case. In addition, we will show that the solutions of this system converge to the solution of the Stefan Problem with the non-equilibrium constitutive equation (system (1.1), (1.4), (1.3)), as $\alpha \to 0$.

2. – Statement of the main results.

Just for the sake of simplicity, assume that $c_\rho = \lambda = k = 1$. Then, the one-dimensional problem reads

$$\begin{cases} \theta_t + \chi_t + q_x = 0, & (x,t) \in \pi_T, \\ \alpha q_t + q + \theta_x = 0, & (x,t) \in \pi_T, \\ \chi_t + H^{-1}(\chi) \ni F(\theta, \chi) & (x,t) \in \pi_T, \end{cases} \tag{2.1}$$

where $\pi_T = \{(x,t) \colon x \in \mathbb{R},\ t \in [0,T]\}$. We consider the Cauchy problem with initial data

$$\theta(x,0) = \theta_0(x), \qquad q(x,0) = q_0(x), \qquad \chi(x,0) = \chi_0(x), \qquad x \in \mathbb{R}. \tag{2.2}$$

Let us give the definition of weak solution for problem (2.1), (2.2). First we introduce a change of the variables θ and q as follows:

$$\Gamma = \frac{\theta + \sqrt{\alpha}\, q}{2}, \qquad S = \frac{\theta - \sqrt{\alpha}\, q}{2}, \tag{2.3}$$

whence, we set

$$f(\Gamma, S, \chi) := F\left(\frac{\Gamma + S}{2}, \chi\right) = F(\theta, \chi).$$

Then (2.1), (2.2) reads equivalently as the following symmetric system:

$$\begin{cases} \Gamma_t + \dfrac{1}{\sqrt{\alpha}}\Gamma_x + \dfrac{\Gamma - S}{2\alpha} = -\dfrac{1}{2}\chi_t, \\ S_t - \dfrac{1}{\sqrt{\alpha}}S_x - \dfrac{\Gamma - S}{2\alpha} = -\dfrac{1}{2}\chi_t, \\ \chi_t + H^{-1}(\chi) \ni f(\Gamma, S, \chi), \\ \Gamma(x, 0) = \Gamma_0(x), \qquad S(x, 0) = S_0(x), \qquad \chi(x, 0) = \chi_0(x), \end{cases} \tag{2.4}$$

where

$$\Gamma_0 = \frac{\theta_0 + \sqrt{\alpha}\, q_0}{2}, \qquad S_0 = \frac{\theta_0 - \sqrt{\alpha}\, q_0}{2}.$$

DEFINITION. – The triplete of (Γ, S, χ) is a weak solution of (2.4) in π_T, if:

D1) $\chi_t \in L^\infty(\pi_T)$ and the function $0 \leqslant \chi \leqslant 1$ satisfies

$$\chi_t + H^{-1}(\chi) \ni f(\Gamma, S, \chi) \quad \text{a.e. in } \pi_T,$$

$$\chi(x, 0) = \chi_0(x) \quad \text{a.e. on } \mathbb{R}.$$

D2) $\Gamma, S \in L^\infty(\pi_T)$ satisfy:

$$\iint_{\pi_T} \left\{ |\Gamma - k| \left(\varphi_t + \frac{1}{\sqrt{\alpha}}\varphi_x\right) - \operatorname{sign}(\Gamma - k)\left[\frac{\Gamma - S}{2\alpha} - \frac{1}{2}\chi_t\right]\varphi \right\} dx\, dt \geqslant 0,$$

$$\iint_{\pi_T} \left\{ |S - l| \left(\psi_t - \frac{1}{\sqrt{\alpha}}\psi_x\right) + \operatorname{sign}(S - l)\left[\frac{\Gamma - S}{2\alpha} - \frac{1}{2}\chi_t\right]\psi \right\} dx\, dt \geqslant 0,$$

for all $\varphi, \psi \in C_0^1(\pi_T)$, $\varphi, \psi \geqslant 0$ and $k, l \in \mathbb{R}$.

D3) There exists a set $E \subset (0, T)$ of measure zero, such that, for each $R > 0$, we have

$$\lim_{\substack{t \to 0 \\ t \in (0, T) \setminus E}} \int_{|x| \leq R} \{ |\Gamma(t, x) - \Gamma_0(x)| + |S(t, x) - S_0(x)| \} \, dx = 0 .$$

THEOREM 2.1 (Existence). – Suppose that

$$\theta_0, q_0, \chi_0 \in L^\infty(\mathbb{R}), \qquad 0 \leq \chi_0 \leq 1 \text{ a.e. in } \mathbb{R}, \tag{2.5}$$

and $F: \mathbb{R} \times [0, 1] \to \mathbb{R}$ be a Lipschitz-continuous function, i.e.,

$$|F(\theta, \chi) - F(\widetilde{\theta}, \widetilde{\chi})| \leq M\{ |\theta - \widetilde{\theta}| + |\chi - \widetilde{\chi}| \} \qquad \forall \theta, \widetilde{\theta} \in \mathbb{R} \text{ and } \forall \chi, \widetilde{\chi} \in [0, 1]. \tag{2.6}$$

Then there exists a unique weak solution (θ, q, χ) of problem (2.1), (2.2).

THEOREM 2.2 (Stability). – Let (θ, q, χ) and $(\widetilde{\theta}, \widetilde{q}, \widetilde{\chi})$ be the weak solutions of problem (2.1), (2.2) with initial data (θ_0, q_0, χ_0) and $(\widetilde{\theta}_0, \widetilde{q}_0, \widetilde{\chi}_0)$, respectively, satisfying (2.5). Then, for every fixed $R > 0$, we have

$$\begin{aligned} \int_{B_t} \{ |\theta - \widetilde{\theta}| + \sqrt{\alpha} |q - \widetilde{q}| + |\chi - \widetilde{\chi}| \}(x, t) \, dx \leq \\ \leq C \int_{B_0} \{ |\theta_0 - \widetilde{\theta}_0| + \sqrt{\alpha} |q_0 - \widetilde{q}_0| + |\chi_0 - \widetilde{\chi}_0| \} \, dx, \qquad \text{for a.e. } t \in [0, T_0], \end{aligned} \tag{2.7}$$

where C is a constant independent of the initial data and t, $T_0 = \min(T, R\sqrt{\alpha})$, $B_t = \{x: |x| \leq R - t/\sqrt{\alpha}\}$.

REMARK 2.1. – Uniqueness clearly follows from Stability Result.

THEOREM 2.3. – Assume that (2.5) and (2.6) hold. Let $(\theta_\alpha, \bar{q}_\alpha, \chi_\alpha)$ be the solution of problem (2.1), (2.2) and let $\theta \in L^\infty(\pi_T)$, $0 \leq \chi \leq 1$ be the weak solution of the Stefan problem with non-equilibrium constitutive relation

$$\begin{cases} \theta_t + \chi_t + \theta_{xx} = 0, & (x, t) \in \pi_T, \\ \chi_t + H^{-1}(\chi) \ni F(\theta, \chi), & (x, t) \in \pi_T, \\ \theta(x, 0) = \theta_0, \ \ \chi(x, 0) = \chi_0, & x \in \mathbb{R}. \end{cases} \tag{2.8}$$

Then,

$$\theta_\alpha \to \theta, \qquad \chi_\alpha \to \chi \quad \text{in } L^1_{\text{loc}}(\pi_T), \ \text{ as } \alpha \to 0. \tag{2.9}$$

3. – Notation and additional properties.

Before proving Theorems 2.1, 2.2 and 2.3, let us introduce some notations and elementary properties.

Let δ be a mollifier function on $\mathbb{R}$ such that $\delta(\sigma) \geq 0$, $\delta(\sigma) \equiv 0$ for $|\sigma| \geq 1$ and $\int_{-\infty}^{+\infty} \delta(\sigma)\, d\sigma \equiv 1$. For any $h > 0$ we set

$$\delta_h(\sigma) = h^{-1}\delta(h^{-1}\sigma).$$

Given $v \in L^1_{\text{loc}}(\mathbb{R})$, let us v^h denote the function

$$v^h(x) = \int_{\mathbb{R}} \frac{1}{h}\delta\left(\frac{x-y}{h}\right) v(y)\, dy\,, \qquad h > 0\,, \qquad x \in \mathbb{R}\,. \tag{3.1}$$

The point x_0 is a point of Lebesgue of $v(x)$ say $x_0 \in \mathcal{L}(v)$ if

$$\lim_{h \to 0} \frac{1}{h} \int_{|x - x_0| \leq h} |v(x) - v(x_0)|\, dx = 0\,.$$

Note that $\lim_{h \to 0} v^h(x_0) = v(x_0)$ for all $x_0 \in \mathcal{L}(v)$. Since meas $(\mathbb{R} \setminus \mathcal{L}(v)) = 0$, then $v^h(x) \to v(x)$ a.e. on $\mathbb{R}$, as $h \to 0$.

Let $\omega(\sigma)$ denote a module of continuity, i.e. a nondecreasing continuous function defined for $\sigma \geq 0$, such that $\omega(0) = 0$.

LEMMA 3.1 (See KRUZKOV, 1970). – Set $K_{r+2\rho} = \{|x| \leq r + 2\rho\}$ for $r > 0$, $\rho > 0$. Let $v \in L^1(K_{r+2\rho})$ satisfy

$$J_s(v, \Delta x) \equiv \int_{K_s} |v(x + \Delta x) - v(x)|\, dx \leq \omega_s(|\Delta x|) \qquad \forall |\Delta x| \leq \rho, \quad s \in [0, r + \rho]\,.$$

Then, for $h \leq \rho$,

$$J_r(v^h, \Delta x) \leq \omega_{r+h}(|\Delta x|) \qquad \text{and} \qquad \int_{K_r} \big|\, |v| - v(\operatorname{sign} v)^h \big|\, dx < 2\omega_r(h)\,.$$

LEMMA 3.2 (See KRUZKOV, 1970). – Set $Q = \{|x| \leq R\} \times [0, T]$. Let $v \in L^\infty(Q)$. If for some $\rho \in (0, \min\{R, T\})$ and any $h \in (0, \rho)$ we set

$$V_h = \frac{1}{h^2} \iint_{Q_h \times Q_h} |v(x, t) - v(y, \tau)|\, dx\, dt\, dy\, d\tau\,,$$

where

$$Q_h = \left\{(x, t): \frac{|t-\tau|}{2} \leqslant h,\ \rho \leqslant \frac{t+\tau}{2} \leqslant T-\rho,\ \frac{|x-y|}{2} \leqslant h,\ \frac{|h+y|}{2} \leqslant R-\rho\right\},$$

then $\lim_{h\to 0} V_h = 0$.

LEMMA 3.3 (See KRUZKOV, 1970). – Let $u \in L^\infty(K_{r+\rho} \times [0, T])$, $0 < 2\rho \leqslant r$. Let for $0 \leqslant t \leqslant T$, $|\Delta x| \leqslant \rho$:

$$J_r(u(x, t), \Delta x) = \int_{|x| \leqslant r} |u(x+\Delta x, t) - u(x, t)|\,dx \leqslant \omega_r^x(|\Delta x|) \tag{3.2}$$

and for every t, $t + \Delta t \in [0, T]$, $\Delta t > 0$ and for every function $g(x) \in C_0^1(K_r)$ the following inequality holds:

$$\left|\int_{|x| \leqslant r} g(x)[u(x, t+\Delta t) - u(x, t)]\,dx\right| \leqslant C_r \Delta t \max_{|x| \leqslant r} [|g| + |g_x|]. \tag{3.3}$$

Then for $0 \leqslant t \leqslant t + \Delta t \leqslant T$

$$I_r(u(x, t), \Delta t) = \int_{|x| \leqslant r} |u(x, t+\Delta t) - u(x, t)|\,dx \leqslant \text{const} \min_{0 < h \leqslant \rho}\left[h + \omega_r^x(h) + \frac{\Delta t}{h}\right]$$

where const depends only on C_r and $\|u\|_{L^\infty(K_{r \times \rho} \times [0, T])}$.

4. – Existence of the weak solution.

1) We will prove existence of a weak solution of problem (2.9) via a finite difference discretization.

For any integer number N ($\tau = T/N$ stands for the time step), the semi-discretization algorithm can be stated as follows.

Suppose the approximate solution $\{\Gamma^n, S^n, \chi^n\}$ is known at the time $n\tau$; then we define $\{\Gamma^{n+1/2}, S^{n+1/2}, \chi^{n+1/2}\}$ and $\{\Gamma^{n+1}, S^{n+1}, \chi^{n+1}\}$ by:

$$\begin{cases} \dfrac{\Gamma^{n+1/2} - \Gamma^n}{\tau} + \dfrac{1}{\sqrt{\alpha}}\Gamma_x^{n+1/2} = -\dfrac{1}{2}\dfrac{\chi^{n+1/2} - \chi^n}{\tau}, \\ \dfrac{S^{n+1/2} - S^n}{\tau} + \dfrac{1}{\sqrt{\alpha}}S_x^{n+1/2} = -\dfrac{1}{2}\dfrac{\chi^{n+1/2} - \chi^n}{\tau}, \\ \dfrac{\chi^{n+1/2} - \chi^n}{\tau} + H^{-1}(\chi^{n+1/2}) \ni f(\Gamma^n, S^n, \chi^n) \end{cases} \tag{4.1}$$

and

$$
(4.2)\qquad
\begin{cases}
\dfrac{\Gamma^{n+1}-\Gamma^{n+1/2}}{\tau}+\dfrac{\Gamma^{n+1}-S^{n+1}}{2\alpha}=0\,,\\[2mm]
\dfrac{S^{n+1}-S^{n+1/2}}{\tau}-\dfrac{\Gamma^{n+1}-S^{n+1}}{2\alpha}=0\,,\\[2mm]
\chi^{n+1}=\chi^{n+1/2}.
\end{cases}
$$

Note that systems (4.1) and (4.2) admit a unique solution which, in explicit form reads:

$$
(4.3)\qquad
\begin{cases}
\Gamma^{n+1/2}(x)=e^{-kx}\displaystyle\int_{-\infty}^{x}e^{k\xi}\left[\Gamma^{n}(\xi)-\dfrac{\chi^{n+1/2}(\xi)-\chi^{n}(\xi)}{2}\right]\cdot k\,d\xi, \qquad k=\sqrt{\alpha}/\tau\,,\\[4mm]
S^{n+1/2}(x)=e^{kx}\displaystyle\int_{x}^{-\infty}e^{-k\xi}\left[S^{n}-\dfrac{\chi^{n+1/2}-\chi^{n}}{2}\right]\cdot k\,d\xi\,,\\[4mm]
\chi^{n+1/2}=\begin{cases}1\,, & \chi^{n}+\tau f(\Gamma^{n},S^{n},\chi^{n})\geqslant 1\,,\\ \chi^{n}+\tau f(\Gamma^{n},S^{n},\chi^{n})\,, & 0<\chi^{n}+\tau f(\Gamma^{n},S^{n},\chi^{n})<1\,,\\ 0\,, & \chi^{n}+\tau f(\Gamma^{n},S^{n},\chi^{n})\leqslant 0\,,\end{cases}
\end{cases}
$$

and

$$
(4.4)\qquad
\begin{cases}
\Gamma^{n+1}=\Gamma^{n+1/2}\dfrac{(1+q)}{2}+S^{n+1/2}\dfrac{(1-q)}{2}\,, \qquad 0<q=\dfrac{\alpha}{\alpha+\tau}<1\,,\\[2mm]
S^{n+1}=\Gamma^{n+1/2}\dfrac{(1-q)}{2}+S^{n+1/2}\dfrac{(1+q)}{2}\,,\\[2mm]
\chi^{n+1}=\chi^{n+1/2}.
\end{cases}
$$

Then, the semidiscrete scheme is well-defined in terms of the initial data $\{\Gamma^0, S^0, \chi^0\}$ defined, for any $h>0$, by

$$
(4.5)\qquad \Gamma^0=(\Gamma_0\xi_h)^h, \qquad S^0=(S_0\xi_h)^h, \qquad \chi^0=(\chi_0\xi_h)^h
$$

where v^h is defined in (3.1) and ξ_h is the cutoff function

$$
\xi_h(x)=1 \quad \text{for } |x|\leqslant 1/h\,, \qquad \xi_h(x)=0 \quad \text{for } |x|>1/h\,.
$$

Let us introduce the functions, defined in π_T,

$$
(4.6)\qquad
\begin{cases}
\Gamma_{i\tau}(t)=\Gamma^{n+i/2},\ S_{i\tau}(t)=S^{n+i/2},\ \chi_{i\tau}(t)=\chi^{n+i/2}, \qquad i=1,2\\
\text{for } t\in]n\tau,(n+1)\tau], \qquad n=0,1,\dots,N-1
\end{cases}
$$

and

$$\begin{cases} \widetilde{\Gamma}_\tau(t),\ \widetilde{S}_\tau(t),\ \widetilde{\chi}_\tau(t) \text{ linear on } (n\tau, (n+1)\tau) \text{ and} \\ \widetilde{\Gamma}_\tau(n\tau) = \Gamma^n,\ \widetilde{S}_\tau(n\tau) = S^n,\ \widetilde{\chi}_\tau(n\tau) = \chi^n. \end{cases} \tag{4.7}$$

2) *A priori estimates in* $L^\infty(\pi_T)$.

LEMMA 4.1. – Set $C_0 = \|\Gamma_0\|_{L^\infty(\mathbb{R})} + \|S_0\|_{L^\infty(\mathbb{R})}$. Then, for all $n = 0, \ldots, N-1$ and $i = 1, 2$, there exists a positive constant K independent of τ and h, such that

$$\|\Gamma^{n+i/2}\|_{L^\infty(\mathbb{R})} + \|S^{n+i/2}\|_{L^\infty(\mathbb{R})} \leq e^{MT} C_0 + K,$$

$$0 \leq \chi^{n+i/2} \leq 1, \qquad \left| \frac{\chi^{n+1} - \chi^n}{\tau} \right| \leq K.$$

PROOF. – Let us set $\lambda^{n+i/2} = \|\Gamma^{n+i/2}\|_{L^\infty(\mathbb{R})}$, $\mu^{n+i/2} = \|S^{n+i/2}\|_{L^\infty(\mathbb{R})}$. Then from (4.3) we have

$$\lambda^{n+1/2} + \mu^{n+1/2} \leq \lambda^n + \mu^n + \max_{x \in \mathbb{R}} |\chi^{n+1/2} - \chi^n| \leq \tag{4.8}$$

$$\leq \lambda^n + \mu^n + \tau \max_{x \in \mathbb{R}} |f(\Gamma^n, S^n, \chi^n)| \leq \lambda^n + \mu^n + \tau M(\lambda^n + \mu^n + 1) + \tau F(0,0),$$

whereas (4.4) gives

$$\lambda^{n+1} + \mu^{n+1} \leq \lambda^{n+1/2} + \mu^{n+1/2}. \tag{4.9}$$

The asserted estimates follow from (4.8) and (4.9). ∎

3) *A module of continuity for the space variable* x.

LEMMA 4.2. – Let $\Delta\varphi(x) = \varphi(x + \Delta x) - \varphi(x)$, $x \in \mathbb{R}$, $\Delta x \in \mathbb{R}$. Then

$$|\Delta\chi^{n+1/2}| + \rho \left| \frac{\Delta\chi^{n+1/2} - \Delta\chi^n}{\tau} \right| \leq |\Delta\chi^n| + 2\tau |\Delta f(\Gamma^n, S^n, \chi^n)|, \tag{4.10}$$

$$\forall x,\ \Delta x \in \mathbb{R}.$$

PROOF. – First, let us prove the estimate

$$|\Delta\chi^{n+1/2}| + |\Delta\chi^n| (1 - \operatorname{sgn}\Delta\chi^n \cdot \operatorname{sgn}\Delta\chi^{n+1/2}) \leq |\Delta\chi^n| + \tau |\Delta f(\Gamma^n, S^n, \chi^n)|. \tag{4.11}$$

From the last equation in (4.1) we have

$$\frac{\Delta\chi^{n+1/2} - \Delta\chi^n}{\tau} + \Delta H^{-1}(\chi^{n+1/2}) \ni \Delta f(\Gamma^n, S^n, \chi^n), \tag{4.12}$$

where

$$\Delta H^{-1}(\chi^{n+1/2}) = \{x - y : x \in H^{-1}(\chi^{n+1/2}(x + \Delta x)),\ y \in H^{-1}(\chi^{n+1/2}(x))\}.$$

We multiply (4.12) by $\operatorname{sgn}\Delta\chi^{n+1/2}$. Noting that $\operatorname{sgn}\Delta\chi^{n+1/2}\cdot\Delta H^{-1}(\chi^{n+1/2})\in\{\xi\geq 0\}$, we readily get inequality (4.11).

The estimate (4.10) follows from (4.11). For example, consider the case $\Delta\chi^{n+1/2}>0>\Delta\chi^{n}$; then

$$|\Delta\chi^{n+1/2}|+\tau\left|\frac{\Delta\chi^{n+1/2}-\Delta\chi^{n}}{\tau}\right|\equiv 2|\Delta\chi^{n+1/2}|+|\Delta\chi^{n}|\leq$$

$$\leq 2(|\Delta\chi^{n+1/2}|+|\Delta\chi^{n}|)+|\Delta\chi^{n}|\leq|\Delta\chi^{n}|+2\tau|\Delta f(\Gamma^{n},S^{n},\chi^{n})|\,.$$

LEMMA 4.3. – There exists a positive constant C independent of τ and h such that for $n=0,1,\dots,N-1$, $i=1,2$

$$\int_{\mathbb{R}}\{|\Delta\Gamma^{n+i/2}|+|\Delta S^{n+i/2}|+|\Delta\chi^{n+i/2}|\}\,dx\leq C\int_{\mathbb{R}}\{|\Delta\Gamma^{0}|+|\Delta S^{0}|+|\Delta\chi^{0}|\}\,dx\,, \tag{4.13}$$

$$\sum_{n=0}^{N-1}\tau\int_{\mathbb{R}}\left|\frac{\Delta\chi^{n+i/2}-\Delta\chi^{n}}{\tau}\right|dx\leq C\int_{\mathbb{R}}\{|\Delta\Gamma^{0}|+|\Delta S^{0}|+|\Delta\chi^{0}|\}\,dx\,. \tag{4.14}$$

PROOF. – For all x, $\Delta x\in\mathbb{R}$, it is easily seen that $\Delta\Gamma^{n+1/2}$ and $\Delta S^{n+1/2}$ satisfy the system

$$\begin{cases}\dfrac{\Delta\Gamma^{n+1/2}-\Delta\Gamma^{n}}{\tau}+\dfrac{1}{\sqrt{\alpha}}\Delta\Gamma_x^{n+1/2}=-\dfrac{1}{2}\,\dfrac{\Delta\chi^{n+1/2}-\Delta\chi^{n}}{\tau}\,,\\[2ex] \dfrac{\Delta S^{n+1/2}-\Delta S^{n}}{\tau}+\dfrac{1}{\sqrt{\alpha}}\Delta S_x^{n+1/2}=-\dfrac{1}{2}\,\dfrac{\Delta\chi^{n+1/2}-\Delta\chi^{n}}{\tau}\,.\end{cases} \tag{4.15}$$

Multiply the first and second equation in (4.15) by $\operatorname{sgn}^{\varepsilon}\Delta\Gamma^{n+1/2}$ and $\operatorname{sgn}^{\varepsilon}\Delta\Gamma^{n+1/2}$, respectively, where $\operatorname{sgn}^{\varepsilon}\varphi(x)=\varphi/\sqrt{\varphi^2+\varepsilon}$. We obtain

$$\operatorname{sgn}_{\varepsilon}\Delta\Gamma^{n+1/2}\Delta\Gamma^{n+1/2}+\operatorname{sgn}_{\varepsilon}\Delta S^{n+1/2}\Delta S^{n+1/2}+\frac{\tau}{\sqrt{\alpha}}\times$$

$$\times\left(\sqrt{(\Delta\Gamma^{n+1/2})^2+\varepsilon}-\sqrt{(\Delta S^{n+1/2})^2+\varepsilon}\right)_x\leq|\Delta\Gamma^{n}|+|\Delta S^{n}|+\tau\left|\frac{\Delta\chi^{n+1/2}-\Delta\chi^{n}}{\tau}\right|.$$

If we integrate on $\mathbb{R}$ and take the limit as $\varepsilon\to 0$ we get

$$\int_{\mathbb{R}}(|\Delta\Gamma^{n+1/2}|+|\Delta S^{n+1/2}|)\,dx\leq \tag{4.16}$$

$$\leq\int_{\mathbb{R}}(|\Delta\Gamma^{n}|+|\Delta S^{n}|)\,dx+\tau\int_{\mathbb{R}}\left|\frac{\Delta\chi^{n+1/2}-\Delta\chi^{n}}{\tau}\right|dx\,.$$

From (4.2) (or (4.4)) we easyly obtain

$$(4.17)\quad \int_{\mathbb{R}} (|\Delta\Gamma^{n+1}| + |\Delta S^{n+1}| + |\Delta\chi^{n+1}|)\,dx \leq$$

$$\leq \int_{\mathbb{R}} (|\Delta\Gamma^{n+1/2}| + |\Delta S^{n+1/2}| + |\Delta\chi^{n+1/2}|)\,dx\,.$$

The asserted estimates then follow from Lemma 4.2 and inequalities (4.16) and (4.17). ■

4) *A module of continuity for the time variable t.*

Lemma 3.3 allows us to estimate the L^1 module of continuity in t, for the approximate solution $\widetilde{\Gamma}_\tau(x, t)$, $\widetilde{S}_\tau(x, t)$, $\widetilde{\chi}_\tau(x, t)$ defined by (4.7). From Lemmas 4.1 and 4.3 it follows that the functions $\widetilde{\Gamma}_\tau$, $\widetilde{S}_\tau$, $\widetilde{\chi}_\tau$ bear the estimates (3.2), (3.3) with the constant $C_r = r\,\text{const}$.

Therefore (we can take $r \geq 2$ and $\rho = 1$ in Lemma 3.3)

$$(4.18)\quad \begin{cases} I_r(\widetilde{\Gamma}(x, t), \Delta t) \leq \omega_r^t(\Delta t) = \text{const} \min\limits_{0<h\leq 1}\left[h + \omega_r^x(h) + \dfrac{\Delta t}{h}\right], \\ I_r(\widetilde{S}(x, t), \Delta t) \leq \omega_r^t(\Delta t), \\ I_r(\widetilde{\chi}(x, t), \Delta t) \leq \omega_r^t(\Delta t). \end{cases}$$

For the function $\widetilde{\chi}_t(x, t) = (\chi^{n+1} - \chi^n)/\tau$ from Lemma 4.2 we have

$$(4.19)\quad \int_0^T \int_{|x|\leq r} |\widetilde{\chi}_t(x + \Delta x, t) - \widetilde{\chi}_t(x, t)|\,dx\,dt \leq \widetilde{\omega}_r^x(\Delta x)$$

and moreover by using the same ideas as in Lemma 4.2 we can obtain that

$$(4.20)\quad \int_0^T \int_{|x|\leq r} |\widetilde{\chi}_t(x, t + \Delta t) - \widetilde{\chi}_t(x, t)|\,dx\,dt \leq \widetilde{\omega}_r^t(\Delta t) \qquad \forall t,\, t + \Delta t \in [0, T]\ (\Delta t > 0)\,.$$

Therefore we have constructed the L^1 modules of continuity $\omega_r^x(\sigma)$, $\omega_r^t(\sigma)$ independent of $\tau(\tau = T/N)$ such that for $0 \leq t \leq T$ $x \in \mathbb{R}$

$$(4.21)\quad J_r(\widetilde{\Gamma}, \Delta x) + J_r(\widetilde{\chi}, \Delta x) + J_r(\widetilde{\chi}. \Delta x) + I_r(\widetilde{\Gamma}, \Delta t) + I_r(\widetilde{S}, \Delta t) + I_r(\widetilde{\chi}, \Delta t) \leq$$

$$\leq \omega_r^x(|\Delta x|) + \omega_r^t(|\Delta t|)$$

and the L^1-modules of continuity $\widetilde{\omega}_r^t$, $\widetilde{\omega}_r^x$ such that estimates (4.19), (4.20) are fulfilled. ■

5) *The limit as* $\tau \to 0$.

From (4.19)-(4.21), (4.6), (4.7) it follows there exists a subsequence $\{\tau_k\} \subseteq \{\tau\}$

such that

$$(4.22)\qquad \begin{cases} (\widetilde{\Gamma}_{\tau_k}; \widetilde{S}_{\tau_k}; \widetilde{\chi}_{\tau_k}; (\widetilde{\chi}_{\tau_k})_t) \xrightarrow{\tau_k \to 0} (\Gamma, S, \chi, \chi_t) & *\text{-weakly in } L^\infty(\pi_T), \text{ a.e. in } \pi_T, \\ (\Gamma_{i\tau_k}; S_{i\tau_k}; \chi_{i\tau_k}) \xrightarrow{\tau_k \to 0} (\Gamma, S, \chi) & \text{strongly in } L^1_{\text{loc}}(\pi_T). \end{cases}$$

Let $\Phi(\xi)$, $\mathfrak{S}(\xi)$ be arbitrary smooth convex functions on the line $-\infty < \xi < +\infty$; if we multiply the first equations in (4.1), (4.2) by $\Phi'(\Gamma^{n+1/2})$, $\Phi'(\Gamma^{n+1})$ respectively and sum the resulting expressions we get

$$\frac{\Phi(\Gamma^{n+1}) - \Phi(\Gamma^n)}{\tau} + \frac{1}{\sqrt{\alpha}}(\Phi(\Gamma^{n+1}))_x + \Phi'(\Gamma^{n+1})\frac{\Gamma^{n+1} - S^{n+1}}{2\alpha} + \frac{1}{2}\widetilde{\chi}_t \Phi'(\Gamma^{n+1/2}) \leq 0 .$$

Here we have used twice the property of convex function, that $\Phi(\xi) - \Phi(\eta) \geq \Phi'(\eta)(\xi - \eta)$, $\forall \xi, \eta \in \mathbb{R}$. Hence for every $\varphi \in C_0^1(\pi_T)$, $\varphi \geq 0$

$$(4.23)\qquad \iint_{\pi_T} \left\{\Phi(\Gamma_{2\tau})\varphi_{\bar{t}} + \Phi(\Gamma_{1\tau})\frac{1}{\sqrt{\alpha}}\varphi_x\right\} - \varphi\left\{\Phi'(\Gamma_{2\tau})\frac{\Gamma_{2\tau} - S_{2\tau}}{2\alpha} + \frac{1}{2}\widetilde{\chi}_t \Phi'(\Gamma_{1t})\right\} dx\, dt \geq 0 ,$$

where

$$\varphi_{\bar{t}} = \frac{\varphi(t) - \varphi(t - \tau)}{\tau} .$$

Taking the limit as $\tau_k \to 0$ we obtain

$$(4.24)\qquad \iint_{\pi_T} \Phi(\Gamma)\left(\varphi_t + \frac{1}{\sqrt{\alpha}}\varphi_x\right) - \varphi\Phi'(\Gamma)\left\{\frac{\Gamma - S}{2\alpha} + \frac{1}{2}\chi_t\right\} dx\, dt \geq 0 .$$

By using the same arguments for every $\psi \in C_0^1(\pi_T)$, $\psi \geq 0$ we obtain

$$(4.25)\qquad \iint_{\pi} \mathfrak{S}(S)\left(\psi_t - \frac{1}{\sqrt{\alpha}}\psi_x\right) + \psi\mathfrak{S}'(S)\left\{\frac{\Gamma - S}{2\alpha} - \frac{1}{2}\chi_t\right\} dx\, dt \geq 0 .$$

After the approximation of the functions $|\Gamma - k|$ and $|S - l|$ by the smooth functions $\Phi(\Gamma)$ and $\mathfrak{S}(S)$ it follows that inequalities (4.24) and (4.25) are valid and for $\Phi = |\Gamma - k|$ and $\mathfrak{S} = |S - l| (\Phi'(\Gamma)) = \operatorname{sgn}(\Gamma - k)$, $\mathfrak{S}'(S) = \operatorname{sgn}(S - k)$.

We readily get that Γ, S and χ satisfy D1) and D2) in the definition of weak solution.

6) *The limit as* $h \to 0$.

In Section 5 we have shown that there exists a weak solution Γ^h, S^h, χ^h of problem (2.4) with initial data (4.5) in $L^\infty(\mathbb{R})$ with a compact support on $\mathbb{R}$.

In the next section we will show the Stability result for arbitrary initial data in

$L^\infty(\mathbb{R})$. Therefore if in Theorem 2.2 we take $(\Gamma_0, S_0, \chi_0) := (\Gamma_0^h, S_0^h, \chi_0^h)$ and

$$(\widetilde{\Gamma}_0, \widetilde{S}_0, \widetilde{\chi}_0) := (\Gamma_0^h(x + \Delta_x),\ S_0^h(x + \Delta x),\ \chi_0^h(x + \Delta x))$$

for a given $\Delta x \in \mathbb{R}$, then for every fixed $r > 0$ there exists a positive constant C independent of h:

$$\int_{B_t} \{ |\Delta \Gamma^h| + |\Delta S^h| + |\Delta \chi^h| \}(x, t)\, dx \leqslant C \int_{B_0} \{ |\Delta \Gamma_0^h| + |\Delta S_0^h| + |\Delta \chi_0^h| \}\, dx \tag{4.26}$$

for a.e. $t \in [0, T_0]$,

where $T_0 = \min(T, r\sqrt{\alpha})$, $B_t = \{x \mid |x| \leqslant r - t/\sqrt{\alpha}\}$.

The modules of continuity of $\Gamma_0^h, S_0^h, \chi_0^h$ are estimated by the module of continuity of Γ_0, S_0, χ_0 (Lemma 3.1). Hence from (4.26) and Lemma 3.3 the set $\{\Gamma^h, S^h, \chi^h\}_{h>0}$ has modules of continuity ω_r^t, ω_r^x independent of h such that for every fixed $r > 0$

$$J_r(\Gamma^h, \Delta x) + J_r(S^h, \Delta x) + J_r(\chi^h, \Delta x) +$$
$$I_r(\Gamma^h, \Delta t) + I_r(S^h, \Delta t) + I_r(\chi^h, \Delta t) \leqslant \omega_r^x(|\Delta x|) + \omega_r^t(\Delta t) \qquad \forall \Delta x \in \mathbb{R},\ \Delta t > 0\,.$$

This estimate allows to find a subsequence of $\{\Gamma^h, S^h, \chi^h\}_{h>0}$ such that

$$\Gamma^h, S^h, \chi^h \longrightarrow \Gamma, S, \chi \qquad *\text{-weakly in } L^\infty(\pi_T),\ \text{a.e. in } \pi_T,\ \text{ strongly in } L^1_{\mathrm{loc}}(\pi_T).$$

Finally, it is easy to see that the limit functions Γ, S, χ are a weak solution of (2.9) with the given initial data Γ_0, S_0, χ_0.

5. – Stability results (Theorem 2.2).

Here we will show the Stability Result, for the variables Γ, S, χ.

Let the triplets $Z = (\Gamma, S, \chi)$ and $\widetilde{Z} = (\widetilde{\Gamma}, \widetilde{S}, \widetilde{\chi})$ be weak solutions of problem (2.4) with initial data $Z_0 = (\Gamma_0, S_0, \chi_0) \in L^\infty(\mathbb{R})$, $0 \leqslant \chi_0 \leqslant 1$ and $\widetilde{Z}_0 = (\widetilde{\Gamma}_0, \widetilde{S}_0, \widetilde{\chi}_0) \in L^\infty(\mathbb{R})$, $0 \leqslant \widetilde{\chi}_0 \leqslant 1$ respectively. Then

$$\int_{B_t} |Z(t, x) - \widetilde{Z}(t, x)|\, dx \leqslant C \int_{B_0} |Z_0(x) - \widetilde{Z}_0(x)|\, dx\,, \tag{5.1}$$

for a.e. $t \in [0, T_0]$, $T_0 = \min(T, R\sqrt{\alpha})$, $B_t = \{x \mid |x| \in R - t/\sqrt{\alpha}\}$.

PROOF. – In the first and second equations of D2) for Γ, S, χ (Definition, § 2) we set $f = \xi(x, t, y, \tau)$ and $g = \xi(x, t, y, \tau)$ respectively, where $\xi \in C_0^1(\pi_T \times \pi_T)$, $\xi \geqslant 0$ and let $k = \widetilde{\Gamma}(y, \tau)$, $l = \widetilde{S}(y, \tau)$. Then we integrate these equations in (y, τ) and add the resulting expressions. We obtain

$$(5.2)\qquad \iiiint_{\pi_T\times\pi_T}\Big\{(|\Gamma-\tilde{\Gamma}|+|S-\tilde{S}|)\xi_t+\frac{1}{\sqrt{\alpha}}(|\Gamma-\tilde{\Gamma}|-|S-\tilde{S}|)\xi_x+ +\xi\Big(\frac{\Gamma-S}{2\alpha}\Big)(\operatorname{sgn}(S-\tilde{S})-\operatorname{sgn}(\Gamma-\tilde{\Gamma}))+ +\frac{1}{2}\xi(\operatorname{sgn}(\Gamma-\tilde{\Gamma})\chi_t-\operatorname{sgn}(S-\tilde{S})\tilde{\chi}_t)\Big\}\,dx\,dy\,dt\,d\tau\geq 0\,.$$

Adding this inequality with the one obtained similarly from $\tilde{\Gamma}$, $\tilde{S}$, $\tilde{\chi}$, we get

$$(5.3)\qquad \iiiint_{\pi_T\times\pi_T}\Big\{(|\Gamma-\tilde{\Gamma}|+|S-\tilde{S}|)(\xi_t+\xi_t)+\frac{1}{\sqrt{\alpha}}(|\Gamma-\tilde{\Gamma}|-|S-\tilde{S}|)(\xi_x+\xi_y)+ +|\chi_t-\tilde{\chi}_\tau|\xi\Big\}\,dx\,dy\,dt\,d\tau\geq 0\,.$$

Here we have used the following inequalities

$$\frac{(\Gamma-\tilde{\Gamma})-(S-\tilde{S})}{2\alpha}(\operatorname{sign}(S-\tilde{S})-\operatorname{sign}(\Gamma-\tilde{\Gamma}))\leq 0\,,$$

$$(\chi_t-\tilde{\chi}_\tau)\frac{(\operatorname{sign}(S-\tilde{S})+\operatorname{sign}(\Gamma-\tilde{\Gamma}))}{2}\leq|\chi_t-\tilde{\chi}_\tau|\,.$$

Let us define

$$\xi(x,t,y,\tau)=\xi_h(x,t,y,\tau)=\delta_h\Big(\frac{t-\tau}{2}\Big)\delta_h\Big(\frac{x-y}{2}\Big)\varphi\Big(\frac{x+y}{2},\,\frac{t+\tau}{2}\Big)$$

where $\varphi\in C_0^1(\pi_T)$, $\varphi\geq 0$. We see that

$$(5.4)\qquad \begin{cases}\xi_t+\xi_\tau=\varphi_t(\ldots)\delta_h\Big(\dfrac{t-\tau}{2}\Big)\delta_h\Big(\dfrac{x-y}{2}\Big),\\[2mm] \xi_x+\xi_y=\varphi_x(\ldots)\delta_h\Big(\dfrac{x-y}{2}\Big)\end{cases}.$$

Taking the limit as $h\to 0$ in (5.3), by using (5.4) from Lemma 3.2 we have

$$(5.5)\qquad \iint_{\pi_T}(|\Gamma-\tilde{\Gamma}|+|S-\tilde{S}|)\varphi_t+ +\frac{1}{\sqrt{\alpha}}(|\Gamma-\tilde{\Gamma}|-|S-\tilde{S}|)\varphi_x+|(\chi-\tilde{\chi})_t|\varphi\}\,dx\,dy\geq 0\,.$$

Let E and $\widetilde{E}$ be the sets of zero measure for which D3) holds for (Γ, S, χ) and $(\widetilde{\Gamma}, \widetilde{S}, \widetilde{\chi})$ respectively. In addition, let us define $E^{\mu} \subseteq [0, T]$ as the set of non Lebesgue points of the bounded measurable function

$$\mu(t) = \int_{B_t} (|\Gamma - \widetilde{\Gamma}| + |S - \widetilde{S}|)\, dx\,. \tag{5.6}$$

We set $E^0 = E \cup \widetilde{E} \cup E^{\mu}$ and note that E^0 has zero measure. We define

$$\alpha_h(\sigma) = \int_{-\infty}^{\sigma} \delta_h(s)\, ds$$

and take $\rho,\ \tau \in (0, T) \setminus E^0$, $\rho < \tau$. In (5.5) let us choose

$$\varphi = [\alpha_h(t-\rho) - \alpha_h(t-\tau)]\, \chi^{\varepsilon}(x, t)\,, \qquad h < \min(\rho, T_0 - \tau)$$

where $\chi^{\varepsilon}(x, t) = 1 - \alpha_{\varepsilon}(|x| - R - t/\sqrt{\alpha} + \varepsilon)$, $\varepsilon > 0$. Note that χ^{ε} satisfies the relations

$$0 \geq \chi_t^{\varepsilon} \pm \frac{1}{\sqrt{\alpha}} \chi_x^{\varepsilon}\,.$$

Then, from (5.5), we obtain the inequality

$$\int\!\!\int_{\pi_T} [\delta_h(t-\rho) - \delta_h(t-\tau)]\, \chi_{\varepsilon}(x, t) \{ |\Gamma - \widetilde{\Gamma}| + |S - \widetilde{S}| \} + |(\chi - \widetilde{\chi})_t|\, \chi_{\varepsilon}(x, t)\}\, dx\, dt \geq 0\,.$$

Taking $h \to 0$ and $\varepsilon \to 0$ we obtain

$$\int_{B_\tau} |\Gamma(\tau, x) - \widetilde{\Gamma}(\tau, x)| + |S(\tau, x) - \widetilde{S}(\tau, x)|\, dx \leq \tag{5.7}$$

$$\leq \int_{B_\rho} \{ |\Gamma(\rho, x) - \widetilde{\Gamma}(\rho, x)| + |S(\rho, x) - \widetilde{S}(\rho, x)| \}\, dx + \int_{\rho}^{\tau} \int_{B_t} |(\chi - \widetilde{\chi})_t|\, dx\, dt\,.$$

In addition, from D1), since $\chi_t,\ \widetilde{\chi}_t \in L^{\infty}(\pi_T)$ we have the following relations:

$$\int_{B_\tau} |\chi(\tau, x) - \widetilde{\chi}(\tau, x)|\, dx - \int_{B_\rho} |\chi(\rho, x) - \widetilde{\chi}(\rho, x)|\, dx + \tag{5.8}$$

$$+ \int_{\rho}^{\tau} \int_{B_t} |(\chi - \widetilde{\chi})_t|\, dx\, dt \leq \int_{\rho}^{\tau} \int_{B_t} |\chi - \widetilde{\chi}|_t + |(\chi - \widetilde{\chi})_t|\, dx\, dt \equiv$$

$$\equiv \int_{\rho}^{\tau}\int_{B_t}(|\chi-\tilde{\chi}|_t + ||\chi-\tilde{\chi}|_t|)\,dx\,dt \leq 2\int_{\rho}^{\tau}\int_{B_t}\sup(0, |\chi-\tilde{\chi}|_t)\,dx\,dt \leq$$

$$\leq 2\int_{\rho}^{\tau}\int_{B_t}|f(\Gamma, S, \chi) - f(\tilde{\Gamma}, \tilde{S}, \tilde{\chi})|\,dx\,dt\,.$$

The last inequality follows from the following inequality (see D1)

$$|\chi-\tilde{\chi}|_t \leq |f(\Gamma, S, \chi) - f(\tilde{\Gamma}, \tilde{S}, \tilde{\chi})| \qquad \text{a.e. in } \pi_T\,.$$

Therefore adding (5.7) and (5.8) and using that f is a Lipschitz function for

$$\tilde{\mu}(t) = \int_{B_t}\{|\Gamma(t,x) - \tilde{\Gamma}(t,x)| + |S(t,x) - \tilde{S}(t,x)| + |\chi(t,x) - \tilde{\chi}(t,x)|\}\,dx\,,$$

we have the following inequality

$$\tilde{\mu}(\tau) \leq \tilde{\mu}(\rho) + K\int_{\rho}^{\tau}\tilde{\mu}(t)\,dt\,.$$

Finally taking $\rho \to 0$ on the set E^0 and applying Gronwall's lemma we prove the statement of Theorem 2.2. ■

6. - Proof of Theorem 2.3.

Denoting by $(\theta_\alpha, \bar{q}_\alpha, \chi_\alpha)$ the weak solution of problem (2.1), (2.2) with initial data θ_0, q_0, χ_0 independent of $\alpha > 0$, we have the following estimates from above results:

(6.1) $\quad \theta_\alpha, \quad \sqrt{\alpha}\,\bar{q}_\alpha, \quad \chi_\alpha \in L^\infty(\pi_T)$,

(6.2)
$$\int_{B_t}|\Delta\theta_\alpha| + \sqrt{\alpha}|\Delta\bar{q}_\alpha| + |\Delta\chi_\alpha|\}\,dx \leq$$

$$\leq C\int_{B_0}|\Delta\theta_0| + \sqrt{\alpha}|\Delta q_0| + |\Delta\chi_0|\}\,dx \qquad \text{for a.e. } t\in[0, T_0]\ \forall \Delta x \in \mathbb{R}\,,$$

where C is a constant independent of α, $\Delta\varphi = \varphi(x+\Delta x) - \varphi(x)$. From (6.1)-(6.2) by using Lemma 3.3 we conclude that $\forall r > 0$, $\Delta x \in \mathbb{R}$, there exist two modules of continuity ω_r^x, ω_r^t independent of α such that

(6.3)
$$J_r(\theta_\alpha, \Delta x) + \sqrt{\alpha}J_r(\bar{q}_\alpha, \Delta x) + J_r(\chi_\alpha, \Delta x) +$$

$$+ I_r(\theta_\alpha, \Delta t) + \sqrt{\alpha}I_r(\bar{q}_\alpha, \Delta t) + I_r(\chi_\alpha, \Delta t) \leq \omega_r^x(|\Delta x|) + \omega_r^t(\Delta t)\,.$$

From (6.1), (6.3) we have

$$\theta_\alpha \to \theta, \quad \chi_\alpha \to \chi, \quad \sqrt{\alpha}\bar{q}_\alpha \to \bar{q},$$

$$\text{as } \alpha \to 0 \quad \text{strongly in } L^1_{loc}(\pi_T) \text{ and } *\text{-weakly in } L^\infty(\pi_T).$$

Therefore easily conclude that θ, χ is a weak solution of the Stefan problem (2.8) with non-equilibrium constitutive relaxation. ■

Acknowledgement. The author would like to thank Professors P. I. PLOTNIKOV and J. F. RODRIGUES for many inspiring discussions.

This paper has been partially written while the author was visiting the CMAF and University of Lisbon, Lisbon, whose stimulating atmosphere is gratefully acknowledged, in the framework of Post-Doctoral Fellowship of Gulbenkian Foundation.

REFERENCES

[1] P. COLLI - GRASSELLI, *An existence result for a hyperbolic phase transition problem with memory*, Appl. Math. Lett., **5**, no. 1 (1992), pp. 99-102.

[2] A. FRIEDMAN - B. HU, *The Stefan problem for a hyperbolic heat equation*, J. Math. Anal. Appl., **37**, no. 1 (1989), pp. 249-279.

[3] J. M. GREENBERG, *A hyperbolic heat transfer problem with phase changes*, IMA J. Appl. Math., **38** (1987), pp. 1-21.

[4] S. N. KRUZKOV, *First order quasilinear equations in several independent variables*, Math. USSR Sb., **10**, no. 2 (1970), pp. 217-273.

[5] S. MAZZULO - M. PAOLINI - C. VERDI, *Polymer crystallization and processing: Free boundary problems and their numerical approximation*, Math. Engng. Ind., **2**, no. 3 (1989), pp. 219-232.

[6] R. H. NOCHETTO - C. VERDI, *Approximation of multidimensional Stefan-like problems via hyperbolic relaxation*, Quart. Numer. Anal. Theory Comput., **25**, no. 3 (1988), pp. 219-232.

[7] N. V. SHEMETOV, *The Stefan problem for a hyperbolic heat equation*, Internat. Series Numer. Mathem., **99** (1991), pp. 365-376.

[8] R. E. SHOWALTER - N. J. WALKINGHTON, *A hyperbolic Stefan problem*, Quart. Appl. Math., **45** (1987), pp. 769-781.

[9] C. VERDI - A. VISINTIN, *Stefan problem with Cattaneo-fourier conduction law and with phase relaxation: existence of a solution and numerical approximation*, to appear.

[10] A. VISINTIN, *Supercooling and superheating effects in phase transitions*, IMA J. Appl. Math., **35** (1985), pp. 233-256.