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Global Existence of Solutions for Perturbed Differential Equations (*).

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Abstract. – *In this paper we consider sufficient conditions for the continuability of solutions for perturbed differential equations. We obtain also some results for the global existence of solutions for differential inclusions and for stochastic differential equations of McShane and Ito type. We give an application to the global inversion of local diffeomorphisms.*

1. – Introduction.

Let us consider the scalar differential equation

$$r' = \phi(t)\, w(r) \tag{1.1}$$

and the perturbed differential equation

$$r' = \phi(t)\, w(r) + \psi(t)\, z(r) \tag{1.2}$$

where $\phi, \psi, w, z \in C(R_+, R_+)$.

HARA, YONEYAMA and SUGIE [45] gave necessary and sufficient conditions for the global existence of the solutions of (1.1). The natural problem arises to give sufficient conditions for the global existence of solutions of (1.2).

Several approaches were made in order to handle differential equations (or differential and integral inequalities) involving two nonlinearities:

(I) DHONGADE and DEO [30, 31] considered nonlinearities belonging to the class of functions $\alpha \in C(R_+, R_+)$, positive and nondecreasing on R_+ and satisfying the condition

$$\frac{1}{v}\alpha(u) \leq \alpha\left(\frac{u}{v}\right), \qquad u \geq 0, \quad v > 0;$$

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(II) BEESACK [7] replaced the preceding condition by the condition

$$\frac{1}{v}\alpha(u) \leqslant \alpha\left(\frac{u}{v}\right), \qquad u > 0\,, \quad v \geqslant 1\,,$$

in order to avoid the triviality $\alpha(u) = \alpha(1)\,u$, $u > 0$;

(III) DANNAN [27, 28] assumed that one of the nonlinearities satisfies a more general submultiplicative condition and the other is bounded by a linear function;

(IV) HARA, YONEYAMA and SUGIE [45] proved that if $w, z \in C(R_+, R_+)$ are such that $z(r) \leqslant Lr$, $r \geqslant \delta > 0$ $(L > 0)$ and $\int_0^\infty ds/w(s) = \infty$ with w nondecreasing, then the solutions of (1.2) are defined in the future;

(V) PINTO [54] considered monotone nondecreasing nonlinearities $w, z \in C(R_+, R_+)$ subject to the condition that w/z is nondecreasing on $(1, \infty)$.

One can see that in all these approaches we have either that one of the nonlinearities is sublinear in a neighborhood of ∞ (this is the case with the classes (I), (II), (III) and (IV)) or the two nonlinearities are comparable in a neighborhood of ∞ (in the sense that there exist $L, M > 0$ with $Mz(r) \leqslant w(r)$ for $r \geqslant L$) as it is the case for the class (V).

We will give a continuation result for the solutions of (1.2) considering two nonlinearities which are not comparable to each other or to a linear function in a neighborhood of ∞. The comparison method of CONTI [22, 23] enables us to apply this result to the general case of ordinary differential equations on R^n, generalizing some results of BERNFELD [9], HARA, YONEYAMA and SUGIE [45] and STOKES [66].

We give a global existence result for the Rayleigh equation, completing a recent result of SOUPLET [65] and our continuation results for the Liénard equation with perturbing term improve some results of BURTON [13], GRAEF [38], HARA, YONEYAMA and SUGIE [44] and NAGABUCHI and YAMAMOTO [52]. Considering the problem of the continuability of solutions for the differential equation

$$(a(t)\,x')' + q(t)\,f(x)\,g(x') = r(t)$$

we generalize some results of BURTON and GRIMMER [15] and GRAEF and SPIKES [39].

We consider also the cases of delay and functional differential equations improving some results of HARA, YONEYAMA and SUGIE [45]. In the case of differential equations in Banach spaces of infinite dimension, our result generalizes some results of ALEXANDROV and DAIRBEKOV [2] and of RADULESCU and RADULESCU [57].

Moreover, we are able to provbe global existence results for differential inclusions obtaining as particular cases some results of SEAH [61] and TANIGUCHI [72].

The method is applicable also in the case of stochastic differential equations of McShane and Ito type. Our results improve some theorems of MCSHANE [62], ELWORTHY [34] and ANGULO IBANEZ and GUTIERREZ JAIMEZ [3] for equations of McShane type and some results of DAPRATO [55], TANIGUCHI [71] and YAMADA [75] for equations of Ito type.

An application of these continuation results to the global inversion of local diffeomorphisms of R^n is also given. As particular cases we obtain some recent results of ZAMPIERI [77].

2. – Continuability of solutions of perturbed ordinary differential equations.

Let us assume that $f \in C(R_+ \times R^n, R^n)$ is such that all solutions of the differential equation

$$x' = f(t, x)$$

are defined in the future. We will give sufficient conditions on $g \in C(R_+ \times R^n, R^n)$ so that all solutions of the perturbed differential equation

$$x' = f(t, x) + g(t, x)$$

are defined in the future.

In our discussion we will impose growth conditions of f and g by requiring the existence of functions $\phi, \psi, z, w \colon R_+ \to R_+$ continuous, $z(r) > 0$ and $w(r) > 0$ for all $r \geq \delta > 0$ such that

$$|f(t, x)| \leq \phi(t) w(|x|), \qquad |g(t, x)| \leq \psi(t) z(|x|), \qquad (t, x) \in R_+ \times R^n .$$

Because of these growth conditions imposed on f and g we have essentially reduced the problem to the study of perturbed scalar differential equations (in view of the comparison method of CONTI [20, 22, 23]).

We define the class of continuous scalar functions

$$\mathfrak{R}_0 = \left\{ w \colon w(r) > 0,\ r \geq \delta;\ \int_\delta^\infty \frac{ds}{w(s)} = \infty \right\}.$$

Let us consider the nonautonomous scalar equations

$$r' = \phi(t) w(r), \tag{2.1}$$

$$r' = \phi(t) w(r) + \psi(t) z(r) \tag{2.2}$$

where $\phi, \psi, z, w \colon R_+ \to R_+$ are continuous, $z(r) > 0$ and $w(r) > 0$ for all $r \geq \delta \geq 0$.

The following result gives necessary and sufficient conditions for the continuability of the solutions of (2.1).

THEOREM 2.1 [45]. – *Suppose that $\phi(t)$ is not identically zero. Then the solutions of* (2.1) *are defined in the future if and only if $w \in \mathfrak{R}_0$.*

In order to state results for (2.2) we consider for each $y \in \mathfrak{R}_0$ with $\liminf_{r \to \infty} y(r) > 0$

the class of continuous scalar functions

$$\mathfrak{R}_y = \left\{ w\colon w(r) > 0,\ r \geqslant \delta;\ \int_\delta^\infty \frac{ds}{w(s) + y(s)} = \infty \right\}.$$

Since

$$\int_\delta^\infty \frac{ds}{w(s)} \geqslant \int_\delta^\infty \frac{ds}{w(s) + y(s)} = \infty$$

we see that $\mathfrak{R}_y \subset \mathfrak{R}_0$ for every $y \in \mathfrak{R}_0$ such that $\liminf_{r\to\infty} y(r) > 0$. We notice that $\mathfrak{R}_y \neq \mathfrak{R}_0$ as it can be seen from Example 2.1.

EXAMPLE 2.1. – Define the function $h\colon [1, \infty) \to R_+$ as follows: for each integer $n \geqslant 1$ such that $n \leqslant t \leqslant n + 1$,

$$h(n) = \frac{1}{n}, \qquad h(n+1) = \frac{1}{n+1},$$

$$h(t) = t^2, \qquad n + \frac{1}{n^2} \leqslant t \leqslant n + 1 - \frac{1}{(n+1)^2},$$

$$\frac{1}{h(t)} \text{ is linear for } n \leqslant t \leqslant n + \frac{1}{n^2} \quad \text{and} \quad n + 1 - \frac{1}{(n+1)^2} \leqslant t \leqslant n + 1 .$$

From the construction of h we have (for more details see [9, page 279]) that

$$\int_1^\infty \frac{ds}{h(s)} = \infty, \qquad \int_1^\infty \frac{ds}{h(s) + 1} < \infty .$$

Since $y(r) > 0$ for $r \geqslant \delta \geqslant 0$ and $\liminf_{r\to\infty} y(r) > 0$ we have that there exists an $\varepsilon \in (0,1)$ such that $y(r) \geqslant \varepsilon$ for $r \geqslant \delta$ thus

$$\int_{1+\delta}^\infty \frac{ds}{y(s) + h(s)} \leqslant \int_{1+\delta}^\infty \frac{ds}{\varepsilon + h(s)} \leqslant \frac{1}{\varepsilon} \int_{1+\delta}^\infty \frac{ds}{1 + h(s)} < \infty$$

i.e. $h \notin \mathfrak{R}_y$. We have so that $\mathfrak{R}_y \neq \mathfrak{R}_0$.

LEMMA 2.1. – *For each $L > 0$ and $M > 0$ we have that*

$$\int_\delta^\infty \frac{ds}{Lw(s) + My(s)} = \infty$$

if and only if $w \in \mathfrak{R}_y$.

PROOF. – If $w \in \mathfrak{R}_y$ we have that

$$\int_{\delta}^{\infty} \frac{ds}{Lw(s)+My(s)} \geqslant \frac{1}{L+M}\int_{\delta}^{\infty} \frac{ds}{w(s)+y(s)} = \infty .$$

Conversely, suppose that

$$\int_{\delta}^{\infty} \frac{ds}{Lw(s)+My(s)} = \infty .$$

We have then that

$$\int_{\delta}^{\infty} \frac{ds}{w(s)+y(s)} \geqslant \min\{L, M\}\int_{\delta}^{\infty} \frac{ds}{Lw(s)+My(s)} = \infty$$

thus $w \in \mathfrak{R}_y$. This completes the proof of the lemma.

We define now the set (for $y \in \mathfrak{R}_0$ with $\liminf_{r\to\infty} y(r) > 0$)

$$A_{z,y}(K) = \{r \geqslant \delta\colon z(r) \leqslant Ky(r)\}.$$

Let $m(A^c_{z,y}(K))$ denote the Lebesgue measure of the complement of $A_{z,y}(K)$ in $[\delta, \infty)$.

THEOREM 2.2. – *Suppose that $\phi(t)\psi(t)$ is not identically zero and suppose that there exists $K > 0$ such that $m(A^c_{z,y}(K)) < \infty$ and*

$$\liminf_{r\to\infty} \frac{z(r)}{y(r)} > 0 .$$

Then the solutions of (2.2) *are defined in the future if and only if $w \in \mathfrak{R}_y$.*

PROOF. – Let us suppose that $w \in \mathfrak{R}_y$.
Observing that

$$\int_{A^c_{z,y}(K)} \frac{ds}{w(s)+Ky(s)} < \infty$$

we obtain (in view of Lemma 2.1) that

$$\int_{A_{z,y}(K)} \frac{ds}{w(s)+Ky(s)} = \int_{\delta}^{\infty} \frac{ds}{w(s)+Ky(s)} - \int_{A^c_{z,y}(K)} \frac{ds}{w(s)+Ky(s)} = \infty .$$

Hence

$$\int_{\delta}^{\infty} \frac{ds}{w(s)+z(s)} = \int_{A_{z,y}(K)} \frac{ds}{w(s)+z(s)} + \int_{A^c_{z,y}(K)} \frac{ds}{w(s)+z(s)} \geqslant \int_{A_{z,y}(K)} \frac{ds}{w(s)+Ky(s)} = \infty .$$

Since $w + z \in \mathfrak{R}_0$ we deduce by Theorem 2.1 that the solutions of the differential equation

$$r' = (\phi(t) + \psi(t))(w(r) + z(r))$$

are all defined in the future. By the comparison method of CONTI [22, 23] we obtain that the solutions of (2.2) are defined in the future.

Let us now prove the necessity of the condition $w \in \mathfrak{R}_y$.

Suppose that $w \notin \mathfrak{R}_y$ and the solutions of (2.2) are defined in the future. Since $\phi(t)\psi(t)$ is not identically zero, there exist $a \geqslant 0$, $b > a$ and $\varepsilon > 0$ such that $\phi(t) \geqslant \varepsilon$ and $\psi(t) \geqslant \varepsilon$ on $[a, b]$. We obtain from (2.2) that

$$r'(t) \geqslant \varepsilon(w(r) + z(r)), \qquad t \in [a, b].$$

Since

$$\liminf_{r \to \infty} \frac{z(r)}{y(r)} > 0$$

we have that there exist $\eta > 0$ and $M > 0$ such that $z(r) \geqslant \eta y(r)$ for all $r \geqslant M$. We deduce that

$$\int_{\delta}^{\infty} \frac{ds}{w(s)+z(s)} = \int_{A_{z,y}(K)} \frac{ds}{w(s)+z(s)} + \int_{A^c_{z,y}(K)} \frac{ds}{w(s)+z(s)} \leqslant$$

$$\leqslant \int_{A_{z,y}(K)\cap[M,\infty)} \frac{ds}{w(s)+\eta y(s)} + \int_{A_{z,y}(K)\cap[0,M]} \frac{ds}{w(s)+z(s)} + \int_{A^c_{z,y}(K)} \frac{ds}{w(s)+z(s)} .$$

Since $m(A^c_{z,y}(K)) < \infty$ and $w \notin \mathfrak{R}_y$ we obtain in view of Lemma 2.1 that

$$\int_{\delta}^{\infty} \frac{ds}{w(s)+z(s)} < \infty .$$

Let $r_0 > \delta$ be such that

$$\int_{r_0}^{\infty} \frac{ds}{w(s)+z(s)} < \varepsilon(b-a).$$

The solution $r(t, a, r_0)$ of (2.2) satisfies

$$r'(t) \geqslant \varepsilon(w(r) + z(r)), \qquad t \in [a, b],$$

thus

$$\int_{r_0}^{r(t)} \frac{ds}{w(s) + z(s)} \geqslant \varepsilon(t-a), \qquad t \in [a, b].$$

We obtain so that

$$\varepsilon(b-a) > \int_{r_0}^{\infty} \frac{ds}{w(s) + z(s)} \geqslant \int_{r_0}^{r(b)} \frac{ds}{w(s) + z(s)} \geqslant \varepsilon(b-a)$$

which is a contradiction.

We have so that $w \in \mathfrak{R}_y$.

This completes the proof of Theorem 2.2.

COROLLARY 2.1. – *Suppose that $\phi(t)\psi(t)$ is not identically zero. Then the solutions of the differential equation*

$$r' = \phi(t)\, w(r) + \psi(t)$$

are defined in the future if and only if

$$\int_{\delta}^{\infty} \frac{ds}{1 + w(s)} = \infty\,.$$

Theorem 2.2 generalizes some results of BERNFELD [9, Theorem 4.1] and HARA, YONEYAMA and SUGIE [45, Theorem 3.2 and Theorem 3.3]. The relation of our result with these results is given by

EXAMPLE 2.2. – Let us consider the scalar differential equation

$$r' = \phi(t)\, r \ln(r+1) + \psi(t)(r + \ln(r+1))$$

where $\phi, \psi \colon R_+ \to R_+$ are continuous and $\phi(t)\psi(t)$ is not identically zero. We can apply Theorem 2.2 with $y\colon R_+ \to R_+$, $y(r) = (r+1)\ln(r+1)$ but we can not apply Theorem 4.1 [9] or Theorem 3.2, Theorem 3.3 [45].

THEOREM 2.3. – *Let $w \in \mathfrak{R}_0$ and suppose there exist constants $K, L, M > 0$ such that*

$$z(r) \leqslant K w(r) \int_{\delta}^{r} \frac{ds}{w(s)} + M w(r), \qquad r \geqslant L \geqslant \delta\,.$$

Then the solutions of (2.2) *are defined in the future.*

PROOF. – Let

$$V(t,r) = -\int_0^t \phi(s)\,ds + \int_\delta^r \frac{ds}{w(s)}, \qquad r \geq L.$$

Since $w \in \mathfrak{R}_0$ we have that

$$V(t,r) \to \infty \text{ as } r \to \infty \text{ for each fixed } t \in R_+.$$

We have that

$$\left.\frac{dV}{dt}\right|_{(2.2)} = \limsup_{h \to 0^+} \frac{V(t+h,\, r + h\phi(t)w(r) + h\psi(t)z(r)) - V(t,r)}{h} =$$

$$= -\phi(t) + \frac{1}{w(r)}(\phi(t)w(r) + \psi(t)z(r)) = \psi(t)\frac{z(r)}{w(r)} \leq K\psi(t)\int_\delta^r \frac{ds}{w(s)} + M\psi(t) =$$

$$= K\psi(t)V(t,r) + M\psi(t) + K\psi(t)\int_0^t \phi(s)\,ds, \qquad r \geq L,\ t \geq 0.$$

By Conti's theorem (see [22, 23, 67]) we deduce that the solutions of

$$r' = \phi(t)w(r) + \psi(t)z(r)$$

are defined in the future. This completes the proof of the theorem.

COROLLARY 2.2. – *If $w \in \mathfrak{R}_0$ is nondecreasing on $[\delta, \infty)$ then the solutions of the differential equation*

$$r' = \phi(t)w(r) + \psi(t)r$$

where $\phi, \psi: R_+ \to R_+$ are continuous, are all defined in the future.

PROOF. – Since w is nondecreasing on $[\delta, \infty)$ we have that

$$w(r)\int_\delta^r \frac{ds}{w(s)} \geq r - \delta$$

thus we can apply Theorem 2.3 with

$$K = 1, \qquad M = \frac{\delta}{w(\delta)}, \qquad L = \delta.$$

Corollary 2.2 is a theorem of STOKES [66]. He arrived at the same result using the Tychonoff fixed point theorem. For a proof of Corollary 2.2 using differential inequalities see [9] (a different method is used in [45]).

COROLLARY 2.3 [9]. – *Assume that* $\phi(t)$ *is not identically zero and all solutions of* (2.1) *exist in the future. Then all solutions of* (2.2) *exist in the future if* z *satisfies*

$$\limsup_{r\to\infty} \frac{z(r)}{w(r)} < \infty .$$

PROOF. – Since all solutions of (2.1) exist in the future we have by Theorem 2.1 that $w \in \mathfrak{R}_0$.

On the other hand there exists $K_0 > 0$ such that

$$z(r) \leqslant K_0 w(r), \qquad r \geqslant \delta .$$

If $L > \delta$ is such that

$$\int_\delta^L \frac{ds}{w(s)} \geqslant K_0$$

we obtain that

$$z(r) \leqslant w(r) \int_\delta^r \frac{ds}{w(s)} \qquad r \geqslant L .$$

We can so apply Theorem 2.3.

COROLLARY 2.4 [9]. – *Assume that* $\psi(t)$ *is not identically zero and all solutions of*

$$r' = \psi(t)\, z(r)$$

exist in the future. If

$$\liminf_{r\to\infty} \frac{z(r)}{w(r)} > 0$$

then all solutions of (2.2) *exist in the future.*

PROOF. – We have that $z \in \mathfrak{R}_0$. Since $\liminf_{r\to\infty} (z(r)/w(r)) > 0$ we obtain that $w \in \mathfrak{R}_0$ and $\limsup_{r\to\infty} (w(r)/z(r)) < \infty$. We can thus apply Corollary 2.3.

EXAMPLE 2.3. – Let us consider the differential equation

$$r' = \phi(t)(r + \ln(r+1)) + \psi(t)(r\ln(r+1) + \ln(r+1)\ln(r+1))$$

where ϕ, ψ: $R_+ \to R_+$ are continuous and $\phi(t)\psi(t)$ is not identically zero. We can apply Theorem 2.3 with w, z: $R_+ \to R_+$, $w(r) = r + \ln(r+1)$, $z(r) = r\ln(r+1) + \ln(r+1)\ln(r+1)$ but we can not apply the theorem of Stokes [66].

REMARK 2.1. – *It should be noted that if* $w_1, w_2 \in \mathfrak{R}_0$ *we can not conclude that* $w_1 + w_2 \in \mathfrak{R}_0$ *even if we suppose that* w_1 *and* w_2 *are nondecreasing (see Example* 3.8 [9]). *Classes of functions for which this is true are given by Theorem* 2.2 *and Theorem* 2.3.

We apply our results in order to study the continuability of solutions of the differential equation

$$u'' + f(u') + g(u) = e(t) \tag{2.3}$$

where e, f, g: $R \to R$ are continuous.

Equation (2.3) is equivalent to the system

$$\begin{cases} x' = y\,, \\ y' = -f(y) - g(x) + e(t)\,. \end{cases} \tag{2.3}$$

THEOREM 2.4. – *Suppose that*

(i) *there exists* $w \in \mathfrak{R}_0$ *nondecreasing on* R_+ *such that*

$$|xf(x)| \leqslant w(x^2)\,, \qquad x \in R\,;$$

(ii) $0 \leqslant xg(x)$ *for* $|x|$ *large enough.*

Then the solutions of (2.3) *are defined on R.*

PROOF. – Let us prove that the solutions of (2.3) are defined in the future. By (ii) we have that there exists $K > 0$ such that

$$\int_0^x g(s)\,ds + K > 0\,, \qquad x \in R\,.$$

Let us define

$$V\colon R \times R \to R_+\,, \qquad V(x, y) = y^2 + 2\int_0^x g(s)\,ds + 2K\,.$$

We have that

$$\frac{dV}{dt}_{(2.3)} = -2yf(y) + 2ye(t) \leqslant 2w(y^2) + (y^2+1)|e(t)| \leqslant$$
$$\leqslant 2w(V(x, y)) + V(x, y)|e(t)| + |e(t)|\,.$$

By Corollary 2.2 we have that

$$\int_\varepsilon^\infty \frac{ds}{2w(s) + s + 1} = \infty\,.$$

From the relation

$$y^2 \leqslant V(x, y), \qquad x, y \in R,$$

we deduce that $y(t)$ can not explode in a finite time $T > t_0$ (t_0 is the initial time) for any solution $(x(t), y(t))$ of (2.3). Since

$$x' = y$$

we deduce also that $x(t)$ can not explode in a finite time $T > t_0$.

Thus the solutions of (2.3) are defined in the future.

In order to prove that the solutions of (2.3) are also defined in the past we reverse the time in order to reduce this problem to the problem of the existence in the future of the solutions of the system

$$\begin{cases} x' = y, \\ y' = -f(-y) - g(x) + e(-t). \end{cases} \tag{2.3'}$$

We observe that

$$\frac{dV}{dt}_{(2.3')} = -2yf(-y) + 2ye(-t) \leqslant 2w(y^2) + (y^2+1)|e(-t)| \leqslant$$
$$\leqslant 2w(V(x, y)) + V(x, y)|e(-t)| + |e(-t)|.$$

We conclude in a similar way to the first part of the proof.

Thus the solutions of (2.3′) are defined in the future i.e. the solutions of (2.3) are defined in the past.

This completes the proof of Theorem 2.4.

The continuability of solutions of the differential equation (2.3) was also considered by SOUPLET [65] in the case $e(t) \equiv 0$.

EXAMPLE 2.4. – Consider the differential equation

$$u'' + u' \ln(1 + |u'|^2) + u^3 = 0.$$

We can apply Theorem 2.4 considering

$$w\colon R_+ \to R_+, \qquad w(r) = r\ln(1+r).$$

The result of SOUPLET [65] is not applicable since we do not have that there exists a constant $M > 0$ such that

$$0 \leqslant xg(x) \leqslant Mx^2 \qquad \text{for } |x| \text{ large enough}.$$

We will give now a global existence result for the solutions of the Liénard equation with perturbing term

$$x'' + f(x)x' + g(x) = e(t, x, x') \tag{2.4}$$

where $f, g: R \to R$ are continuous and $e: R_+ \times R \times R \to R$ satisfies the following conditions:

— $e(t, x, y)$ is piecewise continuous with respect to $t \in R_+$ for arbitrarily fixed $(x, y) \in R \times R$;

— $e(t, x, y)$ is continuous with respect to $(x, y) \in R \times R$ for arbitrarily fixed $t \in R_+$.

Let us denote

$$F(x) = \int_0^x f(s)\,ds, \qquad G(x) = \int_0^x g(s)\,ds, \qquad x \in R.$$

THEOREM 2.5. – *Suppose that the following conditions hold:*

(i) *there exists a constant* $K > 0$ *such that*

$$xF(x) > 0 \quad \text{and} \quad xg(x) > 0 \quad \text{for } |x| \geqslant K;$$

(ii) *there exist continuous functions* $r_1, r_2, z, w: R_+ \to R_+$, $w \in \Re_0$ *nondecreasing, such that*

$$|e(t, x, y - F(x))| \leqslant r_1(t) + r_2(t) z(|y|), \qquad t \in R_+,\ x, y \in R,$$

and

$$|y| z(|y|) \leqslant w(y^2), \qquad y \in R.$$

Then all the solutions of (2.4) *are defined in the future.*

PROOF. – Let $(x(t), x'(t))$ be a solution of (2.4) defined on $[t_0, T)$, $T < \infty$. Then $(x(t), y(t))$ is a solution of the system

$$\begin{cases} x' = y - F(x), \\ y' = -g(x) + \tilde{e}(t), \end{cases}$$

where $y(t) = x'(t) + F(x(t))$ and $\tilde{e}(t) = e(t, x(t), y(t) - F(x(t)))$.
Defining

$$V(x, y) = \frac{y^2}{2} + G(x), \qquad x, y \in R,$$

we obtain that

$$\left.\frac{dV}{dt}\right|_{(2.4)} = -g(x(t)) F(x(t)) + \tilde{e}(t) y(t).$$

Integrating on $[t_0, t]$, $t < T$, we get

$$V(x(t), y(t)) - V(x(t_0), y(t_0)) \leqslant -\int_{t_0}^{t} g(x(s))\, F(x(s))\, ds +$$

$$+\int_{t_0}^{t} |\tilde{e}(s)\, y(s)|\, ds \leqslant -\int_{I_t} g(x(s))\, F(x(s))\, ds + \int_{t_0}^{t} |\tilde{e}(s)\, y(s)|\, ds\,,$$

where $I_t = \{s \in [t_0, t]\colon |x(s)| < K\}$ (in view of (i)).

If $K_1 = \max_{|x| \leqslant K} \{|g(x)\, F(x)|\}$, we obtain that

$$V(x(t), y(t)) \leqslant V(x(t_0), y(t_0)) + K_1 + \int_{t_0}^{t} |\tilde{e}(s)\, y(s)|\, ds\,, \qquad t \in [t_0, T)\,.$$

In view of hypothesis (i), if we denote

$$K_2 = V(x(t_0), y(t_0)) + K_1 - \min_{|x| \leqslant K} \{G(x)\}$$

then we obtain that

$$\frac{y^2(t)}{2} \leqslant K_2 + \int_{t_0}^{t} |\tilde{e}(s)\, y(s)|\, ds\,, \qquad t \in [t_0, T)\,.$$

Using now hypothesis (ii), we get

$$y^2(t) \leqslant 2K_2 + 2\int_{t_0}^{t} \{r_1(s) + r_2(s)\, z(|y(s)|)\}\, |y(s)|\, ds\,, \qquad t \in [t_0, T)\,.$$

We will prove that $y(t)$ is bounded on $[t_0, T)$.

If $R_i = \max_{t_0 \leqslant t \leqslant T} \{r_i(t)\}$, $i = 1,2$ we obtain that

$$y^2(t) \leqslant 2K_2 + 2R_1 \int_{t_0}^{t} |y(s)|\, ds + 2R_2 \int_{t_0}^{t} w(y^2(s))\, ds\,, \qquad t \in [t_0, T)\,,$$

(in view of (ii)), thus (since $2\,|y(s)| \leqslant 1 + y^2(s)$)

$$y^2(t) \leqslant 2K_2 + R_1 T + R_1 \int_{t_0}^{t} y^2(s)\, ds + 2R_2 \int_{t_0}^{t} w(y^2(s))\, ds \leqslant$$

$$\leqslant 2K_2 + R_1 T + (R_1 + 2R_2) \int_{t_0}^{t} (y^2(s) + w(y^2(s)))\, ds\,, \qquad t \in [t_0, T)\,.$$

Let us define

$$W:(0,\,\infty)\to R,\qquad W(x)=\int\limits_1^x \frac{ds}{s+w(s)},\qquad x>0.$$

By Corollary 2.2 and Theorem 2.1 we deduce that $\lim\limits_{x\to\infty} W(x)=\infty$.

By Bihari's inequality [10] we get that

$$y^2(t)\leqslant W^{-1}(W(2K_2+R_1T)+(R_1+2R_2)(t-t_0)),\qquad t\in[t_0,\,T),$$

and therefore $y(t)$ is bounded on $[t_0,\,T)$.

Taking into account the relation

$$x(t)=x(t_0)+\int\limits_{t_0}^t y(s)\,ds-\int\limits_{t_0}^t F(x(s))\,ds,\qquad t\in[t_0,\,T),$$

we get

$$|x(t)|\leqslant|x(t_0)|+\int\limits_{t_0}^t|y(s)|ds+K_3(T-t_0),\qquad t\in[t_0,\,T),$$

where $K_3=\max\limits_{|x|\leqslant K}\{|F(x)|\}$, thus $x(t)$ is bounded on $[t_0,\,T)$.

We have so that $(z(t),y(t))$ is continuable up to T.

This completes the proof of Theorem 2.5.

COROLLARY 2.5 [52]. – *Suppose that the following conditions hold:*

(i) *there exists a constant* $K>0$ *such that*

$$xF(x)>0\quad\textit{and}\quad xg(x)>0\quad\textit{for}\ |x|\geqslant K;$$

(ii) *there exist continuous functions* $r_1,\,r_2\colon R_+\to R_+$ *such that*

$$|e(t,\,x,\,y-F(x))|\leqslant r_1(t)+r_2(t)|y|,\qquad t\in R_+,\ x,\,y\in R.$$

Then the solutions of (2.4) *are defined in the future.*

As a particular case of Corollary 2.5 (if $e(t,x,y)$ depends only on t) we have a continuability result of GRAEF [38] which extends a result of BUSHAW [16].

The relation of Theorem 2.5 with the result of NAGABUCHI and YAMAMOTO [52] is given by

EXAMPLE 2.5. – Let us consider in equation (2.4) the perturbing term of the form

$$e(t,\,x,\,y)=(F(x)+y)\ln(1+(F(x)+y)^2)+r(t)$$

where $r: R_+ \to R_+$ is continuous. If there exists $K > 0$ such that

$$xF(x) > 0 \quad \text{and} \quad xg(x) > 0 \quad \text{for } |x| \geq K$$

we have that all the solutions of (2.4) are defined in the future (applying Theorem 2.5 with $w(r) = r\ln(1+r)$, $r \in R_+$). We can not apply the result from [52].

We will investigate now the continuability of solutions for the forced Liénard equation

$$x'' + f(x)x' + g(x) = e(t) \tag{2.5}$$

with all functions continuous without making the usual assumption ([14], [38])

$$xg(x) > 0 \quad \text{for } |x| \geq K.$$

Define

$$F(x) = \int_0^x f(s)\,ds, \qquad G(x) = \int_0^x g(s)\,ds, \qquad x \in R.$$

Using the Liénard transformation we write (2.5) as the system

$$\begin{cases} x' = y - F(x), \\ y' = -g(x) + e(t). \end{cases} \tag{2.5}$$

THEOREM 2.6. – *Assume that for some positive number P we have*

$$G(x) \geq -P, \qquad x \in R,$$

$$g(x)F(x) \geq -w(G(x) + P + 1), \qquad x \in R,$$

$$|F(x)| \leq g(x)F(x) + w(G(x) + P + 1) + z(|x|), \qquad x \in R,$$

where $w, z: R_+ \to R_+$ are continuous nondecreasing functions satisfying the conditions of Theorem 2.3.

Then every solution of (2.5) exists in the future.

PROOF. – Let $V(x, y) = y^2/2 + G(x) + |x| + P + 1$, $x, y \in R$. We have that

$$\frac{dV}{dt}_{(2.5)} \leq -g(x)F(x) + ye(t) + |y - F(x)| \leq$$

$$\leq y^2 + \frac{e^2(t)+1}{2} + w(G(x) + P + 1) + z(|x|) \leq$$

$$\leq 2V(x, y) + w(V(x, y)) + z(V(x, y)) + \frac{e^2(t)+1}{2}.$$

By Theorem 2.3 we have

$$\int_1^\infty \frac{ds}{w(s) + z(s)} = \infty$$

and by Corollary 2.2 and Theorem 2.1 we deduce that

$$\int_1^\infty \frac{ds}{2s + w(s) + z(s)} = \infty .$$

Using the comparison method we obtain that the solutions of (2.5) are continuable in the future since

$$V(x, y) \to \infty \quad \text{as } |x| + |y| \to \infty .$$

This completes the proof of Theorem 2.6.

COROLLARY 2.6 [13]. – *Assume that for some positive numbers P and Q we have*

$$G(x) \geqslant -P, \qquad x \in R,$$

$$g(x)F(x) \geqslant -Q, \qquad x \in R,$$

$$|F(x)| \leqslant g(x)F(x) + Q + z(|x|), \qquad x \in R,$$

where $z: R_+ \to (0, \infty)$ is some nondecreasing continuous function such that

$$\int_0^\infty \frac{ds}{s + z(s)} = \infty .$$

Then every solution of (2.5) *exists in the future.*

PROOF. – We have that

$$\int_1^\infty \frac{ds}{z(s)} \geqslant \int_1^\infty \frac{ds}{s + z(s)} = \infty$$

so that we can apply Theorem 2.6 with $w(r) = Q$, $r \in R_+$.

REMARK 2.2. – *In view of Corollary* 2.2 *we have that the condition* $\int_0^\infty ds/(s + z(s)) = \infty$ *is equivalent (under the assumption that $z: R_+ \to (0, \infty)$ is nondecreasing) with the condition* $\int_1^\infty ds/(z(s)) = \infty$.

The relation of our result with the result of BURTON [13] is given by

Example 2.6. – Consider the system (2.5) with

$$g(x) = x\,, \qquad F(x) = \ln(1 + x^2)\,, \qquad G(x) = \frac{x^2}{2}\,, \qquad x \in R\,.$$

We can not apply the result of Burton [13] since

$$\lim_{x \to -\infty} g(x)F(x) = -\infty$$

but we can apply Theorem 2.6 considering the functions $w(r) = 2r\ln(1 + 2r) + \ln 2$, $r \in R_+$, and $z(r) = 1$, $r \in R_+$.

An improvement of Conti's continuability theorem [22, 23] was given by Hara, Yoneyama and Sugie [44] using two Lyapunov functions which are not radially unbounded for fixed t. They applied the result to the Liénard equation without making the assumption that $xg(x) > 0$ for $|x| \geqslant K$.

We will show that using Theorem 2.3 and the continuation theorem of Hara, Yoneyama and Sugie [44] we can give a more flexible result for the continuability of solutions of (2.5) than the work done in [44].

We suppose that F and g are regular enough to ensure uniqueness of the solutions of the differential equation (2.5).

Theorem 2.7. – *Suppose that for some positive number P we have*

$$G(x) \geqslant -P\,, \qquad x \in R\,,$$

$$g(x)F(x) \geqslant -w(G(x) + P + 1)\,, \qquad x \in R\,,$$

$$\int_0^\infty \frac{ds}{1 + \max\{0, -F(s)\}} = \infty\,, \qquad \int_{-\infty}^0 \frac{ds}{1 + \max\{0, F(s)\}} = \infty$$

where $w\colon R_+ \to R_+$ is continuous, nondecreasing and $w \in \mathfrak{R}_0$.

Then every solution of (2.5) exists in the future.

Proof. – We define $F_-(x) = \max\{0, -F(s)\}$, $F_+ = \max\{0, F(s)\}$ for $x \in R$. Let $V(x, y) = y^2/2 + G(x) + P + 1$ and $W(x, y) = |x|$ for $x, y \in R$. We have that

$$V(x, y) \to \infty \quad \text{as } |y| \to \infty \text{ uniformly in } x\,;$$

$$W(x, y) \to \infty \quad \text{as } |x| \to \infty \text{ for each fixed } y \in R\,.$$

On the other hand

$$\frac{dV}{dt}_{(2.5)} \leqslant -g(x)F(x) + |ye(t)| \leqslant$$

$$\leqslant \frac{y^2}{2} + \frac{e^2(t)}{2} + w(G(x)+P+1) \leqslant V(x, y) + w(V(x, y)) + \frac{e^2(t)}{2}$$

and by Corollary 2.2 and Theorem 2.1 we have that

$$\int_1^\infty \frac{ds}{s + w(s)} = \infty .$$

If $x \geqslant 0$ and $|y| \leqslant K$, then

$$\frac{dW}{dt}_{(2.5)} = y - F(x) \leqslant K + F_-(x) = K + F_-(W(x, y)) .$$

Similarly, if $x \leqslant 0$ and $|y| \leqslant K$, then

$$\frac{dW}{dt}_{(2.5)} \leqslant K + F_+(-W(x, y)) .$$

Since

$$\int_0^\infty \frac{ds}{1 + F_-(s)} = \infty , \qquad \int_{-\infty}^0 \frac{ds}{1 + F_+(s)} = \infty$$

we have (in view of Lemma 2.1 and Theorem 2.1) that every solution of

$$r' = K + F_-(r)$$

and

$$r' = K + F_+(-r)$$

exists in the future.

By the continuation theorem of HARA, YONEYAMA and SUGIE [44] we deduce that the solutions of (2.5) exist in the future.

COROLLARY 2.7 [44]. – *Suppose that there exist some positive numbers P and Q such that*

$$G(x) \geqslant -P , \qquad x \in R ,$$

$$g(x)F(x) \geqslant -Q(G(x) + P + 1) , \qquad x \in R ,$$

$$\int_0^{\infty} \frac{ds}{1+F_-(s)} = \infty\,, \qquad \int_{-\infty}^{0} \frac{ds}{1+F_+(s)} = \infty\,.$$

Then every solution of (2.5) *exists in the future.*

As a particular case of Corollary 2.7 we have a continuability result of GRAEF [38].

The relation of Theorem 2.7 with the result of HARA, YONEYAMA and SUGIE [44] is given by

EXAMPLE 2.7. – Consider the system (2.5) with

$$g(x) = x\,, \qquad F(x) = -x\ln(1+|x|)\,, \qquad G(x) = \frac{x^2}{2}\,, \qquad x \in R\,,$$

we can not apply Theorem 5.1 [44] but we can apply our result with $w(r) = 2r\ln(1 + 2r) + \ln 2$, $r \in R_+$.

Let us now consider the problem of the continuability of solutions for the differential equation

$$(a(t)\,x')' + q(t)\,f(x)\,g(x') = r(t) \tag{2.6}$$

where $a, q, r: R_+ \to R, f, g: R \to R$, $a(t) > 0$, $q(t) > 0$, $g(x) > 0$, r, f and g are continuous and a, q are differentiable.

We will write equation (2.6) as the system

$$\begin{cases} x' = y\,, \\ y' = \dfrac{-a'(t)\,y - q(t)f(x)g(y) + r(t)}{a(t)}\,. \end{cases} \tag{2.6}$$

Let $q'(t)_+ = \max\{q'(t), 0\}$ and $q'(t)_- = \max\{-q'(t), 0\}$ so that we have $q'(t) = q'(t)_+ - q'(t)_-$. A similar decomposition holds for $a(t)$.

We define

$$F(x) = \int_0^x f(s)\,ds\,, \qquad G(x) = \int_0^x \frac{s}{g(s)}\,ds\,, \qquad x \in R\,,$$

$$p(t) = \exp\left(-\int_0^t \frac{q'(s)_-}{q(s)}\,ds\right), \qquad t \in R_+\,,$$

$$b(t) = \exp\left(-\int_0^t \frac{a'(s)_-}{a(s)}\,ds\right), \qquad t \in R_+\,.$$

THEOREM 2.8. – *Assume that there exist nonnegative constants m and n such that*

$$\frac{|y|}{g(y)} \leqslant m + nw(G(y)), \qquad y \in R,$$

where $w: R_+ \to R_+$ is a nondecreasing continuous function and $w \in \mathfrak{R}_0$.

If $a'(t) \geqslant 0$, $F(x)$ is bounded from below and $G(y) \to \infty$ as $|y| \to \infty$ then all solutions of (2.6) *are defined for all $t \geqslant 0$.*

PROOF. – Let $K > 0$ be such that

$$F(x) + K \geqslant 0, \qquad x \in R.$$

Suppose that there exists a solution $(x(t), y(t))$ of (2.6) and a $T > 0$ such that

$$\lim_{t \to T^-} \{|x(t)| + |y(t)|\} = \infty .$$

Define

$$V(t, x, y) = p(t)\left[\frac{F(x) + K}{a(t)} + \frac{G(y)}{q(t)}\right], \qquad t \in R_+, \ x, y \in R.$$

We have that

$$V'(t) = p(t)\left\{-\frac{(F(x) + K)\,a'(t)}{a^2(t)} + \frac{f(x)\,x'}{a(t)} - \frac{G(y)\,q'(t)}{q^2(t)} + \right.$$

$$\left. + \frac{yy'}{g(y)\,q(t)} - \frac{(F(x) + K)\,q'(t)_-}{a(t)\,q(t)} - \frac{G(y)\,q'(t)_-}{q^2(t)}\right\} \leqslant$$

$$\leqslant p(t)\left\{-\frac{G(y)[q'(t) + q'(t)_-]}{q^2(t)} + \frac{r(t)\,y}{g(y)\,q(t)\,a(t)}\right\} \leqslant \frac{p(t)\,r(t)}{q(t)\,a(t)}\,\frac{|y|}{g(y)} \leqslant$$

$$\leqslant m\frac{p(t)\,r(t)}{q(t)\,a(t)} + n\frac{p(t)\,r(t)}{q(t)\,a(t)}w(G(y)).$$

Denoting

$$M = \sup_{t \in [0, T]}\left\{\frac{q(t)}{p(t)}\right\}$$

we observe that

$$G(y) \leqslant \frac{q(t)}{p(t)}V(t, x, y) \leqslant MV(t, x, y), \qquad t \in [0, T], \ x, y \in R.$$

We obtain that

$$V'(t) \leqslant m\frac{p(t)\,r(t)}{q(t)\,a(t)} + n\frac{p(t)\,r(t)}{q(t)\,a(t)}w(MV(t)), \qquad t \in [0, T).$$

Since $w \in \mathfrak{R}_0$ is nondecreasing we obtain by Bihari's inequality [10] the existence of a constant $K_1 > 0$ such that

$$V(t, x(t), y(t)) \leqslant K_1, \qquad t \in [0, T).$$

We deduce that $y(t) = x'(t)$ is bounded on $[0, T)$ and an integration yields the boundedness of $x(t)$ in $[0, T)$ contradicting the assumption that $(x(t), y(t))$ was a solution of (2.6) with finite escape time.

This completes the proof of Theorem 2.8.

COROLLARY 2.8 [39]. – *Assume that there exist nonnegative constants m and n such that*

$$\frac{|y|}{g(y)} \leqslant m + nG(y), \qquad y \in R.$$

If $a'(t) \geqslant 0$, $F(x)$ is bounded from below and $G(y) \to \infty$ as $|y| \to \infty$ then all solutions of (2.6) *are defined in the future.*

The relation of our result with the result of GRAEF and SPIKES [39] is given by

EXAMPLE 2.8. – Let us consider equation (2.6) with

$$a(t) = q(t) = e^t, \qquad r(t) = 0, \qquad t \in R_+,$$

$$f(x) = \frac{1}{x^2+1}, \qquad g(x) = \frac{1}{2}e^{-x^2}, \qquad x \in R.$$

We observe that in this case we have

$$G(y) = e^{y^2} - 1, \qquad y \in R,$$

so that there are no positive constants m and n with

$$\frac{|y|}{g(y)} = 2|y|e^{y^2} \leqslant m + n(e^{y^2} - 1) = m + nG(y), \qquad y \in R.$$

This makes impossible an application of Corollary 2.8.

We see that we can apply Theorem 2.8 with

$$w\colon R_+ \to R_+, \qquad w(r) = r\ln(1+r),$$

$$m = 6e^9 + 9, \qquad n = 1,$$

since

$$2|y|e^{y^2}+y^2 \leqslant m\,, \qquad |y|\leqslant 3\,,$$

$$2|y|e^{y^2}+y^2 \leqslant 2|y|e^{y^2}+e^{y^2} \leqslant y^2e^{y^2}, \qquad |y|\geqslant 3\,,$$

thus

$$2|y|e^{y^2} \leqslant m + y^2e^{y^2} - y^2 = m + w(e^{y^2}-1)\,, \qquad y\in R\,.$$

The requirement that $a'(t)\geqslant 0$ in Theorem 2.8 can be dropped by imposing a stronger condition on $g(y)$:

THEOREM 2.9. – *Assume that $F(x)$ is bounded from below, $G(y)\to\infty$ as $|y|\to\infty$ and there are positive constants M and k such that*

$$\frac{y^2}{g(y)} \leqslant Mw(G(y))\,, \qquad |y|\geqslant k\,,$$

where $w\colon R_+\to R_+$ is a nondecreasing continuous function and $w\in\mathfrak{R}_0$.
Then all the solutions of (2.6) *are defined in the future.*

PROOF. – Let $-K>0$ be a lower bound for $F(x)$.
We define

$$V(t,x,y) = b(t)p(t)\left[\frac{F(x)+K}{a(t)} + \frac{G(y)}{q(t)}\right], \qquad t\in R_+\,,\;\; x,y\in R\,.$$

Suppose there is a solution $(x(t), y(t))$ of (2.6) and a $T>0$ such that

$$\lim_{t\to T^-}\{|x(t)|+|y(t)|\} = \infty\,.$$

Along this solution we have

$$V'(t) = b(t)p(t)\left\{-\frac{(F(x)+K)\,a'(t)}{a^2(t)} + \frac{f(x)\,y}{a(t)} - \frac{G(y)\,q'(t)}{q^2(t)} - \right.$$

$$\left. -\frac{a'(t)\,y^2}{g(y)\,q(t)\,a(t)} - \frac{f(x)\,y}{a(t)} + \frac{r(t)\,y}{g(y)\,q(t)\,a(t)} - \left[\frac{(F(x)+K)}{a(t)} + \frac{G(y)}{q(t)}\right]\left(\frac{a'(t)_-}{a(t)} + \frac{q'(t)_-}{q(t)}\right)\right\} \leqslant$$

$$\leqslant b(t)p(t)\left\{-\frac{a'(t)}{q(t)\,a(t)}\,\frac{y^2}{g(y)} + \frac{r(t)}{q(t)\,a(t)}\,\frac{|y|}{g(y)}\right\}.$$

If $|y|\leqslant\max\{k,1\}$ we have $(y^2/g(y))\leqslant D$ for some $D>0$, so that

$$\frac{y^2}{g(y)} \leqslant D + Mw(G(y))\,, \qquad y\in R\,.$$

If $|y| \leqslant \max\{k, 1\}$ we have $(|y|/g(y)) \leqslant D_1$ and if $|y| \geqslant \max\{k, 1\}$ we have $|y|/g(y) \leqslant y^2/g(y)$ thus

$$\frac{|y|}{g(y)} \leqslant D_1 + \frac{y^2}{g(y)} \leqslant D_1 + D + Mw(G(y)), \qquad y \in R.$$

We obtain that

$$V'(t) \leqslant \left(\frac{b(t)\,p(t)\,|a'(t)|}{q(t)\,a(t)} + \frac{b(t)\,p(t)\,|r(t)|}{q(t)\,a(t)} \right) \left(\frac{|y|}{g(y)} + \frac{y^2}{g(y)} \right) \leqslant$$

$$\leqslant \left(\frac{b(t)\,p(t)\,|a'(t)|}{q(t)\,a(t)} + \frac{b(t)\,p(t)\,|r(t)|}{q(t)\,a(t)} \right) (D_1 + 2D + 2Mw(G(y))).$$

In a similar way to the final part of the proof of Theorem 2.8 we show that we obtain a contradiction.

This shows that the solutions of (2.6) are defined in the future.

COROLLARY 2.9 [39]. – *Assume that $F(x)$ is bounded from below, $G(y) \to \infty$ as $|y| \to \infty$ and there are positive constants M and k such that*

$$\frac{y^2}{g(y)} \leqslant MG(y), \qquad |y| \geqslant k.$$

Then all the solutions of (2.6) *are defined in the future.*

As a particular case of Corollary 2.9 we obtain a continuability result of BURTON and GRIMMER [15].

The relation of our result with the results from [15, 39] is given by

EXAMPLE 2.9. – Let us consider equation (2.6) with

$$a(t) = q(t) = t^2 + 1, \qquad r(t) = t, \qquad t \in R_+,$$

$$f(x) = \frac{1}{x^2+1}, \qquad g(x) = \frac{1}{3} e^{-x^3}, \qquad x \in R.$$

We have that

$$\frac{y^2}{g(y)} = 3y^2 e^{y^3}, \qquad G(y) = e^{y^3} - 1, \qquad y \in R,$$

so that we can not apply the results from [15, 39] but we can apply Theorem 2.9 with $w(r) = r \ln(1 + r)$, $r \in R_+$ since

$$3y^2 e^{y^3} \leqslant w(G(y)), \qquad |y| \geqslant 4.$$

REMARK 2.3. – *We can formulate results similar to Theorem* 2.2 *and Theorem* 2.3 *for the concept of z-continuability (introduced by Campanini in* [18]) *using the same method. This allows to formulate criteria for the global existence of solutions of ordinary differential equations (see* [18]).

3. – Global existence of solutions for delay and functional differential equations.

In this section we will apply the methods developped in Section 2 in order to give some continuation results for delay and functional differential equations.

Let us first consider the delay differential equation

$$x' = f(t, x) + g(t, x(t - \tau(t))) \tag{3.1}$$

where $f, g: R_+ \times R^n \to R^n$, $\tau: R_+ \to R_+$ are continuous functions.

For $t_0 \geq 0$ the initial interval at t_0 is given by

$$E_{t_0} = \{t_0\} \cup \{s: s = t - \tau(t) \leq t_0 \ \text{ for } t \geq t_0\}.$$

We suppose that E_{t_0} is bounded for every $t_0 \in R_+$.

For any initial continuous function $x_0: E_{t_0} \to R^n$ we say that $x(t)$ is a solution of (3.1) on $[t_0, T)$ if $x(t)$ is continuous on $E_{t_0} \cup [t_0, T)$ and satisfies (3.1) on (t_0, T) with $x(t) = x_0(t)$ for $t \in E_{t_0}$. If f, g are continuous it is known (see DRIVER [33], HALE [41]) that for every $t_0 \geq 0$ and every initial continuous function $x_0: E_{t_0} \to R^n$ equation (3.1) admits a solution on some interval $[t_0, T)$ with $t_0 < T < \infty$.

THEOREM 3.1. – *Suppose that there exist continuous functions $\phi, \psi: R_+ \to R_+$ and w, z satisfying the conditions of Theorem* 2.3 *with w, z nondecreasing on R_+ such that*

$$|f(t, x)| \leq \phi(t) z(|x|), \qquad |g(t, x)| \leq \psi(t) w(|x|), \qquad (t, x) \in R_+ \times R^n.$$

Then the solutions of (3.1) *exist in the future.*

PROOF. – Suppose that there exist a noncontinuable solution $x(t)$ of (3.1). Let $r(t) = |x(t)|$. We have then that

$$r'(t) \leq |x'(t)| \leq \phi(t) z(r(t)) + \psi(t) w(r(t - \tau(t))).$$

If $r(t)$ is not continuable to $T < \infty$, then

$$r(t) \to \infty \qquad \text{as } t \to T^-.$$

Let us prove that $\tau(T) = 0$.

Suppose $\tau(T) > 0$. Then there exists $\tilde{T} \in [t_0, T)$ such that $t - \tau(t) \leq \tilde{T}$ for all $t \in [t_0, T]$. We have thus that $r(t - \tau(t))$ is bounded on $[t_0, T]$ since $r(t)$ is bounded on $E_{t_0} \cup [t_0, \tilde{T}]$.

If we denote

$$M = \sup_{t_0 \leq t \leq T} \{\phi(t)\}, \qquad L = \sup_{t_0 \leq t \leq T} \{\psi(t)\, w(r(t - \tau(t)))\}$$

then

$$r'(t) \leq Mz(r(t)) + L .$$

Since $z \in \mathfrak{N}_0$ is nondecreasing we have by Corollary 2.2 that the solution of the differential equation

$$y' = Mz(y) + L, \qquad y(t_0) = r(t_0),$$

and defined in the future which contradicts the relation

$$r(t) \to \infty \qquad \text{as } t \to T^- .$$

We have so that $\tau(T) = 0$. Thus, there exists $t_1 \in [t_0, T)$ such that $t - \tau(t) \geq t_0$ on $[t_1, T]$. If we denote

$$K = \sup_{t_1 \leq t \leq T} \{\phi(t) + \psi(t)\}$$

we have for $t_1 \leq t < T$ that

$$r'(t) \leq \phi(t)\, z(r(t)) + \psi(t)\, w(r(t)) \leq K(z(r(t)) + w(r(t))) .$$

By Theorem 2.3 we have that $w + z \in \mathfrak{N}_0$ thus the solutions of the differential equation

$$y' = K(z(y) + w(y)), \qquad y(t_0) = r(t_0),$$

are defined in the future. By Conti's comparison method (see [23]) we obtain a contradiction with the relation

$$r(t) \to \infty \qquad \text{as } t \to T^- .$$

Thus every solution of (3.1) is defined in the future,

COROLLARY 3.1 [45]. – *Suppose that there exist continuous functions ϕ, ψ: $R_+ \to R_+$ and a nondecreasing function $w \in \mathfrak{N}_0$ such that*

$$|f(t, x)| \leq \phi(t)|x| + \psi(t)w(|x|), \qquad (t, x) \in R_+ \times R^n .$$

Then the solutions of (3.1) *exist in the future.*

PROOF. – By Corollary 2.2 we have if z: $R_+ \to R_+$, $z(r) = r$ and if $w \in \mathfrak{N}_0$ is nondecreasing, then z, w satisfy the conditions of Theorem 2.3.

We give now sufficient conditions for the continuability of solutions of the generalized Liénard system with time delay

$$\begin{cases} x' = y - F(x), \\ y' = g(t, x(t - \tau(t))), \end{cases} \tag{3.2}$$

where $F\colon R \to R$, $g\colon R_+ \times R \to R$, $\tau\colon R_+ \to R_+$ are continuous functions with E_{t_0} bounded for every $t_0 \in R_+$.

THEOREM 3.2. – *Suppose that there exists w, z nondecreasing on R_+ satisfying the conditions of Theorem 2.3 and $\psi\colon R_+ \to R_+$ continuous such that*

$$|F(x)| \leq z(|x|), \qquad x \in R,$$

$$|g(t, x)| \leq \psi(t)\, w(|x|), \qquad t \in R_+, \ x \in R.$$

Then the solutions of (3.2) exist in the future.

PROOF. – We have that

$$(y - F(x))^2 \leq 2y^2 + 2F^2(x) \leq 2x^2 + 2y^2 + 2z^2(\sqrt{x^2 + y^2}), \qquad x, y \in R,$$

thus

$$|(y - F(x), 0)| \leq 2|(x, y)| + 2z(|(x, y)|), \qquad x,\ y \in R,$$

$$|(0, -g(t, x))| \leq \psi(t)\, w(|(x, y)|), \qquad t \in R_+, \ x, y \in R.$$

By Theorem 2.3 we have that $z + w \in \mathfrak{N}_0$. Since $z + w$ is nondecreasing on R_+, by Corollary 2.2 and Theorem 2.1 we deduce that

$$\int_{\delta}^{\infty} \frac{ds}{s + w(s) + z(s)} = \infty .$$

A repetition of the arguments of the proof of Theorem 3.1 enables us to deduce that the solutions of (3.2) are defined in the future.

The continuability of solutions of (3.2) was also investigated by SUGIE [68] bu Sugie's result is not applicable in the case

$$\{t > 0\colon \tau(t) = 0\} \neq \emptyset .$$

EXAMPLE 3.1. – Consider the system

$$\begin{cases} x' = y - x\ln(1 + |x|), \\ y' = x(t - \tau(t)), \end{cases}$$

where $\tau(t) = \min\{t, 1\}$, $t \in R_+$.

We can apply Theorem 3.2 with

$$z\colon R_+ \to R_+, \qquad z(r) = r\ln(1 + r),$$

$$w\colon R_+ \to R_+, \qquad w(r) = r,$$

but we can not apply the result of SUGIE [68].

We present now a continuation result for the functional differential equation

$$x' = f(t, x) + g(t, x_t) \tag{3.3}$$

where $f\colon R_+ \times R^n \to R^n$, $g\colon R_+ \times C^q \to R^n$ are continuous, $C^q = C([-q, 0], R^n)$ for $q \geq 0$, and $x_t(s) = x(t+s)$ for $s \in [-q, 0]$.

Assume that there exist $\phi, \psi, w, z\colon R_+ \to R_+$ continuous such that

$$|f(t, x)| \leq \phi(t)\, z(|x|), \qquad t \in R_+, \ x \in R^n,$$

$$|g(t, x)| \leq \psi(t)\, w(\|x\|), \qquad t \in R_+, \ x \in C^q,$$

where

$$\|x\| = \sup_{s \in [-q, 0]} \{|x(s)|\}, \qquad x \in C^q.$$

We have the following result

THEOREM 3.3. – *Suppose that $w \in \Re_0$ is nondecreasing and z, w satisfy the condition of Theorem* 2.3. *Then the solutions of* (3.3) *are defined in the future.*

PROOF. – We first show that the solutions of

$$r' = \phi(t)\, z(r) + \psi(t)\, w(\|r_t\|) \tag{3.4}$$

are defined in the future.

Suppose that there exists $t_0 \geq 0$ a continuous initial function $r_0\colon [-q, 0] \to R_+$ and a noncontinuable solution $r(t) = r(t, t_0, r_0)$ of (3.4). Then $r(t) \to \infty$ as $t \to T^-$ for some $T > t_0$ since $r(t)$ is nondecreasing for $t \geq t_0$.

If $t_1 \in [t_0, T)$ is such that

$$r(t_1) \geq \sup_{s \in [-q, 0]} r_0(s)$$

we have that

$$\|r_t\| = r(t), \qquad t_1 \leq t < T,$$

thus

$$r'(t) = \phi(t)\, z(r(t)) + \psi(t)\, w(r(t)), \qquad t_1 \leq t < T.$$

By the hypothesis and by Theorem 2.3 we obtain that $r(t)$ is bounded on $[t_1, T)$ which is a contradiction.

Let now $z(t)$ be a solution of (3.3). If we define $r(t) = |x(t)|$ we have

$$r'(t) \leq \phi(t)\, z(r(t)) + \psi(t)\, w(\|r_t\|)\,.$$

Since the solutions of (3.4) are defined in the future we deduce by the comparison theorem that $x(t)$ is continuable in the future.

This completes the proof of Theorem 3.3.

COROLLARY 3.2 [45]. – *Suppose that* $w \in \mathfrak{R}_0$ *is nondecreasing and* $z(r) = Lr$, $r \in R_+$, *for some* $L > 0$. *Then the solutions of* (3.3) *are defined in the future.*

PROOF. – The result follows from Corollary 2.2 and the preceding theorem.

4. – Continuation results for differential equations in abstract spaces.

In this section we will give a result on the existence and uniqueness of solutions of differential equations in Banach spaces of infinite dimension.

Let X be a Banach space with norm $|\cdot|$ and let $[0, a] \subset R_+$ be a closed interval.

THEOREM 4.1. – *Let* $f\colon [0, a] \times X \to X$ *be a continuous mapping satisfying the following conditions*:

(i) *for every bounded set* $B \subset X$ *there exists a constant* $L(B) > 0$ *such that*

$$|f(t, x) - f(t, y)| \leq L(B)|x - y|\,, \qquad t \in [0, a]\,, \quad x, y \in B\,;$$

(ii) *there is a map* $w \in \mathfrak{R}_0$ *and a continuous function* $\phi \in C([0, a], R_+)$ *such that*

$$|f(t, x)| \leq \phi(t)\, w(|x|)\,, \qquad t \in [0, a]\,, \quad x \in X\,.$$

Then, for every $x_0 \in X$, *the differential equation*

$$x' = f(t, x) \tag{4.1}$$

with initial condition $x(0) = x_0$, *has a unique solution* $x\colon [0, a] \to X$.

PROOF. – By (i) we deduce (see [49] that the equation (4.1) with initial condition $x(0) = x_0$ has a solution $x(t)$ defined for $t \in [0, b)$ for some $b \in (0, a]$.

Let $y(t) = |x(t)|$, $t \in [0, b)$. We have then that

$$\limsup_{h \to 0^+} \frac{y(t+h) - y(t)}{h} \leq |x'(t)| \leq \phi(t)\, w(y(t))\,, \qquad t \in (0, b)\,,$$

and by the comparison method we obtain that

$$y(t) \leq r(t, 0, |x_0|)\,, \qquad t \in [0, b)\,,$$

where $r(t, 0, |x_0|)$ denotes the maximal solution of the differential equation

$$r' = \phi(t)\, w(r)$$

with initial condition $r(0) = |x_0|$ (since $w \in \mathfrak{R}_0$ we have that this maximal solution is defined on $[0, a]$).

Let $M > 0$ be such that

$$r(t, 0, |x_0|) \leqslant M, \qquad t \in [0, b].$$

We obtain

$$|f(t, x(t))| \leqslant \phi(t) \sup_{0 \leqslant u \leqslant M} \{w(u)\}, \qquad t \in [0, b),$$

so that there exists a constant $M_1 > 0$ with

$$|f(t, x(t))| \leqslant M_1, \qquad t \in [0, b).$$

We deduce that for $0 < t_1 \leqslant t_2 < b$,

$$|x(t_2) - x(t_1)| \leqslant M_1 |t_2 - t_1|$$

and therefore $\lim_{t \to b^-} x(t) = x_1 \in X$ exists.

Hypothesis (i) guarantees that there exists a local solution, on some interval $[b, b_1)$ with $b_1 \leqslant a$, of (4.1) with initial condition $x(b) = x_1$. By the uniqueness we have that $x(t)$ may be continued (in a unique way) up to b_1.

This shows that the maximal interval of definition of $x(t)$ is $[0, a]$ and the proof is completed.

REMARK 4.1. – *We see from the proof that condition* (i) *can be replaced by any condition guaranteeing the local existence and uniqueness for* (4.1). *We can not exclude such condition since it is known that if for every continuous function* $f: [0, a] \times X \to X$ *and every* $x_0 \in X$ *the equation* (4.1) *with initial condition* $x(0) = x_0$ *has a solution defined in some neighborhood of zero, then* X *is finite dimensional* (*see* GODUNOV [37]). *We observe also that instead of* $[0, a]$ *we can consider* R_+, *obtaining that all solutions of* (4.1) *are defined in the future.*

COROLLARY 4.1 [2]. – *Let* $f: [0, a] \times X \to X$ *be a continuous mapping satisfying the following conditions*:

(i) *for every bounded set* $B \subset X$ *there exists a constant* $L(B) > 0$ *such that*

$$|f(t, x) - f(t, y)| \leqslant L(B)|x - y|, \qquad t \in [0, a], \quad x, y \in B;$$

(ii) *there is a nondecreasing map* $w \in \mathfrak{R}_0$ *such that*

$$|f(t, x)| \leqslant w(|x|), \qquad t \in [0, a], \quad x \in X.$$

Then, for every $x_0 \in X$, *the differential equation* (4.1) *with initial condition* $x(0) = x_0$, *has a unique solution* $x: [0, a] \to X$.

REMARK 4.2. – *Note that the monotonicity of w is not essential in Corollary* 4.1.

COROLLARY 4.2 [57]. – *Let $f: [0, a] \times X \to X$ be a continuous mapping for which its partial derivative with respect to the second argument, denoted by f'_x, exists and is continuous. Suppose that there exist a constant $c > 0$ and a continuous increasing function $w: R_+ \to (1, \infty)$, which satisfy the following conditions*:

$$\int_0^\infty \frac{ds}{w(s)} = \infty ,$$

$$\|f'_x(t, x)\| \leqslant c \ln (w(|x|)), \qquad t \in [0, a], \quad x \in X.$$

Then, for every $x_0 \in X$, the differential equation (4.1) *with initial condition $x(0) = x_0$, has a unique solution $x: [0, a] \to X$.*

PROOF. – We have that the derivative f'_x is bounded on every bounded set $B \subset X$ thus condition (i) of Theorem 4.1 is satisfied.

On the other hand we have

$$|f(t, x)| \leqslant |f(t, 0)| + \sup_{|y| \leqslant |x|} \{\|f'_y(t, y)\|\} |x| \leqslant$$

$$\leqslant \max_{s \in [0, a]} \{|f(s, 0)|\} + c\{\ln (w(|x|))\} |x| = M + c\{\ln (w(|x|))\} |x|, \quad t \in [0, a], \quad x \in X.$$

If w is bounded on R_+ by a constant $K > 0$, we obtain that

$$|f(t, x)| \leqslant M + c \ln (K) |x|, \qquad t \in [0, a], \quad x \in X,$$

and thus condition (ii) of Theorem 4.1 is fulfilled.

If w is unbounded on R_+, we choose (as ALEXANDROV and DAIRBEKOV did in [2]) a sequence $0 < r_1 < r_2 \dots$ such that $w(r_1) > 1$, $w(r_j) \leqslant r_j^2$ and $\ln (r_j) > j \ln (r_{j-1})$, $j \geqslant 2$. We have then

$$\int_{r_1}^\infty \frac{ds}{s \ln (w(s))} = \sum_{j=2}^\infty \int_{r_{j-1}}^{r_j} \frac{ds}{s \ln (w(s))} \geqslant \sum_{j=2}^\infty \int_{r_{j-1}}^{r_j} \frac{ds}{s \ln (w(r_j))} \geqslant \sum_{j=2}^\infty \int_{r_{j-1}}^{r_j} \frac{ds}{2s \ln (r_j)} =$$

$$= \frac{1}{2} \sum_{j=2}^\infty \left(1 - \frac{\ln (r_{j-1})}{\ln (r_j)}\right) \geqslant \frac{1}{2} \sum_{j=2}^\infty \left(1 - \frac{1}{j}\right) = \infty$$

and (in view of Lemma 2.1) condition (ii) of Theorem 4.1 is fulfilled.

5. – Global existence of solutions of differential inclusions.

We consider now the problem of global existence of solutions of the differential inclusion

$$x' \in F(t, x) \tag{5.1}$$

where F is a multivalued function from $R_+ \times R^n$ into the nonempty, closed and convex subsets of R^n.

By a solution of (5.1) with an initial value (t_0, x_0), $t_0 \geq 0$, $x_0 \in R^n$, we mean an absolutely continuous function $x(t)$: $[t_0, T] \to R^n$ such that $x(t_0) = x_0$ and $x'(t) \in F(t, x(t))$ a.e. on $[t_0, T]$.

In our discussion we will make use of the following existence result of FILIPPOV [35, page 83] which is an extension of a theorem of DAVY [29]:

THEOREM 5.1 [35]. – *Let a multivalued function F: $[t_0, t_1] \times R^n$ satisfy the following conditions:*

(A1) *the set $F(t, x)$ is nonempty, closed and convex for all $x \in R^n$ and for $t \in [t_0, t_1]$ a.e.;*

(A2) *$F(t, x)$ is upper semicontinuous in x for a.e. $t \in [t_0, t_1]$;*

(A3) *$F(t, x)$ is measurable in t for all $x \in R^n$;*

(A4) *there exists a summable function $m(t)$ such that $|F(t, x)| \leq m(t)$ for $(t, x) \in [t_0, t_1] \times R^n$.*

Then the differential inclusion (5.1) *has a solution on* $[t_0, t_1]$.

In the preceding theorem we denoted

$$|U| = \sup_{u \in U} \{|u|\}, \qquad \text{where } U \text{ is a subset of } R^n.$$

THEOREM 5,2. – *Suppose that the multivalued function F: $R_+ \times R^n$ satisfies* (A1)-(A3) *on every compact subinterval of R_+. If there are two functions $w, z \in C(R_+, R_+)$ satisfying the conditions of Theorem* 2.3 *and $\phi, \psi \in L^1_{\mathrm{loc}}(R_+, R_+)$ such that*

$$|F(t, x)| \leq \phi(t)\, w(|x|) + \psi(t)\, z(|x|), \qquad t \in R_+, \quad x \in R^n,$$

then for any initial value (t_0, x_0), the system (5.1) *has a solution defined in the future and satisfying the given initial condition.*

PROOF. – The technique of our proof is adapted from BULGAKOV [11] and SEAH [61].

Let $t_i = t_0 + i$, $i = 1, 2, \ldots$.

We first consider on $[t_0, t_1]$ the problem

$$x' \in F_1(t, x), \qquad x(t_0) = x_0, \tag{5.2}$$

where

$$F_1(t, x) = F(t, x) \quad \text{if } |x| \leqslant K(t_1, x_0),$$

$$F_1(t, x) = F\left(t, \frac{xK(t_1, x_0)}{|x|}\right) \quad \text{if } |x| > K(t_1, x_0).$$

Here $K(t_1, x_0) > 0$ is such that

$$\int_{|x_0|+1}^{K(t_1, x_0)} \frac{ds}{w(s) + z(s)} > \int_{t_0}^{t_1} (\phi(s) + \psi(s))\, ds.$$

We can make such a choice since by Theorem 2.3 and Theorem 2.1 we have that

$$\int_{|x_0|+1}^{\infty} \frac{ds}{w(s) + z(s)} = \infty .$$

From the definition of F_1 we can deduce that

$$|F_1(t, x)| \leqslant (\phi(t) + \psi(t)) \sup_{|x| \leqslant K(t_1, x_0)} \{w(|x|) + z(|x|)\}, \qquad t \in [t_0, t_1], \quad x \in R^n.$$

It can be easily seen that F_1 satisfies conditions (A1)-(A3), thus, by Theorem 5.1, problem (5.2) has a solution $x_1(t)$ on $[t_0, t_1]$.

Let us show that $|x_1(t)| \leqslant K(t_1, x_0)$ for a.e. $t \in [t_0, t_1]$.

Suppose that there exists $T \in (t_0, t_1]$ such that $|x_1(T)| > K(t_1, x_0)$. Since $|x_1(t_0)| = |x_0| < K(t_1, x_0)$ and $x_1(t)$ is continuous on $[t_0, T]$, then exist $t^1, t^2 \in (t_0, T)$ such that

$$|x_1(t^1)| = |x_0| + 1, \qquad |x_1(t^2)| = K(t_1, x_0)$$

and

$$|x_0| + 1 \leqslant |x_1(t)| \leqslant K(t_1, x_0), \qquad t \in [t^1, t^2].$$

We have the relation

$$x_1'(t) \in F_1(t, x_1(t)) = F(t, x_1(t)) \quad \text{a.e. on } [t^1, t^2].$$

In view of the hypothesis we obtain that

$$|x_1'(t)| \leqslant (\phi(t) + \psi(t))(w(|x_1(t)|) + z(|x_1(t)|)) \quad \text{a.e. on } [t^1, t^2].$$

We deduce

$$\limsup_{h \to 0^+} \frac{|x_1(t+h)| - |x_1(t)|}{h} \leqslant (\phi(t) + \psi(t))(w(|x_1(t)|) + z(|x_1(t)|)) \quad \text{a.e. on } [t^1, t^2].$$

It is obvious that the function

$$(t, x) \to (\phi(t) + \psi(t))(w(|x|) + z(|x|)), \qquad t \in R_+, \quad x \in R^n,$$

satisfies the Carathéodory condition [42, page 28] and by comparison theorem of Conti in the case of solutions in the sense of Carathéodory (see [42, page 29]) we deduce from the preceding differential inequality that

$$|x_1(t)| \leqslant W^{-1}\left(\int_{t^1}^{t} (\phi(s) + \psi(s))\, ds \right), \qquad t \in [t^1, t^2],$$

where

$$W: (0, \infty) \to R, \qquad W(r) = \int_{|x_0|+1}^{r} \frac{ds}{w(s) + z(s)}.$$

In particular, we would obtain for $t = t^2$ that

$$\int_{|x_0|+1}^{K(t_1, x_0)} \frac{ds}{w(s) + z(s)} \leqslant \int_{t^1}^{t^2} (\phi(s) + \psi(s))\, ds \leqslant \int_{t_0}^{t_1} (\phi(s) + \psi(s))\, ds$$

which is in contradiction with the way in which we defined $K(t_1, x_0)$.

We have so that $|x_1(t)| \leqslant K(t_1, x_0)$ a.e. on $[t_0, t_1]$ and we obtain

$$x_1'(t) \in F_1(t, x_1(t)) = F(t, x_1(t)) \qquad \text{a.e. on } [t_0, t_1].$$

Consider now the problem (5.3) obtained from (5.2) by replacing t_1 by t_2, t_0 by t_1, x_0 by $x_1(t_1)$ and $K(t_1, x_0)$ by a constant $K(t_2, x_1(t_1)) > 0$ such that

$$\int_{|x_1(t_1)|+1}^{K(t_2, x_1(t_1))} \frac{ds}{w(s) + z(s)} > \int_{t_1}^{t_2} (\phi(s) + \psi(s))\, ds.$$

Proceeding as before we obtain an absolutely continuous function $x_2(t)$ on $[t_1, t_2]$ such that $x_2(t_1) = x_1(t_1)$ and

$$x_2'(t) \in F(t, x_2(t)) \qquad \text{a.e. on } [t_1, t_2].$$

Continuing this process we prove the existence of absolutely continuous functions $x_i(t)$ on $[t_{i-1}, t_i]$, $i = 1, 2, \dots$ with $x_i(t_{i-1}) = x_{i-1}(t_{i-1})$, $i \geqslant 2$, and $x_1(t_0) = x_0$, such that

$$x_i'(t) \in F(t, x_i(t)) \qquad \text{a.e. on } [t_{i-1}, t_i].$$

The function $x:[t_0, \infty) \to R^n$ defined by

$$x(t) = x_i(t), \qquad t \in [t_{i-1}, t_i],$$

is a solution of (5.1) defined on $[t_0, \infty)$ and we have that $x(t_0) = x_0$.

This completes the proof of Theorem 5.2.

THEOREM 5.3. – *Ubder the hypotheses of Theorem* 5.2 *we have that any solution of* (5.1) *exists in the future* (*there is a continuation in the future*).

PROOF. – Let $x(t)$ be a solution of (5.1) defined on $[t_0, T]$.

By the preceding method we are able to construct a solution $y(t)$ of (M) on $[T, \infty)$ satisfying the initial condition $y(T) = x(T)$.

We have that $u:[t_0, \infty) \to R^n$ defined by

$$u(t) = x(t), \qquad t \in [t_0, T],$$

$$u(t) = y(t), \qquad t \in [T, \infty)$$

satisfies

$$u'(t) \in F(t, u(t)) \quad \text{a.e. on } [t_0, \infty).$$

If $x(t)$ is a solution of (5.1) on some interval $[t_0, T)$ with $T \in R_+$, we observe that

$$|x'(t)| \leqslant (\phi(t) + \psi(t))(w(|x(t)|) + z(|x(t)|)) \quad \text{a.e. on } [t_0, T).$$

From the preceding relation we can easily deduce that $|x(t)|$ is bounded on $[t_0, T)$ using the comparison method (as we did in the proof of Theorem 5.2).

We obtain that for some constant $K > 0$ we have

$$|x'(t)| \leqslant K(\phi(t) + \psi(t)) \quad \text{a.e. on } [t_0, T).$$

By the preceding relation we deduce that for $t_0 < t_1 < t_2 < T$,

$$|x(t_2) - x(t_1)| \leqslant K \int_{t_1}^{t_2} (\phi(s) + \psi(s))\, ds .$$

Therefore $x(t_2) - x(t_1) \to 0$ as $t_1, t_2 \to T$ which implies that we can define

$$\lim_{t \to T^-} x(t) = x(T)$$

obtaining thus a solution of (5.1) on $[t_0, T]$.

In a similar way to the first part of the proof we show that this solution is continuable in the future.

This completes the proof of Theorem 5.3.

COROLLARY 5.1 [72]. – *Suppose that the multivalued function* $F: R_+ \times R^n$ *satisfies* (A1)-(A3) *on every compact subinterval of* R_+. *If the inequality*

$$|F(t, x)| \leq \phi(t) + \psi(t)|x|, \qquad t \in R_+, \quad x \in R^n,$$

holds, where $\phi, \psi \in L^1_{\mathrm{loc}}(R_+, R_+)$, *then any solution of* (5.1) *exists in the future.*

COROLLARY 5.2 [72]. – *Suppose that the multivalued function* $F: R_+ \times R^n$ *satisfies* (A1)-(A3) *on every compact subinterval of* R_+. *If the inequality*

$$|F(t, x)| \leq \phi(t) w(|w|), \qquad t \in R_+, \quad x \in R^n,$$

$$\int_\delta^\infty \frac{ds}{w(s)} = \infty,$$

holds, where $\phi \in L^1_{\mathrm{loc}}(R_+, R_+)$ *and* $w \in C(R_+, R_+)$ *with* $w(r) > 0$ *for* $r \geq \delta > 0$, *then any solution of* (5.1) *exists in the future.*

REMARK 5.1. – *Corollary* 5.2 *improves a result of* SEAH [61] *which asks for the supplementary condition* $\int_0^\infty \phi(s)\, ds < \infty$.

COROLLARY 5.3 [72]. – *Suppose that the multivalued function* $F: R_+ \times R^n$ *satisfies* (A1)-(A3) *on every compact subinterval of* R_+. *If there exist* $\phi, \psi \in L^1_{\mathrm{loc}}(R_+, R_+)$ *and a monotone nondecreasing function* $w \in C(R_+, R_+)$ *with* $w(r) > 0$ *for* $r \geq \delta > 0$ *satisfying*

$$|F(t, x)| \leq \phi(t) w(|x|) + \psi(t), \qquad t \in R_+, \quad x \in R^n,$$

$$\int_\delta^\infty \frac{ds}{w(s)} = \infty,$$

then any solution of (5.1) *exists in the future.*

COROLLARY 5.4. – *Let the multivalued function* $F: R_+ \times R^n$ *satisfy conditions* (A1)-(A3) *on every compact subinterval of* R_+. *If there exist* $\phi, \psi \in L^1_{\mathrm{loc}}(R_+, R_+)$ *and a monotone nondecreasing function* $w \in C(R_+, R_+)$ *with* $w(r) > 0$ *for* $r \geq \delta > 0$ *satisfying*

$$|F(t, x)| \leq \phi(t) w(|x|) + \psi(t)|x|, \qquad t \in R_+, \quad x \in R^n,$$

$$\int_\delta^\infty \frac{ds}{w(s)} = \infty,$$

then any solution of (5.1) *exists in the future.*

EXAMPLE 5.1. – Any solution of the differential inclusion $x' \in F(t, x)$ exists in the future, where

$$F(t, x) = t(t^2 + |x| \ln(|x| + 1)) + U(t, x), \qquad t \in R_+, \quad x \in R^n,$$

with $U(t, x) = \{y \in R^n : |y| \leq |x| + 1\}$, $t \in R_+$, $x \in R^n$.

Indeed, we have that

$$|F(t, x)| \leq t^3 + t|x| \ln(|x| + 1) + |x| + 1, \qquad t \in R_+, \quad x \in R^n,$$

and we are able to conclude in view of Corollary 5.4.

6. – Global existence and uniqueness of solutions of McShane stochastic integral equations.

We will use the techniques developped in the previous sections in order to give a result on the global existence and uniqueness of the solution processes of the stochastic integral system

$$(6.1) \qquad x^i(t) = \alpha^i(t) + \sum_{j=1}^{r} \int_0^t g_j^i(s, x(s))\, dz_j(s) + \\ + \sum_{j,k=1}^{r} \int_0^t h_{jk}^i(s, x(s))\, dz_j(s)\, dz_k(s), \qquad 0 \leq t, \quad i = 1, \ldots, n$$

where the stochastic integrals are interpreted as McShane stochastic integrals.

In the special case in which α^i is not depending on time t we have the stochastic differential system

$$(6.2) \qquad dx^i(t) = \sum_{j=1}^{r} g_j^i(s, x(s))\, dz_j(s) + \\ + \sum_{j,k=1}^{r} h_{jk}^i(s, x(s))\, dz_j(s)\, dz_k(s), \qquad 0 \leq t, \quad i = 1, \ldots, n$$

with the initial condition

$$(6.3) \qquad x^i(0) = \alpha^i, \qquad i = 1, \ldots, n.$$

Let (Ω, F, P) be a complete probability space and let $\{F_t, 0 \leq t\}$ be a family of complete σ-subalgebras of F such that if $0 \leq s \leq t$ then $F_s \subseteq F_t$.

Let L_2 be the space of all random variables $y: \Omega \to R$ with finite L_2-norm $\|\cdot\|$ and let L_2^n be the space of all random variables $x: \Omega \to R^n$ with finite norm $\|\cdot\|_n$,

$$\|x\|_n^2 = \sum_{i=1}^{n} \|x_i\|^2, \qquad x = (x_1, \ldots, x_n) \in L_2^n.$$

We say that the real valued second order stochastic process z satisfies a K-condition if z is adapted to the F_t (i.e. $z(t)$ is F_t-measurable for every $t \geq 0$) and

$$|E[(z(t)-z(s))^p/F_s]| \leq K|t-s| \quad \text{a.s. whenever } 0 \leq s \leq t, \ p = 1, 2, 4.$$

An example is a Wiener process with respect to the F_t, $0 \leq t$.

We refer to ELWORTHY [34] and MCSHANE [62] for the basic elements of the McShane stochastic calculus theory. We remind only that if $f: [0, a] \to L_2$ is a measurable process adapted to the F_t and if $t \to \|f(t)\|^2$ is Lebesgue integrable on $[0, a]$ then [34, 62], if z_1 and z_2 satisfy a K-condition, the McShane integrals

$$\int_0^a f(s)\, dz_1(s), \qquad \int_0^a f(s)\, dz_1(s)\, dz_2(s),$$

exist and the following estimates hold

$$\left\| \int_0^a f(s)\, dz_1(s) \right\| \leq C \left\{ \int_0^a \|f(s)\|^2 ds \right\}^{1/2},$$

$$\left\| \int_0^a f(s)\, dz_1(s)\, dz_2(s) \right\| \leq C \left\{ \int_0^a \|f(s)\|^2 ds \right\}^{1/2},$$

where $C = (2 + 8Ka)^{1/2}$.

Let $C[0, a]$ denote the space of all processes $x: [0, a] \to L_2^n$ which are continuous and adapted to the F_t, $0 \leq t \leq a$.

A solution to (6.1) on $[0, a]$ is a process $x \in C[0, a]$ which satisfies (6.1) on $[0, a]$.

In this section we sill use the following result

LEMMA 6,1. – *Let $w, z: R_+ \to R_+$ be continuous, $w(x) > 0$, $z(x) > 0$ for $x > 0$ and suppose that*

$$\lim_{r \to 0} \int_r^1 \frac{ds}{w(s)} = \infty, \qquad \lim_{r \to \infty} \int_1^r \frac{ds}{w(s)} = \infty.$$

If there exist constants $L, M > 0$ such that

$$z(r) \leq Lw(r) \left| \int_1^r \frac{ds}{w(s)} \right| + Mw(r), \qquad r > 0,$$

then

$$\lim_{r\to 0}\int_r^1 \frac{ds}{w(s)+z(s)} = \infty\,, \qquad \lim_{r\to\infty}\int_1^r \frac{ds}{w(s)+z(s)} = \infty\,.$$

PROOF. – By Theorem 2.3 we have that

$$\lim_{r\to\infty}\int_1^r \frac{ds}{w(s)+z(s)} = \infty\,.$$

Let us define

$$w_1,\, z_1 : (0,\,\infty) \to R_+\,, \qquad w_1(s) = s^2 w\left(\frac{1}{s}\right), \qquad z_1(s) = s^2 z\left(\frac{1}{s}\right), \qquad s > 0\,.$$

We see that

$$(\bullet) \qquad \int_r^1 \frac{ds}{w(s)+z(s)} = \int_1^{1/r} \frac{ds}{w_1(s)+z_1(s)}\,, \qquad 0 < r \leq 1\,.$$

By the relation

$$z(r) \leq Lw(r)\int_r^1 \frac{ds}{w(s)} + Mw(r)\,, \qquad 0 < r \leq 1\,,$$

we have that

$$z_1\left(\frac{1}{r}\right) \leq Lw_1\left(\frac{1}{r}\right)\int_1^{1/r} \frac{ds}{w_1(s)} + Mw_1\left(\frac{1}{r}\right), \qquad 0 < r \leq 1\,.$$

We obtain in view of Theorem 2.3 that

$$\lim_{r\to 0}\int_1^{1/r} \frac{ds}{w_1(s)+z_1(s)} = \infty$$

and by ($\bullet$) we deduce that

$$\lim_{r\to 0}\int_r^1 \frac{ds}{w(s)+z(s)} = \infty\,.$$

This completes the proof of Lemma 6.1.

We assume that

(H1) the noise processes z_j, $j = 1, \ldots, r$ satisfy on R_+ a K-condition;

(H2) if f is any one of the functions g_j^i, h_{jk}^i: $R_+ \times L_2^n \to L_2$, $i = 1, \ldots, n$; $k = 1, \ldots, r$, then $f(s, x)$ is continuous in x on L_2^n for every $s \in R_+$ and for any $x \in C[0, a]$, the process $t \to f(t, x(t))$ is measurable and F_t-adapted with $t \to \|f(t, x(t))\|^2$ bounded on $[0, a]$ for $a > 0$;

(H3) there exist w, z: $R_+ \to R_+$ continuous, nondecreasing with $w(r) > 0$, $z(r) > 0$ for $r > 0$ and

$$\lim_{r \to 0} \int_r^1 \frac{ds}{w(s)} = \infty, \qquad \lim_{r \to \infty} \int_1^r \frac{ds}{w(s)} = \infty.$$

such that for some constants $L, M > 0$ we have

$$z(r) \leq Lw(r) \left| \int_1^r \frac{ds}{w(s)} \right| + Mw(r), \qquad r > 0,$$

and

$$\|g_j^i(t, x) - g_j^i(t, y)\|^2 \leq w(\|x - y\|_n^2), \qquad i = 1, \ldots, n, \ j = 1, \ldots, r, \ t \in R_+, \ x, y \in L_2^n,$$

$$\|h_{jk}^i(t, x) - h_{jk}^i(t, y)\|^2 \leq z(\|x - y\|_n^2), \qquad i = 1, \ldots, n, \ j, k = 1, \ldots r, \ t \in R_+, \ x, y \in L_2^n;$$

(H4) the initial condition α belongs to $C[0, a]$ for every $a > 0$.

THEOREM 6.1. – *Let us suppose that the hypotheses* (H1)-(H4) *are satisfied. Then there exists a unique solution of* (6.1) *on* R_+.

PROOF. – Let $a > 0$. We will first prove the existence of a solution of the equation (6.1) on $[0, a]$.

We define the operator T: $C[0, a] \to C[0, a]$ by

$$Tx(t) = \alpha(t) + \sum_{j=1}^r \int_0^t g_j(s, x(s))\, dz_j(s) + \sum_{j,k=1}^r \int_0^t h_{jk}(s, x(s))\, dz_j(s)\, dz_k(s), \qquad 0 \leq t \leq a.$$

Let

$$P = \max\left\{ \sup_{t \in [0, a],\, i = 1, \ldots, n;\, j = 1, \ldots r} \|g_j^i(t, 0)\|^2, \sup_{t \in [0, a],\, i = 1, \ldots, n;\, k = 1, \ldots r} \|h_{jk}^i(t, 0)\|^2 \right\}$$

and denote

$$Q = 3n \sup_{t \in [0, a]} \{\|\alpha(t)\|_n^2\} + 12nC^2(r^2 + r^4)Pa.$$

By Lemma 6.1 we have that the maximal solution (which we will denote again

$m(t)$) of the differential equation

$$m'(t) = 6nr^2C^2w(m(t)) + 6nr^4C^2z(m(t)), \qquad 0 \leq t,$$

with initial condition $m(0) = Q$ is defined on R_+.

We consider the set

$$B = \{x \in C[0, a]: \|x(t)\|_n^2 \leq m(t),\ 0 \leq t \leq a\}.$$

The set B is a closed, bounded and convex subset of the Banach space $C[0, a]$.

Let us show that $T(B) \subset B$. For $x \in B$ we have that

$$\begin{aligned}\left\|\int_0^t g_j^i(s, x(s))\,dz_j(s)\right\| &\leq C\left\{\int_0^t \|g_j^i(s, x(s))\|^2\,ds\right\}^{1/2} \leq \\ &\leq C\left\{\int_0^t (\|g_j^i(s, x(s)) - g_j^i(s, 0)\| + \|g_j^i(s, 0)\|)^2\,ds\right\}^{1/2} \leq \\ &\leq C\left\{\int_0^t (2\|g_j^i(s, x(s)) - g_j^i(s, 0)\|^2 + 2\|g_j^i(s, 0)\|^2)\,ds\right\}^{1/2} \leq \\ &\leq C\left\{\int_0^t w(\|x(s)\|_n^2)\,ds + 2Pt\right\}^{1/2}, \qquad 0 \leq t \leq a,\end{aligned}$$

and in a similar way we obtain that

$$\left\|\int_0^t h_{jk}^i(s, x(s))\,dz_j(s)\,dz_k(s)\right\| \leq C\left\{\int_0^t z(\|x(s)\|_n^2)\,ds + 2Pt\right\}^{1/2}, \qquad 0 \leq t \leq a.$$

We deduce that

$$\begin{aligned}\|Tx(t)\|_n \leq \sqrt{n} \sup_{t \in [0, a]} \|\alpha(t)\|_n &+ \sqrt{2n}\,rC\left\{\int_0^t w(\|x(s)\|_n^2)\,ds + 2\,Pt\right\}^{1/2} + \\ &+ \sqrt{2n}\,r^2C\left\{\int_0^t z(\|x(s)\|_n^2)\,ds + 2\,Pt\right\}^{1/2}, \qquad 0 \leq t \leq a,\end{aligned}$$

and since $x \in B$ we obtain that

$$\|Tx(t)\|_n \leqslant \left\{\sqrt{n} \sup_{t \in [0,a]} \|\alpha(t)\|_n + \sqrt{2n}\, rC \left\{ \int_0^t w(m(s))\, ds + 2Pa \right\}^{1/2} + \right.$$

$$\left. + \sqrt{2n}\, r^2 C \left\{ \int_0^t z(m(s))\, ds + 2Pa \right\}^{1/2} \right\}^2 \leqslant 3n \sup_{t \in [0,a]} \|\alpha(t)\|_n^2 +$$

$$+ 12nC^2 (r^2 + r^4) Pa + 6nr^2 C^2 \int_0^t w(m(s))\, ds + 6nr^4 C^2 \int_0^t z(m(s))\, ds =$$

$$= Q + 6nr^2 C^2 \int_0^t w(m(s))\, ds + 6nr^4 C^2 \int_0^t z(m(s))\, ds = m(t), \qquad 0 \leqslant t \leqslant a.$$

We proved so that $T(B) \subset B$.

The same method enables us to deduce that

$$\|Tx(t) - Tx(s)\|_n^2 \leqslant 6nr^2 C^2 \int_s^t w(m(s))\, ds + 6nr^4 C^2 \int_s^t z(m(s))\, ds +$$

$$+ 3n \sup_{t \in [0,a]} \|\alpha(t) - \alpha(s)\|_n^2 + 12nC^2 (r^2 + r^4) P(t - s), \qquad 0 \leqslant s \leqslant t \leqslant a,$$

thus the set $T(B)$ is equicontinuous.

For $x, y \in B$ we have that

$$\|Tx^i(t) - Ty^i(t)\| \leqslant C \sum_{j=1}^r \left\{ \int_0^t \|g_j^i(s, x(s)) - g_j^i(s, y(s))\|^2 \right\}^{1/2} +$$

$$+ C \sum_{j,k=1}^r \left\{ \int_0^t \|h_{jk}^i(s, x(s)) - h_{jk}^i(s, y(s))\|^2 \right\}^{1/2}, \qquad 0 \leqslant t \leqslant a,$$

so that (taking into account (H2) and (H3)) T is continuous by the Lebesgue convergence theorem.

Applying Schauder's fixed point theorem (see also [50]) we deduce that T has a fixed point in B. This fixed point is a solution of (6.1) on $[0, a]$.

Let us now prove the uniqueness of solutions (6.1).

Suppose that there exist two solutions $x, y \in C[0, a]$ of (6.1) on some interval $[0, a_1]$ with $0 < a_1 \leq a$. We have then that

$$\|x^i(t) - y^i(t)\| \leq C \sum_{j=1}^{r} \left\{ \int_0^t \|g_j^i(s, x(s)) - g_j^i(s, y(s))\|^2 \right\}^{1/2} +$$

$$+C \sum_{j,k=1}^{r} \left\{ \int_0^t w(\|x(s) - y(s)\|_n^2)\, ds \right\}^{1/2} + Cr^2 \left\{ \int_0^t z(\|x(s) - y(s)\|_n^2)\, ds \right\}^{1/2}, \qquad 0 \leq t \leq a_1,$$

thus

$$\|x(t) - y(t)\|_n^2 \leq 2nC^2 r^2 \int_0^t w(\|x(s) - y(s)\|_n^2)\, ds +$$

$$+2nC^2 r^4 \int_0^t z(\|x(s) - y(s)\|_n^2)\, ds, \qquad 0 \leq t \leq a_1.$$

Since $x(0) = y(0)$, in view of Lemma 6.1, we deduce by Osgood's uniqueness criterion (see [51]) that

$$\|x(t) - y(t)\|_n^2 = 0, \qquad 0 \leq t \leq a_1.$$

Thus equation (6.1) has a unique solution on $[0, a]$ for every $a > 0$. We deduce that (6.1) has a unique solution on R_+.

This completes the proof of Theorem 6.1.

REMARK 6.1. – *Theorem* 6.1 *enables us to give a bound for the unique solution of equation* (6.1):

$$\|x(t)\|_n^2 \leq m(t), \qquad 0 \leq t.$$

REMARK 6.2. – *For* $w(t) = z(t) = Mt$, $t \in R_+$ $(M > 0)$ *we obtain the existence and uniqueness theorem of Angulo Ibanez and Gutierrez Jaimez* [3].

When α *is not depending on time we obtain the existence and uniqueness theorem of McShane* [62]. *Note also that our requirements are weaker is some aspects than those made in* [34] (*Elworthy requires Lipschitz conditions on* g_j^i *and* $h_{ij,k}$).

7. – Successive approximations to solutions, global existence and uniqueness for Ito stochastic differential equations.

Let (Ω, F, P) be a complete probability space and let $\{F_t, 0 \leq t\}$ be a family of complete σ-subalgebras of F such that $F_s \subset F_t$ if $0 \leq s \leq t$.

Let $W(t)$, $t \geq 0$, be the m-dimensional (F_t)-Brownian motion on $(\Omega, F, P, (F_t))$. Let $f(t, x)$ be an R^n-valued measurable function defined on $R_+ \times R^n$ and let $g(t, x) = (g_i(t, x))_{i=\overline{1,m}}$, where each $g_i(t, x)$, $i = \overline{1, m}$, is a R^n-valued measurable function on $R_+ \times R^n$.

We consider the Ito stochastic integral equation

$$X(t) = X_0 + \int_0^t f(s, X(s))\,ds + \int_0^t g(s, X(s))\,dW(s), \tag{7.1}$$

where X_0 is an F_0-measurable, R^n-valued function independent of the Brownian motion $W(t)$, $t \geq 0$, and with $E|X_0|^2 < \infty$.

Equation (7.1) is equivalent to the stochastic Ito type initial value problem

$$dX(t) = f(t, X(t)) + g(t, X(t))\,dW(t), \tag{7.2}$$

$$X(0) = X_0 \quad \text{a.e.}$$

Let us consider the sequence of stochastic processes which are defined by the successive approximations

$$X_i(t) = X_0 + \int_0^t f(s, X_{i-1}(s))\,ds + \int_0^t g(s, X_{i-1}(s))\,dW(s), \qquad i = 1, 2, \dots. \tag{7.4}$$

As ITO [48] proved, the convergence of the sequence of stochastic processes dedfined by the successive approximations (7.4) is guaranteed under the conditions ($L > 0$ is a constant)

$$|f(t, x) - f(t, y)| + |g(t, x) - g(t, y)| \leq L|x - y| \text{ (Lipschitz condition)},$$

$$|f(t, x)|^2 + |g(t, x)|^2 \leq L^2(1 + |x|^2) \text{ (growth condition)}.$$

YAMADA [75] proved the convergence of successive approximations to solutions under more general condictions than Ito's and TANIGUCHI [71] extended Yamada's result.

We will show that the results of the previous sections enable us, using a method similar to the method of Taniguchi [71], to give more general conditions under which on any finite interval $[0, T]$, the sequence of stochastic processes defined by the successive approximations (7.4) converges uniformly to a unique solution of (7.1) (by uniqueness we mean *pathwise uniqueness*, i.e. if $X(t)$ and $Y(t)$ are two solutions, then $P(\sup_{t \in [0, T]} \{|X(t) - Y(t)|\} = 0) = 1$, and not *uniqueness in the law sense*, i.e. solutions have the same distributions; note that pathwise uniqueness is stronger that uniqueness in the law sense—see YAMADA and WATANABE [74]).

THEOREM 7.1. – *Suppose that the following conditions are satisfied:*

(i) *the functions* $f(t, x)$ *and* $g(t, x)$ *are measurable functions on* $R_+ \times R^n$ *and continuous in* x *for each fixed* $t \in R_+$, *with* $|f(t, 0)|$, $|g(t, 0)| \in L^2_{\text{loc}}(R_+, R_+)$;

(ii) *there exist* $\phi, \psi \in L^1_{\text{loc}}(R_+, R_+)$ *and* $w, z \in C(R_+, R_+)$ *nondecreasing,* $w(r) > 0$, $z(r) > 0$ *for* $r > 0$ *and*

$$\lim_{r \to 0} \int_r^1 \frac{ds}{w(s)} = \infty, \qquad \lim_{r \to \infty} \int_1^r \frac{ds}{w(s)} = \infty,$$

such that for some constant $L > 0$ *we have that*

$$z(r) \leq Lw(r) \left| \int_1^r \frac{ds}{w(s)} \right| + Lw(r), \qquad r > 0,$$

and

$$E|f(t, X) - f(t, Y)|^2 \leq \phi(t)\, w(E|X - Y|^2), \qquad t \in R_+, \quad X, Y \in L^2(\Omega, R^n),$$

$$E|g(t, X) - g(t, Y)|^2 \leq \psi(t)\, z(E|X - Y|^2), \qquad t \in R_+, \quad X, Y \in L^2(\Omega, R^n).$$

Then, for any finite interval $[0, T]$, *the sequence* $\{X_i(t)\}$, $0 \leq t \leq T$, *defined by the successive approximations* (7.4), *converges uniformly a unique solution of* (7.1).

PROOF. – If $X \in L^2(\Omega, R^n)$ we have that

$$E|f(t, X)|^2 \leq 2E|f(t, X) - f(t, 0)|^2 + 2|f(t, 0)|^2 \leq$$
$$\leq 2\phi(t)\, w(E|X|^2) + 2|f(t, 0)|^2, \qquad 0 \leq t \leq T,$$

and similarly

$$E|g(t, X)|^2 \leq 2\psi(t)\, z(E|X|^2) + 2|g(t, 0)|^2, \qquad 0 \leq t \leq T.$$

Fix $T > 0$. We will prove first that $\{E|X_i(t)|^2\}$, $i \geq 0$, are uniformly bounded on $[0, T]$.

The conditions imposed on w and z guarantee (in view of Lemma 6.1) that the differential equation

$$u' = 6(1 + T)(\phi(t)\, w(u) + \psi(t)\, z(u)) + 6(1 + T)(|f(t, 0)|^2 + |g(t, 0)|^2)$$

with initial condition $u(0) = u_0 > 3E|X_0|^2$, has a solution $u(t)$ (in the sense of Carathéodory) defined on R_+.

We have that

$$E|X_1(t)|^2 \leqslant 3E\left[|X_0|^2 + \left| \int_0^t f(s, X_0)\, ds \right|^2 + \left| \int_0^t g(s, X_0)\, dW(s) \right|^2 \right] \leqslant$$

$$\leqslant 3E|X_0|^2 + 3TE\int_0^t |f(s, X_0)|^2 ds + 3E\int_0^t |g(s, X_0)|^2 ds \leqslant 3E|X_0|^2 +$$

$$+6(1+T)\int_0^t (\phi(s)\, w(E|X_0|^2) + \psi(s)\, z(E|X_0|^2) + |f(s, 0)|^2 + |g(s, 0)|^2)\, ds\,,$$

for all $t \in [0, T]$.

Since

$$u(t) > E|X_0|^2, \qquad t \geqslant 0\,,$$

we have that

$$3E|X_0|^2 + 6(1+T)\int_0^t (\phi(s)\, w(E|X_0|^2) + \psi(s)\, z(E|X_0|^2) + |f(s, 0)|^2 + |g(s, 0)|^2)\, ds \leqslant$$

$$\leqslant u_0 + 6(1+T)\int_0^t (\phi(s)\, w(u(s)) + \psi(s)\, z(u(s)) +$$

$$+ |f(s, 0)|^2 + |g(s, 0)|^2)\, ds = u(t)\,, \qquad t \in [0, T]\,,$$

thus

$$E|X_1(t)|^2 \leqslant u(t)\,, \qquad t \in [0, T]\,.$$

Let us prove by recurrence that

$$E|X_i(t)|^2 \leqslant u(t)\,, \qquad t \in [0, T]\,,\ \ i = 1, 2, \ldots.$$

Suppose this is true for $i = k$ and let us prove it for $k+1$.

We observe that

$$E|X_{k+1}(t)|^2 \leqslant 3E\left[|X_0|^2 + \left| \int_0^t f(s, X_k(s))\, ds \right|^2 + \left| \int_0^t g(s, X_k(s))\, dW(s) \right|^2 \right] \leqslant$$

$$\leqslant 3E|X_0|^2 + 6(1+T)\int_0^t (\phi(s)\, w(E|X_k(s)|^2)\, ds +$$

$$+6(1+T)\int_0^t \psi(s)\,z(E|X_k(s)|^2)+|f(s,0)|^2+|g(s,0)|^2)\,ds \leq$$

$$\leq u_0+6(1+T)\int_0^t (\phi(s)\,w(u(s))+\psi(s)\,z(u(s))+|f(s,0)|^2+|g(s,0)|^2)\,ds = u(t)$$

for $t\in[0,T]$, thus

$$E|X_{k+1}(t)|^2 \leq u(t), \qquad t\in[0,T].$$

We proved so by recurrence that

$$E|X_i(t)|^2 \leq u(T), \qquad t\in[0,T], \qquad i=1,2,\ldots.$$

Define now the functions

$$a_{mn}(t)=E|X_m(t)-X_n(t)|^2,$$

$$b_n(t)=\sup_{p\geq q\geq n}\{a_{pq}(t)\}.$$

We will show that the sequence $\{b_n(t)\}$ has a subsequence $\{b_{n_j}(t)\}$ which converges uniformly on $[0,T]$ to a continuous function $b(t)$.

Since $\{E|X_i(t)|^2\}$ is uniformly bounded on $[0,T]$, there exists a constant $M>0$ such that

$$a_{mn}(t)\leq 2E(|X_n(t)|^2+|X_m(t)|^2)<M, \qquad t\in[0,T],$$

(we can take for instance $M=4u(T)$).

Observe that

$$|a_{mn}(t)-a_{mn}(s)| = |E|X_m(t)-X_n(t)|^2-E|X_m(s)-X_n(s)|^2| =$$

$$= E(|X_m(t)-X_n(t)|+|X_m(s)-X_n(s)|)\,|\,|X_m(t)-X_n(t)|-|X_m(s)-X_n(s)|\,| \leq$$

$$\leq E(|X_m(t)-X_n(t)|+|X_m(s)-X_n(s)|)(|X_m(t)-X_m(s)|+|X_n(t)-X_n(s)|)\leq$$

$$\leq [E(|X_m(t)-X_n(t)|+|X_m(s)-X_n(s)|)^2]^{1/2}[E(2|X_m(t)-X_m(s)|^2+$$

$$+2|X_n(t)-X_n(s)|^2)]^{1/2}\leq [E(2|X_m(t)-X_n(t)|^2+2|X_m(s)-X_n(s)|^2)]^{1/2}\cdot$$

$$\cdot[E(2|X_m(t)-X_m(s)|^2+2|X_n(t)-X_n(s)|^2)]^{1/2}=$$

$$=2[a_{mn}(t)+a_{mn}(s)]^{1/2}[E(|X_m(t)-X_m(s)|^2+|X_n(t)-X_n(s)|^2)]^{1/2}.$$

On the other hand

$$E|X_m(t)-X_m(s)|^2 \leqslant E\left[\left|\int_s^t f(\tau, X_{m-1}(\tau))\,d\tau\right| + \left|\int_s^t g(\tau, X_{m-1}(\tau))\,dW(\tau)\right|\right]^2 \leqslant$$

$$\leqslant 2TE\int_s^t |f(\tau, X_{m-1}(\tau))|^2\,d\tau + 2E\int_s^t |g(\tau, X_{m-1}(\tau))|^2\,d\tau \leqslant$$

$$\leqslant 4(1+T)\int_s^t \left(|f(\tau,0)|^2 + |g(\tau,0)|^2 + \phi(\tau)\,w(u(\tau)) + \psi(\tau)\,z(u(\tau))\right)d\tau \leqslant$$

$$\leqslant 4(1+T)(u(t)-u(s))\,, \qquad 0 \leqslant s \leqslant t \leqslant T,$$

so that we obtain

$$|a_{mn}(t) - a_{mn}(s)| \leqslant 8\sqrt{M(1+T)}(u(t)-u(s))^{1/2}, \qquad 0 \leqslant s \leqslant t \leqslant T.$$

We deduce that

$$0 \leqslant b_n(t) < M\,, \qquad 0 \leqslant t \leqslant T,$$

$$|b_n(t) - b_n(s)| \leqslant 8\sqrt{M(1+T)}(u(t)-u(s))^{1/2}, \qquad 0 \leqslant s \leqslant t \leqslant T.$$

By the theorem of Ascoli-Arzelà (see [42, page 2]) there exists a subsequence $\{b_{n_j}(t)\}$ which converges uniformly on $[0, T]$ to a continuous function $b(t)$.

If $m, n \geqslant n_{j+1}$ we have that $m-1, n-1 \geqslant n_j$, so that

$$a_{mn}(t) = E|X_m(t) - X_n(t)|^2 \leqslant$$

$$\leqslant 2E\left(\left|\int_0^t (f(s, X_{m-1}(s)) - f(s, X_{n-1}(s)))\,ds\right|^2 +\right.$$

$$\left. + \left|\int_0^t (g(s, X_{m-1}(s)) - g(s, X_{n-1}(s)))\,dW(s)\right|^2\right) \leqslant$$

$$\leqslant 2(1+T)\int_0^t E|f(s, X_{m-1}(s)) - f(s, X_{n-1}(s))|^2\,ds +$$

$$+2\int_0^t E|g(s, X_{m-1}(s)) - g(s, X_{n-1}(s))|^2\,ds \leqslant$$

$$\leqslant 2(1+T)\int_0^t (\phi(s)\,w(E|X_{m-1}(s)-X_{n-1}(s)|^2)+\psi(s)\,z(E|X_{m-1}(s)-X_{n-1}(s)|^2))\,ds \leqslant$$

$$\leqslant 2(1+T)\int_0^t (\phi(s)\,w(b_{n_j}(s))+\psi(s)\,z(b_{n_j}(s)))\,ds\,, \qquad 0\leqslant t\leqslant T\,,$$

thus

$$b_{n_{j+1}}(t)\leqslant 2(1+T)\int_0^t (\phi(s)\,w(b_{n_j}(s))+\psi(s)\,z(b_{n_j}(s)))\,ds\,, \qquad 0\leqslant t\leqslant T\,.$$

Letting $j\to\infty$, by the continuity of w and z and the dominated convergence theorem of Lebesgue, we obtain

$$b(t)\leqslant 2(1+T)\int_0^t (\phi(s)\,w(b(s))+\psi(s)\,z(b(s)))\,ds\,, \qquad 0\leqslant t\leqslant T\,.$$

Taking into account Lemma 6.1 and Osgood's uniqueness criterion (see [60, page 102]), we deduce that $b(t)=0$, $t\in[0,T]$.

Observe now that if $m, n\geqslant n_{j+1}$, then

$$E\Big(\sup_{0\leqslant t\leqslant T}\{|X_m(t)-X_n(t)|^2\}\Big)\leqslant$$

$$\leqslant 2E\left(\sup_{0\leqslant t\leqslant T}\left\{\left|\int_0^t [f(s,X_{m-1}(s))-f(s,X_{n-1}(s))]\,ds\right|^2\right\}+\right.$$

$$\left.+\sup_{0\leqslant t\leqslant T}\left\{\left|\int_0^t [g(s,X_{m-1}(s))-g(s,X_{n-1}(s))]\,dW(s)\right|^2\right\}\right)\leqslant$$

$$\leqslant 2T\int_0^T E|f(s,X_{m-1}(s))-f(s,X_{n-1}(s))|^2\,ds+$$

$$+2\int_0^T E|g(s,X_{m-1}(s))-g(s,X_{n-1}(s))|^2\,ds\leqslant$$

$$\leq 2(1+T)\int_0^T \phi(s)\, w(E|X_{m-1}(s)-X_{n-1}(s)|^2)\, ds +$$

$$+2(1+T)\int_0^T \psi(s)\, z(E|X_{m-1}(s)-X_{n-1}(s)|^2)\, ds \leq$$

$$\leq 2(1+T)\int_0^T (\phi(s)\, w(b_{n_j}(s)) + \psi(s)\, z(b_{n_j}(s)))\, ds \to 0$$

as $j \to \infty$, thus $\{X_n(t)\}$ is a Cauchy sequence in the Banach space B of functions $h(t, \omega)$: $R_+ \times \Omega \to R^n$, measurable in ω for each fixed $t \in [0, T]$, continuous in t for a.e. $\omega \in \Omega$ and with

$$E\Big(\sup_{0\leq t\leq T}\{|h(t, \omega)|^2\}\Big) < \infty ;$$

this is a Banach space (see RODKINA [59]) with the norm

$$\|h(t, \omega)\|_B = \Big\{E\Big(\sup_{0\leq t\leq T}\{|h(t, \omega)|^2\}\Big)\Big\}^{1/2}.$$

We deduce that there exists a stochastic process $X(t)$, $0 \leq t \leq T$, such that

$$E\Big(\sup_{0\leq t\leq T}\{|X_i(t)-X(t)|^2\}\Big) \to 0$$

as $i \to \infty$.

Furthermore

$$E\left(\sup_{0\leq t\leq T}\left\{\left| X_i(t) - \left[X_0 + \int_0^t f(s, X(s))\, ds + \int_0^t g(s, X(s))\, dW(s)\right]\right|^2\right\}\right) =$$

$$= E\left(\sup_{0\leq t\leq T}\left\{\left|\int_0^t (f(s, X_{i-1}(s)) - f(s, X(s)))\, ds +\right.\right.\right.$$

$$\left.\left.\left. + \int_0^t (g(s, X_{i-1}(s)) - g(s, X(s)))\, dW(s)\right|^2\right\}\right) \leq$$

$$\leq 2T\int_0^T E|f(s, X_{i-1}(s)) - f(s, X(s))|^2\, ds +$$

$$+2\int_0^T E|g(s, X_{i-1}(s)) - g(s, X(s))|^2\, ds \leq$$

$$\leqslant 2(1+T)\int_0^T \phi(s)\,w(E|X_{i-1}(s)-X(s)|^2)\,ds +$$

$$+2(1+T)\int_0^T \psi(s)\,z(E|X_{i-1}(s)-X(s)|^2)\,ds \to 0$$

as $i \to \infty$, thus

$$X(t) = X_0 + \int_0^t f(s, X(s))\,ds + \int_0^t g(s, X(s))\,dW(s), \qquad 0 \leqslant t \leqslant T.$$

We proved so that the sequence of successive approximations converges uniformly to a solution of (7.1) on $[0, T]$.

All we are left to prove is the uniqueness of this solution.

Let $X(t)$ and $Y(t)$ be two solutions on $[0, t_0]$ with $X(0) = Y(0) = X_0$. We obtain

$$E|X(t)-Y(t)|^2 \leqslant 2(1+T)\int_0^t (\phi(s)\,w(E|X(s)-Y(s)|^2) + \psi(s)\,z(E|X(s)-Y(s)|^2))\,ds,$$

$$0 \leqslant t \leqslant t_0,$$

and by Lemma 6.1 we conclude that $E|X(t)-Y(t)|^2 = 0$, $0 \leqslant t \leqslant t_0$. Thus, for any $t \in [0, t_0]$, $P(X(t) = Y(t)) = 1$, and so for a countable dense subset $S \subset [0, t_0]$,

$$P(X(t) = Y(t),\ t \in S) = 1.$$

Using the a.e. continuity of X and Y, we get

$$P\Big(\sup_{0 \leqslant t \leqslant t_0} \{|X(t) - Y(t)|\} > 0\Big) = 0$$

which implies the pathwise uniqueness of solutions of (7.1).

This completes the proof of Theorem 7.1.

REMARK 7.1. – *Instead of the assumption*

$$E|f(t,X) - f(t,Y)|^2 \leqslant \phi(t)\,w(E|X-Y|^2), \qquad t \in R_+,\ \ X, Y \in L^2(\Omega, R^n),$$

we can suppose that

$$|f(t,x) - f(t,y)|^2 \leqslant \phi(t)\,w(|x-y|^2), \qquad t \in R_+,\ \ x, y \in R^n,$$

is the function w is a concave on R_+ (a simple application of Jensen's inequality [36] *shows that the second condition implies the first one). The same remark applies if $r \to w^2(r)/r$ is concave on R_+ (see* [70]).

COROLLARY 7.1 [71]. – *For the stochastic differential equation* (7.1), *suppose that the following conditions are satisfied:*

(i) $|f(t, x) - f(t, y)|^2 + |g(t, x) - g(t, y)|^2 \leq \lambda(t)\alpha(|x-y|^2)$;

(ii) $|f(t, 0)|, \ |g(t, 0)| \in L^2_{\mathrm{loc}}(R_+, R_+)$;

where $\lambda \in L^1_{\mathrm{loc}}(R_+, R_+)$ *and* $\alpha \in C(R_+ R_+)$ *is monotone nondecreasing, concave, with* $\alpha(0) = 0$ *and such that*

$$\lim_{r \to 0} \int_r^1 \frac{ds}{\alpha(s)} = \infty .$$

Then, on any finite interval $[0, T]$, *the sequence* $\{X_i(t)\}$, $0 \leq t \leq T$, *defined by the successive approximations* (7.4), *converges uniformly to a unique solution of* (7.1).

PROOF. – Since α is concave on R_+, there exist positive constants $a > 0$, $b > 0$, with

$$\alpha(r) \leq a + br, \qquad r \in R_+ .$$

We can conclude in view of Theorem 7.1 and Remark 7.1.

REMARK 7.2. – *If* $\lambda(t) = 1$, $t \in R_+$, *Corollary* 7.1 *becomes Yamada's theorem* [75]. *In the particular case* $\alpha(r) = r$, $r \in R_+$, *and* $\lambda(t) = L > 0$, $t \in R_+$, *we obtain Ito's conditions* [48]. *Note that the result of Da Prato* [55] *is a particular case of Corollary* 7.1.

COROLLARY 7.2 [70]. – *For the stochastic differential equation* (7.1), *suppose that the following conditions are satisfied:*

(i) $|f(t, x) - f(t, y)|^2 + |g(t, x) - g(t, y)|^2 \leq \lambda(t)\alpha(|x-y|^2)$;

(ii) $|f(t, 0)|, \ |g(t, 0)| \in L^2_{\mathrm{loc}}(R_+ R_+)$;

where $\lambda \in L^1_{\mathrm{loc}}(R_+, R_+)$ *and* $\alpha \in C(R_+, R_+)$ *is monotone nondecreasing, concave, with* $\alpha(0) = 0$ *and such that*

$$r \to \frac{\alpha^2(r)}{r}$$

is concave on R_+ *and*

$$\lim_{r \to 0} \int_r^1 \frac{ds}{\alpha(s)} = \infty .$$

Then, on any finite interval $[0, T]$, *the sequence* $\{X_i(t)\}$, $0 \leq t \leq T$, *defined by the successive approximations* (7.4), *converges uniformly to a unique solution of* (7.1).

PROOF. – If $a > 0$, $b > 0$ are such that

$$\frac{\alpha^2(r)}{r} \leqslant a + br\,, \qquad r \in R_+\,,$$

we deduce that

$$\alpha(r) \leqslant (a + \sqrt{b})\,r + 1\,, \qquad r \in R_+\,,$$

and we can apply Theorem 7.1 (taking into account Remark 7.1).

REMARK 7.3. – *A nontrivial example of a function* α *satisfying the conditions of Corollary* 7.2 *is*

$$\alpha(r) = r\left[\ln\left(\frac{1}{r}\right)\right]^{1/2}, \qquad \frac{1}{e} > r > 0\,; \qquad \alpha(r) = \sqrt{\frac{r}{e}}\,, \qquad r \geqslant \frac{1}{e}\,.$$

In both Corollaries, we have a single nonlinear function of $E|X - Y|^2$ *which bounds the expression*

$$E|f(t, X) - f(t, Y)|^2 + E|g(t, X) - g(t, Y)|^2\,.$$

Theorem 7.1 *enables us to consider two separate bounds for*

$$E|f(t, X) - f(t, Y)|^2$$

and respectively

$$E|g(t, X) - g(t, Y)|^2$$

which are not comparable (*in the sense stated in the Introduction*) *in a neighborhood of* ∞.

7. – Applications to global inversion of local diffeomorphisms.

If $f\colon R^n \to R^n$ is of class C^1 and

$$\det f'(x) \neq 0\,, \qquad x \in R^n\,,$$

we have by the inverse function theorem [56, page 32] that f is a local diffeomorphism at every point of R^n.

In this section we will show that the continuation method of Conti with the results stated in Section 2 can be applied in order to give sufficient conditions for f to be injective and so a global diffeomorphism $f\colon R^n \to f(R^n)$ and a sufficient condition for f to be bijective and so a global diffeomorphism onto R^n. As particular cases of our result we have the celebrated theorem of Hadamard [40] and the theorem of Caccioppoli-Banach-Mazur [6, 17] in the particular case of local diffeomorphisms $f\colon R^n \to R^n$ as well as some recent results of ZAMPIERI [77].

In our theorems we will use a nonnegative auxiliary scalar coercive function, that is, a continuous mapping $k: R^n \to R_+$ such that

$$k(x) \to \infty \qquad \text{as } |x| \to \infty .$$

The use of such functions in the context of global inversion theorems is due to ZAMPIERI [77]. Our method is similar to Zampieri's method.

If $k: R^n \to R_+$ we define

$$D_v^+ k(x) = \limsup_{h \to 0^+} \frac{k(x+hv) - k(x)}{h}$$

for all $x, v \in R^n$.

THEOREM 8.1. – *The mapping $f \in C^1(R^n, R^n)$ is a global diffeomorphism $R^n \to \to f(R^n)$ if*

(i) $\det f'(x) \neq 0$, $x \in R^n$;

(ii) *there exists a point $x_0 \in R^n$ and a locally Lipschitzian coercive function $k: R^n \to R_+$ such that*

$$D_v^+ k(x) \leq w(k(x)), \qquad v = -f'(x)^{-1}(f(x) - f(x_0)), \qquad x \in R^n,$$

where $w \in \mathfrak{R}_0$.

PROOF. – Consider the Cauchy problem

$$x' = F(x), \qquad x(0) = \tilde{x}, \tag{8.1}$$

with $F(x) = -f'(x)^{-1}(f(x) - f(x_0))$, $x \in R^n$.

Since F is continuous there is local existence for (8.1).

Let us prove that we have also uniqueness for the Cauchy problem (8.1).

If $x(t, \tilde{x})$ is a solution of (8.1) defined on some neighborhood of zero, we observe that for all t for which this solution is defined, we have

$$f'(x(t, \tilde{x}))(x'(t, \tilde{x})) = -\big(f(x(t, \tilde{x})) - f(x_0)\big)$$

thus

$$[e^t (f(x(t, \tilde{x})) - f(x_0))]' = 0$$

and an integration yields

$$f(x(t, \tilde{x})) - f(x_0) = e^{-t}(f(\tilde{x}) - f(x_0))$$

for all t for which this solution is defined.

We have so that

$$f(x(t, \tilde{x})) = f(x_0) + e^{-t}(f(\tilde{x}) - f(x_0))$$

and since f is a local diffeomorphism at $\tilde{x}$ we obtain that there is uniqueness for (6.1).

From the continuity of F and from the uniqueness for (8.1) we have also continuous dependence on the initial conditions for (8.1) (see [25, page 17]).

Let now $A(x_0)$ be the «basin of attraction» of x_0, i.e. the set of all points $\tilde{x} \in R^n$ such that the maximal solution $t \to x(t, \tilde{x})$ of (8.1) with $x(0, \tilde{x}) = \tilde{x}$ is defined in the future and $\lim_{t \to \infty} x(t, \tilde{x}) = x_0$.

The set $A(x_0)$ is not void since $x_0 \in A(x_0)$—the Cauchy problem (8.1) with $\tilde{x} = x_0$ has the unique solution $x(t, x_0) = x_0$, $t \geq 0$.

We prove now that $A(x_0)$ is open and $f/A(x_0)$ is one-to-one.

Let $x_1 \in A(x_0)$. We denote $x(t, x_1) = x_1(t)$, $t \geq 0$.

Let U and V be respectively neighborhoods of x_0 and $f(x_0)$ so that f/U: $U \to V$ is a diffeomorphism (this choice is possible by the inverse function theorem).

We choose $\varepsilon > 0$, $\delta > 0$ such that

$$|f(x) - f(x_0)| < \varepsilon \quad \text{if } |x - x_0| < \delta,$$

$$\{x \in R^n \colon |x - x_0| < 2\delta\} \subset U,$$

$$\{y \in R^n \colon |y - f(x_0)| < 2\varepsilon\} \subset V.$$

Since $\lim_{t \to \infty} x_1(t) = x_0$, there is a $T > 0$ such that

$$|x_1(t) - x_0| < \delta, \qquad t \geq T.$$

By the continuous dependence on the initial conditions for (8.1) there is a $\gamma = \gamma(\delta, T)$ such that the solution $x_2(t) = x(t, x_2)$ exists on $[0, T]$ and $|x_1(t) - x_2(t)| < \delta$ for $t \in [0, T]$ provided that $|x_1 - x_2| < \gamma$.

We deduce that if $|x_1 - x_2| < \min\{\delta, \gamma\}$ then $x_2(t)$ is defined on $[0, T]$ and

$$|x_2(T) - x_0| \leq |x_2(T) - x_1(T)| + |x_1(T) - x_0| < 2\delta,$$

$$|f(x_2(t)) - f(x_1(t))| \leq e^{-t}|f(x_2) - f(x_1)| < \varepsilon, \qquad t \in [0, T].$$

Define

$$x_2(t) = f^{-1}\big(f(x_1(t)) + e^{-t}(f(x_2) - f(x_1))\big), \qquad t > T.$$

This can be done since

$$|f(x_1(t)) + e^{-t}(f(x_2) - f(x_1)) - f(x_0)| \leq |f(x_1(t)) - f(x_0)| +$$
$$+ e^{-t}|f(x_2) - (f(x_1)| < 2\varepsilon, \qquad t \geq T,$$

and

$$|x_2(T) - x_0| < 2\delta.$$

We proved so that $x_2(t)$ is defined in the future (recall that we have uniqueness for (8.1)).

Since

$$x_2(t) = f^{-1}(f(x_1(t)) + e^{-t}(f(x_2) - f(x_1))), \qquad t > T,$$

we deduce that

$$\lim_{t \to \infty} x_2(t) = x_0.$$

The preceding facts can be resumed as follows: if $|x_1 - x_2| < \min\{\delta, \gamma\}$ then $x_2 \in A(x_0)$. We proved so that $A(x_0)$ is open.

In order to prove that $f/A(x_0)$ is one-to-one, suppose that $x_1, x_2 \in A(x_0)$ are such that $f(x_1) = f(x_2)$. We deduce that

$$f(x(t, x_1)) - f(x(t, x_2)) = e^{-t}(f(x_1) - f(x_2)) = 0.$$

Let $T > 0$ be such that

$$|x(t, x_1) - x_0| + |x(t, x_2) - x_0| < \varepsilon, \qquad t \geqslant T.$$

Since $f(x(t, x_1)) = f(x(t, x_2))$, $t \geqslant T$, by the way in which we chose ε (remember that $|f(x(t, x_1)) - f(x_0)| < \delta$ and $|f(x(t, x_2)) - f(x_0)| < \delta$ for $t \geqslant T$) we deduce that

$$x(t, x_1) = x(t, x_2), \qquad t \geqslant T.$$

From the uniqueness property of (8.1) we obtain that $x_1 = x_2$, thus $f/A(x_0)$ is one-to-one.

The next step in the proof is to show that if $\tilde{x} \in bA(x_0)$ (the boundary of $A(x_0)$), then the maximal solution $x(t, \tilde{x})$ cannot exist in the future.

We proved that $f/A(x_0)$ is one-to-one so $f/A(x_0)$: $A(x_0) \to f(A(x_0))$ is a homeomorphism. We obtain the existence of an $\beta > 0$ such that

$$|f(x) - f(x_0)| > \beta, \qquad x \in b(A(x_0)).$$

Suppose there exists an $\tilde{x} \in b(A(x_0))$ such that the maximal solution $x(t, \tilde{x})$ is defined for all $t \in R_+$. It is obvious that $x(t, \tilde{x}) \notin A(x_0)$, $t \geqslant 0$.

From the relation

$$f(x(t, \tilde{x})) - f(x_0) = e^{-t}(f(\tilde{x}) - (x_0)), \qquad t \geqslant 0,$$

we obtain that

$$e^{-t}|f(\tilde{x}) - f(x_0)| \geqslant \beta, \qquad t \geqslant 0,$$

thus

$$t \leqslant \ln\left(\frac{|f(\tilde{x}) - f(x_0)|}{\beta}\right), \qquad t \geqslant 0\,,$$

which is a contradiction.

We have so that if $\tilde{x} \in b(A(x_0))$, then $x(t, \tilde{x})$ does not exist in the future.

On the other hand, the conditions required on k imply by Conti's comparison result (with the slight extension obtained in [20]) that every solution of (8.1) can be defined in the future.

We obtain that $b(A(x_0)) = \emptyset$ and thus $A(x_0) = R^n$. This implies that $f\colon R^n \to f(R^n)$ is a global diffeomorphism.

COROLLARY 8.1 [77]. – *The mapping $f \in C^1(R^n, R^n)$ is a global diffeomorphism $R^n \to f(R^n)$ if*

(i) $\det f'(x) \neq 0$, $x \in R^n$;

(ii) *there exists a point $x_0 \in R^n$ and a locally Lipschitzian coercive function $k\colon R^n \to R_+$ such that for every $x, v \in R^n$,*

$$\lim_{h \to \infty^+} \frac{k(x + hv) - k(x)}{h}$$

exists and

$$\sup\left\{\lim_{h \to 0^+} \frac{k(x + hv) - k(x)}{h} : v = -f'(x)^{-1}(f(x) - f(x_0)),\ x \in R^n\right\} < \infty\,.$$

PROOF. – We apply Theorem 8.1 with $w(r) = M$, $r \in R_+$, where

$$M = \sup\left\{\lim_{h \to 0^+} \frac{k(x + hv) - k(x)}{h} : v = -f'(x)^{-1}(f(x) - f(x_0)),\ x \in R^n\right\}.$$

REMARK 8.1. – *The hypotheses of Theorem 8.1 do not guarantee that f is onto R^n.*

EXAMPLE 8.1. – Let us consider

$$f\colon R^2 \to R^2, \qquad f(x_1, x_2) = \left(\frac{e^{x_1}}{\sqrt{1 + x_2^2}}, \frac{e^{x_1} x_2}{\sqrt{1 + x_2^2}}\right).$$

We have that (take $x_0 = (0, 0)$)

$$x \cdot F(x) \leqslant 1 + |x|^2, \qquad x \in R^2,$$

(for more details, see [77, page 931]).

Defining

$$k(x) = \ln(1 + |x|^2), \qquad x \in R^2,$$

we have that

$$D^+_{F(x)} k(x) \leq 1, \qquad x \in R^2,$$

thus $f: R^2 \to f(R^2)$ is a global diffeomorphism.

On the other hand it is clear that $f(R^2) \neq R^2$ since

$$\frac{e^{x_1}}{\sqrt{1 + x_2^2}} > 0, \qquad x = (x_1, x_2) \in R^2.$$

Let us now consider the problem of giving sufficient conditions that $f: R^n \to R^n$ of class C^1 with

$$\det f'(x) \neq 0, \qquad x \in R^n,$$

be a global diffeomorphism onto R^n.

THEOREM 8.2. – *The mapping $f \in C^1(R^n, R^n)$ is a global diffeomorphism onto R^n if*

(i) $\det f'(x) \neq 0$, $x \in R^n$;

(ii) *there exists a locally Lipschitzian coercive function $k: R^n \to R_+$ such that*

$$D_v^+ k(x) \leq w(k(x)), \qquad v = f'(x)^{-1} u, \qquad u \in R^n, \quad |u| = 1,$$

where $w \in \mathfrak{R}_0$.

PROOF. – As in the proof of Theorem 8.1 we consider the Cauchy problem

$$x' = F(x), \qquad x(0) = \tilde{x}, \tag{8.1}$$

with $F(x) = -f'(x)^{-1}(f(x) - f(x_0))$, $x \in R^n$, where $x_0 \in R^n$ is fixed.

By Theorem 8.1 we have that f is injective.

Considering the opposite vector field to (8.1), namely the differential equation

$$y' = -F(y) \tag{8.2}$$

we have by Conti's comparison theorem (with k as a Lyapunov function) that all solutions of (8.2) are defined in the future thus all solutions of (8.1) are defined in the past.

If $x(t, \tilde{x})$ is a solution of (8.1), then we have that it is continuable in the past and in the future and

$$f(x(t, \tilde{x}) - f(x_0)) = e^{-t}(f(\tilde{x}) - f(x_0)), \qquad t \in R,$$

thus f maps $x(t, \tilde{x})$ into

$$\{y \in R^n : y = f(x_0) + s(f(\tilde{x}) - f(x_0)),\ 0 < s < \infty\}.$$

Since f is locally surjective (thus the image of a neighborhood of x_0 will be a neighborhood of $f(x_0)$) we can conclude the proof.

COROLLARY 8.2 [77]. – *The mapping $f \in C^1(R^n, R^n)$ is a global diffeomorphism onto R^n if*

(i) $\det f'(x) \neq 0$, $x \in R^n$;

(ii) *there exists a coercive function $k \in C^1(R^n, R_+)$ such that*

$$\sup\{\|k'(x) f'(x)^{-1}\| : x \in R^n\} < \infty .$$

PROOF. – We apply Theorem 8.2 with $w(r) = Mr$, $r \in R_+$, where

$$M = \sup\{\|k'(x) f'(x)^{-1}\| : x \in R^n\}.$$

COROLLARY 8.3 [40]. – *Let $f \in C^1(R^n, R^n)$ with* $\det f'(x) \neq 0$, $x \in R^n$. *Then f is a diffeomorphism onto R^n if there exists a function $w \in \mathfrak{R}_0$ such that*

$$\|f'(x)^{-1}\| \leq w(|x|), \qquad x \in R^n.$$

PROOF. – Taking $k: R^n \to R_+$, $k(x) = |x|$, we obtain that

$$D^+_{f'(x)^{-1}u} k(x) \leq \|f'(x)^{-1}\| \leq w(|x|) = w(k(x)), \qquad x \in R^n.$$

An application of Theorem 8.2 yields the result.

COROLLARY 8.4. – *The mapping $f \in C^1(R^n, R^n)$ is a global diffeomorphism onto R^n if*

(i) $\det f'(x) \neq 0$, $x \in R^n$;

(ii) *f is coercive.*

PROOF. – Defining $k: R^n \to R_+$, $k(x) = \ln(1 + |f(x)|^2)$, we obtain that

$$D^+_{f'(x)^{-1}u} k(x) \leq 1, \qquad x \in R^n,$$

and so we can apply Theorem 8.2.

REMARK 8.2. – *Since a global surjective diffeomorphism is coercive we have that the conditions of Theorem 8.2 are also necessary (for the necessity part, we define k as in the proof of Corollary 8.4). It should be also noted that the theorem of Caccioppoli-Banach-Mazur* [6, 17] *in the particular case of a local diffeomorphism $f: R^n \to R^n$ is a consequence of our results since the properness of f (i.e. $f^{-1}(C)$ is compact for any compact C) is equivalent in this case to coerciveness.*

REMARK 8.3. – *Our results are more flexible than the results of Zampieri since we look for a nonnegative auxiliary coercive function k which enables in Corollary* 8.2 *for example that*

$$\sup \{\|k'(x)\, f'(x)^{-1}\|: x \in R^n\} = \infty$$

provided that

$$\|k'(x)\, f'(x)^{-1}\| \leqslant w(k(x))\,, \qquad x \in R^n,$$

for some $w \in \mathfrak{R}_0$.

EXAMPLE 8.2. – Let us consider the nonlinear rotation f: $R^2 \to R^2$ defined by

$$\begin{pmatrix} \cos\phi(r) & -\sin\phi(r) \\ \sin\phi(r) & \cos\phi(r) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

where $r = \sqrt{x_1^2 + x_2^2}$. The mapping f is of class C^1 if ϕ: $R \to R$ is even and of class C^2.

We have that $f'(x)^{-1}$ is given by

$$\begin{pmatrix} \cos\phi(r) + x_2\phi'(r)\cos(\theta + \phi(r)) & \sin\phi(r) + x_2\phi'(r)\sin(\theta + \phi(r)) \\ -\sin\phi(r) - x_1\phi'(r)\cos(\theta + \phi(r)) & \cos\phi(r) - x_1\phi'(r)\sin(\theta + \phi(r)) \end{pmatrix}$$

where $x_1 = r\cos\theta$, $x_2 = r\sin\theta$.

It is easy to see that

$$\|f'(x)^{-1}\| \leqslant \sqrt{2}(1 + r|\phi'(r)|) = \sqrt{2}(1 + |x|\,|\phi'(|x|)|)\,, \qquad x \in R^2.$$

If

$$\int_1^\infty \frac{ds}{1 + s|\phi'(s)|} = \infty$$

we deduce by Corollary 8.3 that f is a diffeomorphism onto R^2.

If in addition we have

$$\frac{\pi}{2} < \phi(r) \leqslant \pi\,, \qquad \phi'(r) \geqslant 0\,, \qquad r \geqslant 0\,,$$

then

$$\operatorname{tr} f'(x) < 0\,, \qquad \det f'(x) = 1 > 0\,, \qquad x \in R^2,$$

and, since f is globally one-to-one, we obtain that $x = 0$ is a global sink for $x' = f(x)$ (see [78]).

REMARK 6.4. – *In the particular case of Example* 8.2 *when* $r \to r\phi'(r)$ *is bounded* *on R_+, ZAMPIERI and GORNI [78] showed that f is injective. One can see that in this-case actually f is a diffeomorphism onto R^2.*

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