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Some Inclusion Theorems for Orlicz and Musielak-Orlicz Type Spaces (*).

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Introduction.

In various papers ([6-10, 12, 13]) some extensions of the classical Schur-Hardy inequality are studied in the case of integral operators of type

$$(\mathcal{K}f)(t) = \int_0^{+\infty} K(t, s) f(s)\, ds \tag{I}$$

where K is a homogeneous kernel and f belongs to some Köthe functional space.

In these papers the estimates are taken with respect to the Köthe norm of the space.

Recently in [2] we obtained analogous estimates for functions belonging to Orlicz or Musielak-Orlicz type spaces L^φ, with respect to the canonical modular functional.

These results enable us to say that, for example,

$$\mathcal{K}f \in L^\varphi(\mathbb{R}^+) \quad \text{if } A(\cdot)^{1+\gamma} f(\cdot) \in L^\varphi(\mathbb{R}^+),$$

where A is a suitable constant and γ is the degree of homogeneity of K. In general $f \in L^\varphi(\mathbb{R}^+)$ does not imply $\mathcal{K}f \in L^\varphi(\mathbb{R}^+)$.

Then this result has been extended in [3] to the case of fractional Musielak-Orlicz spaces; such spaces were introduced in [3] by using the Riemann-Liouville fractional integral of f, according with an idea developed in [19, 20] for the fractional Jordan variation.

A different definition of fractional Orlicz spaces is given in [15] by using a fractional derivative.

In this paper we consider integral operators of type (I) with not necessarily homogeneous kernel in which the domain is a bounded interval $]a, b[\subset \mathbb{R}^+$ instead of $]0, +\infty[$.

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In this case we obtain some estimates that imply the property

$$f \in L^{\varphi}(]a, b[) \Rightarrow \mathcal{K}f \in L^{\varphi}(]a, b[) \quad \text{(Section 2)}.$$

An interesting consequence is the following inclusion:

$$L^{\varphi}(]a, b[) \subset \bigcap_{\alpha \in]0, 1[} L^{\varphi, \alpha}(]a, b[),$$

where $L^{\varphi, \alpha}$ is the fractional Orlicz space of order $\alpha \in]0, 1[$.

This inclusion is strict in general, as we show by an example (Section 3). Moreover, it is no longer true if the base interval is unbounded; this is pointed out by means of two examples (Section 5).

The last section is concerned with some partial extension to the case in which the generating function of the space L^{φ} depends on a parameter and satisfies the h-boundedness condition (see [2-4]).

In this setting too, some estimates of operators (I) with homogeneous kernel are obtained (Theorem 5) and we have some inclusion properties.

However, the situation becomes different when K is the kernel of the fractional integral of order α of f, because it does not satisfy all the assumptions of Theorem 5.

Nevertheless, we show (Theorem 6) that it is possible to obtain again analogous results by suitably restricting the space L^{φ}.

1. - Notations and definitions.

Let $I =]a, b[\subset \mathbb{R}^+$ be a bounded interval, $\mathcal{L}$ the class of Lebesgue measurable sets of I. We will put $\mathbb{R}^+ =]0, +\infty[$, $\mathbb{R}_0^+ = [0, +\infty[$. Let Φ be the class of all functions $\varphi: \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that

(i) φ is convex,

(ii) $\varphi(0) = 0$ and $\varphi(u) > 0$ if $u > 0$.

Let us denote by $\mathcal{K}$ the class of all $\mathcal{L} \times \mathcal{L}$-measurable functions $K: I \times I \to \mathbb{R}_0^+$ such that, for every $(t, z) \in I \times I$

$$\text{(K.1)} \qquad 0 < \int_a^b K(t, z)\, dz \leq A_K; \qquad 0 < \int_a^b K(t, z)\, dt \leq A_K$$

for a constant $A_K > 0$.

Let $K \in \mathcal{K}$ be fixed and let X_K be the space of all classes of equivalence of measurable functions $f: I \to \mathbb{R}$ for which the integral

$$\text{(1)} \qquad (\mathcal{K}f)(t) = \int_a^b K(t, z) f(z)\, dz$$

is well-defined for almost all $t \in I$, as a Lebesgue integral.

We will assume the following condition:

(K.2) $$\mathcal{K}f = 0 \quad \text{if and only if } f = 0 .$$

We will consider the functionals on X_K defined by:

$$\rho(f) = \int_a^b \varphi(|f(t)|)\,dt , \qquad \rho^{\mathcal{K}}(f) = \rho(\mathcal{K}f), \qquad \rho_*^{\mathcal{K}}(f) = \rho(\mathcal{K}|f|).$$

It is well-known that ρ and $\rho_*^{\mathcal{K}}$ are convex modulars on X (see [16]); moreover by (K.2), $\rho^{\mathcal{K}}$ is a modular too. The corresponding modular spaces (of Orlicz type) are denoted by $L^{\varphi}(a, b)$, $L^{\varphi,\mathcal{K}}(a, b)$, $L_*^{\varphi,\mathcal{K}}(a, b)$ respectively. The terms «ρ-convergence», «$\rho^{\mathcal{K}}$-convergence», «$\rho_*^{\mathcal{K}}$-convergence», denote the respective modular convergences.

Note that (K.2) is only used in order to make $\rho^{\mathcal{K}}$ a modular; most of the results contained in this paper can be obtained also if $\rho^{\mathcal{K}}$ is a pseudomodular.

2. - Some relation between $L^{\varphi}, L^{\varphi,\mathcal{K}}, L_*^{\varphi,\mathcal{K}}$.

It is clear that $L_*^{\varphi,\mathcal{K}}(a, b) \subset L^{\varphi,\mathcal{K}}(a, b)$ since by (i), $\rho^{\mathcal{K}}(f) \leq \rho_*^{\mathcal{K}}(f)$. Thus the most interesting relations will relate L^{φ}, $L_*^{\varphi,\mathcal{K}}$ or $L^{\varphi,\mathcal{K}}$.

THEOREM 1. – *For every $f \in X_K$ we have:*

(2) $$\rho_*^{\mathcal{K}}(f) \leq \rho(A_K f).$$

PROOF. – Without loss of generality we can assume $\rho(A_K f) < +\infty$. For $t \in]a, b[$, we write:

$$\int_a^b K(t, z) f(z)\,dz = \int_a^b f(z)\,d\mu_t(z)$$

where μ_t is the measure on $]a, b[$ defined by

$$\mu_t(E) = \int_E K(t, z)\,dz .$$

Moreover we put $h(t) = \mu_t(]a, b[)$, $t \in]a, b[$, and $\tilde{\mu}_t := (1/h(t))\mu_t$. Then $\tilde{\mu}_t(]a, b[) = 1$ and so by (i) and Jensen inequality,

$$\varphi\left(\int_a^b K(t, z)|f(z)|\,dz\right) = \varphi\left(h(t)\int_a^b |f(z)|\,d\tilde{\mu}_t(z)\right) \leq \int_a^b \varphi(h(t)|f(z)|)\,d\tilde{\mu}_t(z) =$$

$$= \frac{1}{h(t)}\int_a^b \varphi(h(t)|f(z)|)\,K(t, z)\,dz \leq \frac{1}{h(t)}\int_a^b \varphi\left(\frac{h(t)}{A_K} A_K\,|f(z)|\right) K(t, z)\,dz .$$

Again by (i), we have

$$\varphi\left(\int_a^b K(t,z)\,|f(z)|\,dz\right) \leqslant \frac{1}{A_K}\int_a^b \varphi(A_K\,|f(z)|)\,K(t,z)\,dz\,.$$

Then by Fubini Theorem

$$\rho_*^{\mathcal{K}}(f) \leqslant \frac{1}{A_K}\int_a^b \varphi(A_K\,|f(z)|)\left\{\int_a^b K(t,z)\,dt\right\}dz \leqslant \rho(A_K f)\,.$$

As a consequence of Theorem 1, we obtain also:

$$\rho^{\mathcal{K}}(f) \leqslant \rho(A_K f), \qquad f \in X_K\,,$$

and we have the inclusions:

$$L^{\varphi}(a,b) \subset L^{\varphi,\mathcal{K}}(a,b), \qquad L^{\varphi}(a,b) \subset L_*^{\varphi,\mathcal{K}}(a,b),$$

In section 3 we will prove that the inclusions are strict in general.

COROLLARY 1. – *Let $\{f_n\}$ be a sequence in $L^{\varphi}(]a,b[)$. Suppose that $\{f_n\}$ is ρ-convergent to $f \in L^{\varphi}(]a,b[)$. Then $\{f_n\}$ is $\rho^{\mathcal{K}}$ and $\rho_*^{\mathcal{K}}$ convergent to f.*

PROOF. – The assertion follows by

$$\rho^{\mathcal{K}}(\lambda(f_n - f)) \leqslant \rho_*^{\mathcal{K}}(\lambda(f_n - f)) \leqslant \rho(\lambda A_K(f_n - f))\,,$$

for $\lambda > 0$.

3. – A particular case: the fractional Orlicz spaces.

Let $X = \{f\colon I \to \mathbb{R}\colon f \in L^1]a,c[$, for every $c \in]a,b[\}$ and for each $f \in X$ let us denote by $f_{(1-\alpha)}$ the Riemann-Liouville fractional integral of order $1-\alpha$, $\alpha \in]0,1[$ (see [5, 17]):

$$f_{(1-\alpha)}(t) = \frac{1}{\Gamma(1-\alpha)}\int_a^t \frac{f(z)}{(t-z)^{\alpha}}\,dz\,. \tag{3}$$

The operator $f_{(1-\alpha)}$ is well-defined for every $f \in X$ and it can be written in the form:

$$f_{(1-\alpha)}(t) = \int_a^b K_\alpha(t,z) f(z)\,dz\,, \qquad \alpha \in]0,1[$$

where

$$K_\alpha(t, z) = \frac{1}{\Gamma(1-\alpha)} \frac{1}{(t-z)^\alpha} \chi_{]a, t[}(z), \qquad (t, z) \in I \times I .$$

It is easy to show that (K.1) and (K.2) are satisfied ((K.2) follows by Titchmarsh Theorem [14]). In this case $X_{K_\alpha} = X$ and the respective modulars will be denoted by ρ^α and ρ_*^α.

The corresponding Orlicz type spaces $L^{\varphi, \alpha}(I)$, $L_*^{\varphi, \alpha}(I)$ are called «fractional Orlicz spaces». As corollary of Theorem 1 we have

COROLLARY 2. – *For every* $f \in X$, $\alpha \in]0, 1[$, *we have*

$$\rho_*^\alpha(f) \leqslant \rho\left(\frac{(b-a)^{1-\alpha}}{\Gamma(2-\alpha)} f\right).$$

PROOF. – It is sufficient to remark that:

$$\int_a^b K_\alpha(t, z)\, dz \leqslant \frac{(b-a)^{1-\alpha}}{\Gamma(2-\alpha)} .$$

Analogously, taking into account that $\chi_{]a, t[}(z) = \chi_{]z, b[}(t)$ we have,

$$\int_a^b K_\alpha(t, z)\, dt \leqslant \frac{(b-a)^{1-\alpha}}{\Gamma(2-\alpha)}$$

and hence we can take $A_\alpha := \dfrac{(b-a)^{1-\alpha}}{\Gamma(2-\alpha)}$.

COROLLARY 3. – $L^\varphi(I) \subset \bigcap_{\alpha \in]0, 1[} L_*^{\varphi, \alpha}(I)$.

The following example shows that the inclusion in Corollary 3 is strict in general.

EXAMPLE 1. – Let $I =]0, 1[$, $\varphi(u) = u^p$, $p \geqslant 1$. Let f be defined on I by $f(t) = t^{-1/p}$. Then

$$f_{(1-\alpha)}(t) = \frac{\Gamma(1-1/p)}{\Gamma(2-1/p-\alpha)} t^{1-(\alpha+1/p)} .$$

So $f \notin L^\varphi(I)$, and since $p(1-\alpha-1/p) > -1$, for every $\alpha \in]0, 1[$, we have $f \in L_*^{\varphi, \alpha}(I)$ for every $\alpha \in]0, 1[$.

The following result relates the spaces $L_*^{\varphi, \alpha}(I)$ and $L_*^{\varphi, \beta}(I)$, $\alpha, \beta \in]0, 1[$, $\alpha \neq \beta$.

THEOREM 2. – *Let $f \in X$. If $\alpha < \beta$, then:*

$$\rho_*^\alpha(f) \leq \rho_*^\beta\left(\frac{\Gamma(1-\beta)}{\Gamma(1-\alpha)}(b-a)^{\beta-\alpha} f\right). \tag{4}$$

PROOF. – We can assume that the right hand side of (4) is finite. If $\alpha < \beta$, we have:

$$\int_a^t \frac{|f(v)|}{(t-v)^\alpha} dv \leq (b-a)^{\beta-\alpha} \int_a^t \frac{|f(v)|}{(t-v)^\beta} dv,$$

for a.e. $t \in]a, b[$ and so, by (i), the inequality (4) follows.

From Theorem 2, we deduce $L_*^{\varphi,\beta}(I) \subset L_*^{\varphi,\alpha}(I)$ and ρ_*^β-convergence implies ρ_*^α-convergence in $L_*^{\varphi,\beta}(I)$.

REMARK 1. – We remark that our setting contains also certain «weighted» cases of the Riemann-Liouville fractional integral. For example we can consider kernels of the form:

$$K(x, t) = \sigma x^{-\sigma\eta} t^{\sigma\eta+\sigma-1} \chi_{]a, x[}(t), \qquad (x, t) \in]a, b[\times]a, b[$$

for suitable constants σ and η and kernels of the fractional power of the operators defined by K (see e.g. [18]).

4. – The case of homogeneous kernels.

Here, as a simple consequence of Theorem 1, we state a «bounded» version of estimates for integral operators (1), where K: $\mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}_0^+$ is Lebesgue measurable and homogeneous of degree $\gamma \in \mathbb{R}$.

Let K be such a function, $]a, b[$ a bounded interval in $\mathbb{R}^+$ and let us denote again by X_K the class of all functions f: $]a, b[\to \mathbb{R}$ which are measurable and the integral

$$(\mathcal{K}f)(t) = \int_a^b K(t, z) f(z)\, dz$$

is well-defined as a Lebesgue integral.

We have the following:

THEOREM 3. – *Suppose that K: $\mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}_0^+$ satisfies the following assumptions:*

(K.3) *K is homogeneous of degree $\alpha \geq -1$;*

(K.4) $K(1, \cdot)$, $K(\cdot, 1) \in L^1(\mathbb{R}^+)$.

Then, for the interval $]a, b[\subset \mathbb{R}^+$, *K verifies* (K.1), *and for every* $f\in X_K$ *it results:*

$$\rho_*^{\mathcal{K}}(f) \leq \rho(Af), \tag{5}$$

where A is defined by

$$A = \max\left\{ b^{1+\alpha}\int_{a/b}^{+\infty} K(1, v)\,dv,\ \ b^{1+\alpha}\int_{a/b}^{+\infty} K(v, 1)\,dv \right\}.$$

PROOF. – For $t, z \in]a, b[$ we have:

$$\int_a^b K(t, z)\,dz = t^{1+\alpha}\int_{a/t}^{b/t} K(1, v)\,dv \leq b^{1+\alpha}\int_{a/b}^{+\infty} K(1, v)\,dv < +\infty .$$

The second condition in (K.1) follows by a symmetry argument.

REMARK 2. – (2.*a*) From the proof we can see that if $a > 0$ then the assertion of Theorem 3 is valid for kernels of general homogeneity degree $\alpha \in \mathbb{R}$ and such that $K(1, \cdot), K(\cdot, 1) \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$.

(2.*b*) Theorem 3 represents a bounded version of Theorem 2 in [2]. For unbounded intervals, inequality (5) may fail. As an example we can consider the kernel of Example *a*) in [2]:

$$K(t, z) = \left(\frac{z}{t}\right)^{\beta}\frac{1}{z^{\alpha}}\chi_{]0,t[}(z) \qquad \beta > 1, \quad 0 < \alpha < 1, \quad (t, z) \in]0, +\infty[\times]0, +\infty[.$$

Then K satisfies (K.3) and (K.4) of Theorem 3, but there is $f \in L^{\varphi}(]0, +\infty[)$ such that $\mathcal{K}f \notin L^{\varphi}(]0, +\infty[)$, with suitable φ.

(2.*c*) The kernel K_α of the Riemann-Liouville fractional integral is homogeneous of degree $-\alpha > -1$, but it does not satisfy the assumption (K.4) of Theorem 3. However, the functions

$$g(t) = t^{1-\alpha}\int_{a/t}^{b/t} K(1, v)\,dv, \qquad h(t) = t^{1-\alpha}\int_{a/t}^{b/t} K(v, 1)\,dv$$

are bounded by the constant $A = (b-a)^{1-\alpha}/\Gamma(2-\alpha)$ (see the proof of Corollary 2) and so, for $K = K_\alpha$, the estimate (5) holds.

A related result is given by the following theorem:

THEOREM 4. – *Let* $K\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}_0^+$ *be a homogeneous kernel of degree* $\alpha \geq -1$. *Let* $]a, b[$ *a bounded interval in* $\mathbb{R}^+$.

Suppose that:

$$0 < A_K := \int_{a/b}^{+\infty} K(1, v)\,dv < +\infty\,; \qquad B_K := \int_{a/b}^{+\infty} v^{-1} K(1, v)\,dv < +\infty\,.$$

Then, for every $f \in X_K$ *we have:*

$$\rho_*^{\mathcal{K}}(f) \leqslant \frac{B_K}{A_K}\,\rho(A_K b^{1+\alpha} f)\,. \tag{6}$$

PROOF. – Putting $h(t) := t^{1+\alpha}\int_{a/t}^{b/t} K(1, v)\,dv$, $t \in]a, b[$, we have $h(t) \leqslant A_K b^{1+\alpha}$, for every $t \in]a, b[$.

Thus, using the same methods as in Theorems 1 and 3, we deduce:

$$\varphi\left(\int_a^b K(t, z)\,|f(z)|\,dz\right) \leqslant \frac{1}{A_K}\int_{a/t}^{b/t} \varphi(A_K b^{1+\alpha}\,|\tilde f(tv)|)\,K(1, v)\,dv\,.$$

Here, $\tilde f(t) = f(t)$, if $t \in]a, b[$; $\tilde f(t) = 0$, if $t \notin]a, b[$.

Then by Fubini Theorem,

$$\rho_*^{\mathcal{K}}(f) \leqslant \frac{1}{A_K}\int_a^b \left\{\int_{a/t}^{b/t} \varphi(A_K b^{1+\alpha}\,|\tilde f(tv)|)\,K(1, v)\,dv\right\} dt \leqslant$$

$$\leqslant \frac{1}{A_K}\int_{a/b}^{+\infty} \left\{\int_a^b \varphi(A_K b^{1+\alpha}\,|\tilde f(tv)|)\,dt\right\} K(1, v)\,dv \leqslant$$

$$\leqslant \frac{1}{A_K}\int_{a/b}^{+\infty} \left\{\int_a^b \varphi(A_K b^{1+\alpha}\,|f(z)|)\,dz\right\} \frac{K(1, v)}{v} \leqslant \frac{B_K}{A_K}\,\rho(A_K b^{1+\alpha} f)\,.$$

REMARK 3. – If $a > 0$, the condition $A_K < +\infty$, implies $B_K < +\infty$. If $a = 0$, then $A_K < +\infty$ does not imply $B_K < +\infty$; moreover, $A_K + B_K < +\infty$ does not imply that $K(v, 1)$ is integrable on $\mathbb{R}^+$. Indeed, consider the following example:

$$\left(\frac{z}{t}\right)^{\beta} \frac{1}{z^{\alpha}}\,\chi_{]0, t[}(z)\,, \qquad (t, z) \in \mathbb{R}^+ \times \mathbb{R}^+\,,$$

with $0 < \alpha < \beta < 1$.

This kernel is homogeneous of degree $-\alpha > -1$, $A_K < +\infty$, $B_K < +\infty$, but $\int_0^{+\infty} K(t, 1)\,dt = +\infty$.

5. – Unbounded case: two counterexamples in fractional case.

Here we prove that for unbounded domains, the fractional Orlicz spaces $L_*^{\varphi,\alpha}(I)$ and $L^\varphi(I)$ are not comparable.

Let $I =]0, +\infty[$, $\varphi \in \Phi$, $X = \{f\colon]0, +\infty[\to \mathbb{R}\colon f \in L^1(]a, b[)$, for every $b > a\}$ and denote again by $\rho, \rho^\alpha, \rho_*^\alpha$, $\alpha \in]0, 1[$ the corresponding modulars and by $L^\varphi(I)$, $L^{\varphi,\alpha}(I)$, $L_*^{\varphi,\alpha}(I)$ the corresponding Orlicz spaces.

Here,

$$f_{(1-\alpha)}(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f(v)}{(t-v)^\alpha}\, dv, \qquad f \in X.$$

EXAMPLE 2. – Let $\varphi(u) = u^p$, $p \geq 1$ and let $f\colon I \to \mathbb{R}$ be the function defined by:

$$f(t) = \begin{cases} 0 & 0 < t < 1, \\ t^{-\beta} & t > 1, \end{cases}$$

where $\beta > 0$ is chosen in such a way that $p\beta > 1$.

Thus $f \in L^\varphi$. Moreover

$$f_{(1-\alpha)}(t) = \frac{t^{1-(\alpha+\beta)}}{\Gamma(1-\alpha)} \int_{1/t}^1 v^{-\beta}(1-v)^{-\alpha}\, dv.$$

Now,

$$\lim_{t \to +\infty} f_{(1-\alpha)}(t) = \begin{cases} +\infty & \text{if } \alpha + \beta < 1, \\ \Gamma(1-\beta) & \text{if } \alpha + \beta = 1. \end{cases}$$

So, for $\alpha + \beta \leq 1$, $f_{(1-\alpha)} \notin L^\varphi$ and hence $f \notin L^{\varphi,\alpha}$.

EXAMPLE 3. – Let $\varphi(u) = u^p$, $p \geq 1$ and

$$f(t) = \begin{cases} t^{-\beta} & 0 < t < 1, \\ 0 & t > 1, \end{cases}$$

where β is chosen in such a way that $p\beta > 1$, $0 < \beta < 1$.

Thus $f \notin L^{\varphi}$. Moreover:

$$f_{(1-\alpha)}(t) = \begin{cases} \dfrac{\Gamma(1-\beta)}{\Gamma(2-\alpha-\beta)} t^{1-(\alpha+\beta)} & \text{if } 0 < t < 1\,, \\ \dfrac{t^{1-(\alpha+\beta)}}{\Gamma(1-\alpha)} \displaystyle\int_0^{1/t} v^{-\beta}(1-v)^{-\alpha}\, dv & \text{if } t > 1\,. \end{cases}$$

Let us put $g(t) = \int_0^{1/t} v^{-\beta}(1-v)^{-\alpha}\, dv$.

Then $\lim_{t \to +\infty} t^{1-\beta} g(t) = 1/(1-\beta) \neq 0$.

Suppose now that $p(1-\alpha-\beta) > -1$ and $p\alpha > 1$. Then, we have:

$$\rho_*^{\alpha}(f) = \int_0^{+\infty} \varphi(|f|_{(1-\alpha)}(t))\, dt = \left\{ \int_0^1 + \int_1^{+\infty} \right\} \varphi(|f|_{(1-\alpha)}(t))\, dt = I_1 + I_2\,.$$

Then $I_1 < +\infty$, and there is $M > 0$ such that

$$I_2 = \frac{1}{\Gamma(1-\alpha)} \int_1^{+\infty} t^{-p\alpha}[t^{1-\beta} g(t)]^p\, dt \leq M \int_1^{+\infty} t^{-p\alpha}\, dt < +\infty\,.$$

Thus $f \in L_*^{\varphi,\alpha}$.

6. – Some partial extensions to fractional Musielak-Orlicz spaces.

Let ψ: $\mathbb{R}^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a function such that $\psi(\cdot, u)$ is measurable for every $u \in \mathbb{R}_0^+$, $\psi(t, \cdot)$ is convex, and $\psi(t, 0) = 0$, for every $t \in \mathbb{R}^+$ and $\psi(t, u) > 0$ for $u > 0$. Let $I =]a, b[\subset \mathbb{R}^+$ be a bounded interval.

We will assume that ψ is also «h-bounded» in I ([16, 2-4]), i.e. there exists a constant $H > 0$ such that, for every $u \in \mathbb{R}_0^+$, $t, z \in \mathbb{R}^+$ with $t \in]az, bz[$, we have

$$\psi(tz^{-1}, u) \leq \psi(t, Hu) + \Delta(t, z)\,, \tag{7}$$

where Δ is a measurable function on $\mathbb{R}^+ \times \mathbb{R}^+$ and such that the function

$$h(z) = \int_{az}^{bz} \Delta(t, z)\, dt$$

is well-defined a.e. $z \in \mathbb{R}^+$.

Note that this condition is different from the classical Δ_2-condition (see e.g. [16, 11]). Indeed, for example $\varphi(t, u) \equiv \varphi(u) = e^u - 1$ clearly satisfies (7) but, as it is well-known, it does not satisfy the Δ_2-condition; other nontrivial examples can be found in [3].

Here we consider the case of non trivial homogeneous kernels $K\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}_0^+$. Let, as before, X_K be the classes of all (classes of equivalence of) measurable functions $f\colon I \to \mathbb{R}$ for which the integral (1) is well-defined for almost all $t \in I$, as a Lebesgue integral.

As before we consider the functionals on X_K defined by:

$$\eta(f) = \int_a^b \psi(t, |f(t)|)\,dt, \qquad \eta^{\mathcal{K}}(f) = \eta(\mathcal{K}f), \qquad \eta_*^{\mathcal{K}}(f) = \eta(\mathcal{K}|f|).$$

The functionals η and $\eta_*^{\mathcal{K}}$ are modulars on X_K; denote by $L^{\psi}(I)$ and $L_*^{\psi,\mathcal{K}}(I)$ the corresponding Musielak-Orlicz spaces. If K verifies (K.2), $\eta^{\mathcal{K}}$ is a modular too and $L^{\psi,\mathcal{K}}(I)$ denotes the respective Musielak-Orlicz space.

Theorems 5 and 6 below are generalized versions of Theorem 4.

THEOREM 5. – *Let $K\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}_0^+$ be a homogeneous kernel of degree $\alpha \in \mathbb{R}$. Then, for every $f \in X_K$ the following statements hold:*

(5.*a*) *Suppose $I =]a, b[$ with $a > 0$ and $\alpha \geq -1$. If*

$$0 < A_K := \int_{a/b}^{b/a} K(1, v)\,dv < +\infty, \qquad \text{and} \qquad C_K := \int_{a/b}^{b/a} h(v)v^{-1}K(1, v)\,dv < +\infty,$$

then we have

$$\eta_*^{\mathcal{K}}(f) \leq B_K A_K^{-1}\eta(HA_K b^{1+\alpha}f) + C_K A_K^{-1}, \tag{8}$$

where $B_K = \int_{a/b}^{b/a} K(1, v)v^{-1}\,dv.$

(5.*b*) *Suppose $a > 0$, and $\alpha < -1$. Then* (8) *holds with $a^{1+\alpha}$ instead of $b^{1+\alpha}$.*

(5.*c*) *Suppose $a = 0$, and $\alpha \geq -1$. If* $0 < D_K := \int_0^{+\infty} K(1, v)\,dv < +\infty$, $E_K := \int_0^{+\infty} v^{-1}K(1, v)\,dv < +\infty$, *and* $F_K := \int_0^{+\infty} h(v)v^{-1}K(1, v)\,dv < +\infty$, *then we have*

$$\eta_*^{\mathcal{K}}(f) \leq E_K D_K^{-1}\eta(b^{1+\alpha}D_K Hf) + F_K D_K^{-1}. \tag{8'}$$

PROOF. – (5.*a*) We can suppose that $\eta(HA_K b^{1+\alpha}f) < +\infty$.

By using the same notations and techniques as in Theorem 4, we have

$$\psi\left(t, \int_a^b K(t, z)|f(z)|\,dz\right) \leq \frac{1}{A_K}\int_{a/b}^{b/a} \psi(t, b^{1+\alpha}A_K|\tilde{f}(tv)|)K(1, v)\,dv.$$

Thus, by (7) and Fubini Theorem, we have

$$\eta_*^{\mathcal{K}}(f) \leqslant \frac{1}{A_K}\int_a^b \left\{ \int_{a/b}^{b/a} \psi(t, b^{1+\alpha}A_K|\tilde{f}(tv)|)K(1,v)\,dv \right\} dt =$$

$$= \frac{1}{A_K}\int_{a/b}^{b/a} \left\{ \int_{av}^{bv} \psi(zv^{-1}, b^{1+\alpha}A_K|\tilde{f}(z)|)\,dz \right\} \frac{K(1,v)}{v}\,dv \leqslant$$

$$\leqslant \frac{1}{A_K}\int_{a/b}^{b/a} \left\{ \int_{av}^{bv} \psi(z, b^{1+\alpha}A_K H|\tilde{f}(z)|)\,dz \right\} \frac{K(1,v)}{v}\,dv + \frac{1}{A_K}\int_{a/b}^{b/a} h(v)v^{-1}K(1,v)\,dv$$

This implies that

$$\eta_*^{\mathcal{K}}(f) \leqslant \frac{B_K}{A_K}\eta(A_K H b^{1+\alpha} f) + \frac{C_K}{A_K}\,.$$

Note that $B_K < +\infty$.

(5.*b*) This follows taking into account that $t^{1+\alpha} \leqslant a^{1+\alpha}$, for every $t \in]a, b[$, and so

$$\psi\left(t, \int_a^b K(t,z)f(z)\,dz\right) \leqslant \psi\left(t, a^{1+\alpha}\int_{a/b}^{b/a} K(1,v)|\tilde{f}(tv)|\,dv\right).$$

(5.*c*) We can suppose that $\eta(b^{1+\alpha}D_K Hf) < +\infty$.
As in (5.*a*), we deduce:

$$\psi\left(t, \int_0^b K(t,z)|f(z)|\,dz\right) \leqslant \frac{1}{D_K}\int_0^{+\infty} \psi(t, b^{1+\alpha}D_K|\tilde{f}(tv)|)K(1,v)\,dv$$

and, by similar reasonings, we have

$$\eta_*^{\mathcal{K}}(f) \leqslant \frac{1}{D_K}\int_0^{+\infty} \left\{ \int_0^{bv} \psi(z, b^{1+\alpha}D_K H|\tilde{f}(z)|)\,dz \right\} \frac{K(1,v)}{v}\,dv + \frac{1}{D_K}\int_0^{+\infty} h(v)v^{-1}K(1,v)\,d$$

and hence (8)′ follows.

As a Corollary, we have the inclusions

$$L^{\psi}(I) \subset L_*^{\psi,\mathcal{K}}(I) \subset L^{\psi,\mathcal{K}}(I)\,.$$

Unfortunately the kernel $K(t, z)$ of the fractional integral of a locally integrable function does not satisfy the assumption $E_K < +\infty$.

Thus the proof of inequality (8)′ cannot be repeated in this case. In order to give some results in this direction, we denote by Y_γ, $0<\gamma<1$, the class:

$$Y_\gamma = \{f \in X_K:\ \text{there is } \lambda > 0 \text{ with } \psi(t, \lambda|f(t)|) = O(t^{-\gamma}),\ t \to 0^+\}.$$

Here, as usual, ψ verifies the previous assumptions and K is a homogeneous kernel of degree α.

We have the following:

THEOREM 6. – *Let* $I =]0, b[$ *and let* Y_γ, $0<\gamma<1$ *the corresponding subset of* X_K. *Suppose that* K *is homogeneous of degree* $\alpha > \gamma - 2$, $\int_0^{+\infty} v^{-\gamma} K(1, v)\, dv < +\infty$, $F_K := \int_0^{+\infty} K(1, v) v^{-1} h(v)\, dv < +\infty$ *and* $A_K := \int_0^{+\infty} K(1, v)\, dv < +\infty$.
Then, for every $f \in Y_\gamma$, *we have*

$$\eta_*^{\mathcal{K}}\left(\frac{\lambda}{A_K H} f\right) \leq A\,\eta(\lambda f) + B,$$

where $\lambda > 0$ *is the constant corresponding to* f *in the definition of* Y_γ, H *is the constant of the h-boundedness and* A, B *are suitable constants.*

PROOF. – Let $\lambda > 0$ such that $\psi(t, \lambda|f(t)|) = O(t^{-\gamma})$, $t \to 0^+$ and H the constant of the h-boundedness of ψ.

With similar methods as before, we have

$$\eta_*^{\mathcal{K}}\left(\frac{\lambda}{A_K H} f\right) \leq \frac{1}{A_K} \int_0^{+\infty} K(1, v) v^{-(2+\alpha)} \left\{ \int_0^{bv} z^{1+\alpha} \psi(z, \lambda|\tilde{f}(z)|)\, dz \right\} dv + \\ + \frac{b^{1+\alpha}}{A_K} \int_0^{+\infty} K(1, v) v^{-1} h(v)\, dv = I_1 + I_2.$$

Now, $I_2 = b^{1+\alpha}(F_K/A_K)$. Next, let $\delta > 0$ be a constant such that $z^\gamma \psi(z, \lambda|f(z)|) \leq M$, for every $z \in]0, \delta[$.

Then,

$$I_1 \leq \frac{1}{A_K} \left\{ \int_0^{\delta/b} + \int_{\delta/b}^{+\infty} \right\} K(1, v) v^{-(2+\alpha)} \left[\int_0^{bv} z^{1+\alpha} \psi(z, \lambda|\tilde{f}(z)|)\, dz \right] dv = I_1^1 + I_1^2.$$

We have

$$I_1^1 \leq \frac{M}{A_K} \int_0^{\delta/b} K(1, v) \frac{1}{v^{2+\alpha}} \left\{ \int_0^{bv} z^{1+\alpha-\gamma}\, dz \right\} dv = \frac{M}{A_K} \frac{b^{2+\alpha-\gamma}}{2+\alpha-\gamma} \int_0^{\delta/b} K(1, v) v^{-\gamma}\, dv < +\infty.$$

Moreover,

$$I_1^2 \leqslant \frac{1}{A_K} \int_{\delta/b}^{+\infty} K(1, v) v^{-1} b^{1+\alpha} \left\{ \int_0^{bv} \psi(z, \lambda |\tilde{f}(z)|)\, dz \right\} dv \leqslant \frac{b^{2+\alpha}}{\delta} \eta(\lambda f).$$

Finally,

$$\eta_*^{\mathcal{K}}\left(\frac{\lambda}{A_K H} f\right) \leqslant \frac{b^{2+\alpha}}{\delta} \eta(\lambda f) + \frac{M}{A_K} \frac{b^{2+\alpha-\gamma}}{2+\alpha-\gamma} \int_0^{\delta/b} K(1, v) v^{-\gamma} dv + b^{1+\alpha} \frac{F_K}{A_K}.$$

Let now, $K(t, z) = K_\alpha(t, z)$, be the kernel of the fractional integration of order $\alpha \in]0, 1[$.

As consequence of Theorems 5 and 6 we have the following Corollaries. Here the spaces $L_*^{\psi, \alpha}(I)$, $L^{\psi, \alpha}(I)$ are the Musielak-Orlicz spaces of order α (see e.g. [3,4]).

COROLLARY 4. – *Let* $I =]a, b[\subset \mathbb{R}^+$ *with* $a > 0$. *Then, if* $C_K < +\infty$,

$$L^\psi(I) \subset \bigcap_{\alpha \in]0, 1[} L_*^{\psi, \alpha}(I) \subset \bigcap_{\alpha \in]0, 1[} L^{\psi, \alpha}(I).$$

PROOF. – The kernel of the fractional integration of order α, $\alpha \in]0, 1[$ satisfies the assumption (5.*a*) of Theorem 5, and so the assertion follows.

COROLLARY 5. – *Let* $I =]0, b[$, $b > 0$. *Let* Y_γ *be the corresponding subset of* $X_K = X_\alpha = L^1_{\mathrm{loc}}(I)$. *Then, if* $F_K < +\infty$, $A_K < +\infty$,

$$L^\psi(I) \cap Y_\gamma(I) \subset \bigcap_{\alpha \in]0, 1[} L_*^{\psi, \alpha}(I) \subset \bigcap_{\alpha \in]0, 1[} L^{\psi, \alpha}(I).$$

PROOF. – The other assumptions of Theorem 6 are satisfied for the kernel $K_\alpha(t, z)$. Indeed, e.g.

$$\int_0^{+\infty} v^{-\gamma} K(1, v)\, dv = \frac{\Gamma(1-\gamma)}{\Gamma(2-\alpha-\gamma)} < +\infty.$$

The question if in general $L^\psi(I) \subset \bigcap_{\alpha \in]0, 1[} L_*^{\psi, \alpha}(I)$, for $I =]0, b[$ is at the moment an open problem.

Moreover, it is easy to see that also in this setting we have that, if $\alpha < \beta$ then $L_*^{\psi, \beta}(I) \subset L_*^{\psi, \alpha}(I)$.

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