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Bounded Solutions for $\bar{\partial}$-Problem in Pseudo-Siegel Domains (*)(**)

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Sunto. – *Si studia il problema dell'esistenza di soluzioni limitate per l'equazione* $\bar{\partial}u = f$ *sui domini pseudo-Siegel* $S_p = \left\{\zeta \in \mathbb{C}^n : \sum_{j=1}^{n-1} |\zeta_j|^{2p_j} + \Im m\, \zeta_n^{p_n} - 1 < 0\right\}$ *quando il dato* $f \in C^{\infty}_{(0,1)}(\bar{S}_p)$ *soddisfa alla condizione* $|\zeta|^k |f| < +\infty$ *per* $|\zeta| \to \infty$.

Summary. – *We study the problem of the existence of bounded solutions for the equation* $\bar{\partial}u = f$ *on pseudo-Siegel domains* $S_p = \left\{\zeta \in \mathbb{C}^n : \sum_{j=1}^{n-1} |\zeta_j|^{2p_j} + \Im m\, \zeta_n^{p_n} - 1 < 0\right\}$ *when the data* $f \in C^{\infty}_{(0,1)}(\bar{S}_p)$ *satisfies the condition* $|\zeta|^k |f| < +\infty$ *for* $|\zeta| \to \infty$

Introduction.

Let S_p be the domain

$$\left\{(\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n : \sum_{j=1}^{n-1} |\zeta_j|^{2p_j} + \Im M\, \zeta_n^{p_n} - 1 < 0\right\},$$

$p = (p_1, \ldots, p_n) \in (\mathbb{Z}^+)^n$. S_p is a generalization of the classical Siegel domain

$$S = \left\{(\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n : \sum_{j=1}^{n-1} |\zeta_j|^2 + \Im m\, \zeta_n - 1 < 0\right\}$$

and we refer to it is a *pseudo-Siegel* domain.

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In this paper we deal with the existence of smooth bounded solutions of $\bar\partial u = f$, when f is a smooth bounded $\bar\partial$-closed (0,1)-form on $\bar{S}_p$.

S_p is biholomorphic to the bounded pseudoconvex domain

$$\mathbb{E}_p = \left\{(\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n : \sum_{j=1}^{n} |\zeta_j|^{2p_j} - 1 < 0\right\}$$

by $B\colon \mathbb{E}_p \to S_p$ (section 1). Thus we are led back to study $\bar\partial u = B^* f$ on $\mathbb{E}_p$. $B^* f$ is no longer bounded (it has a finite number of singular points on $\partial\mathbb{E}_p$) but if f satisfies the condition

$$|\zeta|^k |f| < +\infty$$

as

$$|\zeta| \to \infty , \qquad \text{for } k > 1 + p_n\left(1 - \frac{1}{\prod_{j=1}^{n} p_j}\right),$$

then a bounded solution exists for $\bar\partial v = B^* f$ such that $u = B_* v$ is bounded, smooth on S_p and $\bar\partial u = f$.

The analogous problem for unbounded domains was first considered in [11] where a bounded solution is obtained for a special class of pseudoconvex domains assuming, at least, $k > 1$ and with the additional hypothesis that supp $f \cap \partial\Omega$ is compact.

1. - The biholomorphism.

Let us consider the unbounded domain

$$S_p = \left\{\zeta \in \mathbb{C}^n : \rho(\zeta) = \sum_{j=1}^{n-1} |\zeta_j|^{2p_j} + \Im m\, \zeta_n^{p_n} - 1 < 0\right\}$$

where $p = (p_1, \ldots, p_n) \in (\mathbb{Z}^+)^n$: when $p = (1, \ldots, 1)$ $S_p = S$ is the Siegel domain which is strongly pseudoconvex and biholomorphic to the unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$ by the Cayley map. If $p_j > 1$, for some j, $1 \leq j \leq n$, S_p is weakly pseudoconvex.

Let us take

$$\mathbb{E}_p = \{z \in \mathbb{C}^n : r(z) = |z_1|^{2p_1} + \ldots + |z_n|^{2p_n} - 1 < 0\};$$

then, for every choice of the p_j-th roots, the map $B\colon \mathbb{E}_p \to S_p$ given by

$$B(z_1, \ldots, z_n) = \left(\left(\frac{i}{1 + z_n^{p_n}}\right)^{1/p_1} z_1, \ldots, \left(\frac{2i}{1 + z_n^{p_n}}\right)^{1/p_n} z_n\right) = (B_1(z), \ldots, B_n(z))$$

is defined and holomorphic on $\mathbb{C}^n - (\mathbb{C}^{n-1} \times \{e^{i\pi(1+2\lambda)/p_n}\})$, $\forall \lambda = 0, \ldots, p_n - 1$, where $\mathbb{C}^n \times \{e^{i\pi(1+2\lambda)/p_n}\}$ is the holomorphic tangent space to $\partial\mathbb{E}_p$ in $z^{(\lambda)} = (0, \ldots, e^{i\pi(1+2\lambda)/p_n})$.

We choose principal roots. B is a biholomorphism and it has the inverse

$$b(\zeta_1, \ldots, \zeta_n) = \left(\left(\frac{2}{2i - \zeta_n^{p_n}}\right)^{1/p_1} \zeta_1, \ldots, \left(\frac{1}{2i - \zeta_n^{p_n}}\right)^{1/p_n} \zeta_n\right) = (b_1(\zeta), \ldots, b_n(\zeta))$$

where the p_j-th root is the principal one, $1 \leq j \leq n$. When $p = (1, \ldots, 1)$, $b(\zeta)$ is the Cayley map.

If $\zeta \in \bar{S}_p$ is such that $|\zeta| \to \infty$ then it happens either $|\zeta_n| \to \infty$ for $1 \leq j \leq n-1$, and in this case, since $1 - \Im m\, \zeta_n^{p_n} > \sum_{j=1}^{n-1} |\zeta_j|^{2p_j}$, it follows that $|\zeta_j|^{2p_j}/|\zeta_n^{p_n}|$ is bounded and $|\zeta_n| \to \infty$. Hence one has:

$$\lim_{|\zeta| \to \infty} |b_r(\zeta)| = \lim_{|\zeta_n| \to \infty} |b_r(\zeta)| \leq C \lim_{|\zeta_n| \to \infty} \left(\frac{|\zeta_n|^{p_n/2}}{|2i - \zeta_n^{p_n}|}\right)^{1/p_r} = 0$$

for $1 \leq r \leq n-1$, and

$$\lim_{|\zeta| \to \infty} b_n^{p_n}(\zeta) = \lim_{|\zeta_n| \to \infty} b_n^{p_n}(\zeta) = -1$$

therefore $\lim_{|\zeta| \to \infty} b(\zeta) = (0, \ldots, 0, e^{i\pi(1+2\lambda)/p_n}) = z^{(\lambda)}$, for $0 \leq \lambda \leq p_n - 1$.

Characterization of Proper Holomorphic Mappings. Since in [8] the group $\mathrm{Aut}(\mathbb{E}_p)$ of the automorphism of $\mathbb{E}_p$ and the proper holomorphic mappings from $\mathbb{E}_p$ to $\mathbb{E}_q$ are completely described, the explicit expression of the biholomorphism $B\colon \mathbb{E}_p \to S_p$ naturally gives

1) *every automorphism Φ of S_p is conjugate to an automorphism of $\mathbb{E}_p$ in the sense that $\Phi = B \circ \Psi \circ b$ for $\Psi \in \mathrm{Aut}(\mathbb{E}_p)$;*

2) *every biholomorphic map from S_p is given by $B \circ \Psi$, with $\Psi \in \mathrm{Aut}(\mathbb{E}_p)$;*

3) *every proper holomorphic mapping $f\colon S_p \to \mathbb{E}_p$ is a biholomorphism;*

4) *there exist a proper holomorphic map $f\colon S_p \to S_q$ if and only if $p/q = (p_1/q_1, \ldots, p_n/q_n) \in (\mathbb{Z}^+)^n$ and it is, up to biholomorphisms of S_q,*

$$f(\zeta_1, \ldots, \zeta_n) = (\zeta_1^{p_1/q_1}, \ldots, \zeta_n^{p_n/q_n});$$

5) *every proper holomorphic self-mapping of S_p is a biholomorphism.*

2. – The $\bar\partial$-problem: the case with compact support.

We denote by $C^k_{(p,q)}(D)$ the vector space of (p,q)-forms with C^k-coefficients on a domain D, $0 \leq k \leq \infty$, and for $f \in C^k_{(p,q)}(D)$ let $|f(\zeta)| = \sum_{\substack{|I|=p\\|J|=q}} |f_{IJ}(\zeta)|$, $\|f\|_\infty = \sup_{\zeta \in D} |f(\zeta)|$.

If $f \in C^k_{(0,1)}(\bar{S}_p)$ then for the pull-back $F(z) = B^*f(\zeta) = \sum_{s=1}^{n} F_s(z)\, d\bar z_s$ of f by B, we have

$$(2.1)\qquad F_r(z) = \left(\frac{-i}{1+\bar z_n^{p_n}}\right)^{1/p_r} f_r(B(z)), \qquad 1 \leq r \leq n-1,$$

$$(2.2)\qquad F_n(z) = \sum_{r=1}^{n-1}\left[-\frac{p_n}{p_r}\left(\frac{-i}{1+\bar z_n^{p_n}}\right)^{1/p_n}\frac{\bar z_r \bar z_n^{p_n-1}}{1+\bar z_n^{p_n}}\right] f_r(B(z)) + $$
$$+\left(\frac{-2i}{1+\bar z_n^{p_n}}\right)^{1/p_n}\left(\frac{1}{1+\bar z_n^{p_n}}\right) f_n(B(z))$$

and it is $\bar\partial_z$-closed if f is $\bar\partial_\zeta$-closed; furthermore if U is the solution of the $\bar\partial$-equation in $\mathbb{E}_p$ then $u(\zeta) = U \circ b(\zeta)$ solves $\bar\partial_\zeta u = f$ in S_p and

$$(2.3)\qquad \lim_{|\zeta|\to\infty} |u(\zeta)| = \lim_{|\zeta|\to\infty} |U(b(\zeta))| = \lim_{z\to z^{(\lambda)}} |U(z)|.$$

Let us assume that $f \in C^1_{(0,1)}(\bar S_p)$ has compact support, that is $\operatorname{supp} f \cap \bar S_p$ is contained in a ball $B(0, r)$ and so there exists a neighbourhood B_λ of $z^{(\lambda)}$ such that the form $F = B^*f$ is identically zero on $B_\lambda \cap \bar{\mathbb{E}}_p$; moreover $F \in C^1_{(0,1)}(\bar{\mathbb{E}}_p)$, $\operatorname{supp} F \cap \partial\mathbb{E}_p \neq \emptyset$ and $\|F\|_\infty \leq C\|f\|_\infty$.

On the Siegel domain S, by Theorem 3.2 of [11], the ∂-equation $\bar\partial u = f$ has a bounded solution $u \in C^1(\bar S)$ which goes to 0 as $|\zeta| \to \infty$.

We prove that this is still true the domains S_p and for the Siegel domain S there is a Hölder solution with exponent $\alpha = 1/2$.

PROPOSITION 2.1. – *If $f \in C^1_{(0,1)}(\bar S_p)$ with compact support is $\bar\partial$-closed then there exists a bounded and Hölder solution of the equation $\bar\partial u = f$ such that*

$$\lim_{|\zeta|\to\infty} |u(\zeta)| = 0.$$

PROOF. – Let $\tilde f(\zeta) = (2i - \zeta_n)^h f(\zeta)$, where $h > 0$; since the form $\tilde F(z) = B^*\tilde f(z) = [2i - (B_n(z))^{p_n}]^h F(z) = (2i/(1+z_n^{p_n}))^h F(z)$ vanishes on B_λ, it is $C^1(\bar{\mathbb{E}}_p)$ and $\bar\partial$-closed because F is $\bar\partial$-closed. By [3] and [9], there exists a solution $\tilde U \in C^\infty(\mathbb{E}_p)$ of the equation $\bar\partial \tilde U = \tilde F$ which is a Hölder function with exponent $\alpha = 1/(2\max\{p_j\})$. Then the function

$$U(z) = \left(\frac{1+z_n^{p_n}}{2i}\right)^h \tilde U(z)$$

solves $\bar\partial U = F$ and

$$\text{i)}\quad |U(z)| \leq \left(\frac{1+|z_n|^{p_n}}{2}\right)^h |\tilde U(z)| \leq \|\tilde U\|_\infty,$$

$$\text{ii)}\quad \lim_{z\to z^{(\lambda)}} |U(z)| = \lim_{z\to z^{(\lambda)}} \frac{|1+z_n^{p_n}|^h}{2^h} |\tilde U(z)| = 0.$$

It follows that the function $u(\zeta) = U \circ b(\zeta)$ is bounded in S_p and by (2.3) it goes to 0 as $|\zeta| \to \infty$ with order $h > 0$. Moreover for $z, z' \in \mathbb{E}_p$ one has:

$$|U(z) - U(z')| \leq c|z - z'|^\alpha$$

for $\alpha = 1/(2\max\{p_j\})$, hence, since b is a Lipschitz function, one gets, for $\zeta, \zeta' \in S_p$:

$$|u(\zeta) - u'(\zeta)| = |U \circ b(\zeta) - U \circ b(\zeta')| \leq c|b(\zeta) - b(\zeta')|^\alpha \leq C|\zeta - \zeta'|^\alpha .$$

so u is α-Hölder continuous.

REMARK 2.1. – *When $f \in C^{m+1}_{(0,1)}(\bar S_p)$, by* [6], *for every $m \in \mathbb{N}$, there is a solution $\widetilde U \in C^m(\overline{\mathbb{E}}_p)$ of $\bar\partial \widetilde U = \widetilde F$, therefore, using the above arguments, one can obtain a solution $u \in C^m(\overline{S}_p)$ of $\bar\partial u = f$ such that* $\lim\limits_{|\zeta| \to \infty} |u(\zeta) = 0$.

3. – Analytic coverings and $\bar\partial$-problem.

The biholomorphic equivalence between S_p and $\mathbb{E}_p$ leads us to find bounded solutions for $\bar\partial$-equation on $\mathbb{E}_p$ when the data is singular on the boundary $\partial\mathbb{E}_p$. Since $\mathbb{E}_p$ is an analytic covering via $\pi_p: \mathbb{E}_p \to \mathbb{B}^n$, $\pi_p(z) = (z_1^{p_1}, \ldots, z_n^{p_n})$, we study the problem for a smoothly bounded pseudoconvex domain $\Omega \subset \mathbb{C}^n$ which is an analytic covering of a smoothly bounded strongly pseudoconvex domain $D \subset \mathbb{C}^n$, $n \geq 2$, that is there exists a proper holomorphic mapping $\pi: \Omega \to D$; assume that $\pi \in C^\infty(\bar\Omega)$.

To solve this problem we need an integral representation formula for the solution of the $\bar\partial$-equation on strongly pseudoconvex domains introduced by Dautov and Henkin [2]. To write the integral solution we construct in Ω the kernel as in [10] by means of the Henkin function. Following the results of Theorem 9 and Theorem 16 of [4], for simplicity, we can present the formula for D strictly convex.

For $\xi, w \in \overline{D}$ let

$$\widetilde\Phi_D(\xi, w) = \Phi_D(\xi, w) - s(\xi)$$

where $s \in C^2(\widetilde D, \mathbb{R})$ is a function strictly plurisubharmonic in a neighbourhood $\widetilde D$ of $\overline{D}$ defining D, $\operatorname{grad} s \neq \underline{0}$ on ∂D and $\Phi_D(\xi, w) = \sum_{j=1}^{n} (\partial s(\xi)/\partial \xi_j)(w_j - \xi_j)$ is the Henkin function of D. From Taylor's formula and strict convexity of D we have:

$$2\,\Re e\,\widetilde\Phi_D(\xi, w) \geq -s(\xi) - s(w) + \gamma|\xi - w|^2$$

with γ depending only on D, hence

$$|\widetilde\Phi_D(\xi, w)| \geq C_D[\,|s(\xi) - s(w)| + |\Im m\,\Phi_D(\xi, w)| - s(w) + |\xi - w|^2].$$

We define, for $\zeta, z \in \bar\Omega$,

$$\widetilde\Phi(\zeta, z) = \widetilde\Phi_D(\pi(\zeta), \pi(z)) .$$

For every $w \in \overline{D}$, $\pi^{-1}(w) = \{z^{(1)}, \ldots, z^{(k)}\}$ consists of a finite number of points in Ω each of which has its multiplicity m_j, $1 \leq j \leq k$ and $\sum_{j=1}^{k} m_j = M$ is the branching order of π. We consider the set

$$A_{j,w} = \{\zeta \in \Omega \colon |\zeta - z^{(j)}| = \min\{|\zeta - z^{(1)}|, \ldots |\zeta - z^{(k)}|\}\}$$

so that $\Omega = \bigcup_{j=1}^{k} A_{j,w}$. For fixed $\zeta \in \overline{\Omega}$, by Theorem 2.3 in [10], it follows that, for every $z \in \overline{\Omega}$

$$|\pi(\zeta) - \pi(z)|^2 \geq c \prod_{j=1}^{k} |\zeta - z^{(j)}|^{2m_j}, \tag{3.1}$$

where $\pi(z) = \pi(z^{(j)})$.

Taking into account (3.1) and the defining function $r = s \circ \pi$ of Ω, we get the following estimate:

$$|\widetilde{\Phi}(\zeta, z)| \geq C_D\left[|r(\zeta) - r(z)| + |\Im m\, \Phi(\zeta, z)| - r(z) + c \prod_{j=1}^{k} |\zeta - z^{(j)}|^{2m_j}\right] \tag{3.2}$$

for $z^{(j)} \in \pi^{-1}(\pi(z))$.

Let us consider the functions $a_h(\zeta, z) = [1 - (\Phi(\zeta, z)/\widetilde{\Phi}(\zeta, z))^{2n}]^h$, for $h \geq 0$, and $\eta = \eta(\zeta, z, \lambda) = \lambda(p(\zeta)/\Phi(\zeta, z)) + (1 - \lambda)((\bar\zeta - \bar z)/|\zeta - z|^2)$, where, by Lemma 1.3 in [10], $p(\zeta) = (p_1(\zeta), \ldots, p_n(\zeta)$ is such that $\Phi(\zeta, z) = \Phi_D(\pi(\zeta), \pi(z)) = \sum_{j=1}^{n} p_j(\zeta)(z_j - \zeta_j)$. Take the forms $\omega(\zeta) = \bigwedge_{i=1}^{n} d\zeta_i$, $\omega(\zeta + z) = \sum_{p=0}^{n} \omega_p(\zeta + z)$, where $\omega_p(\zeta + z)$ is a $(p, 0)$-form in z, an $(n - p, 0)$-form in ζ, and $\omega'(\eta) = \sum_{i=1}^{n} (-1)^{i-1} \eta_i \bigwedge_{j \neq i} d\eta_j$: using the same techniques as in [2] one can prove the following

PROPOSITION 3.1. – *Let $F \in C^{\infty}_{(p,q)}(\Omega)$, $0 \leq p \leq n - 1$, $1 \leq q \leq n$, $\bar\partial$-closed such that $r(z)^{h-1} F(\zeta)$ has integrable coefficients. Then the $(p, q - 1)$-forms*

$$U_h(z) = \frac{(-1)^q (n-1)!}{(2\pi i)^n}\left[\int_{\zeta \in \Omega} a_h(\zeta, z) F(\zeta) \wedge \omega'\left(\frac{\bar\zeta - \bar z}{|\zeta - z|}\right) \wedge \omega_p(\zeta + z) + \right.$$
$$\left. + \int_{(\zeta, \lambda) \in \Omega \times [0,1]} F(\zeta) \wedge \bar\partial_\zeta a_h(\zeta, z) \wedge \omega'(\eta) \wedge \omega_p(\zeta + z)\right]$$

for $h \geq 1$ and $z \in \Omega$, are solutions of the equation $\bar\partial U = F$ and $U \in C^{\infty}_{(p, q-1)}(\Omega)$.

REMARK 3.1. – *The Proposition is still true under the weaker hypothesis that F is a (p, q)-form with regular measure coefficients on the domain $\Omega \subset \mathbb{C}^n$.*

By the definition of $a_h(\zeta, z)$ and calculating $\bar\partial_\zeta a_h(\zeta, z)$, Proposition 3.1 implies the following integral representation formula for the solution of the $\bar\partial$-problem on Ω:

$$(3.3)\qquad U_h(z) = C_{n,q}\left[\int\limits_{\zeta\in\Omega}\frac{r(\zeta)^h F(\zeta)\wedge\psi(\zeta, z)}{\tilde\Phi(\zeta, z)^h|\zeta - z|^{2n-1}} + \right.$$

$$\left. + \int\limits_{\zeta\in\Omega}\frac{r(\zeta)^{h-1}F(\zeta)\wedge\bar\partial r(\zeta)\wedge\psi'(\zeta, z)}{\tilde\Phi(\zeta, z)^{h+1}|\zeta - z|^{2n-3}} + \int\limits_{\zeta\in\Omega}\frac{r(\zeta)^h F(\zeta)\wedge\psi''(\zeta, z)}{\tilde\Phi(\zeta, z)^{h+2}|\zeta - z|^{2n-4}}\right]$$

where $C_{n,q} = ((-1)^q(n-1)!)/(2\pi i)^n$ and $\psi(\zeta, z)$, $\psi'(\zeta, z)$, $\psi''(\zeta, z)$ are forms with coefficients in $C^\infty(\Omega\times\Omega)\cap L^\infty(\Omega\times\Omega)$.

If $F\in C^\infty_{(p,q)}(\Omega)$, $q\geq 1$, satisfies the following condition:

there exists $\alpha\in\mathbb{R}$, $\alpha < 1/M$, *such that*

$$\sup_{\zeta\in\Omega}|r(\zeta)|^\alpha[|F(\zeta)| + |r(\zeta)|^{-1/2M}|F(\zeta)\wedge\bar\partial r(\zeta)|] < +\infty,$$

then $r(\zeta)^{h-1}F(\zeta)$ has integrable coefficients for every $h\geq 1$ and (3.3) yields

$$|U_h(z)| \leq C_{n,q}\left[\int\limits_{\zeta\in\Omega}\frac{|r(\zeta)|^{h-\alpha}dV(\zeta)}{|\tilde\Phi(\zeta, z)|^h|\zeta - z|^{2n-1}} + \right.$$

$$\left. + \int\limits_{\zeta\in\Omega}\frac{|r(\zeta)^{h-1-\alpha+1/2M}dV(\zeta)}{|\tilde\Phi(\zeta, z)|^{h+1}|\zeta - z|^{2n-3}} + \int\limits_{\zeta\in\Omega}\frac{|r(\zeta)|^{h-\alpha}dV(\zeta)}{|\tilde\Phi(\zeta, z)|^{h+2}|\zeta - z|^{2n-4}}\right]$$

(where dV is the Lebesgue measure). Therefore, in this case, to get bounded solutions for the $\bar\partial$-equation we estimate integrals of type:

$$I_{a,b,c} = \int\limits_{\zeta\in\Omega}\frac{|r(\zeta)|^a dV(\zeta)}{|\tilde\Phi(\zeta, z)|^b|\zeta - z|^c}.$$

By the definition, $\tilde\Phi(\zeta, z^{(j)}) = \tilde\Phi(\zeta, z)$, $\forall j$, $1\leq j\leq k$, hence (3.2) and the property of $A_{j,w}$ imply that

$$I_{a,b,c}\leq C_D\sum_{j=1}^{k}\int\limits_{\zeta\in A_{j,\omega}}\frac{|r(\zeta)|^a dV(\zeta)}{[|r(\zeta) - r(z^{(j)})| + |\Im m\,\Phi(\zeta, z^{(j)})| - r(z^{(j)}) + c|\zeta - z^{(j)}|^{2M}]^b|\zeta - z^{(j)}|^c}.$$

LEMMA 3.1. - *Let* $M\geq 1$ *and*

$$J_{a,b,c} = \int\limits_{\zeta\in\Omega}\frac{|r(\zeta)|^a dV(\zeta)}{[|r(\zeta) - r(z)| + |\Im m\,\Phi(\zeta, z)| - r(z) + c|\zeta - z|^{2M}]^b|\zeta - z|^c}.$$

Then $J_{a,b,c}$ is bounded for $c = 2n-1$ if $a-b+1>0$ and for $c = 2n-3, 2n-4$ if $2M(a-b+2)+2(n-1)-c>0$.

$J_{a,b,c} \leq C\log(-r(z))$ for $c = 2n-1$ if $a-b+1=0$ and for $c = 2n-3, 2n-4$ if $2M(a-b+2)+2(n-1)-c=0$.

PROOF. – For the sake of simplicity we can assume that $\operatorname{diam}(\Omega) < 1$.

Choose, as usual, coordinates $t_1, \ldots, t_n$ with $t_1 = r(z) - r(\zeta) \geq 0$, $t_2 = \Im m\, \Phi(\zeta, z)$, and $t = |\zeta - z|$, $\varepsilon = -r(z)$ so that we have

$$|r(\zeta) - r(z)| - r(z) = |t_1| + \varepsilon$$

therefore

$$J_{a,b,c} \leq \int\limits_{\substack{|t|<1 \\ t_1+\varepsilon>0}} \frac{(t_1+\varepsilon)^a\, dt_1 \ldots dt_n}{[|t_1| + |t_2| + \varepsilon + c|t|^{2M}]^b\, |t|^c}.$$

One can use polar coordinates $t_1 = \rho\cos\phi_1$, $t_2 = \rho\cos\phi_2 \operatorname{sen}\phi_1$ and following [2] we put $s_1 = su$, $s_2 = s(1-|u|)$ with $s_1 = \cos\phi_1$, $s_2 = |\cos\phi_2|$ so that $ds_1\, ds_2 = -s\, ds\, du$ for $0 \leq su \leq 1$. Therefore one gets

$$J_{a,b,c} \leq C_0 \int\limits_{\substack{0\leq\rho,\, s\leq 1 \\ -1\leq u\leq 1 \\ \rho su+\varepsilon>0}} \frac{(\rho su+\varepsilon)^a s \rho^{2n-1-c}}{(\rho s + \varepsilon + r^{2M})^b}\, d\rho\, ds\, du.$$

When $c = 2n-1$, one takes $\rho s = \varepsilon v$, which implies $s\, d\rho\, ds\, du = \varepsilon\, dv\, ds\, du$ and with the method of [2] one gets $J_{a,b,c}$ bounded for $a-b+1>0$, $J_{a,b,c} \leq C\log\varepsilon$ if $a-b+1=0$.

When $c = 2n-3$ or $2n-4$ one puts $2n-1-c = m+2$ getting:

$$J_{a,b,c} \leq C_0 \int_0^1 \rho^m\, d\rho \int_0^1 \frac{\rho\, ds}{(\rho s + \varepsilon + c\rho^{2M})^b} \int_{-\varepsilon/\rho s}^1 (\rho su + \varepsilon)^a \rho s\, du \leq$$

$$\leq C_1 \int_0^1 \rho^m\, d\rho \int_0^1 \rho(\rho s + \varepsilon + c\rho^{2M})^{a-b+1}\, ds.$$

If $a-b+2 \geq 0$ then

$$J_{a,b,c} \leq C_2 \int_0^1 \rho^m [(\rho + \varepsilon + c\rho^{2M})^{a-b+2} - (\varepsilon + c\rho^{2M})^{a-b+2}]\, d\rho \leq$$

$$\leq C_3 \int_0^1 \rho^m (\rho + \varepsilon + c\rho^{2M})^{a-b+2}\, d\rho \leq C_3'(1+c+\varepsilon)^{a-b+2} \int_0^1 \rho^m\, d\rho \leq C_3(1+c+\varepsilon)^{a-b+2}.$$

If $a - b + 2 < 0$ then

$$J_{a,b,c} \leqslant C_2' \int_0^1 \rho^m [(\varepsilon + c\rho^{2M})^{a-b+2} - (\rho + \varepsilon + c\rho^{2M})^{a-b+2}]\, d\rho \leqslant$$

$$\leqslant C_3' \int_0^1 \rho^m (\varepsilon + c\rho^{2M})^{a-b+2} d\rho \leqslant C_4' \int_0^1 \left[\left(\frac{\varepsilon}{c}\right)^{1/2M} + \rho\right]^{m + 2M(a-b+2)} d\rho$$

where one uses the Hölder inequality $(A+B)^p \leqslant C(A^p + B^p)$, $p \in \mathbb{R}$, $p \geqslant 0$. Hence

$$J_{a,b,c} \leqslant C_5' \left[\left(1 + \left(\frac{\varepsilon}{c}\right)^{1/2M}\right)^{2M(a-b+2)+m+1} - \left(\frac{\varepsilon}{c}\right)^{a-b+2+(m+1)/2M}\right]$$

which is bounded when $\varepsilon \to 0$ if $2M(a-b+2)+m+1 > 0$, $m = 0, 1$.

If $2M(a-b+2)+m+1 = 0$ we get $J_{a,b,c} \leqslant C \log \varepsilon$.

Now it follows

THEOREM 3.1. – *Let $\Omega \subset \mathbb{C}^n$, $n \geqslant 2$, be a smoothly bounded pseudoconex domain, let $\pi\colon \Omega \to D$ be an analytic covering with branching order M on a smoothly bounded strongly pseudoconvex domain $D \subset \mathbb{C}^n$, and $\pi \in C^\infty(\bar\Omega)$.*

Let $F \in C^\infty_{(p,q)}(\Omega)$ be $\bar\partial$-closed. If there is $\alpha \in \mathbb{R}$ such that $\alpha < 1/M$ and

$$\sup_{\zeta \in \Omega} |r(\zeta)|^\alpha [|F(\zeta)| + |r(\zeta)|^{-1/2M} |F(\zeta) \wedge \bar\partial r(\zeta)|] < +\infty \tag{3.4}$$

then there exists $U \in C^\infty_{(p,q-1)}(\Omega) \cap L^\infty_{(p,q-1)}(\Omega)$ which is a solution of the equation $\bar\partial U = F$.

PROOF. – The functions $U_h(z)$, defined by (3.3), are $C^\infty_{p,q-1}(\Omega)$ and solve $\bar\partial U = F$, for $h \geqslant 1$. Hence (3.4) give integrals of type $J_{h-\alpha,h,2n-1}$, $J_{h-1-\alpha+(1/2M),h+1,2n-3}$, $J_{h-\alpha,h+2,2n-4}$. Since $\alpha < 1/M$ and $h \geqslant 1$ applying Lemma 3.1, it follows that $J_{h-\alpha,h,2n-1}$, $J_{h-1-\alpha+(1/2M),h+1,2n-3}$, $J_{h-\alpha,h+2,2n-4}$ are bounded when $r(z) \to 0$.

REMARK 3.2. – *When F has bounded coefficients,* (3.4) *is satisfied for every α such that $1/2M \leqslant \alpha < 1/M$.*

The condition (3.4) *is sharp.* For this we give two examples. With the first one we point out the significance of the behaviour of the $\bar\partial$-closed $(0,1)$-form F at the boundary of the domain. The second one shows that the condition on the exponent $\alpha < 1/M$ is sharp.

Both examples are taken in the unit ball $\mathbb{B}^2 \subset \mathbb{C}^2$; obviously in this case $M = 1$.

1) Let us take the (0,1)-form with C^∞ coefficients in $\mathbb{B}^2$

$$F(z_1, z_2) = \frac{d\bar{z}_1}{(1+z_2)^\beta}, \qquad \frac{1}{2} < \beta < 1$$

then

$$|r(z)|^\alpha |F(z)| = \frac{(1-|z_1|^2-|z_2|^2)^\alpha}{|1+z_2|^\beta} \leqslant \frac{(1-|z_2|^\alpha)}{|1+z_2|^\beta} \leqslant 2^\alpha |1+z_2|^{\alpha-\beta}$$

which is bounded for $\beta \leqslant \alpha < 1$, while

$$|r(z)|^{\alpha-1/2}|F(z) \wedge \bar{\partial} r(z)| = \frac{(1-|z_1|^2-|z_2|^2)^{\alpha-1/2}|z_2|}{|1+z_2|^\beta} \leqslant 2^{\alpha-1/2}|1+z_2|^{\alpha-\beta-1/2}$$

which is bounded for $\alpha \geqslant \beta + 1/2 > 1$.

Hence the condition (3.4) is not satisfied. The function

$$U(z) = \frac{\bar{z}_1}{(1+z_2)^\beta}$$

is a solution of $\bar{\partial} U = F$, but it is not bounded and $U \notin L^p(\mathbb{B}^2)$ for $p > 6/(2\beta - 1)$. One can prove as in [7] that $U(z)$ is the unique solution of the $\bar{\partial}$-equation which is orthogonal to the holomorphic functions in $L^2(\mathbb{B}^2)$ so that any other solution V of $\bar{\partial} U = F$ can be decomposed as $V = P(V) + U$, where P is the Bergman projection of the ball. In this case P maps L^p into itself (cf. [14]), so it follows that if V is a bounded solution then we would have $U = V - P(V) \in L^p(\mathbb{B}^2)$, which is not possible. Hence the equation $\bar{\partial} U = F$ has no bounded solution.

2) Consider the C^∞, (0,1)-form on $\mathbb{B}^2$

$$G(z_1, z_2) = \frac{-z_1}{1-|z_1|^2-|z_2|^2} d\bar{z}_1 - \frac{z_2}{1-|z_1|^2-|z_2|^2} d\bar{z}_2 .$$

Then $|r(z)|\,|G(z)|$ is bounded and $G(z) \wedge \bar{\partial} r(z) = 0$. So condition (3.4) is satisfied for $\alpha = 1$, but in $\mathbb{B}^2$ the C^∞ function

$$U(z) = \log(1 - | - |z_1|^2 - |z_2|^2)$$

is a solution of $\bar{\partial} U = G$ which is not bounded. Moreover there is no holomorphic function $H(z)$ such that $V(z) = U(z) + H(z)$ is a bounded solution of $\bar{\partial} U = G$: in fact if there exists such an $H(z)$, the function $W(z) = (1 - |z_1|^2 - |z_2|^2)\, e^{H(z)}$ is bounded by below in a neighbourhood of $\partial \mathbb{B}^2$ so that $e^{-H(z)}$ is a holomorphic function which goes to 0 as z goes to $\partial \mathbb{B}^2$: this is impossible by the maximum principle.

4. – The existence of bounded solutions.

Now we assume that $f \in C^{\infty}_{(0,1)}(\overline{S}_p)$ and

$$|f(\zeta)| \simeq \frac{1}{|\zeta|^k} \quad \text{when } |\zeta| \to \infty .$$

This condition implies, for each components f_r of f, that there exists a constant C_r such that

$$|\zeta|^k |f_r(\zeta)| \leq C_r \quad \text{as } |\zeta| \to \infty ; \tag{4.1}$$

from section 1 it follows that, for $\zeta = B(z)$,

$$\left[\sum_{s=1}^{n-1} |B_s(z)|^2 + |B_n(z)|^2\right]^{k/2} |f_r(B(z))| \leq$$

$$\leq \left[\sum_{s=1}^{n-1} \frac{|z_s|^2}{|1 + z_n^{p_n}|^{2/p_s}} + 4^{1/p_n} \frac{|z_n|^2}{|1 + z_n^{p_n}|^{1/p_n}}\right]^{k/2} |f_r(B(z))| \leq C_r$$

for $z \to z^{(\lambda)}$.

Since $\sum_{s=1}^{n-1} |z_s|^{2p_s} < 1 - |z_n|^{2p_n}$ then the function

$$\sum_{s=1}^{n-1}\left[\frac{|z_s|^2}{|1 + z_n^{p_n}|^{2/p_s}} + 4^{1/p_n} \frac{|z_n|^2}{|1 + z_n^{p_n}|^{2/p_n}}\right]^{k/2}$$

goes to infinity as the function $|1 + z_n^{p_n}|^{-k/p_n}$ when $z \to z^{(\lambda)}$, so $|f_r(B(z))|$ goes to 0 with the same order as $|1 + z_n^{p_n}|^{k/p_n}$, $\forall r$, $1 \leq r \leq n$.

The (0,1)-form $F(z) = B^* f(z)$ has C^{∞}-smooth coefficients on $\mathbb{E}_p - \bigcup_{\lambda=0}^{p_n-1} B(z^{(\lambda)}, \delta)$, where $B(z^{(\lambda)}, \delta)$ is a ball with center in $z^{(\lambda)}$, radius a suitable δ, $0 < \delta < 1$ and from (2.2), (2.3), one has:

$$|F_r(z)| = \frac{|f_r(B(z))|}{|1 + z_n^{p_r}|^{1/p_r}} \leq C_r |1 + z_n^{p_n}|^{(k/p_n) - (1/p_r)} \tag{4.2}$$

for $z \to z^{(\lambda)}$, $1 \leq r \leq n-1$ and

$$|F_n(z)| \leq \sum_{r=1}^{n-1} \frac{p_n}{p_r} \frac{|z_r|\,|z_n|^{p_n-1}}{|1 + z_n^{p_n}|^{1+(1/p_r)}} |f_r(B(z))| + 2^{1/p_n} \frac{|f_n(B(z))|}{|1 + z_n^{p_n}|^{1+(1/p_n)}} \leq \tag{4.3}$$

$$\leq \sum_{r=1}^{n-1} C_r \frac{p_n}{p_r} |1 + z_n^{p_n}|^{(k/p_n) - (1/p_r)} + 2^{1/p_n} C_n |1 + z_n^{p_n}|^{[(k-1)/p_n] - 1} .$$

LEMMA 4.1. – *Let $r(z)$ be the defining function of $\mathbb{E}_p$, $M = \prod_{j=1}^{n} p_j$ and $p_0 = \min\{p_1, \ldots, p_n\}$. If $F \in C^{\infty}_{(0,1)}(\mathbb{E}_p)$ is the form defined above and it satisfies* (4.2), (4.3) *for $k \geq p_n/p_0$ then*

$$\sup_{z \in \mathbb{E}_p} |r(z)|^{\alpha}[|F(z)| + |r(z)|^{-1/2M}|F(z) \wedge \bar\partial r(z)|] < +\infty \tag{4.4}$$

for $\alpha = \max(1/2M, 1 - [(k-1)/p_n])$.

PROOF. – Using the previous notations, when $z \in \overline{\mathbb{E}}_p - \bigcup_{\lambda=0}^{p_n - 1} B(z^{(\lambda)}, \delta)$, the condition (4.4) is obviously satisfied; when $z \in \mathbb{E}_p \cap B(z^{(\lambda)}, \delta)$ we have:

$$|z_r|^{2p_r - 1} = \left(1 - \sum_{\substack{s=1\\ s\neq r}}^{n} |z_s|^{2p_s}\right)^{(2p_r-1)/2p_r} \leq (1 - |z_n|^{2p_n})^{1-(1/2p_r)} =$$

$$= [(1 + |z_n|^{p_n})(1 - |z_n|^{p_n})]^{1-(1/2p_r)} \leq 2^{1-(1/2p_r)}|1 + z_n^{p_n}|^{1-(1/2p_r)}$$

for $1 \leq r \leq n-1$ and

$$|r(z)| = 1 - \sum_{r=1}^{n} |z_r|^{2p_r} \leq 1 - |z_n|^{2p_n} \leq 2|1 + z_n^{p_n}|. \tag{4.5}$$

By (4.2) and (4.3) we get

$$|F(z)| = \sum_{r=1}^{n} |F_r(z)| \leq c_1| + z_n^{p_n}|^{[(k-1)/p_n]-1}, \tag{4.6}$$

$$|F(z) \wedge \bar\partial r(z)| \leq \sum_{\substack{r,s=1\\ r\neq s}}^{n-1} [|z_s|^{2p_s-1}|F_r(z)| + p_r|z_r|^{2p_r-1}|F_s(z)|] + \tag{4.7}$$

$$+ |F_n(z)|\sum_{r=1}^{n-1} p_r|z_r|^{2p_r-1} + p_n|z_n|^{2p_n-1}\sum_{r=1}^{n-1}|F_r(z)| \leq$$

$$\leq c_2|1 + z_n^{p_n}|^{[(k-1)/p_n]-(1/2p_0)} = c_2|1 + z_n^{p_n}|^{[(k-1)/p_n]-(1/2p_0)}.$$

When $z \in \mathbb{E}_p \cap B(z^{(\lambda)}, \delta)$ we have

$$|r(z)|^{\alpha}[|F(z)| + |r(z)|^{-1/2M}|F(z) \wedge \bar\partial r(z)|] =$$

$$= |r(z)|^{\alpha}|F(z)| + |r(z)|^{\alpha-(1/2M)}|F(z) \wedge \bar\partial r(z)| \leq$$

$$\leq 2^{\alpha}c_1|1 + z_n^{p_n}|^{\alpha - 1 + [(k-1)/p_n]} + 2^{\alpha-(1/2M)}c_2|1 + z_n^{p_n}|^{\alpha-(1/2M)+[(k-1)/p_n]-(1/2p_0)} <$$

$$< 2^{\alpha-(1/2M)}c_3|1 + z_n^{p_n}|^{\alpha+[(k-1)/p_n]-1}$$

and this is bounded because $\alpha \geq 1 - [(k-1)/p_n]$.

We can prove the following

PROPOSITION 4.1. – *Let* $M = \prod_{j=1}^{n} p_j$. *If* $f \in C^{\infty}_{(0,1)}(\bar{S}_p)$ *is* $\bar{\partial}$*-closed and such that*

$$|f| \simeq \frac{1}{|\zeta|^k} \quad as\ |\zeta| \to \infty, \qquad \zeta \in \bar{S}_p$$

with $k > 1 + p_n(1 - 1/M)$, *then there exists a bounded solution* $u \in C^{\infty}(S_p)$ *of the* $\bar{\partial}$*-equation* $\bar{\partial}u = f$.

PROOF. – Let $B\colon \mathbb{E}_p \to S_p$ be the biholomorphism of section 1 with its inverse $b\colon S_p \to \mathbb{E}_p$. Since $k > 1 + p_n(1 - 1/M)$ is obviously greater than p_n/p_0, $\forall p_1, \ldots, p_n$, then the (0,1)-form $F = B^*f$ verifies the hypothesis of Lemma 4.1; moreover there exists $\alpha \in \mathbb{R}$, with $\max\{1/2M, 1 - [(k-1)/p_n]\} \leq \alpha < 1/M$ such that (4.4) continues to be true. Hence Theorem 3.1 gives a bounded solution $U \in C^{\infty}(\mathbb{E}_p)$ of $\bar{\partial}U = F$, so the function $u(\zeta) = U \circ b(\zeta) \in C^{\infty}(S_p)$ is a bounded solution of the $\bar{\partial}$-equation $\bar{\partial}u = f$ on S_p.

REMARK 4.1. – *We note that when* $k > 1 + p_n$, *using the same techniques of the proof of Proposition* 4.1, *one gets a solution which goes to* 0 as $|\zeta| \to \infty$.

REMARK 4.2. – *For the Siegel domain S,* $M = 1$ *and* $\alpha \geq 2 - k$. *Hence if* $f \in C^{1}_{(0,1)}(\bar{S})$ *is* $\bar{\partial}$*-closed and verifies the condition of the Proposition* 4.1 *then the* $\bar{\partial}$*-equation* $\bar{\partial}u = f$ *has a bounded solution for* $k > 1$. *By the example in* [11] *it follows that in* **S** *this condition is, in a certain sense, sharp.*

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