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A General Theory of Hypersurface Potentials (*).

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Summary. – *A general theory of hypersurface potentials in n-dimensional space is proposed. Not only smooth densities but also potentials generated either by L^1 functions or by measures are considered.*

In the last years methods for solving boundary value problems by means of boundary integral representations of the solution have acquired a large interest after the advent of the computational methods founded on boundary finite elements.

While in the case of two independent variables starting from the paper [10] potential theoretic methods for the representation of the solutions of general elliptic linear partial differential equations have been fully developed (see [12], [13], [16]) in the case of n greater then two variables, only very particular results are known for equations of order $2m$, $m > 1$. This is due to the fact that in spaces of dimension $n > 2$ a general theory of *hypersurface potentials* has not yet been developed. The aim of this paper is to fill this gap by proposing a general theory of hypersurface potentials in n-dimensional space. The theory will be expanded not only for smooth (i.e. hölder continuous) densities but also for hypersurface potentials generated either by L^1 functions or by measures.

1. – Basic results.

In this section $x = (x_1, \dots, x_{n-1})$ denotes a point of $\mathbb{R}^{n-1}$ $(n \geq 2)$. We assume $t \in \mathbb{R}$. By $\mathcal{H}^\lambda(\mathbb{R}^{n-1})$ we denote the space of the functions which are bounded and measurable in $\mathbb{R}^{n-1}$ and have a compact support. Moreover we suppose that each function of $\mathcal{H}^\lambda(\mathbb{R}^{n-1})$ satisfies a Hölder condition of exponent λ in the origin 0 of $\mathbb{R}^{n-1}$ $(0 < \lambda \leq 1)$.

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I. *Let* $K(x; t) \in C^0(\mathbb{R}^n - \{0\})$ *be a function such that*

$$|K(x; t)| \leq C(|x|^2 + t^2)^{(1-n)/2} \qquad \forall (x; t) \in \mathbb{R}^n - \{0\}. \tag{1.1}$$

Let us suppose

(i) *for any* $t > 0$ *the following integral exists*

$$\int_{\mathbb{R}^{n-1}} K(x; t)\, dx = \lim_{M \to \infty} \int_{|x| < M} K(x; t)\, dx\,;$$

(ii) *there exists* $\gamma \in \mathbb{R}$ *such that*

$$\int_{\mathbb{R}^{n-1}} K(x; t)\, dx = \gamma \qquad \forall\, t > 0\,; \tag{1.2}$$

(iii) *for any* $\delta > 0$

$$\lim_{t \to 0^+} \int_{|x| > \delta} K(x; t)\, dx = 0\,. \tag{1.3}$$

Then, for any $\varphi \in \mathcal{H}^\lambda(\mathbb{R}^{n-1})$, *we have*

$$\lim_{t \to 0^+} \int_{\mathbb{R}^{n-1}} \varphi(x) K(x; t)\, dx = \gamma \varphi(0) + \int_{\mathbb{R}^{n-1}} \varphi(x) K(x; 0)\, dx$$

where the last integral exists as a singular integral (*i.e. as* $\lim_{\delta \to 0^+} \int_{|x| > \delta}$).

Define

$$A(t) = \int_{\mathbb{R}^{n-1}} \varphi(x) K(x; t)\, dx - \gamma \varphi(0)\,; \qquad B(\delta) = \int_{|x| > \delta} \varphi(x) K(x; 0)\, dx\,.$$

Because of (1.2) we may write

$$A(t) - B(\delta) = \int_{|x| < \delta} [\varphi(x) - \varphi(0)] K(x; t)\, dx - \varphi(0) \int_{|x| > \delta} K(x; t)\, dx + \\ + \int_{|x| > \delta} \varphi(x)[K(x; t) - K(x; 0)]\, dx\,.$$

Observing that

$$|[\varphi(x) - \varphi(0)] K(x; t)| \leq C |\varphi(x) - \varphi(0)| (|x|^2 + t^2)^{(1-n)/2} \leq C' |x|^{1-n+\lambda}, \tag{1.4}$$

for any fixed $\delta > 0$ we have

$$\lim_{t \to 0^+} \int_{|x| < \delta} [\varphi(x) - \varphi(0)] K(x; t)\, dx = \int_{|x| < \delta} [\varphi(x) - \varphi(0)] K(x; 0)\, dx\,.$$

Since $K(x; t) \in C^0(\mathbb{R}^n - \{0\})$ and the support of φ is contained in a ball

$\{x \in \mathbb{R}^{n-1} \mid |x| \leqslant M\}$, we have

$$\lim_{t\to 0^+} \int_{|x|>\delta} \varphi(x)[K(x;\, t) - K(x;\, 0)]\, dx = \tag{1.6}$$

$$= \lim_{t\to 0^+} \int_{\delta<|x|<M} \varphi(x)[K(x;\, t) - K(x;\, 0)]\, dx = 0\,.$$

From (1.3), (1.5), (1.6) it follows that there exists the following limit and

$$\lim_{t\to 0^+} A(t) = B(\delta) + \int_{|x|<\delta} [\varphi(x) - \varphi(0)]\, K(x;\, 0)\, dx\,.$$

On the other hand, because of (1.4),

$$\lim_{\delta\to 0^+} \int_{|x|<\delta} [\varphi(x) - \varphi(0)]\, K(x;\, 0)\, dx = 0\,.$$

This completes the proof of the Theorem.

We say that $K(x;\, t)$ is *homogeneous of degree* α if $K(\rho x;\, \rho t) = \rho^\alpha K(x;\, t)$, $\forall (x;\, t) \in \mathbb{R}^n - \{0\}$, $\rho > 0$. We write $K(x;\, t) \in C^\lambda(\mathbb{R}^n - \{0\})$ if $K(x;\, t)$ satisfies a uniform Hölder condition on any compact set in $\mathbb{R}^n - \{0\}$.

II) *If* $K(x;\, t) \in C^\lambda(\mathbb{R}^n - \{0\})$ *is homogeneous of degree* $1 - n$, *the conditions* (i), (ii), (iii) *of Theorem I are satisfied if and only if*(1)

$$\int_{|\eta|=1} K(\eta;\, 0)\, d\sigma_\eta = 0 \tag{1.7}$$

where $d\sigma_\eta$ *represents the area element on the* $(n-1)$*-dimensional unit sphere* $|\eta| = 1$.

Let us suppose $0 < \delta < M$; because of (1.7) we may write

$$\int_{\delta<|x|<M} K(x;\, t)\, dx = \int_\delta^M \rho^{n-2} d\rho \int_{|\eta|=1} K(\rho\eta;\, t)\, d\sigma_\eta = \tag{1.8}$$

$$= \int_\delta^M \rho^{-1} d\rho \int_{|\eta|=1} [K(\eta;\, t/\rho) - K(\eta;\, 0)]\, d\sigma_\eta\,.$$

If $0 \leqslant t \leqslant 1$, $\rho \geqslant \delta$, then $1 \leqslant |\eta|^2 + (t/\rho)^2 \leqslant 1 + 1/\delta^2$. In other words, $(\eta;\, t/\rho)$ be-

(1) From now on (1.7) must be understood as $K(-1;\, 0) + K(1;\, 0) = 0$, if $n = 2$.

longs to a compact set contained in $\mathbb{R}^n - \{0\}$. This implies

$$\rho^{-1}\,|K(\eta;\, t/\rho) - K(\eta;\, 0)| \leq Ht^{\lambda}\rho^{-1-\lambda}, \qquad \forall \rho \geq \delta. \tag{1.9}$$

Therefore the limit

$$\lim_{M\to\infty} \int\limits_{\delta<|x|<M} K(x;\, t)\, dx = \lim_{M\to\infty} \int_{\delta}^{M} \rho^{-1} d\rho \int\limits_{|\eta|=1} [K(\eta;\, t/\rho) - K(\eta;\, 0)]\, d\sigma_\eta$$

exists and it is finite. On the other hand

$$\int\limits_{|x|<\delta} K(x;\, t)\, dx$$

is finite ($t > 0$) and therefore condition (i) is satisfied. Moreover from (1.8), (1.9) it follows that

$$\left| \int\limits_{|x|>\delta} K(x;\, t)\, dx \right| = \left| \int_{\delta}^{+\infty} \rho^{-1} d\rho \int\limits_{|\eta|=1} [K(\eta;\, t/\rho) - K(\eta;\, 0)]\, d\sigma_\eta \right| \leq$$

$$\leq Ht^{\lambda} \int_{\delta}^{+\infty} \rho^{-1-\lambda} d\rho \int\limits_{|\eta|=1} d\sigma_\eta = H'\, t^{\lambda} \delta^{-\lambda},$$

H' being a constant independent of t and δ; then (iii) is satisfied. Condition (ii) holds, because

$$\int\limits_{\mathbb{R}^{n-1}} K(x;\, t)\, dx = \int_0^{+\infty} \rho^{n-2} d\rho \int\limits_{|\eta|=1} K(\rho\eta;\, t)\, d\sigma_\eta = t^{1-n} \int_0^{+\infty} \rho^{n-2} d\rho \int\limits_{|\eta|=1} K(\rho\eta/t;\, 1)\, d\sigma_\eta$$

and, by making the substitution $\rho = \tilde{\rho} t$, the last term is equal to

$$\int_0^{+\infty} \tilde{\rho}^{n-2} d\tilde{\rho} \int\limits_{|\eta|=1} K(\tilde{\rho}\eta;\, 1)\, d\sigma_\eta = \int\limits_{\mathbb{R}^{n-1}} K(x;\, 1)\, dx\,.$$

Conversely, since

$$\int\limits_{\delta<|x|<M} K(x;\, t)\, dx = \int_{\delta}^{M} \rho^{-1} d\rho \int\limits_{|\eta|=1} [K(\eta;\, t/\rho) - K(\eta;\, 0)]\, d\sigma_\eta + \log \frac{M}{\delta} \int\limits_{|\eta|=1} K(\eta;\, 0)\, d\sigma_\eta$$

condition (i) and (1.9) imply that

$$\lim_{M\to\infty} \log \frac{M}{\delta} \int\limits_{|\eta|=1} K(\eta;\, 0)\, d\sigma_\eta$$

is finite. This is possible if and only if (1.7) holds.

We remark that (1.7) is just the well known necessary and sufficient condition for the existence of the singular integral

$$\int_{\mathbb{R}^{n-1}} \varphi(x) K(x; 0)\, dx$$

(see e.g. [1], [2], [3], [15]).

The next Lemma gives a simple sufficient condition under which (1.7) is satisfied.

III) *If* $h(x; t) \in C^1(\mathbb{R}^n - \{0\})$ *is even (i.e.* $h(-x; -t) = h(x; t)$*) and* $K(x; t) = \partial h(x; t)/\partial t$*, then* (1.7) *is satisfied.*

If $h(x; t)$ is even then $K(x; t)$ is odd. Thus (1.7) holds, because $K(-x; 0) = - K(x; 0)$.

In order to unify the study for $n = 2$ and $n \geqslant 3$, it will be convenient to consider the following definition. We say that $h(x; t)$ is *essentially homogeneous of degree* α if $h(x; t)$ is homogeneous in case $\alpha < 0$ or, if α is an integer $\geqslant 0$, $h(x; t)$ has the form $h(x; t) = h_1(x; t) \log \sqrt{|x|^2 + t^2} + h_2(x; t)$, where $h_1(x; t)$ is a homogeneous polynomial of degree α and $h_2(x; t)$ is homogeneous of degree α.

The next Theorem provides a necessary and sufficient condition for some kernels under which (1.7) holds.

IV) *If* $h(x; t) \in C^2(\mathbb{R}^n - \{0\})$ *is essentially homogeneous of degree* $2 - n$ *and* $K(x; t) = \partial h(x; t)/\partial t$*, then* (1.7) *holds if and only if*

$$\int_{\substack{|\omega| = 1 \\ \omega_n > 0}} \Delta_2 h(\omega)\, d\Sigma_\omega = 0 \tag{1.10}$$

where Δ_2 *represents the Laplacean* $\sum_{k=1}^{n-1} \partial^2/\partial x_k^2 + \partial^2/\partial t^2$*, and* $d\Sigma_\omega$ *represents the area element on the n-dimensional unit sphere* $|\omega| = 1$.

We remark that, obviously, if $h(x; t)$ is essentially homogeneous of degree $2 - n$, then $K(x; t)$ is homogeneous of degree 1-n. Let D be the domain $\{(x; t) | 1 < |x|^2 + t^2 < e^2,\ t > 0\}$. We have

$$\int_D \Delta_2 h\, dx\, dt = - \int_{\partial D} \frac{\partial h}{\partial \nu} d\Sigma$$

ν being the inward unit normal to ∂D and $d\Sigma$ the area element on ∂D. If $X = (x; t)$, $R = |X|$, $\omega = X/R$, we have

$$h(X) = h_1 \log R + R^{2-n} h_2(\omega); \qquad \frac{\partial h}{\partial R} = R^{-1} h_1 + (2 - n) R^{1-n} h_2(\omega) \tag{1.11}$$

(where h_1 is a constant if $n = 2$ or $h_1 = 0$ if $n \geq 3$). Thus

$$\int_{\partial D} \frac{\partial h}{\partial \nu} d\Sigma = \int_1^e \rho^{n-2} d\rho \int_{|\eta|=1} K(\rho\eta; 0)\, d\sigma_\eta + \int_{\substack{|X|=1 \\ t>0}} \frac{\partial h}{\partial R} d\Sigma - \int_{\substack{|X|=e \\ t>0}} \frac{\partial h}{\partial R} d\Sigma =$$

$$= \int_1^e \rho^{-1} d\rho \int_{|\eta|=1} K(\eta; 0)\, d\sigma_\eta + \int_{\substack{|\omega|=1 \\ \omega_n>0}} [h_1 + (2-n) h_2(\omega)]\, d\Sigma_\omega -$$

$$- \int_{\substack{|\omega|=1 \\ \omega_n>0}} [e^{-1} h_1 + (2-n) e^{1-n} h_2(\omega)]\, e^{n-1} d\Sigma_\omega = \int_{|\eta|=1} K(\eta; 0)\, d\sigma_\eta .$$

On the other hand

$$\int_D \Delta_2 h\, dx\, dt = \int_1^e R^{n-1} dR \int_{\substack{|\omega|=1 \\ \omega_n>0}} \Delta_2 h(R\omega)\, d\Sigma_\omega =$$

$$= \int_1^e R^{-1} dR \int_{\substack{|\omega|=1 \\ \omega_n>0}} \Delta_2 h(\omega)\, d\Sigma_\omega = \int_{\substack{|\omega|=1 \\ \omega_n>0}} \Delta_2 h(\omega)\, d\Sigma_\omega .$$

Consequently

$$\int_{\substack{|\omega|=1 \\ \omega_n>0}} \Delta_2 h(\omega)\, d\Sigma_\omega = - \int_{|\eta|=1} K(\eta; 0)\, d\sigma_\eta$$

and the conclusion follows immediately.

Now we are in a position to calculate the constant γ of Theorem I for the kernels we are considering.

V) *If* $h(x; t) \in C^2(\mathbb{R}^n - \{0\})$ *is essentially homogeneous of degree* $2 - n$ *and it satisfies* (1.10), *then*

$$\gamma = \begin{cases} \pi h_1 - \displaystyle\int_{\substack{|\omega|=1 \\ \omega_2>0}} \Delta_2 h_2(\omega) \log|\omega_2|\, d\Sigma_\omega & n = 2 , \\ \displaystyle\int_{\substack{|\omega|=1 \\ \omega_n>0}} [(2-n) h(\omega) - \Delta_2 h(\omega) \log|\omega_n|\,]\, d\Sigma_\omega & n \geq 3 , \end{cases} \tag{1.12}$$

where $\gamma = \int_{\mathbb{R}^{n-1}} K(x; t)\, dx$ *and* $K(x; t) = \partial h(x; t)/\partial t$.

Define $B_M = \{(x;\tau) \mid |x|^2 + \tau^2 < M^2 + t^2;\ \tau > t\}$, $\partial^+ B_M = \{(x;\tau) \mid |x|^2 + \tau^2 = M^2 + t^2;\ \tau \geq t\}$. We have

$$(1.13)\qquad \int_{B_M} \Delta_2 h\, dx\, dt = -\int_{\partial B_M} \frac{\partial h}{\partial \nu} d\Sigma = -\int_{|x|<M} K(x;t)\, dx - \int_{\partial^+ B_M} \frac{\partial h}{\partial \nu} d\Sigma .$$

Let us introduce the spherical coordinates in the following way

$$\begin{cases} x_h = \rho \eta_h \sin \vartheta & h = 1, \dots, n-1, \\ t = \rho \cos \vartheta, \end{cases}$$

where $\vartheta \in [0, \pi]$, $\rho \geq 0$ and $\eta = (\eta_1, \dots, \eta_{n-1})$ varies on the $(n-1)$-dimensional unit sphere if $n \geq 3$, and $\eta_1 = \pm 1$ if $n = 2$.

Setting $u = t(M^2 + t^2)^{-1/2}$ and recalling (1.11), we have

$$\int_{\partial^+ B_M} \frac{\partial h}{\partial \nu} d\Sigma =$$

$$= -\int_0^{\arccos u} \sin^{n-2}\vartheta\, d\vartheta \int_{|\eta|=1} [(M^2+t^2)^{-1/2} h_1 + (2-n)(M^2+t^2)^{(1-n)/2} h_2(\eta \sin\vartheta; \cos\vartheta)] \cdot$$

$$\cdot (M^2+t^2)^{(n-1)/2} d\sigma_\eta = -\int_0^{\arccos u} \sin^{n-2}\vartheta\, d\vartheta \int_{|\eta|=1} [h_1 + (2-n) h_2(\eta \sin\vartheta; \cos\vartheta)]\, d\sigma_\eta$$

and then

$$\lim_{M\to\infty} \int_{\partial^+ B_M} \frac{\partial h}{\partial \nu} d\Sigma = -\int_0^{\pi/2} \sin^{n-2}\vartheta\, d\vartheta \int_{|\eta|=1} [h_1 + (2-n) h_2(\eta \sin\vartheta; \cos\vartheta)]\, d\sigma_\eta =$$

$$= -\int_{\substack{|\omega|=1\\ \omega_n>0}} [h_1 + (2-n) h_2(\omega)]\, d\Sigma_\omega = \begin{cases} -\pi h_1 & n = 2, \\ (n-2) \displaystyle\int_{\substack{|\omega|=1\\ \omega_n>0}} h(\omega)\, d\Sigma_\omega & n \geq 3 . \end{cases}$$

On the other hand

$$\int_{B_M} \Delta_2 h\, dx\, dt = \int_0^{\arccos u} \sin^{n-2}\vartheta\, d\vartheta \int_{t/\cos\vartheta}^{\sqrt{M^2+t^2}} \rho^{n-1} d\rho \int_{|\eta|=1} \Delta_2 h(\rho\eta\sin\vartheta;\, \rho\cos\vartheta)\, d\sigma_\eta = \tag{1.14}$$

$$= \int_0^{\arccos u} \sin^{n-2}\vartheta\, d\vartheta \int_{t/\cos\vartheta}^{\sqrt{M^2+t^2}} \rho^{-1} d\rho \int_{|\eta|=1} \Delta_2 h(\eta\sin\vartheta;\, \cos\vartheta)\, d\sigma_\eta =$$

$$= -\log u \int_0^{\arccos u} \sin^{n-2}\vartheta\, d\vartheta \int_{|\eta|=1} \Delta_2 h(\eta\sin\vartheta;\, \cos\vartheta)\, d\sigma_\eta +$$

$$+ \int_0^{\arccos u} \sin^{n-2}\vartheta \log|\cos\vartheta|\, d\vartheta \int_{|\eta|=1} \Delta_2 h(\eta\sin\vartheta;\, \cos\vartheta)\, d\sigma_\eta .$$

We may write

$$\log u \int_0^{\arccos u} \sin^{n-2}\vartheta\, d\vartheta \int_{|\eta|=1} \Delta_2 h(\eta\sin\vartheta;\, \cos\vartheta)\, d\sigma_\eta = (u\log u)\frac{1}{u} \int_0^{\arccos u} \Phi(\vartheta)\, d\vartheta$$

where

$$\Phi(\vartheta) = \sin^{n-2}\vartheta \int_{|\eta|=1} \Delta_2 h(\eta\sin\vartheta;\, \cos\vartheta)\, d\sigma_\eta .$$

Since, by assumption,

$$\int_0^{\pi/2} \Phi(\vartheta)\, d\vartheta = \int_{\substack{|\omega|=1\\ \omega_n>0}} \Delta_2 h(\omega)\, d\Sigma_\omega = 0 ,$$

we have

$$\lim_{u\to 0^+} \frac{1}{u} \int_0^{\arccos u} \Phi(\vartheta)\, d\vartheta = -\lim_{u\to 0^+} \frac{\Phi(\arccos u)}{\sqrt{1-u^2}} = -\Phi(\pi/2)$$

and then

$$\lim_{u\to 0^+} (u\log u)\frac{1}{u} \int_0^{\arccos u} \Phi(\vartheta)\, d\vartheta = 0 .$$

From (1.14) it follows

$$\lim_{M\to\infty}\int_{B_M}\Delta_2 h\,dx\,dt=\int_0^{\pi/2}\sin^{n-2}\vartheta\log|\cos\vartheta|\,d\vartheta\int_{|\eta|=1}\Delta_2 h(\eta\sin\vartheta;\,\cos\vartheta)\,d\sigma_\eta=$$

$$=\int_{\substack{|\omega|=1\\ \omega_n>0}}\Delta_2 h(\omega)\log|\omega_n|\,d\Sigma_\omega.$$

Now (1.12) follows from (1.13).

As far as the kernels $\partial h(x;\,t)/\partial x_j$ are concerned, the situation is different.

VI) *If* $h(x;\,t)\in C^1(\mathbb{R}^n-\{0\})$ *is essentially homogeneous of degree* $2-n$, *the functions* $K_j(x;\,t)=\partial h(x;\,t)/\partial x_j$ $(j=1,\dots,n-1)$ *satisfy* (1.7) *and*

$$\gamma_j=\begin{cases} h_2(1;\,0)-h_2(-1;\,0) & n=2\,,\\ \displaystyle\int_{|\eta|=1} h(\eta;\,0)\,\eta_j\,d\sigma_\eta & n\geq 3\,,\end{cases}\tag{1.15}$$

where $\gamma_j=\int_{\mathbb{R}^{n-1}}K_j(x;\,t)\,dx$.

If $n=2$, we have

$$K(-1;\,0)+K(1;\,0)=-h_1+\frac{\partial h_2}{\partial x}(-1;\,0)+h_1+\frac{\partial h_2}{\partial x}(1;\,0)=$$

$$=\int_1^e\left[\frac{\partial h_2}{\partial x}(-1;\,0)+\frac{\partial h_2}{\partial x}(1;\,0)\right]\frac{dx}{x}=\int_1^e\left[\frac{\partial h_2}{\partial x}(-x;\,0)+\frac{\partial h_2}{\partial x}(x;\,0)\right]dx=$$

$$=-h_2(-e;\,0)+h_2(e;\,0)+h_2(-1;\,0)-h_2(1;\,0)=0\,;$$

$$\int_{|x|<M}K_1(x;\,t)\,dx=\int_{-M}^{M}\frac{\partial h}{\partial x}(x;\,t)\,dx=h(M;\,t)-h(-M;\,t)=h_2(M;\,t)-$$

$$-h_2(-M;\,t)=h_2(1;\,t/M)-h_2(-1;\,t/M)\to h_2(1;\,0)-h_2(-1;\,0)\qquad(M\to+\infty).$$

In order to obtain (1.7) if $n\geq 3$, it is sufficient to observe that $K_j(x;\,0)=$

$= (\partial/\partial x_j)[h(x;0)]$ and apply a known result (see e.g. [1], [15]). Moreover

$$\int_{|x|<M} K_j(x;t)\,dx = \int_{|x|=M} h(x;t)\,x_j/|x|\,dx = \int_{|\eta|=1} h(M\eta;t)\,\eta_j M^{n-2}\,d\sigma_\eta =$$

$$= \int_{|\eta|=1} h(\eta;t/M)\,\eta_j\,d\sigma_\eta \to \int_{|\eta|=1} h(\eta;0)\,\eta_j\,d\sigma_\eta \qquad (M\to+\infty).$$

REMARK. – If $h(x;t)$ is even, then $\gamma_j = 0$ $(j=1,\dots,n-1)$.

The final result is given by the following Theorem

VII) *Let $h(x;t) \in C^1(\mathbb{R}^n - \{0\})$ be essentially homogeneous of degree $2-n$. Let us set $K(x;t) = \partial h(x;t)/\partial t$, $K_j(x;t) = \partial h(x;t)/\partial x_j$ $(j=1,\dots,n-1)$. For any $\varphi \in \mathcal{H}^\lambda(\mathbb{R}^{n-1})$, we have that*

$$\lim_{t\to 0^+} \int_{\mathbb{R}^{n-1}} \varphi(x) K_j(x;t)\,dx = \gamma_j \varphi(0) + \int_{\mathbb{R}^{n-1}} \varphi(x) K_j(x;0)\,dx \tag{1.16}$$

where the last integral exists as a singular integral and γ_j are given by (1.15).

If $h(x;t) \in C^2(\mathbb{R}^n - \{0\})$ satisfy (1.10) *then for any $\varphi \in \mathcal{H}^\lambda(\mathbb{R}^{n-1})$*

$$\lim_{t\to 0^+} \int_{\mathbb{R}^{n-1}} \varphi(x) K(x;t)\,dx = \gamma\varphi(0) + \int_{\mathbb{R}^{n-1}} \varphi(x) K(x;0)\,dx \tag{1.17}$$

where the last integral exists as a singular integral and γ is given by (1.12).

Moreover if $h(x;t)$ is even, then it satisfies (1.10). *In this case the constants in* (1.16), (1.17) *are*

$$\gamma_j = 0 \qquad (j=1,\dots,n-1),$$

$$\gamma = \begin{cases} \pi h_1 - \dfrac{1}{2}\displaystyle\int_{|\omega|=1} \Delta_2 h_2(\omega)\log|\omega_2|\,d\Sigma_\omega & n=2, \\ \dfrac{1}{2}\displaystyle\int_{|\omega|=1} [(2-n)h(\omega) - \Delta_2 h(\omega)\log|\omega_n|]\,d\Sigma_\omega & n\geqslant 3. \end{cases}$$

This Theorem follows immediately from the previous ones.

It is worthwhile to remark that the cases of logarithmic and Newtonian potentials, i.e. $h(x;t) = \log(|x|^2+t^2)$ if $n=2$, $=(|x|^2+t^2)^{-n/2}$ if $n\geqslant 3$, in which the kernels $K(x;0)\equiv 0$ are not singular, are very special. In the general case, while the singular integrals

$$\int_{\mathbb{R}^{n-1}} \varphi(x) K_j(x;0)\,dx$$

exist for any $h(x; t)$ essentially homogeneous of degree $2 - n$, the integral

$$\int_{\mathbb{R}^{n-1}} \varphi(x) K(x; 0)\, dx$$

may fail to exist, even as a singular integral. An example is given by $h(x; t) = t(|x|^2 + t^2)^{(1-n)/2}$, $K(x; 0) = |x|^{1-n}$.

2. - Boundary values of some potentials with Hölder continuous density.

Let Ω be a bounded domain of $\mathbb{R}^n$ such that its boundary Σ is a Lyapunov manifold, i.e. Σ has a uniform Hölder continuous normal field of exponent λ $(0 < \lambda \leq 1)$.

By $C^\lambda(\Sigma)$ we denote the space of all the continuous functions satisfying in Σ a uniform Hölder condition of exponent λ.

The aim of this section is to study the boundary values of the integral

$$\int_\Sigma \varphi(y) \frac{\partial}{\partial x_k} h(x - y)\, d\sigma_y$$

where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, $h(x)$ is even and essentially homogeneous of degree $2 - n$.

VIII) *If* $K(x) \in C^1(\mathbb{R}^n - \{0\})$ *is a homogeneous function of degree* $-m$ $(m \in \mathbb{N})$, *then there exists a constant* Γ *such that*

$$|K(x) - K(y)| \leq \Gamma |x - y| \sum_{h=0}^{m} |x|^{-1-h} |y|^{h-m} \qquad \forall\, x, y \in \mathbb{R}^n - \{0\}. \tag{2.1}$$

We may write

$$K(x) - K(y) = |x|^{-m} K(x/|x|) - |y|^{-m} K(y/|y|) = \\ |x|^{-m}[K(x/|x|) - K(y/|y|)] + K(y/|y|)(|x|^{-m} - |y|^{-m}).$$

The result follows from the inequalities

$$|K(x/|x|) - K(y/|y|)| \leq C_1 \big| x|x|^{-1} - y|y|^{-1} \big| = \\ = C_1 \big| (x - y)|x|^{-1} + y(|x|^{-1} - |y|^{-1}) \big| \leq 2C_1 |x - y|/|x|;$$

$$\big| |x|^{-m} - |y|^{-m} \big| = \big| |x|^{-1} - |y|^{-1} \big| \sum_{h=0}^{m-1} |x|^{-h} |y|^{h+1-m} \leq |x - y| \sum_{h=0}^{m-1} |x|^{-h-1} |y|^{h-m}.$$

IX) *Let* $h(x) \in C^2(\mathbb{R}^n$ *be an essentially homogeneous function of degree* $2 - n$. *We suppose* $h(x)$ *even, i.e.*

$$h(-x) = h(x). \tag{2.2}$$

Let ν_{x_0} be the inward unit normal vector to Σ at the point $x_0 \in \Sigma$. If $\varphi \in C^\lambda(\Sigma)$, then the singular integral

$$\int_\Sigma \varphi(y) \frac{\partial}{\partial \nu_{x_0}} h(x_0 - y)\, d\sigma_y \tag{2.3}$$

exist and

$$\lim_{\substack{x \to x_0 \\ x \in \nu_{x_0}\Sigma}} \int \varphi(y) \frac{\partial}{\partial \nu_{x_0}} h(x - y)\, d\sigma_y = \gamma(x_0)\,\varphi(x_0) + \int_\Sigma \varphi(y) \frac{\partial}{\partial \nu_{x_0}} h(x_0 - y)\, d\sigma_y \tag{2.4}$$

where

$$\gamma(x_0) = \begin{cases} \pi h_1 - \dfrac{1}{2} \displaystyle\int_{|\omega|=1} \Delta_2 h_2(\omega) \log |\omega \cdot \nu_{x_0}|\, d\Sigma_\omega & n = 2\,, \\ \dfrac{1}{2} \displaystyle\int_{|\omega|=1} [(2-n)\,h(\omega) - \Delta_2 h(\omega) \log |\omega \cdot \nu_{x_0}|\,]\, d\Sigma_\omega & n \geq 3\,, \end{cases} \tag{2.5}$$

(the dot denotes the scalar product in $\mathbb{R}^n$). The limit relation (2.4) is uniform with respect to x_0.

Let $(\tau_1, \ldots, \tau_{n-1}, \nu_{x_0})$ be an orthonormal system and consider the coordinate system $(\eta; t) = (\eta_1, \ldots, \eta_{n-1}, t)$ with the origin in x_0, corresponding to the basis $(\tau_1, \ldots, \tau_{n-1}, \nu_{x_0})$. Let Σ_d be the part of Σ which admits the representation $t = \gamma(\eta)$, with (2)

$$\gamma(\eta) \in C^{1+\lambda}(B_d)\,, \quad \gamma(0) = 0\,, \quad \operatorname{grad}\gamma(0) = 0\,, \quad \|\operatorname{grad}\gamma(\eta)\|_{C^\lambda(B_d)} \leq G\,, \tag{2.6}$$

where $B_d = \{\eta \in \mathbb{R}^{n-1} \mid \leq d\}$ and the constants d, G are independent of x_0.

Let t_ε be such that for $|x - x_0| < t_\varepsilon$

$$\left| \int_{\Sigma - \Sigma_d} \varphi(y) \left[\frac{\partial}{\partial \nu_{x_0}} h(x - y) - \frac{\partial}{\partial \nu_{x_0}} h(x_0 - y) \right] d\sigma_y \right| < \varepsilon\,. \tag{2.7}$$

t_ε can be chosen independent of x_0.

Since $x = x_0 + t\nu_{x_0}$, $y = x_0 + \sum_{h=1}^{n-1} \eta_h \tau_h + \gamma(\eta)\,\nu_{x_0}$, we have $x - y = -\sum_{h=1}^{n-1} \eta_h \tau_h + (t - \gamma(\eta))\,\nu_{x_0}$.

---

(2) By $\|f\|_{C^\lambda(B_d)}$ we mean: $\max |f(x)| + \sup [\,|f(x) - f(y)|/|x - y|^\lambda]$, where max and sup are taken in B_d and in $\{(x, y) \mid x, y \in B_d,\, x \neq y\}$ respectively.

Let us introduce the functions:

$$h_{x_0}(\eta;\, t) = h\left[-\sum_{h=1}^{n-1}\eta_h\tau_h + t\nu_{x_0}\right]; \qquad K(\eta;\, t) = \frac{\partial}{\partial t}h_{x_0}(\eta;\, t)\,. \tag{2.8}$$

We remark that $h_{x_0}(\eta;\, t)$ belongs to $C^2(\mathbb{R}^n - \{0\})$, it is essentially homogeneous of degree $2-n$ and $K(\eta; t)$ satisfies condition (1.7), because of (2.2) (Lemma III). Since

$$\frac{\partial}{\partial\nu_{x_0}}h(x-y) = K(\eta;\, t-\gamma(\eta))\,,$$

we have

$$\int_{\Sigma_d}\varphi(y)\frac{\partial}{\partial\nu_{x_0}}h(x-y)\,d\sigma_y = \int_{|\eta|<d}\Phi(\eta)K(\eta;\, t-\gamma(\eta))\,d\eta\,,$$

where $\Phi(\eta) = \varphi\left[x_0 + \sum_{h=1}^{n-1}\eta_h\tau_h + \gamma(\eta)\nu_{x_0}\right]$ $(1 + |\text{grad}\,\gamma(\eta)|^2)^{1/2}$. Defining $\Phi(\eta) = 0$ $(|\eta| \geq 1)$, we have that

$$\|\Phi\|_{L^\infty(\mathbb{R}^{n-1})} \leq (1+G^2)^{1/2}\|\varphi\|_{L^\infty(\Sigma)}; \qquad |\Phi(\eta) - \Phi(0)| \leq G\|\varphi\|_{C^\lambda(\Sigma)}|\eta|^\lambda. \tag{2.9}$$

Moreover

$$\int_{\Sigma_d}\varphi(y)\frac{\partial}{\partial\nu_{x_0}}h(x-y)\,d\sigma_y = \int_{\mathbb{R}^{n-1}}\Phi(\eta)[K(\eta;\, t-\gamma(\eta)) - K(\eta;\, t)]\,d\eta + \int_{\mathbb{R}^{n-1}}\Phi(\eta)K(\eta;\, t)\,d\eta\,.$$

Setting $F(v) = K(\eta;\, t+v) - K(\eta;\, v)$, we have

$$|K(\eta;\, t-\gamma(\eta)) - K(\eta;\, -\gamma(\eta)) - K(\eta;\, t) + K(\eta;\, 0)| =$$

$$= |F(-\gamma(\eta)) - F(0)| = |F'(-\sigma\gamma(\eta))|\,|\gamma(\eta)| \qquad (\sigma\in(0,1))\,.$$

On the other hand, $F'(v) = K_t(\eta;\, t+v) - K_t(\eta;\, v)$, and from Lemma VIII

$$|F'(-\sigma\gamma(\eta))| = |K_t(\eta;\, t-\sigma\gamma(\eta)) - K_t(\eta;\, -\sigma\gamma(\eta))| \leq \Gamma t|\eta|^{-n-1},$$

where Γ is independent of t, η, x_0. But (2.6) implies $|\gamma(\eta)| \leq G|\eta|^{1+\lambda}$ and thus $|K(\eta;\, t-\gamma(\eta)) - K(\eta;\, -\gamma(\eta)) - K(\eta;\, t) + K(\eta;\, 0)| \leq \Gamma G t|\eta|^{\lambda-n-1}$. Thus, recalling (2.9),

$$\lim_{t\to 0^+}\int_{\mathbb{R}^{n-1}}\Phi(\eta)[K(\eta;\, t-\gamma(\eta)) - K(\eta;\, t)]\,d\eta = \int_{\mathbb{R}^{n-1}}\Phi(\eta)[K(\eta;\, -\gamma(\eta)) - K(\eta;\, 0)]\,d\eta$$

uniformly with respect to x_0.

$h_{x_0}(\eta;\, t)$ satisfy all the conditions of Theorem VII and then

$$\lim_{t\to 0^+} \int_{\mathbb{R}^{n-1}} \Phi(\eta)\, K(\eta;\, t)\, d\eta = \gamma(x_0)\, \Phi(0) + \int_{\mathbb{R}^{n-1}} \Phi(\eta)\, K(\eta;\, 0)\, d\eta \tag{2.11}$$

where

$$\gamma(x_0) = \begin{cases} \pi h_1 - \dfrac{1}{2} \displaystyle\int_{|\omega|=1} \Delta_2 h_{x_0 2}(\omega) \log|\omega_2|\, d\Sigma_\omega & n = 2\,, \\ \dfrac{1}{2} \displaystyle\int_{|\omega|=1} [(2-n)\, h_{x_0}(\omega) - \Delta_2 h_{x_0}(\omega) \log|\omega_n|\,]\, d\Sigma_\omega & n \geq 3\,. \end{cases}$$

Since

$$\int_{|\eta|>\delta} \Phi(\eta)\, K(\eta;\, -\gamma(\eta))\, d\eta = \tag{2.12}$$

$$= \int_{|\eta|>\delta} \Phi(\eta)[K(\eta;\, -\gamma(\eta)) - K(\eta;\, 0)]\, d\eta + \int_{|\eta|>\delta} \Phi(\eta)\, K(\eta;\, 0)\, d\eta$$

it follows that there exists

$$\lim_{\delta\to 0^+} \int_{|\eta|>\delta} \Phi(\eta)\, K(\eta;\, -\gamma(\eta))\, d\eta$$

i.e. there exists the singular integral (2.3). (2.4) follows from (2.7), (2.10), (2.11). Since $h_{x_0}(\omega) = h\left[-\sum_{h=1}^{n-1} \omega_h \tau_h + \omega_n \nu_{x_0}\right]$, by making the substitution $\xi = -\sum_{h=1}^{n-1} \omega_h \tau_h + \omega_n \nu_{x_0}$, we obtain

$$\int_{|\omega|=1} h_{x_0}(\omega)\, d\Sigma_\omega = \int_{|\xi|=1} h(\xi)\, d\Sigma_\xi;$$

$$\int_{|\omega|=1} \Delta_2 h_{x_0}(\omega) \log|\omega_{\mathrm{n}}|\, \mathrm{d}\Sigma_\omega = \int_{|\xi|=1} \Delta_2 \mathrm{h}(\xi) \log|\xi\cdot \nu_{\mathrm{x}_0}|\, \mathrm{d}\Sigma_\xi.$$

In order to complete the proof, taking into account (2.7) and (2.10), we need to show that the limit relation (2.11) is uniform with respect to x_0. If we denote

$$A(t,\, x_0) = \int_{\mathbb{R}^{n-1}} \Phi(\eta)\, K(\eta;\, t)\, d\eta - \gamma(x_0)\, \Phi(0)\,; \qquad B(\delta,\, x_0) = \int_{|\eta|>\delta} \Phi(\eta)\, K(\eta;\, 0)\, d\eta$$

we have

$$A(t, x_0) - B(\delta, x_0) = \int_{|\eta|<\delta} [\Phi(\eta) - \Phi(0)] K(\eta; t)\, d\eta - \Phi(0) \int_{|\eta|>\delta} K(\eta; t)\, d\eta +$$

$$+ \int_{|\eta|>\delta} \Phi(\eta)[K(\eta; t) - K(\eta; 0)]\, d\eta .$$

Taking $\delta < d$, because of (2.9) and Lemma VIII, we may write

$$\left| \int_{|\eta|<\delta} [\Phi(\eta) - \Phi(0)][K(\eta; t) - K(\eta; 0)]\, d\eta \right| \leqslant$$

$$\leqslant G\|\varphi\|_{C^\lambda(\Sigma)} \int_{|\eta|<\delta} |\eta|^\lambda \,|K(\eta; t) - K(\eta; 0)|\, d\eta \leqslant$$

$$\leqslant \Gamma\|\varphi\|_{C^\lambda(\Sigma)} \int_{|\eta|<\delta} t|\eta|^{\lambda-n}\, d\eta \leqslant \Gamma'\|\varphi\|_{C^\lambda(\Sigma)}\, t ;$$

$$\left| \Phi(0) \int_{|\eta|>\delta} K(\eta; t)\, d\eta \right| \leqslant$$

$$\leqslant (1 + G^2)^{1/2} \|\varphi\|_{L^\infty(\Sigma)} \int_\delta^{+\infty} \rho^{-1} d\rho \int_{|\eta|=1} |K(\eta; t/\rho) - K(\eta; 0)|\, d\sigma_\eta \leqslant H'\|\varphi\|_{L^\infty(\Sigma)}\, t^\lambda ;$$

$$\left| \int_{|\eta|>\delta} \Phi(\eta)[K(\eta; t) - K(\eta; 0)]\, d\eta \right| \leqslant$$

$$\leqslant (1 + G^2)^{1/2} \|\varphi\|_{L^\infty(\Sigma)} \int_{\delta<|\eta|<M} |[K(\eta; t) - K(\eta; 0)]|\, d\eta \leqslant$$

$$\leqslant \Lambda\|\varphi\|_{L^\infty(\Sigma)} \int_{\delta<|x|<M} t|\eta|^{-n}\, d\eta \leqslant \Lambda' \|\varphi\|_{L^\infty(\Sigma)}\, t .$$

Since all the constants in the last inequalities are independent of x_0, it follows that

$$\lim_{t\to 0^+} A(t, x_0) = B(\delta, x_0) + \int_{|\eta|<\delta} [\Phi(\eta) - \Phi(0)] K(\eta; 0)\, d\eta$$

uniformly with respect to x_0.

Let us consider now the tangential operators

$$M^{ik}_{x_0} = \nu_i(x_0)\frac{\partial}{\partial x_k} - \nu_k(x_0)\frac{\partial}{\partial x_i} \qquad (i, k = 1, \dots, n).$$

X) *If $h(x)$ satisfy the hypothesis of Theorem IX and $\varphi \in C^\lambda(\Sigma)$, then*

$$\lim_{\substack{x \to x_0 \\ x \in \nu_{x_0}}} \int_\Sigma \varphi(y) M^{ik}_{x_0}[h(x-y)]\, d\sigma_y = \int_\Sigma \varphi(y) M^{ik}_{x_0}[h(x_0 - y)]\, d\sigma_y \qquad (i, k = 1, \dots, n),$$

where the last integral exists as a singular integral. The limit relation is uniform with respect to x_0.

If $i = k$ or $\nu_i(x_0) = \nu_k(x_0) = 0$ the result is trivial. Otherwise let $(\tau_1, \dots, \tau_{n-1}, \nu_{x_0})$ be an orthogonal system, where $\tau_1 = (\tau_{11}, \dots, \tau_{1n})$ is given by $\tau_{1k} = -\nu_i(x_0)$, $\tau_{1i} = \nu_k(x_0)$, $\tau_{1j} = 0$ if $j \neq i, k$. We have $M^{ik}_{x_0}[h(x-y)] = K(\eta; t - \gamma(\eta))$, where $K(\eta; t) = -(\partial/\partial\eta_1) h_{x_0}(\eta; t)$ and $h_{x_0}(\eta; t)$ is given by (2.8). Now the result follows by using the same arguments employed in Theorem IX.

XI) *If $h(x)$ satisfy the hypothesis of Theorem IX and $\varphi \in C^\lambda(\Sigma)$, then*

$$\lim_{\substack{x \to x_0 \\ x \in \nu_{x_0}}} \int_\Sigma \varphi(y) h_{x_k}(x-y)\, d\sigma_y = \nu_k(x_0)\gamma(x_0)\varphi(x_0) + \int_\Sigma \varphi(y) h_{x_k}(x_0 - y)\, d\sigma_y,$$

$$(K = 1, \dots, n)$$

where the last integral exists as a singular integral and $\gamma(x_0)$ is given by (2.5). The limit relation is uniform with respect to x_0.

This Theorem follows immediately from Theorems IX, X because

$$\frac{\partial}{\partial x_k} = \nu_k(x_0)\frac{\partial}{\partial \nu} + \sum_{i=1}^{n} \nu_i(x_0)\left[\nu_i(x_0)\frac{\partial}{\partial x_k} - \nu_k(x_0)\frac{\partial}{\partial x_i}\right].$$

REMARK. – In the Theorems of this section we have considered only the case in which x tends to x_0 while remaining on ν_{x_0}. It is obvious how these results have to be modified if x tends to x_0 while remaining on $-\nu_{x_0}$. For example, the relation in Theorem XI becomes

$$\lim_{\substack{x \to x_0 \\ x \in -\nu_{x_0}}} \int_\Sigma \varphi(y) h_{x_k}(x-y)\, d\sigma_y = -\nu_k(x_0)\gamma(x_0)\varphi(x_0) + \int_\Sigma \varphi(y) h_{x_k}(x_0 - y)\, d\sigma_y,$$

$$(k = 1, \dots, n).$$

We remark also that since all the limit relations obtained in this section are uniform, then they continue to hold if x tends to x_0 while remaining in Ω or in $\mathbb{R}^n - \bar{\Omega}$ (see [11], p. 269-271).

3. – Boundary values of some potentials with integrable density.

In this section we want to study the boundary behavior for potentials with more general densities. Specifically we shall prove that the formulas we have found in previous sections hold almost everywhere, when $\varphi \in L^p(\Sigma)$ $(p > 1)$.

We recall that x_0 is a Lebesgue point for φ if

$$\lim_{d \to 0^+} d^{1-n} \int_{\Sigma_d} |\varphi(x) - \varphi(x_0)| \, d\sigma_x = 0 .$$

It is very well known that the set in which this condition is not satisfied is a set of zero Lebesgue $(n-1)$-dimensional measure.

Let $f(x)$ be a function defined in Ω (in $\mathbb{R}^n - \bar{\Omega}$) and $x_0 \in \Sigma$. We say that the limit

$$\lim_{x \to x_0} f(x) = L$$

is an *internal (external) angular boundary value* if, given $\alpha \in (0, \pi)$, x tends, to x_0 while remaining in the set

$$\Omega \cap \{x \in \mathbb{R}^n \,|\, (x - x_0)\cdot \nu_{x_0} > 0, \ |x - x_0|^2 - [(x - x_0)\cdot \nu_{x_0}]^2 \leq [(x - x_0)\cdot \nu_{x_0}]^2 \tan^2(\alpha/2)\},$$

$$(\Omega \cap \{x \in \mathbb{R}^n \,|\, (x - x_0)\cdot \nu_{x_0} < 0, \ |x - x_0|^2 - [(x - x_0)\cdot \nu_{x_0}]^2 \leq [(x - x_0)\cdot \nu_{x_0}]^2 \tan^2(\alpha/2)\}).$$

This means that x belongs to a «circular cone» with the vertex at x_0, the axis coinciding with ν_{x_0} and the plane angle at the vertex equal to α. Moreover L has to be independent of α.

The idea of proving formulas like the next ones in Lebesgue points was used for the first time in [7] (see also [8], [9]) and later used in more general situations (see [14], p. 293-300).

XII) *If $h(x)$ satisfy the hypothesis of Theorem IX, $\varphi \in L^1(\Sigma)$ and $x_0 \in \Sigma$ is a Lebesgue point for φ, then*

$$\lim_{x \to x_0} \left\{ \int_{\Sigma} \varphi(y) \frac{\partial}{\partial \nu_{x_0}} h(x - y)\, d\sigma_y - \int_{\Sigma - \Sigma_{|x - x_0|}} \varphi(y) \frac{\partial}{\partial \nu_{x_0}} h(x_0 - y)\, d\sigma_y \right\} = \gamma(x_0)\, \varphi(x_0) \tag{3.1}$$

where $\Sigma_{|x - x_0|} = \{\xi \in \Sigma \,|\, |\xi - x_0| \leq |x - x_0|\}$, $\gamma(x_0)$ is given by (2.5) *and the limit must be understood as an internal angular boundary value* (3).

If $\varphi(x) \equiv 1$, the limit relation (2.4) holds and it is uniform with respect to x_0. This

(3) The idea of the proof is substantially the one contained in [14] (pp. 293-298), but with some modifications.

implies the following internal boundary value:

$$\lim_{x\to x_0}\varphi(x_0)\left\{\int_\Sigma \frac{\partial}{\partial\nu_{x_0}}h(x-y)\,d\sigma_y - \int_{\Sigma-\Sigma_{|x-x_0|}}\frac{\partial}{\partial\nu_{x_0}}h(x_0-y)\,d\sigma_y\right\} = \gamma(x_0)\,\varphi(x_0)$$

for any φ. Then in order to obtain the Theorem, it will suffice to show that

$$\lim_{x\to x_0}\left\{\int_\Sigma \chi(y)\frac{\partial}{\partial\nu_{x_0}}h(x-y)\,d\sigma_y - \int_{\Sigma-\Sigma_{|x-x_0|}}\chi(y)\frac{\partial}{\partial\nu_{x_0}}h(x_0-y)\,d\sigma_y\right\} = 0\,,$$

where $\chi(y)=\varphi(y)-\varphi(x_0)$.

Let d be such that (2.6) holds. If $|x-x_0|<d$, then

$$(3.2)\qquad \int_\Sigma \chi(y)\frac{\partial}{\partial\nu_{x_0}}h(x-y)\,d\sigma_y - \int_{\Sigma-\Sigma_{|x-x_0|}}\chi(y)\frac{\partial}{\partial\nu_{x_0}}h(x_0-y)\,d\sigma_y =$$

$$= \int_{\Sigma-\Sigma_d}\chi(y)\left[\frac{\partial}{\partial\nu_{x_0}}h(x-y) - \frac{\partial}{\partial\nu_{x_0}}h(x_0-y)\right]d\sigma_y +$$

$$+ \int_{\Sigma_d-\Sigma_{|x-x_0|}}\chi(y)\left[\frac{\partial}{\partial\nu_{x_0}}h(x-y) - \frac{\partial}{\partial\nu_{x_0}}h(x_0-y)\right]d\sigma_y + \int_{\Sigma_{|x-x_0|}}\chi(y)\frac{\partial}{\partial\nu_{x_0}}h(x-y)\,d\sigma_y\,.$$

It is obvious that

$$(3.3)\qquad \lim_{x\to x_0}\int_{\Sigma-\Sigma_d}\chi(y)\left[\frac{\partial}{\partial\nu_{x_0}}h(x-y) - \frac{\partial}{\partial\nu_{x_0}}h(x_0-y)\right]d\sigma_y = 0\,.$$

As far the second term in the right hand side of (3.2) is concerned, let us observe that, by using the same notations of Theorem IX and setting $t=|x-x_0|$, $x = x_0 + \sum_{h=1}^{n-1}\xi_h\tau_h + t\nu_{x_0}$, $\xi=(\xi_1,\dots,\xi_{n-1})$ we have in view of Lemma VIII

$$\left|\frac{\partial}{\partial\nu_{x_0}}h(x-y) - \frac{\partial}{\partial\nu_{x_0}}h(x_0-y)\right| = |K(\eta-\xi;\,t-\gamma(\eta)) - K(\eta;\,-\gamma(\eta))| \leqslant$$

$$\leqslant \Gamma t\sum_{k=0}^{n-1}[\,|\eta-\xi|^2+(t-\gamma(\eta))^2]^{-(1+h)/2}[\,|\eta|^2+\gamma^2(\eta)]^{(h-n+1)/2} \leqslant$$

$$\leqslant n\Gamma t[\,|\eta-\xi|^2+(t-\gamma(\eta))^2]^{-1/2}\,|\eta|^{1-n}\,.$$

On the other hand, since we are considering an internal angular boundary value, we have $|\xi|\leq t\tan(\alpha/2)$ and, because of (2.6), $|\gamma(\eta)|\leqslant Gd^\lambda|\eta|$. By elementary arguments [4], it follows $|\xi|(|\xi|^2+t^2)^{-1/2} \leqslant \sin(\alpha/2)$, $|\gamma(\eta)|(|\eta|^2+\gamma^2(\eta))^{-1/2} \leqslant Gd^\lambda(1+G^2d^{2\lambda})^{-1/2}$. Thus

$$|\xi\cdot\eta + t\gamma(\eta)|[(|\xi|^2+t^2)(|\eta|^2+\gamma^2(\eta))]^{-1/2} \leqslant \sin(\alpha/2) + Gd^\lambda(1+G^2d^{2\lambda})^{-1/2}.$$

We may chose d in such a way

$$\beta \equiv \sin(\alpha/2) + Gd^\lambda(1+G^2d^{2\lambda})^{-1/2} < 1.$$

This implies

$$\begin{aligned}(3.4)\qquad |\eta-\xi|^2 + (t-\gamma(\eta))^2 &= |\xi|^2+|\eta|^2+t^2+\gamma^2(\eta) - 2\xi\cdot\eta - 2t\gamma(\eta) \geqslant \\ &\geqslant |\xi|^2+|\eta|^2+t^2+\gamma^2(\eta) - 2\beta[(|\xi|^2+t^2)(|\eta|^2+\gamma^2(\eta))]^{-1/2} \geqslant \\ &\geqslant (1-\beta)(|\xi|^2+|\eta|^2+t^2+\gamma^2(\eta)) \geqslant (1-\beta)(|\eta|^2+t^2)\end{aligned}$$

and then

$$|K(\eta-\xi;\, t-\gamma(\eta)) - K(\eta;\, -\gamma(\eta))| \leqslant \Gamma' t(|\eta|^2+t^2)^{-1/2}|\eta|^{1-n},$$

Γ' being a constant independent of η, t and x_0. Moreover

$$\left|\int\limits_{\Sigma_d - \Sigma_{|x-x_0|}} \chi(y)\left[\frac{\partial}{\partial\nu_{x_0}} h(x-y) - \frac{\partial}{\partial\nu_{x_0}} h(x_0-y)\right] d\sigma_y\right| =$$

$$= \left|\int\limits_{t<|\eta|<d} \Phi(\eta)[K(\eta-\xi;\, t-\gamma(\eta)) - K(\eta;\, -\gamma(\eta))]\, d\eta\right| \leqslant \Gamma' \cdot$$

$$\cdot \int\limits_{t<|\eta|<d} |\Phi(\eta)|\, t(|\eta|^2+t^2)^{-1/2}|\eta|^{1-n} d\eta = \Gamma'\int_t^d t\psi'(\rho)(\rho^2+t^2)^{-1/2}\rho^{1-n} d\rho$$

where

$$\Phi(\eta) = \chi\left[x_0 + \sum_{h=1}^{n-1}\eta_h\tau_h + \gamma(\eta)\nu_{x_0}\right](1+|\operatorname{grad}\gamma(\eta)|^2)^{1/2},$$

$$\psi(\rho) = \int\limits_{|\eta|<\rho} |\Phi(\eta)|\, d\eta = \int_0^\rho r^{n-1} dr \int\limits_{|\omega|=1} |\Phi(r\omega)|\, d\Sigma_\omega.$$

---

(4) If $a, b \geqslant 0$ and $k > 0$ are such that $a \leqslant kb$, then $a^2+b^2 \geqslant a^2(1+k^{-2})$ and therefore $a(a^2+b^2)^{-1/2} \leqslant k(1+k^2)^{-1/2}$.

Since x_0 is a Lebesgue point for φ, we have

$$\lim_{\rho \to 0^+} \rho^{1-n} \psi(\rho) = 0 \; ; \tag{3.5}$$

we may suppose that, given $\varepsilon > 0$, d is such that $\rho^{1-n}\psi(\rho) \leqslant \varepsilon$, $\forall \rho \in (0, d]$. This implies

$$\int_t^d t\psi'(\rho)(\rho^2 + t^2)^{-1/2}\rho^{1-n} d\rho = [t\psi(\rho)(\rho^2 + t^2)^{-1/2}\rho^{1-n}]_{\rho=t}^{\rho=d} -$$

$$- \int_t^d t\psi(\rho)(\rho^2 + t^2)^{-3/2}\rho^{-n}[t^2 - n(\rho^2 + t^2)]\, d\rho \leqslant t\psi(d)(d^2 + t^2)^{-1/2} d^{1-n} +$$

$$+ \psi(t) 2^{-1/2} t^{1-n} + \int_t^d t\psi(\rho)(\rho^2 + t^2)^{-3/2}\rho^{-n}[t^2 + n(\rho^2 + t^2)]\, d\rho \leqslant$$

$$\leqslant (1 + 2^{-1/2})\,\varepsilon + \varepsilon \int_t^d t\rho(\rho^2 + t^2)^{-3/2}(2n + 1)\, d\rho \leqslant$$

$$\leqslant \varepsilon[1 + 2^{-1/2} + (2n + 1)] \int_t^d t(\rho^2 + t^2)^{-1} d\rho \leqslant \varepsilon[1 + 2^{-1/2} + (2n + 1)\pi/4] .$$

Therefore, given $\varepsilon > 0$, there exists d_ε such that

$$\left| \int_{\Sigma_d - \Sigma_{|x - x_0|}} \chi(y) \left[\frac{\partial}{\partial \nu_{x_0}} h(x - y) - \frac{\partial}{\partial \nu_{x_0}} h(x_0 - y) \right] d\sigma_y \right| \leqslant \varepsilon \tag{3.6}$$

for $0 < |x - x_0| < d < d_\varepsilon$.

Recalling (3.4), we may write

$$\left| \int_{\Sigma_{|x - x_0|}} \chi(y) \frac{\partial}{\partial \nu_{x_0}} h(x - y)\, d\sigma_y \right| \leqslant \int_{|\eta| < t} |\Phi(\eta)|\, |K(\eta - \xi; t - \gamma(\eta))|\, d\eta \leqslant$$

$$k \int_{|\eta| < t} |\Phi(\eta)| [\, |\eta - \xi|^2 + (t - \gamma(\eta))^2]^{(1-n)/2} d\eta \leqslant k(1 - \beta)^{(1-n)/2} t^{1-n} \int_{|\eta| < t} |\Phi(\eta)|\, d\eta .$$

Because of (3.5) we have

$$\lim_{x \to x_0} \int_{\Sigma_{|x - x_0|}} \chi(y) \frac{\partial}{\partial \nu_{x_0}} h(x - y)\, d\sigma_y = 0 .$$

The result follows from this relation and from (3.2), (3.3), (3.6).

XIII) *If* $h(x)$ *satisfy the hypothesis of Theorem IX,* $\varphi \in L^1(\Sigma)$ *and* $x_0 \in \Sigma$ *is a*

Lebesgue point for φ, then

$$\lim_{x \to x_0} \left\{ \int_\Sigma \varphi(y) M^{ik}_{x_0}[h(x-y)]\, d\sigma_y - \int_{\Sigma - \Sigma_{|x-x_0|}} \varphi(y) M^{ik}_{x_0}[h(x_0-y)]\, d\sigma_y \right\} = 0\,;$$

$$\lim_{x \to x_0} \left\{ \int_\Sigma \varphi(y)\, h_{x_k}(x-y)\, d\sigma_y - \int_{\Sigma - \Sigma_{|x-x_0|}} \varphi(y)\, h_{x_h}(x_0-y)\, d\sigma_y \right\} = \nu_k(x_0)\, \gamma(x_0)\, \varphi(x_0)\,.$$

These relations follow from Theorems X, XI and they are proved in the same way as Theorem XII.

XIV) *If $h(x)$ satisfy the hypothesis of Theorem IX, $\varphi \in L^p(\Sigma)$ $(p > 1)$, then*

$$\lim_{x \to x_0} \int_\Sigma \varphi(y) \frac{\partial}{\partial \nu_{x_0}} h(x-y)\, d\sigma_y = \gamma(x_0)\, \varphi(x_0) + \int_\Sigma \varphi(y) \frac{\partial}{\partial \nu_{x_0}} h(x_0-y)\, d\sigma_y\,;$$

$$\lim_{x \to x_0} \int_\Sigma \varphi(y) M^{ik}_{x_0}[h(x-y)]\, d\sigma_y = \int_\Sigma \varphi(y) M^{ik}_{x_0}[h(x_0-y)]\, d\sigma_y\,;$$

$$\lim_{x \to x_0} \int_\Sigma \varphi(y)\, h_{x_k}(x-y)\, d\sigma_y = \nu_k(x_0)\, \gamma(x_0)\, \varphi(x_0) + \int_\Sigma \varphi(y)\, h_{x_k}(x_0-y)\, d\sigma_y$$

almost everywhere on Σ, where $\gamma(x_0)$ is given by (2.5), *the limits must be understood as internal angular boundary values and the integrals on the right hand sides exist as singular integrals.*

At first let us show that

$$\int_\Sigma \varphi(y) \frac{\partial}{\partial \nu_{x_0}} h(x_0-y)\, d\sigma_y$$

exists almost everywhere as a singular integral. This follows form (2.12), because the singular integral

$$\int_{\mathbb{R}^{n-1}} \Phi(\eta)\, K(\eta;\, 0)\, d\eta$$

exists almost everywhere in view of a classical Theorem of Calderon and Zygmund (see [4]). Then Theorem XII implies the first limit relation. The other two relations are proved in a similar way, by using Theorem XIII.

REMARKS. – In the Theorems of this section we have considered only internal angular boundary values. It is obvious how these results have to be modified for external angular boundary values. For example, the last relation in Theorem XIV becomes

$$\lim_{x\to x_0}\int_\Sigma \varphi(y)\,h_{x_k}(x-y)\,d\sigma_y = -\nu_k(x_0)\,\gamma(x_0)\,\varphi(x_0)+\int_\Sigma \varphi(y)\,h_{x_k}(x_0-y)\,d\sigma_y .$$

In sections 2, 3 for sake of simplicity we have supposed Σ is the boundary of a domain. But it is evident from the proofs that the limit relations we gave continue to hold in the interior of a compact bordered Lyapunov manifold.

4. – Boundary values of some potentials generated by measures.

The next Theorem provides the boundary behavior of potentials of the following kind

$$\int_\Sigma h_{x_k}(x-y)\,d\mu_y \tag{4.1}$$

where $\mu\in M(\Sigma)$ is a measure defined on the family of all the Borel sets of Σ. The approach we follow is the one introduced in [9] and considered in some other particular cases in [5], [6].

In order to study the boundary behavior of (4.1) it will convenient to introduce some «parallel» surface to Σ. Let $\zeta(x)$ be a unit vector defined and continuously differentiable on Σ such that $\zeta(x)\cdot\nu_x\geqslant p>0$, (for the existence of such a vector, see [11] pp. 273-275). Let Σ_ρ be the surface $x_\rho = x+\rho\zeta(x)$, $x\in\Sigma$, where $|\rho|\leqslant\rho_0$ (ρ_0 small enough).

XV) *If $h(x)$ satisfy the hypothesis of Theorem IX and $\mu\in M(\Sigma)$, for any $f\in C^\lambda(\mathbb{R}^n)$ we have*

$$\lim_{\rho\to 0^\pm}\int_{\Sigma_\rho} f(x_\rho)\,d\sigma_{x_\rho}\int_\Sigma h_{x_k}(x_\rho-y)\,d\mu_y = \pm\int_\Sigma \nu_k\gamma f\,d\sigma+\int_\Sigma d\mu_y\int_\Sigma f(x)\,h_{x_k}(x-y)\,d\sigma_x$$

where γ is given by (2.5).

Let $\rho>0$ and $f\in C^\lambda(\mathbb{R}^n)$. We may write

$$\int_{\Sigma_\rho} f(x_\rho)\,h_{x_k}(x_\rho-y)\,d\sigma_{x_\rho} = \int_{\Sigma_\rho} f(x_\rho)\,h_{x_k}(x_\rho-y)\,d\sigma_{x_\rho} -$$

$$-\int_\Sigma f(x)\,h_{x_k}(x-y_{-\rho})\,d\sigma_x+\int_\Sigma f(x)\,h_{x_k}(x-y_{-\rho})\,d\sigma_x .$$

From Theorem XI (see Remarks at the end of section 2) it follows that

$$\lim_{\rho\to 0^+}\int_{\Sigma} f(x)\, h_{x_k}(x-y_{-\rho})\, d\sigma_x = -\lim_{\rho\to 0^+}\int_{\Sigma} f(x)\, h_{y_k}(y_{-\rho}-x)\, d\sigma_x =$$

$$= \nu_k(y)\,\gamma(y)\,\varphi(y) + \int_{\Sigma} f(x)\, h_{x_k}(x-y)\, d\sigma_x$$

uniformly with respect to $y\in\Sigma$. In order to obtain the result we have to show that

$$\lim_{\rho\to 0^+}\int_{\Sigma_\rho} f(x_\rho)\, h_{x_k}(x_\rho-y)\, d\sigma_{x_\rho} - \int_{\Sigma} f(x)\, h_{x_k}(x-y_{-\rho})\, d\sigma_x = 0 \tag{4.2}$$

uniformly with respect to $y\in\Sigma$.

Let $(\tau_1,\ldots,\tau_{n-1},\nu_y)$ be an orthonormal system and consider the coordinate system $(\eta;t)=(\eta_1,\ldots,\eta_{n-1},t)$ with the origin in a fixed $y\in\Sigma$, corresponding to the basis $(\tau_1,\ldots,\tau_{n-1},\nu_y)$. Let Σ_d be the part of Σ which admits the representation $t=\gamma(\eta)$, where $\gamma(\eta)$ satisfy (2.6). We have

$$|x-y+\rho\zeta(x)|^2 \geqslant H(|x-y|^2+\rho^2); \tag{4.3}$$

$$|x-y+\rho\zeta(y)|^2 \geqslant H(|x-y|^2+\rho^2) \qquad \forall\, x\in\Sigma_d$$

H being a positive constant independent of y (see [5], p. 196).

We denote by $\Sigma_{\rho,d}$ the part of Σ_d which admits the representation $x_\rho = x+\rho\zeta(x)$, $x\in\Sigma_d$. Let $\{c_{ij}(\rho,\eta)\}$ $(i=1,\ldots,n-1;\, j=1,\ldots,n)$ be the matrix whose elements are $c_{ij}(\rho,\eta)=\delta_{ij}+\rho(\partial\zeta_j/\partial\eta_i)$ $(i,j=1,\ldots,n-1)$, $c_{ij}(\rho,\eta)=\partial\gamma/\partial\eta_i+\rho(\partial\zeta/\partial\eta_j)$ $(j=1,\ldots,n)$, where $(\zeta_1,\ldots,\zeta_n)$ are the components of ζ with respect to the system $(\tau_1,\ldots,\tau_{n-1},\nu_y)$. If $H_\rho(\eta)=\left[\det\left\{\sum_{h=1}^{n} c_{ih}(\rho,\eta)\,c_{jh}(\rho,\eta)\right\}\right]^{1/2}$, we have $d\sigma_x = H_0(\eta)\,d\eta$ (on Σ_d), $d\sigma_{x_\rho}=H_\rho(\eta)\,d\eta$ (on $\Sigma_{\rho,d}$). Therefore, considering $H_\rho(\eta)$ as a function of $x\in\Sigma_d$ (which we denote by $H_\rho(x)$), we may write

$$\int_{\Sigma_{\rho,d}} f(x_\rho)\, h_{x_k}(x_\rho-y)\, d\sigma_{x_\rho} - \int_{\Sigma_d} f(x)\, h_{x_k}(x-y_{-\rho})\, d\sigma = \tag{4.4}$$

$$=\int_{\Sigma_d}\left[f(x_\rho)\frac{H_\rho(x)}{H_0(x)} - f(x)\right] h_{x_k}(x_\rho-y)\, d\sigma_x + \int_{\Sigma_d} f(x)[h_{x_k}(x_\rho-y)-h_{x_k}(x-y_{-\rho})]\, d\sigma_x .$$

Since

$$\left| f(x_\rho)\frac{H_\rho(x)}{H_0(x)} - f(x)\right| \leqslant |f(x_\rho)-f(x)|\,\frac{H_\rho(x)}{H_0(x)} + \frac{|f(x)|}{H_0(x)}\,|H_\rho(x)-H_0(x)|$$

and $|H_\rho(x) - H_0(x)| = \mathcal{O}(\rho)$ (5), in view of (4.3), it follows that

$$\left[f(x_\rho)\frac{H_\rho(x)}{H_0(x)} - f(x)\right] h_{x_k}(x_\rho - y) = \mathcal{O}(\rho^\lambda |x_\rho - y|^{1-n}) = \mathcal{O}(\rho^{\lambda/2}|x - y|^{1-n+\lambda/2})$$

this relation being uniform with respect to $y \in \Sigma$. It follows

$$\lim_{\rho\to 0^+} \int_{\Sigma_d} \left[f(x_\rho)\frac{H_\rho(x)}{H_0(x)} - f(x)\right] h_{x_k}(x_\rho - y)\, d\sigma_x = 0 \tag{4.5}$$

uniformly with respect to $y \in \Sigma$.

Moreover because of Lemma VIII and (4.3), we have

$$|h_{x_k}(x_\rho - y) - h_{x_k}(x - y_{-\rho})| \leq \Gamma |x_\rho - y - x + y_{-\rho}| \sum_{h=0}^{n-1} |x_\rho - y|^{-1-h} |x - y_{-\rho}|^{h-n+1} \leq$$

$$\leq \Gamma' \rho |\zeta(x) - \zeta(y)| (|x - y|^2 + \rho^2)^{-n/2}.$$

Thus $h_{x_k}(x_\rho - y) - h_{x_k}(x - y_{-\rho}) = \mathcal{O}(\rho^{1/2}|x - y|^{1-n+1/2})$ and this implies

$$\lim_{\rho\to 0^+} \int_{\Sigma_d} f(x)[h_{x_k}(x_\rho - y) - h_{x_k}(x - y_{-\rho})]\, d\sigma_x = 0 \tag{4.6}$$

uniformly with respect to $y \in \Sigma$.

Finally observing that

$$\lim_{\rho\to 0^+} \int_{\Sigma - \Sigma_{\rho,d}} f(x_\rho)\, h_{x_k}(x_\rho - y)\, d\sigma_{x_\rho} - \int_{\Sigma - \Sigma_d} f(x)\, h_{x_k}(x - y_{-\rho})\, d\sigma_x = 0$$

uniformly with respect to $y \in \Sigma$, from (4.4), (4.5) and (4.6) it follows (4.2), i.e. the Theorem.

In the same way we may prove the result if $\rho < 0$.

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(5) The derivative of $H_\rho(\eta)$ with respect to ρ is bounded. This implies the relation in the text, which, on the other hand, may be proved by direct calculation.

REFERENCES

[1] S. AGMON, *Lectures on Elliptic Boundary Value Problems*, Van Nostrand, Princeton, New Jersey (1965).

[2] L. BERS-M. SCHECHTER, *Elliptic equations*, in *Partial Differential Equations*, Lect. Appl. Math., Vol. III, Interscience Publishers (1957), pp. 131-299.

[3] A. P. CALDERON - A. ZYGMUND, *On the existence of certain singular integrals*, Acta Math, 88 (1952), pp. 85-139.

[4] A. P. CALDERON - A. ZYGMUND, *On the singular integrals*, Amer. J. Math., **78** (1956), pp. 289-309.

[5] A. CIALDEA, *Sul problema della derivata obliqua per le funzioni armoniche e questioni connesse*, Rend. Accad. Naz. Sc. XL, Mem. Mat., **12** (1988), pp. 181-200.

[6] A. CIALDEA, *Elastostatics with non absolutely continuous measures*, J. Elasticity, **23** (1990), pp. 23-51.

[7] G. FICHERA, *Teoremi di completezza sulla frontiera di un dominio per taluni sistemi di funzioni*, Ann. Mat. Pura Appl. (IV), **27** (1948), pp. 1-28.

[8] G. FICHERA, *Una introduzione alla teoria delle equazioni integrali singolari*, Rend. Matem. (V), **17** (1958), pp. 82-191.

[9] G. FICHERA, *Approssimazione uniforme delle funzioni olomorfe mediante funzioni razionali aventi poli semplici prefissati. Nota I e II*, Rend. Acc. Lincei (1959), 193-201, 317-323.

[10] G. FICHERA, *Linear elliptic equations of higher order in two independent variables and singular integral equations, with application to anisotropic inhomogeneous elasticity.* in: *Partial Differential Equations and Continuum Mechanics*, R. E. LANGER (editor), Madison (1961), pp. 55-80.

[11] G. FICHERA - L. DE VITO, *Funzioni analitiche di una variabile complessa*, III Ediz., Veschi, Roma (1971).

[12] G. FICHERA - P. E. RICCI, *The single layer potential approach in the theory of boundary value problems for elliptic equations,* in: *Function Theoretic Methods for Part. Diff. Equat.*, Lecture Notes in Mathem., **561** (Springer-Verlag, Berlin, Heidelberg, New York (1976), pp. 40-50.

[13] G. HSIAO - R. C. MACCAMY, *Solution of boundary value problems by integral equations of the first kind,* SIAM Review, **15**, 1 (1973), pp. 687-705.

[14] V. D. KUPRADZE (Editor), *Three-Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity*, North-Holland, Amsterdam (1979).

[15] C. B. MORREY, *Multiple Integrals in the Calculus of Variations*, Springer-Verlag, Berlin, Heidelberg, New York (1966).

[16] W. L., WENDLAND, *Elliptic Systems in the Plane*, Pitman, London (1979).