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Multivalued Linear Operators and Degenerate Evolution Equations (*).

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Sunto. – *L'equazione differenziale lineare degenere in uno spazio di Banach X*

$$d(M(t)v)/dt + L(t)v = f(t), \qquad 0 < t \leq T,$$

viene trasformata nell'equazione multivoca $du/dt + A(t)u \ni f(t)$, $0 < t \leq T$. *Sotto opportune ipotesi sulla norma di* $M(t)(zM(t) + L(t))^{-1}$ *nello spazio* $L(X)$, *z numero complesso, si dimostra che* $-A(t) = -L(t)M(t)^{-1}$ *genera un semigruppo infinitamente differenziabile in X e si costruisce la soluzione fondamentale per tali problemi. I risultati vengono applicati sia a molte equazioni differenziali paraboliche degeneri nella derivata temperale, sia alla equazione di Stokes.*

1. – Introduction.

Let

$$\begin{cases} d(M(t)v)/dt + L(t)v = f(t), & 0 < t \leq T, \\ \lim\limits_{t \to 0+} M(t)v(t) = u_0, \end{cases} \tag{1.1}$$

and

$$\begin{cases} M(t)\,du/dt + L(t)u = M(t)f(t), & 0 < t \leq T, \\ u(0) = u_0, \end{cases} \tag{1.2}$$

be initial value problems of parabolic type in a Banach space X with a degeneration in the time derivative, that is, $-L(t)$, $0 \leq t \leq T$, are the generators of infinitely differ-

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entiable linear semigroups on X and $M(t)$, $0 \leqslant t \leqslant T$, are closed linear operators in X, the inverse of which may not be bounded operators.

We shall make use of the notion of multivalued linear operator to present a new method for constructing a fundamental solution for (1.1) and (1.2).

The first systematic reserches for these problems were made by CARROLL-SHOWALTER [24]; they studied the weak solution using the theory of sesquilinear forms jointed with the energy method. More recently, the strict solution was studied by FAVINI-PLAZZI [8, 9, 10] (cf. also FAVINI [5, 6, 7]); they reduced the problem (1.1) to seeking the «inverse» of a linear operator $BM + L$ acting in some Hölder continuous function space $\mathcal{C}^\theta([0, T]; X)$, $\theta > 0$, where $Bu = du/dt$ with the initial condition $u(0) = 0$, generalizing the technique in DA PRATO-GRISVARD [3]. For further detailed historical references, we refer the reader to the references in [8, 9, 10].

However, our starting point of the present work is rather a result obtained in [4] by the first author of this paper. In [4, Sect. 5] he handled a time homogeneous equation

$$\text{(1.3)} \qquad \begin{cases} d(Mv)/dt + Lv = f(t), & 0 < t \leqslant T, \\ \lim\limits_{t \to 0+} Mv(t) = u_0, \end{cases}$$

in X and indicated that the operator defined by

$$Z_0(t) = \frac{1}{2\pi i} \int_\Gamma e^{-\lambda t} (\lambda M - L)^{-1} d\lambda, \qquad t > 0,$$

plays a role like a fundamental solution under the assumption that

(1.4) the bounded inverse $(\lambda M - L)^{-1}$, called M-modified resolvent of L, exists for any $\lambda \in \Sigma = \{\lambda \in C; |\arg \lambda| \geqslant \omega\}$, $0 < \omega < \pi/2$, and $(\lambda M - L)^{-1}$ satisfies $\|L(\lambda M - L)^{-1}\|_{\mathcal{L}(X)} \leqslant \text{Const}$, $\lambda \in \Sigma$.

To understand this result intuitively, let us rewrite (1.3) into a non degenerate form by putting $u(t) = Mv(t)$ (such a change of unknown function was already seen in SHOWALTER [15], in which M is a non linear operator). It then turns out that

$$\text{(1.5)} \qquad \begin{cases} du/dt + Au \ni f(t), & 0 < t \leqslant T, \\ u(0) = u_0, \end{cases}$$

were $A = LM^{-1}$ and M^{-1} is the inverse of M. Of course M^{-1} is no longer defined as a univalent operator, but conserves its linearity; such an operator is called a multivalued linear operator (see Section 2). Let us next examine the resolvent of A. It is verified (at least formally, since $\lambda - A = (\lambda M - L)M^{-1}$) that $(\lambda - A)^{-1} = M(\lambda M - L)^{-1}$ for $\lambda \in \Sigma$; so that, it follows from (1.4) that

the bounded inverse $(\lambda - A)^{-1}$ exists as a univalent operator for any $\lambda \in \Sigma$, and $(\lambda - A)^{-1}$ satisfies:

$$\|(\lambda - A)^{-1}\|_{\mathcal{L}(X)} \leq \mathrm{Const}/|\lambda|, \qquad \lambda \in \Sigma.$$

We are then led naturally to define an operator

$$e^{-tA} = \frac{1}{2\pi i}\int_\Gamma e^{-t\lambda}(\lambda - A)^{-1}d\lambda = \frac{1}{2\pi i}\int_\Gamma e^{-t\lambda}M(\lambda M - L)^{-1}d\lambda, \qquad t > 0,$$

and to expect that e^{-tA} is a fundamental solution for (1.5), or a semigroup generated by A (see Section 3). Since $e^{-tA} = MZ_0(t)$, the role of the operator $Z_0(t)$ is now clarified.

Our method is thus to rewrite the equation (1.1) or (1.2) into non degenerate form using multivalued linear operators. Indeed, (1.1) (resp. (1.2)) is reduced to the problem

$$\begin{cases} du/dt + A(t)u \ni f(t), & 0 < t \leq T, \\ u(0) = u_0, \end{cases} \tag{1.6}$$

where $A(t) = L(t)M(t)^{-1}$ (resp. $M(t)^{-1}L(t)$), by putting $u(t) = M(t)v(t)$ (resp. by operating $M(t)^{-1}$). The machinery of constructing a fundamental solution for the parabolic equation (1.6) has then been, although we have still the minor but somewhat long task to adapt it to the multivalued case, established satisfactorily by an extensive literature, including SOBOLEVSKI [16, 17], KATO-TANABE [14], TANABE [18, 19], AMANN [2], ACQUISTAPACE [1], YAGI [20, 21, 22]. But we may notice that the domains $\mathcal{D}(A(t))$ of $A(t)$ in (1.6) are not dense in general (on the contrary, they can be $\{0\}$, cf. Example 6.5 in Section 6); in addition, we may notice that $-A(t)$ may be merely generators of infinitely differential semigroups even if $-L(t)$ generate analytic semigroups (cf. Examples 6.3 and 4). By this reason we shall use among other the results obtained in [21, 22] in which such general (univalent) cases were considered. Essential things in generalizing them to the multivalued case are that all the results in [21, 22] are described in terms of the resolvents, and that, on the other hand, even though $A(t)$ themselves in (1.6) are multivalued, their resolvents $(\lambda - A(t))^{-1}$ can exist as univalent operators for suitable λ (see Section 4).

The fundamental solution for (1.1) or (1.2) we shall construct in Section 5 under suitable assumptions then provides the existence and uniqueness of the strict solution of the problem as well its representation formula. Especially it will be noticed that the initial value u_0 can be taken arbitrarily in X (without any compatibly condition), provided that we impose on the initial condition of (1.1) (resp. (1.2)) only a weak sense, i.e. $M(t)v(t)$ (resp. $u(t)) \to u_0$, as $t \to 0$, in a certain seminorm specifically determined by $M(0)$ and $L(0)$ (see Theorems 5.2 and 5.3). While, when u_0 is more regular, the maximal regularity results in [8] can be applied (this will be discussed systematically in the paper [11]).

It is possible to apply our results to many parabolic differential equations of degenerate type with respect to time derivative. Among others only immediate applications will be presented in Section 6; for example the Stokes equation also falls in the scope (see Example 6.2). Further applications will be, however, published in the forthcoming paper.

Not only for the parabolic equations, our method is available also for the hyperbolic equations. For example, a generation theorem of semigroup for a multivalued linear operator A in a Banach space X can be proved (although this semigroup e^{-tA} is, now, far from C_0 semigroup, cf. Example 6.1) under the condition of Hille-Yosida type

$$\|(\lambda - A)^{-n}\|_{\mathcal{L}(X)} \leq M/|\lambda - \beta|^n, \qquad \lambda < \beta, \qquad n = 1, 2, 3, \ldots,$$

see [23].

NOTATIONS. – Throughout this paper, X denotes a Banach space; the norm is denoted by $\|\cdot\|_X$; a seminorm defined on X is denoted by $p(\cdot)$. Let Y be another Banach space; $\mathcal{L}(X, Y)$ is the space of all bounded linear operators from X to Y; $\|\cdot\|_{\mathcal{L}(X, Y)}$ denotes the uniform operator norm in $\mathcal{L}(X, Y)$. We shall use the abbreviation $\mathcal{L}(X)$ (resp. $\|\cdot\|_{\mathcal{L}(X)}$) instead of $\mathcal{L}(X, X)$ (resp. $\|\cdot\|_{\mathcal{L}(X, X)}$). For a closed interval $[a, b]$, $\mathcal{C}([a, b]; X)$, $\mathcal{C}^\sigma([a, b]; X)$ $(0 < \sigma < 1)$ and $\mathcal{C}^1([a, b]; X)$ denote respectively the space of all functions on $[a, b]$ with values in X which are continuous, which are σ-Hölder continuous, and which are continuously differentiable. $L^p(a, b; X)$ $(1 \leq p \leq \infty)$ denotes the space of all measurable functions f on an open interval (a, b) with values in X such that $\|f(\cdot)\|_{X^p}$ are integrable if $1 \leq p < \infty$, such that f are essentially bounded if $p = \infty$.

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2. – Multivalued linear operators.

Let X be a Banach space over the complex numbers $\boldsymbol{C}$. For two subsets F, G of X, we define: $F + G = \{f + g; f \in F, g \in G\}$ and, for a number $\lambda \in \boldsymbol{C}$, $\lambda F = \{\lambda f; f \in F\}$.

DEFINITION. – A mapping A from X into 2^X is called a *multivalued linear operatori in X* if the domain $\mathcal{D}(A) = \{u \in X; Au \neq \emptyset\}$ is a linear subspace of X and A satisfies:

$$\begin{cases} Au + Av \subset A(u+v) & \text{for } u, v \in \mathcal{D}(A), \\ \lambda Au \subset A(\lambda u) & \text{for } \lambda \in \boldsymbol{C} \text{ and } u \in \mathcal{D}(A). \end{cases}$$

$\mathcal{R}(A) = \bigcup_{u \in \mathcal{D}(A)} Au$ is called the range of A.

This section is devoted to establish basic results of the multivalued linear operators. Theorems 2.1-2.7, excluding 2.3 and 2.7, are quite analogous to those of the univalent linear operators. To the contrary, Theorems 2.3 and 2.7 seem to be very proper to the multivalued linear operators, which we shall use in an essential way in the subsequent sections.

Throughout this section A and B denote multivalued operators in X.

THEOREM 2.1. – *$Au + Av = A(u+v)$ for $u, v \in \mathcal{D}(A)$. $\lambda Au = A(\lambda u)$ for $u \in \mathcal{D}(A)$ if $\lambda \neq 0$.*

PROOF. – By definition, $A(u+v) - Av \subset A(u+v) + A(-v) \subset Au$; hence $A(u+v) \subset \subset Au + Av$. Similarly, $A(\lambda u) = \lambda\lambda^{-1}A(\lambda u) \subset \lambda Au$ if $\lambda \neq 0$.

THEOREM 2.2. – *$A0$ is linear subspace of X. $Au = f + A0$ for any $f \in Au$, $u \in \mathcal{D}(A)$; in particular, A is univalent if and only if $A0 = \{0\}$.*

PROOF. – From definition $A0 + A0 = A0$ and $\lambda A0 = A0$, hence the first assertion. Clearly, $f + A0 \subset Au$ for $f \in Au$; on the other hand, if $g \in Au$, $g - f \in Au - Au = = A0$.

The inverse A^{-1} of a multivalued linear operator A is defined as

$$\begin{cases} \mathcal{D}(A^{-1}) = \mathcal{R}(A), \\ A^{-1}f = \{u \in \mathcal{D}(A); Au \ni f\}. \end{cases}$$

THEOREM 2.3. – *A^{-1} is also a multivalued linear operator in X. $f \in Au$ if and only if $u \in A^{-1}f$; in particular, $(A^{-1})^{-1} = A$.*

PROOF. – If $u \in A^{-1}f$ and $v \in A^{-1}g$, then $f \in Au$ and $g \in Av$; so that $f + g \in A(u+v)$; hence $u + v \in A^{-1}(f+g)$. Similarly, $u \in A^{-1}f$ implies $\lambda f \in A(\lambda u)$ and $\lambda u \in A^{-1}(\lambda f)$, hence $\lambda A^{-1}f \subset A^{-1}(\lambda f)$. The second assertion is obvious from the definition.

If, for two multivalued linear operators A and B, $\mathcal{D}(A) \subset \mathcal{D}(B)$ and $Au \subset Bu$ for every $u \in \mathcal{D}(A)$, then B is called an extension of A and it is denoted by

$A \subset B$. Obviously $A \subset B$ defines an order relation and, in particular, $A \subset B$ and $B \subset A$ imply $A = B$.

If a univalent linear operator A° satisfies $\mathcal{D}(A^\circ) = \mathcal{D}(A)$ and $A^\circ \subset A$, then A° is called a linear section of A.

Let now U be a univalent linear operator in X. Summation $A + U$ and multiplication AU and UA are defined respectively as follows:

$$\begin{cases} \mathcal{D}(A + U) = \mathcal{D}(A) \cap \mathcal{D}(U), \\ (A + U)u = Au + Uu \quad \text{for } u \in \mathcal{D}(A + U); \end{cases}$$

$$\begin{cases} \mathcal{D}(AU) = \{u \in \mathcal{D}(U);\ Uu \in \mathcal{D}(A)\}, \\ AUu = A(Uu); \end{cases}$$

$$\begin{cases} \mathcal{D}(UA) = \{u \in \mathcal{D}(A);\ Au \cap \mathcal{D}(U) \neq \emptyset\}, \\ UAu = \{Uv;\ v \in Au \cap \mathcal{D}(U)\}. \end{cases}$$

THEOREM 2.4. – *The operator $A+U$, AU and UA defined above are multivalued linear operators.*

PROOF. – The proofs for $A+U$ and AU are immediate from the definition. Let u_1, $u_2 \in \mathcal{D}(UA)$ and let $Uv_1 \in UAu_1$, $Uv_2 \in UAu_2$, then $v_1 + v_2 \in A(u_1 + u_2) \cap \mathcal{D}(U)$, so that $U(v_1 + v_2) \in UA(u_1 + u_2)$ i.e. $UAu_1 + UAu_2 \subset UA(u_1 + u_2)$. Similarly, it is observed that $\lambda UAu \subset UA(\lambda u)$.

THEOREM 2.5. – *Let U, V be two univalent linear operators in X. Then, $(UV^{-1})^{-1} = VU^{-1}$ and $(U^{-1}V)^{-1} = V^{-1}U$.*

PROOF. – Let $u \in (UV^{-1})^{-1}f$; then, $UV^{-1}u \ni f$ or $Uv = f$ for some $v \in \mathcal{D}(U) \cap V^{-1}u$; so that, $Vv = u$ with $v \in \mathcal{D}(V) \cap U^{-1}f$; hence $u \in VU^{-1}f$. The converse is also true. The second assertion is verified similarly noting that $u \in (U^{-1}V)^{-1}f$ if and only if $Vu = Uf$ with $u \in \mathcal{D}(V)$ and $f \in \mathcal{D}(U)$.

For a multivalued linear operator A, the set of all numbers $\lambda \in \mathbf{C}$ such that $\mathcal{R}(\lambda - A) = \mathcal{D}((\lambda - A)^{-1}) = X$ and $(\lambda - A)^{-1}$ is a univalent bounded operator on X is called the resolvent set of A and is denoted by $\rho(A)$. $(\lambda - A)^{-1}$, $\lambda \in \rho(A)$, is called the resolvent of A.

THEOREM 2.6. – *$\rho(A)$ is an open set of $\mathbf{C}$. The resolvent $(\lambda - A)^{-1}$ is a holomorphic function in $\rho(A)$ with values in $\mathcal{L}(X)$.*

PROOF. – I) We first consider the case that $(\lambda - A)^{-1} \neq 0$ for every $\lambda \in \rho(A)$. Let $\lambda_0 \in \rho(A)$ and let $|\lambda - \lambda_0| < 1/\|(\lambda_0 - A)^{-1}\|_{\mathcal{L}(X)}$. For any $f \in X$, put

$$g = \{1 + (\lambda - \lambda_0)(\lambda_0 - A)^{-1}\}^{-1} f \quad \text{and} \quad u = (\lambda_0 - A)^{-1} g;$$

since $g + (\lambda - \lambda_0) u = f$ and $g \in (\lambda_0 - A) u$, we have: $f \in (\lambda - A) u$; hence $\mathcal{R}(\lambda - A) = X$. Conversely, let $(\lambda - A)^{-1} 0 \ni u$ or $(\lambda - A) u \ni 0$; then there is an element $g \in (\lambda_0 - A) u$ such that $g + (\lambda - \lambda_0) u = 0$; so that $\{1 + (\lambda - \lambda_0)(\lambda_0 - A)^{-1}\} u = 0$; this shows that $u = 0$. Thus we have proved that $\lambda \in \rho(A)$ and $(\lambda - A)^{-1} = (\lambda_0 - A)^{-1}\{1 + (\lambda - \lambda_0)(\lambda_0 - A)^{-1}\}^{-1}$.

II) Let now $(\lambda_0 - A)^{-1} = 0$ for some $\lambda_0 \in \rho(A)$. Then $\lambda_0 - A = O_\infty$ is the inverse of the 0 operator and we observe that $\mathcal{D}(O_\infty) = \{0\}$ and $O_\infty 0 = X$; so that, $A = O_\infty$. It is also verified easily that $\rho(O_\infty) = \boldsymbol{C}$ and $(\lambda - O_\infty)^{-1} = 0$ identically, hence the result.

THEOREM 2.7. – $(\lambda - A)^{-1} A \subset \lambda(\lambda - A)^{-1} - 1 \subset A(\lambda - A)^{-1}$ *for* $\lambda \in \rho(A)$. *In particular,* $(\lambda - A)^{-1} A$ *is univalent on* $\mathcal{D}(A)$ *and* $(\lambda - A)^{-1} A u = (\lambda - A)^{-1} f$ *for any* $f \in Au$.

PROOF. – Let $f \in Au$; then $\lambda u - f \in (\lambda - A) u$ and $(\lambda - A)^{-1} f = \lambda(\lambda - A)^{-1} u - u$; this means that $(\lambda - A)^{-1} A \subset \lambda(\lambda - A)^{-1} - 1$. On the other hand, let $v = (\lambda - A)^{-1} f$; then $\lambda v - f \in Av$ and $\lambda(\lambda - A)^{-1} f - f \in A(\lambda - A)^{-1} f$; hence $\lambda(\lambda - A)^{-1} - 1 \subset$ $\subset A(\lambda - A)^{-1}$.

DEFINITION. – We denote by $A^\circ(\lambda - A)^{-1}$ the linear section $\lambda(\lambda - A)^{-1} - 1$ of the multivalued operator $A(\lambda - A)^{-1}$ for $\lambda \in \rho(A)$.

We shall conclude this section by proving the resolvent equation for the multivalued linear operators.

Let $\lambda, \mu \in \rho(A)$. Putting $\lambda_\mu = \lambda - \mu$ and $A_\mu = A - \mu$, we apply Theorem 2.7 to A_μ. Then, noting $\lambda_\mu, 0 \in \rho(A_\mu)$, we obtain that

$$(\lambda_\mu - A_\mu)^{-1} A_\mu \subset \lambda_\mu (\lambda_\mu - A_\mu)^{-1} - 1,$$
$$(\lambda_\mu - A_\mu)^{-1} A_\mu (A_\mu)^{-1} \subset \lambda_\mu (\lambda_\mu - A_\mu)^{-1} (A_\mu)^{-1} - (A_\mu)^{-1}.$$

Since the inclusion $1 \subset A_\mu (A_\mu)^{-1}$ holds, it follows from Theorem 2.7 again that $(\lambda_\mu - A_\mu)^{-1} A_\mu (A_\mu)^{-1} = (\lambda_\mu - A_\mu)^{-1}$. Therefore

$$(2.1) \quad (\lambda - A)^{-1} - (\mu - A)^{-1} = -(\lambda - \mu)(\lambda - A)^{-1}(\mu - A)^{-1} \qquad \text{for } \lambda, \mu \in \rho(A).$$

Consider now two multivalued operators A and B, and let $\lambda, 0 \in \rho(A) \cap \rho(B)$. Writing:

$$(\lambda - A)^{-1} - (\lambda - B)^{-1} = \{\lambda(\lambda - A)^{-1} - 1\}(\lambda - B)^{-1} - (\lambda - A)^{-1}\{\lambda(\lambda - B)^{-1} - 1\},$$

we replace $(\lambda - A)^{-1}$ (resp. $(\lambda - B)^{-1}$) by $\lambda\{(\lambda - A)^{-1} - 1\} A^{-1}$ (by respectively,

$B^{-1}\{\lambda(\lambda-B)^{-1}-1\}$), then it turns out that

$$(2.2)\quad (\lambda-A)^{-1}-(\lambda-B)^{-1}=-\{\lambda((\lambda-A)^{-1}-1\}(A^{-1}-B^{-1})\{\lambda(\lambda-B)^{-1}-1\}= \\ =-A^{\circ}(\lambda-A)^{-1}(A^{-1}-B^{-1})B^{\circ}(\lambda-B)^{-1}\qquad \text{for } \lambda\in\rho(A)\cap\rho(B).$$

3. – Time homogeneous equations.

In this section we consider an initial value problem for the time homogeneous evolution equation of parabolic type

$$(\text{H.E})\qquad \begin{cases} du/dt+Au\ni f(t), & 0<t\leq T,\\ u(0)=u_0, \end{cases}$$

in a Banach space X, where A is a multivalued linear operator in X, $f:[0,T]\to X$ is a given continuous function, u_0 is an initial value in X, and $u:[0,T]\to X$ is the unknown function of the problem.

Our main assumption on A is that A satisfies:

(H.P) The resolvent set $\rho(A)$ contains a region Σ

$$\rho(A)\supset\Sigma=\{\lambda\in\boldsymbol{C};\ \operatorname{Re}\lambda\leq c(|\operatorname{Im}\lambda|+1)^{\alpha}\}$$

and there the resolvents $(\lambda-A)^{-1}$ satisfy

$$\|(\lambda-A)^{-1}\|_{\mathcal{L}(X)}\leq M/(|\lambda|+1)^{\beta},\qquad \lambda\in\Sigma$$

with some exponents $0<\beta\leq\alpha\leq 1$ and constants c, $M>0$.

Using the semigroup generated by A, we shall prove the existence and uniqueness of the strict solution of (H.E). By the strict solution we mean.

DEFINITION. – A function $u:[0,T]\to X$, $u\in\mathcal{C}^1((0,T];X)$, and $u(t)\in\mathcal{D}(A)$ for $0<t\leq T$, is called *a strict solution* of (H.E) if u satisfies the equation in (H.E) for every $0<t\leq T$ and satisfies the initial condition: $u(0)=u_0$ with respect to the seminorm $p_A(\cdot)=\|A^{-1}\cdot\|_X$.

Let a multivalued linear operator A satisfy Condition (H.P). We define a family of bounded operators on X

$$(3.1)\quad \exp(-\tau A)=\frac{1}{2\pi i}\int_{\Gamma}e^{-\tau\lambda}(\lambda-A)^{-1}\,d\lambda\qquad \text{for } \tau>0, \text{ and } \exp(-0A)=1,$$

by the Dunford integrals in $\mathcal{L}(X)$, where Γ: $\lambda=c(|y|+1)^{\alpha}+iy$, $-\infty<y<\infty$, is a contour lying in Σ. $\exp(-\tau A)$, $\tau\geq 0$, is called the semigroup generated by A. In the same way as for the univalent case (using the resolvent equation (2.1)), it is verified that $\exp(-\tau A)$ has the semigroup property $\exp(-\tau A)\exp(-\sigma A)=\exp(-(\tau+\sigma)A)$.

In addition, $\exp(-\tau A)$ is infinitely differentiable for $\tau > 0$ and

$$\frac{d^k \exp(-\tau A)}{d\tau^k} = \frac{(-1)^k}{2\pi i}\int_\Gamma \lambda^k e^{-\tau\lambda}(\lambda - A)^{-1}d\lambda, \qquad \tau > 0,\ k = 1, 2, 3, \dots.$$

We then define bounded operators $A^\circ \exp(-\tau A)$, $\tau > 0$, by

$$A^\circ \exp(-\tau A) = \frac{1}{2\pi i}\int_\Gamma \lambda e^{-\tau\lambda}(\lambda - A)^{-1}d\lambda, \qquad \tau > 0.$$

Since $A^{-1}\{\lambda(\lambda-A)^{-1}-1\}=(\lambda-A)^{-1}$, it follows that $A^{-1}A^\circ\exp(-\tau A) = \exp(-\tau A)$; this then shows that $A^\circ \exp(-\tau A)$, $\tau > 0$, is a linear section of $A \exp(-\tau A)$. Hence

$$-\frac{d\exp(-\tau A)}{d\tau} = A^\circ \exp(-\tau A) \subset A \exp(-\tau A), \qquad \tau > 0.$$

In the multivalued case, however, the convergence of $\exp(-\tau A)$ to 1, as $\tau \to 0$, becomes considerably worse; it is no longer possible to expect it on the whole space in any norm; indeed, if $f \in A0$, then $(\lambda - A)^{-1}f = 0$ and $\exp(-\tau A)f = 0$ for all $\tau > 0$. What we can simply observe is that

$$\exp(-\tau A)A^{-1} \to A^{-1} = \frac{1}{2\pi i}\int_\Gamma \lambda^{-1}(\lambda - A)^{-1}d\lambda, \qquad \text{as } \tau > 0, \text{ in } \mathcal{L}(X). \tag{3.2}$$

In other words, $\exp(-\tau A)$ converges to 1 in the norm X only on the domain $\mathcal{D}(A)$ of A, and only in the seminorm $p_A(\cdot)$ sense on the whole space X.

For a multivalued operator A satisfying (H.P), we can define the fractional powers $A^{-\theta}$, $\theta > 1 - \beta$, of A by the integrals

$$A^{-\theta} = \frac{1}{2\pi i}\int_\Gamma \lambda^{-\theta}(\lambda - A)^{-1}d\lambda.$$

$A^{-\theta}$, $\theta > 1 - \beta$, are bounded operators on X and $A^{-\theta}A^{-\theta'} = A^{-(\theta+\theta')}$ for $\theta, \theta' > 1 - \beta$. As was defined above for $\theta = 1$, we define, for every $\theta \geq 0$, a bounded operator $(A^\theta)^\circ \exp(-\tau A)$ by

$$(A^\theta)^\circ \exp(-\tau A) = \frac{1}{2\pi i}\int_\Gamma \lambda^\theta e^{-\tau\lambda}(\lambda - A)^{-1}d\lambda, \qquad \tau > 0.$$

By (3.1), $(A^0)^\circ \exp(-\tau A) = \exp(-\tau A)$. It is easily verified from (H.P) that

$$\|(A^\theta)^\circ \exp(-\tau A)\|_{\mathcal{L}(X)} \leq CD^\theta \Gamma\left(\frac{\theta + 1 - \beta}{\alpha}\right)\tau^{(\beta-\theta-1)/\alpha}, \qquad \theta \geq 0,\ \tau > 0, \tag{3.3}$$

where C, D are some constants independent of θ, τ and $\Gamma(\cdot)$ is the gamma function with the convention that $\Gamma(0)$ is understood as $\Gamma(0) = 1$.

Let A still satisfy (H.P). The bounded operators on X defined by

$$A_n = A^\circ (1 + n^{-1}A)^{-1} = n - n^2(n + A)^{-1} \qquad \text{for } n = 1, 2, 3, \ldots \tag{3.4}$$

are called the Yosida approximations of A. If $n\lambda/(n - \lambda) \in \rho(A)$, then $\lambda \in \rho(A_n)$ and

$$(\lambda - A_n)^{-1} = \frac{1}{\lambda - n} + \left(\frac{n}{n - \lambda}\right)^2 \left(\frac{n\lambda}{n - \lambda} - A\right)^{-1}. \tag{3.5}$$

In fact, this relation is well known when A is univalent; in the present case, this is then verified by a direct calculation using (2.1), i.e. we put $(\lambda - A_n)^{-1}$ the right hand side of (3.5) and show directly that $(\lambda - A_n)(\lambda - A_n)^{-1} = (\lambda - A_n)^{-1}(\lambda - A_n) = 1$. Since there exists an integer n_0 such that $\{n/(n - \lambda);\ n \geq n_0,\ \lambda \in \Sigma\}$ is a bounded set, and since $n\lambda/(n - \lambda) = |n/(n - \lambda)|^2 \lambda - |\lambda/(n - \lambda)|^2 n$, we observe that, for some $0 < \tilde{c} < c$, the resolvent sets $\rho(A_n)$ contain

$$\rho(A_n) \supset \tilde{\Sigma} = \{\lambda \in \boldsymbol{C};\ \operatorname{Re}\lambda \leq \tilde{c}(|\operatorname{Im}\lambda| + 1)^\alpha\} \qquad \text{for all } n \geq n_0 .$$

Similarly,

$$\|(\lambda - A_n)^{-1}\|_{\mathcal{L}(X)} \leq \tilde{M}/(|\lambda| + 1)^\beta \qquad \text{for all } \lambda \in \tilde{\Sigma}$$

with some constant $\tilde{M}$ independent of $n \geq n_0$.

We do not know, in general, whether, as $n \to \infty$, A_n converges to some linear section of A or not. But we can easily verify (and we shall use the fact essentially in the sequel) that, for each $\lambda \in \tilde{\Sigma}(\subset \Sigma)$, the resolvents $(\lambda - A_n)^{-1}$ converge to $(\lambda - A)^{-1}$ in $\mathcal{L}(X)$ as $n \to \infty$. Consequently, it follows that $A_n^\theta \exp(-\tau A)$ converges, for each $\theta \geq 0$ and $\tau > 0$, to $(A^\theta)^\circ \exp(-\tau A)$ in $\mathcal{L}(X)$ with an estimate

$$\|A_n^\theta \exp(-\tau A_n)\|_{\mathcal{L}(X)} \leq \tilde{C}\tilde{D}^\theta \Gamma\left(\frac{\theta + 1 - \beta}{\alpha}\right) \tau^{(\beta - \theta - 1)/\alpha}, \qquad \theta \geq 0 \text{ and } \tau > 0, \tag{3.6}$$

where $\tilde{C}$ and $\tilde{D}$ are constants independent of $n \geq n_0$. More strongly, we can prove:

THEOREM 3.1. – *Let $J_n = (1 + n^{-1}A)^{-1}$. A similar estimate*

$$\|A_n^\theta J_n \exp(-\tau A_n)\|_{\mathcal{L}(X)} \leq \tilde{C}\tilde{D}^\theta \Gamma\left(\frac{\theta + 1 - \beta}{\alpha}\right) \tau^{(\beta - \theta - 1)/\alpha}, \qquad \theta \geq 0 \text{ and } \tau > 0,$$

holds for every $n \geq n_0$ with uniform constants $\tilde{C}$ and $\tilde{D}$; in addition, as $n \to \infty$, $A_n^\theta J_n \exp(-\tau A_n)$ converges to $(A^\theta)^\circ \exp(-\tau A)$ in $\mathcal{L}(X)$ for each $\theta \geq 0$ and $\tau > 0$.

PROOF. – Using the resolvent equation (2.1), in view of (3.5) we have:

$$J_n(\lambda - A_n)^{-1} = \frac{n}{\lambda - n}(n + A)^{-1} + \frac{n^3}{(n-\lambda)^2}(n + A)^{-1}\left(\frac{n\lambda}{n-\lambda} - A\right)^{-1}$$

$$= \frac{n}{n-\lambda}\left(\frac{n\lambda}{n-\lambda} - A\right)^{-1};$$

therefore $J_n(\lambda - A_n)^{-1}$ converges to $(\lambda - A)^{-1}$ for each $\lambda \in \tilde{\Sigma}$ with an estimate

$$\|J_n(\lambda - A_n)^{-1}\|_{\mathcal{L}(X)} \leq \tilde{M}/(|\lambda| + 1)^{\beta}, \qquad \lambda \in \tilde{\Sigma},$$

with some $\tilde{M}$ independent of $n \geq n_0$. Then the proof is the same as for $A_n^{\theta}\exp(-\tau A_n)$.
We are now in a position to state the main theorem of this section.

THEOREM 3.2. – *Let A satisfy the Condition* (H.P) *with $2\alpha + \beta > 2$. For any Hölder continuous function $f \in \mathcal{C}^{\sigma}([0, T]; X)$, $(2 - \alpha - \beta)/\alpha < \sigma (\leq 1)$, and any initial value $u_0 \in X$, the function u given by*

$$u(t) = \exp(-tA)u_0 + \int_0^t \exp(-(t-\tau)A)f(\tau)\,d\tau, \qquad 0 < t \leq T, \tag{3.7}$$

is a strict solution of (H.E). *Conversely, any strict solution of* (H.E) *with $f \in \mathcal{C}([0, T]; X)$ and $u_0 \in X$ is necessarily of the form* (3.7).

PROOF. – Let f be the Hölder continuous function above. Since $\alpha + \beta > 1$, the integral in (3.7) is well defined (with $\theta = 0$) by (3.3). We consider a sequence of functions u_n, $n \geq n_0$, defined by

$$u_n(t) = \exp(-tA_n)u_0 + \int_0^t \exp(-(t-\tau)A_n)f(\tau)\,d\tau, \qquad 0 < t \leq T.$$

As $n \to \infty$, $u_n(t)$ converges obviously to $u(t)$ pointwisely on $[0, T]$. Moreover, operate A_n and write:

$$A_n u_n(t) = A_n \exp(-tA_n)u_0 + \int_0^t A_n \exp(-(t-\tau)A_n)\{f(\tau) - f(t)\}\,d\tau + \{1 - \exp(-tA_n)\}f(t).$$

Then $A_n u_n(t)$, in view of (3.6) (with $\theta = 1$), converges to a continuous function $g(t)$ pointwisely on $(0, T]$. Therefore, from $u_n(t) = A_n^{-1}A_n u_n(t)$ it follows that $u(t) = A^{-1}g(t)$, hence $u(t) \in \mathcal{D}(A)$ and $g(t) \in Au(t)$. Letting $n \to \infty$ in

$$u_n(t) - u_n(\varepsilon) = \int_{\varepsilon}^t \{f(\tau) - A_n u_n(\tau)\}\,d\tau, \qquad \varepsilon \leq t \leq T,$$

with any $\varepsilon > 0$, we also obtain that $u \in \mathcal{C}^1((0, T]; X)$ and u satisfies the equation for every $0 < t \leq T$. The initial condition of (H.E) is an immediate consequence of (3.2), since, as $t \to 0$, $\int_0^t \exp(-(t-\tau)A) f(\tau)\,d\tau \to 0$ in X. Conversely, let now u be any strict solution of (H.E). Writing for any $\varepsilon > 0$:

$$u(t) - \exp(-(t-\varepsilon)A_n)u(\varepsilon) = \int_\varepsilon^t \frac{\partial}{\partial\tau}\exp(-(t-\tau)A_n)u(\tau)\,d\tau =$$

$$= \int_\varepsilon^t \exp(-(t-\tau)A_n)\{(A_n u(\tau) - g(\tau)\}\,d\tau + \int_\varepsilon^t \exp(-(t-\tau)A_n) f(\tau)\,d\tau,$$

where $g(t) = f(t) - du/dt \in Au(t)$, we use Theorem 3.1. Then, since $A_n u(t) = J_n g(t)$ from (3.4), the convergence of $\exp(-\tau A_n)J_n$ yields that

$$u(t) - \exp(-(t-\varepsilon)A)u(\varepsilon) = \int_\varepsilon^t \exp(-(t-\tau)A) f(\tau)\,d\tau, \qquad \varepsilon \leq t \leq T.$$

Noting $A^\circ \exp(-(t-\varepsilon)A)A^{-1} = \exp(-(t-\varepsilon)A)$, we next make ε to converge to 0. Then from the initial condition on u we conclude (3.7).

REMARK. – If, for a multivalued linear operator A, $\rho(A) \supset \{\lambda \in \boldsymbol{C};\ \mathrm{Re}\,\lambda \leq 0\}$ and the resolvent $(\lambda - A)^{-1}$ satisfies:

$$\|(\lambda - A)^{-1}\|_{\mathcal{L}(X)} \leq M/(|\lambda| + 1)^\beta \quad \text{for } \mathrm{Re}\,\lambda \leq 0 \tag{3.8}$$

with some $0 < \beta \leq 1$, then by Theorem 2.6 $(\lambda - A)^{-1}$ can be continued to a region $\Sigma = \{\lambda \in \boldsymbol{C};\ \mathrm{Re}\,\lambda \leq c(|\mathrm{Im}\,\lambda| + 1)^\beta\}$, $c > 0$, as a holomorphic function with a similar estimate as (3.8) on Σ. This then shows that if the Condition (H.P) holds for an operator A with some exponent $0 < \alpha, \beta \leq 1$, then the relation $\beta \leq \alpha$ always takes place.

REMARK. – We notice from (3.2) that the function u given by (3.7) is continuous at $t = 0$ in the norm of X if $u_0 \in \mathcal{D}(A)$. Therefore, if the initial value u_0 of the problem (H.E) $\in \mathcal{D}(A)$, then there exists a strict solution which belongs to $\mathcal{C}([0, T]; X)$.

4. – Time non homogeneous equations.

We consider in this section the initial value problem for a time dependent equation of parabolic type:

$$\text{(E)} \quad \begin{cases} du/dt + A(t)u \ni f(t), & 0 < t \leq T, \\ u(0) = u_0, \end{cases}$$

in a Banach space X. Here, $A(t)$, $0 \leq t \leq T$, are multivalued linear operators in X which generate infinitely differentiable semigroups on X, i.e. $A(t)$ satisfies, for each $0 \leq t \leq T$, the Condition (H.P) in Section 3. $f: [0, T] \to X$ is a given continuous function, $u_0 \in X$ is the initial value, and $u: [0, T] \to X$ is the unknown function of the problem.

We shall prove the existence and uniqueness of the strict solution of (E) by constructing the evolution (fundamental) operator $U(t, s)$, $0 \leq s \leq t \leq T$. The strict solution of (E) is defined analogously to the time homogeneous problem according:

DEFINITION If the function $u: [0, T] \to X$, $u \in \mathcal{C}^1((0, T]; X)$, and $u(t) \in \mathcal{D}(A(t))$ for $0 < t \leq T$, satisfies the equation (E) for every $0 < t \leq T$ and satisfies the initial condition: $u(0) = u_0$ with continuity at $t = 0$ with respect to the seminorm $p_{A(0)}(\cdot) = \|A(0)^{-1} \cdot\|_X$, the function u is called *a strict solution* of (E).

By the evolution operator for $A(t)$ we mean bounded operators $U(t, s)$, $0 \leq s \leq t \leq T$, on X which have the properties at least: *a)* $U(t, s)\, U(s, r) = U(t, r)$ for $0 \leq r \leq s \leq t \leq T$, $U(s, s) = 1$ for $0 \leq s \leq T$; *b)* $U(t, s)$ is strongly continuous for $0 \leq s < t \leq T$ with an estimate $\|U(t, s)\|_{\mathcal{L}(X)} \leq C_1 (t - s)^{(\beta - 1)/\alpha}$; *c)* $U(t, s)$ is strongly differentiable in t for $0 \leq s < t \leq T$, and $\partial U(t, s)/\partial t$ is strongly continuous for $0 \leq s < t \leq T$ with an estimate $\|\partial U(t, s)/\partial t\|_{\mathcal{L}(X)} \leq C_2 (t - s)^{(\beta - 2)/\alpha}$; and *d)* the range $\mathcal{R}(U(t, s))$ of $U(t, s)$ is contained in $\mathcal{D}(A(t))$ for $0 \leq s < t \leq T$, and there exists a bounded linear section $A(t)^\circ U(t, s)$ of $A(t)\, U(t, s)$ such that $-\partial U(t, s)/\partial t = A(t)^\circ U(t, s) \subset A(t)\, U(t, s)$.

As was mentioned already, the following condition is a basic assumption throughout this section:

(P) The resolvent sets $\rho(A(t))$, $0 \leq t \leq T$, of $A(t)$ contain a region Σ

$$\rho(A(t)) \supset \Sigma = \{\lambda \in C;\ \mathrm{Re}\,\lambda \leq c(|\mathrm{Im}\,\lambda| + 1)^\alpha\}$$

and there the resolvents $(\lambda - A(t))^{-1}$ satisfy:

$$\|(\lambda - A(t))^{-1}\|_{\mathcal{L}(X)} \leq M/(|\lambda| + 1)^\beta, \qquad \lambda \in \Sigma$$

with some exponents $0 < \beta \leq \alpha \leq 1$ and constants c, $M > 0$ independent of t.

On the other hand, assumptions on the regularity of $A(t)$ with respect to t depend, as in the case of univalent coefficients $A(t)$, on the nature of variation of the domains $\mathcal{D}(A(t))$ of $A(t)$. We state them separating in the two cases that 1) $\mathcal{D}(A(t))$ varies temperately with t (inclusing the case that $\mathcal{D}(A(t))$ is independent of t) and that 2) $\mathcal{D}(A(t))$ varies completely with t.

Assumptions for the case of temperately variable domains:

(T.1) For some $0 < \mu, \nu \leqslant 1$, $A(\cdot)$ satisfies:

$$\|A(t)^\circ (\lambda - A(t))^{-1}\{A(t)^{-1} - A(s)^{-1}\}\|_{\mathcal{L}(X)} \leqslant$$

$$\leqslant K|t-s|^{\mu}/(|\lambda| + 1)^{\nu} \quad \text{for all } \lambda \in \Sigma \text{ and } 0 \leqslant s, t \leqslant T.$$

(T.2) The exponents α, β, μ and ν satisfy the relation $2(\alpha + \beta) + \alpha\mu + \nu > 5$. (In particular, in order that such a relation takes place it is necessary that $3\alpha + 2\beta > 4$).

Assumption for the case of completely variable domains:

(C.1) $A(\cdot)^{-1}$ is strongly continuously differentiable in t, and the derivative satisfies:

$$\|A(t)^\circ (\lambda - A(t))^{-1} dA(t)^{-1}/dt\|_{\mathcal{L}(X)} \leqslant$$

$$\leqslant N/(|\lambda| + 1)^{\nu} \quad \text{for all } \lambda \in \Sigma \text{ and } 0 \leqslant t \leqslant T,$$

with some exponent $0 < \nu \leqslant 1$.

(C.2) The exponents α, β and ν satisfy a relation $2(\alpha + \beta) + \nu > 4$. (In particular, $2(\alpha + \beta) > 3$).

(For the definition of $A(t)^\circ (\lambda - A(t))^{-1}$, see Section 2.)

Let $J_n(t) = (1 + n^{-1}A(t))^{-1}$, $n = 1, 2, 3, \ldots$, and let

$$A_n(t) = A(t)^\circ J_n(t) = n - n^2(n + A(t))^{-1}, \qquad n = 1, 2, 3, \ldots,$$

be the Yosida approximation of $A(t)$. Since we have from the resolvent equation (2.2):

$$(\lambda - A(t))^{-1} - (\lambda - A(s))^{-1} =$$

$$= -A(t)^\circ (\lambda - A(t))^{-1}\{A(t)^{-1} - A(s)^{-1}\} A(s)^\circ (\lambda - A(s))^{-1}, \qquad \text{for } \lambda \in \Sigma,$$

any condition (T.1) or (C.1) implies that $A_n(\cdot)$ is a Hölder continuous function with values in $\mathcal{L}(X)$. Therefore there exists an evolution operator $U_n(t, s)$ for $A_n(t)$, $n = 1, 2, 3, \ldots$. We can then prove in each case:

THEOREM 4.1 (*Case of temperately variable domains*). – *Let* (P) *and* (T.1 and 2) *be satisfied. Then* $U_n(t, s)$ *and* $U_n(t, s)J_n(s)$ *have a common strong limit* $U(t, s)$ *as* $n \to \infty$ *for* $0 \leqslant s \leqslant t \leqslant T$, *which has the properties a)-d) announced above. Moreover, for any* f *and any* u_0 *such that*

$$f \in \mathcal{C}^{\sigma}([0, T]; X), \ \sigma > (2 - \alpha - \beta)/\alpha, \quad \textit{and } u_0 \in X, \tag{4.1}$$

the function u defined by

$$u(t) = U(t, 0) u_0 + \int_0^1 U(t, \tau) f(\tau) d\tau, \qquad 0 \leq t \leq T, \tag{4.2}$$

gives a strict solution of (E) *with*

$$t^{(1-\beta)/\alpha} u, \; t^{(2-\beta)/\alpha} du/dt \in L^\infty((0, T); X). \tag{4.3}$$

Conversely, for

$$f \in \mathcal{C}([0, T]; X) \quad \textit{and} \quad u_0 \in X, \tag{4.4}$$

any strict solution of (E) *with*

$$t^\gamma du/dt \in L^\infty((0, T); X), \qquad \gamma < (\alpha + \nu - 1)/\alpha + \mu, \tag{4.5}$$

must be necessarily of the form (4.2).

THEOREM 4.2 (*Case of completely variable domains*). – *Let* (P) *and* (C.1 and 2) *be satisfied. Then $U_n(t, s)$ and $U_n(t, s) J_n(s)$ converge strongly to an evolution operator $U(t, s)$, $0 \leq s \leq t \leq T$, with the properties a)-d) and giving for any* (4.1) *a strict solution of* (E) *with* (4.3). *Conversely, for* (4.4), *any strict solution of* (E) *with*

$$t^\gamma u \in L^\infty((0, T); X), \qquad \gamma < (\alpha + \beta + \nu - 2)/\alpha, \tag{4.6}$$

must be necessarily of the form (4.2).

As will be verified soon, the Condition (T.1) (resp. (C.1)) implies the similar condition (4.8) (resp. (4.15)) for the Yosida approximation $A_n(\cdot)$ with the same exponents as $A(\cdot)$ as well strong convergence (4.9) (resp. (4.16)). In the univalent case, in fact by [22, Theorem 3.2 and 3] (resp. by [21, Theorems 4.3 and 4]), a machinary of constructing the evolution operator starting from these two things was already established. Therefore what we have to do here may be only to modify it to the multivalued case. Detailed calculations at each stage of the proof described below may be hence often omitted, for which we refer the reader to the papers quoted above and to other papers in the References.

PROOF OF THEOREM 4.1. – As it has be en shown in Section 3, the resolvent sets of the Yosida approximation $A_n(t)$, $n \geq n_0$, contain a fixed region $\tilde{\Sigma}$ and there the uniform estimate

$$\|(\lambda - A_n(t))^{-1}\|_{\mathcal{L}(X)} \leq \tilde{M}/(|\lambda| + 1)^\beta, \qquad \lambda \in \tilde{\Sigma} \tag{4.7}$$

holds with the same α and β as in (P). And, as $n \to \infty$, the resolvent $(\lambda - A_n(t))^{-1}$ converges to $(\lambda - A(t))^{-1}$ in $\mathcal{L}(X)$ for each $\lambda \in \tilde{\Sigma}$. On the other hand, since we have from (3.5):

$$A_n(t)(\lambda - A_n(t))^{-1} = \lambda(\lambda - A_n(t))^{-1} - 1 = \frac{n}{n-\lambda}\left\{\frac{n\lambda}{n-\lambda}\left(\frac{n\lambda}{n-\lambda} - A(t)\right)^{-1} - 1\right\} =$$

$$= \frac{n}{n-\lambda}A(t)^{\circ}\left(\frac{n\lambda}{n-\lambda} - A(t)\right)^{-1}, \qquad \text{for } n \geq n_0 \text{ and } \lambda \in \tilde{\Sigma},$$

the condition (T.1) implies the same uniform estimate

$$\|A_n(t)(\lambda - A_n(t))^{-1}\{A_n(t)^{-1} - A_n(s)^{-1}\}\|_{\mathcal{L}(X)} \leq \tilde{K}|t-s|^{\mu}/(|\lambda|+1)^{\nu}, \qquad \lambda \in \tilde{\Sigma} \tag{4.8}$$

for $n \geq n_0$ with the same μ and ν. In addition, as $n \to \infty$,

$$A_n(t)(\lambda - A_n(t))^{-1}\{A_n(t)^{-1} - A_n(s)^{-1}\} \to A(t)^{\circ}(\lambda - A(t))^{-1}\{A(t)^{-1} - A(s)^{-1}\} \tag{4.9}$$

in $\mathcal{L}(X)$ for each $\lambda \in \tilde{\Sigma}$.

Let $U_n(t, s)$ be the evolution operator for $A_n(\cdot)$, and put $V_n(t, s) = U_n(t, s)A_n(s)^{1-\rho}$ and $W_n(t, s) = A_n(t)U_n(t, s) - A_n(t)\exp(-(t-s)A_n(t))$, where ρ is a fixed number such that $3 - \alpha - \beta - \alpha\mu < \rho < \nu$. As was verified in the proof of [22, Proposition 3.1], the following three integral equations in $U_n(t, s)$, $V_n(t, s)$ and $W_n(t, s)$ hold:

$$U_n(t, s) = \exp(-(t-s)A_n(s)) + \int_s^t V_n(t, \tau)D_n(\tau, s)A_n(s)\exp(-(\tau-s)A_n(s))\,d\tau, \tag{4.10}$$

$$V_n(t, s) = A_n(s)^{1-\rho}\exp(-(t-s)A_n(s)) + \int_s^t V_n(t, \tau)D_n(\tau, s)A_n(s)^{2-\rho}\exp(-(\tau-s)A_n(s))\,d\tau, \tag{4.11}$$

$$W_n(t, s) = R_n(t, s) - \int_s^t A_n(t)^{2-\rho}\exp(-(t-\tau)A_n(t))D_n(t, \tau)W_n(\tau, s)\,d\tau, \tag{4.12}$$

where $D_n(t, s) = A_n(t)^{\rho}\{A_n(t)^{-1} - A_n(s)^{-1}\}$ and

$$R_n(t, s) = -\int_s^t A_n(t, s)(t)^{2-\rho}\exp(-(t-\tau)A_n(t))D_n(t, \tau)A_n(\tau)\exp(-(\tau-s)A_n(\tau))\,d\tau.$$

Since all the integral kernels in these equations can be written in terms of $(\lambda - A_n(t))^{-1}$ and of $A_n(t)(\lambda - A_n(t))^{-1}\{A_n(t)^{-1} - A_n(s)^{-1}\}$ alone, it is possible to estimate them by using (4.7) and (4.8) and to conclude from that they have all integrable singularities at $t = s$ which are uniform in $n \geq n_0$. In addition, we also conclude from (4.9) that they converge strongly in $\mathcal{L}(X)$ as $n \to \infty$ for each $0 \leq s < t \leq T$.

As a consequence, we obtain that the solutions $V_n(t, s)$ and $W_n(t, s)$ of (4.11) and

(4.12) respectively converge strongly in $\mathcal{L}(X)$ for $0 \leq s < t \leq T$; from (4.10) it is the same of $U_n(t, s)$. Let us define $U(t, s)$, $0 \leq s < t \leq T$, as the limit of $U_n(t, s)$ and $U(s, s) = 1$, and denote the limit of $A_n(t) U_n(t, s)$ by $A(t)^\circ U(t, s)$. Then it is easy to verify that this $U(t, s)$ has the properties *a)*-*d)*; for example, $U_n(t, s) = A_n(t)^{-1} A_n(t) U_n(t, s)$ implies that $U(t, s) = A(t)^{-1} A(t)^\circ U(t, s)$; this shows that $A(t)^\circ U(t, s)$ is a section of $A(t) U(t, s)$, i.e. $A(t)^\circ U(t, s) \subset A(t) U(t, s)$. Letting $n \to \infty$ in

$$U_n(t, s) = U_n(s + \varepsilon, s) - \int_{s+\varepsilon}^{t} A_n(\tau) U_n(\tau, s)\, d\tau, \qquad s + \varepsilon \leq t \leq T,\ \varepsilon > 0,$$

we observe that $\partial U(t, s)/\partial t = -A(t)^\circ U(t, s)$. Convergence of $U_n(t, s) J_n(s)$ to $U(t, s)$ is proved by operating $J_n(s)$ to (4.10) from the right and by noting the fact shown by Theorem 3.1.

We shall next prove that the function u defined by (4.2) is under (4.1) a strict solution of (E). Put for $n \geq n_0$:

$$u_n(t) = U_n(t, 0) u_0 + \int_0^t U_n(t, \tau) f(\tau)\, d\tau, \qquad 0 \leq t \leq T, \tag{4.13}$$

and write, after operating $A_n(t)$, $A_n(t) u_n(t)$ in the form

$$A_n(t) u_n(t) = A_n(t) U_n(t, 0) u_0 + \int_0^t A_n(t) U_n(t, \tau)\{f(\tau) - f(t)\}\, d\tau + \\ + \int_0^t W_n(t, \tau) f(t)\, d\tau + \{1 - \exp(-tA_n(t))\} f(t).$$

Then $A_n(t) u_n(t)$ is observed from *c)* and *d)* and from (4.1) to converge pointwisely on $(0, T]$ to a continuous function $g(t)$. Since $u_n(t)$ clearly converges to $u(t)$ pointwisely on $[0, T]$, we have: $u(t) = A(t)^{-1} g(t)$, i.e. $g(t) \in A(t) u(t)$. On the other hand, letting $n \to \infty$ in

$$u_n(t) - u_n(\varepsilon) = \int_\varepsilon^t \{f(\tau) - A_n(\tau) u_n(\tau)\}\, d\tau, \qquad \varepsilon \leq t \leq T, \text{ with any } \varepsilon > 0,$$

we observe that $du(t)/dt = f(t) - g(t)$, $0 < t \leq T$, hence $u(t)$ satisfies the equation in (E). The continuity of u at $t = 0$ in the seminorm $p_{A(0)}(\cdot)$ is verified from

$$\|U(t, 0) - \exp(-tA(0))\|_{\mathcal{L}(X)} \leq Ct^{(\alpha + 2\beta + \rho - 4)/\alpha + \mu} \to 0, \qquad \text{as } t \to 0, \tag{4.14}$$

(which follows from (4.10)) and from the fact that, as $t \to 0$, $\exp(-tA(0)) \to 1$ on X in $p_{A(0)}(\cdot)$ (see (3.2)). Finally, (4.3) follows from *b)* and *c)*.

Conversely, let us prove that under (4.4) any strict solution u of (E) with (4.5) is of the form (4.2). Since $\partial U_n(t, s)/\partial s = U_n(t, s) A_n(s)$, we have for any $\varepsilon > 0$:

$$u(t) - U_n(t, \varepsilon) u(\varepsilon) = \int_{\varepsilon}^{t} U_n(t, \tau) \left\{ A_n(\tau) u(\tau) + \frac{du}{d\tau}(\tau) \right\} d\tau, \qquad \varepsilon \leqslant t \leqslant T.$$

Put $g(t) = f(t) - du/dt \in A(t) u(t)$; then, since $J_n(t) g(t) = A_n(t) u(t)$ by virtue of Theorem 2.7, it follows that

$$u(t) - U_n(t, \varepsilon) u(\varepsilon) = \int_{\varepsilon}^{t} U_n(t, \tau)\{J_n(\tau) - 1\} g(\tau)\, d\tau + \int_{\varepsilon}^{t} U_n(t, \tau) f(\tau)\, d\tau.$$

So that letting $n \to \infty$, we obtain in view of $U_n(t, \tau) J_n(\tau) \to U(t, \tau)$ that

$$u(t) - U(t, \varepsilon) u(\varepsilon) = \int_{\varepsilon}^{t} U(t, \tau) f(\tau)\, d\tau, \qquad \varepsilon \leqslant t \leqslant T.$$

To finish the proof therefore it suffices to prove $U(t, \varepsilon) u(\varepsilon) \to U(t, 0) u_0$ in X as $\varepsilon \to 0$; or more strongly $u(\varepsilon) - U(\varepsilon, 0) u_0 \to 0$ in X. Inserting $\exp(-(\varepsilon A(0)) u_0$, then we first observe from (4.14) that $\{\exp(-\varepsilon A(0)) - U(\varepsilon, 0)\} u_0 \to 0$ in X. On the other hand, writing for any $0 < \delta < \varepsilon$:

$$u(\varepsilon) - \exp(-(\varepsilon - \delta) A_n(0)) u(\delta) = \int_{\delta}^{\varepsilon} \exp(-(\varepsilon - \tau) A_n(0)) \left\{ A_n(0) u(\tau) + \frac{du}{d\tau}(\tau) \right\} d\tau,$$

and letting $n \to \infty$, we have:

$$u(\varepsilon) - \exp(-(\varepsilon - \delta) A(0)) u(\delta) =$$

$$= \int_{\delta}^{\varepsilon} A(0)^{\circ} \exp(-(\varepsilon - \tau) A(0))\{A(\tau)^{-1} - A(0)^{-1}\} g(\tau)\, d\tau + \int_{\delta}^{\varepsilon} \exp(-(\varepsilon - \tau) A(0)) f(\tau)\, d\tau$$

(note that $A(0)^{\circ} \exp(-\tau A(0)) A(0)^{-1} = \exp(-\tau A(0))$, $\tau > 0$). Let here $\delta \to 0$, then from the initial condition of (E) it follows that

$$u(\varepsilon) - \exp(-(\varepsilon A(0)) u_0 =$$

$$= \int_{0}^{\varepsilon} A(0)^{\circ} \exp(-(\varepsilon - \tau) A(0))\{A(\tau)^{-1} - A(0)^{-1}\} g(\tau)\, d\tau + \int_{0}^{\varepsilon} \exp(-(\varepsilon - \tau) A(0)) f(\tau)\, d\tau.$$

With the aid of the estimate

$$\|A(0)^{\circ} \exp(-\tau A(0))\{A(t)^{-1} - A(0)^{-1}\}\|_{\mathcal{L}(X)} \leqslant C\tau^{(\nu - 1)/\alpha} t^{\mu}, \qquad \tau > 0 \text{ and } 0 \leqslant t \leqslant T,$$

which follows from (T.1), we conclude from (4.5) that

$$\|u(\varepsilon) - \exp(-\varepsilon A(0))\, u_0\|_X \leqslant C\{\varepsilon^{(\alpha+\nu-1)/\alpha+\mu-\gamma} + \varepsilon^{(\alpha+\beta-1)/\alpha}\},$$

and hence the result to be proved.

PROOF OF THEOREM 4.2. – By the same argument as for (4.8), we verify that the condition (C.1) implies the same estimate for the Yosida approximation $A_n(t)$, $n \geqslant n_0$:

$$\|A_n(t)(\lambda - A_n(t))^{-1} dA_n(t)^{-1}/dt\|_{\mathcal{L}(X)} \leqslant \tilde{N}/(|\lambda|+1)^{\nu}, \qquad \lambda \in \tilde{\Sigma}, \tag{4.15}$$

and that, as $n \to \infty$,

$$A_n(t)(\lambda - A_n(t))^{-1} dA_n(t)^{-1}/dt \to A(t)^{\circ}(\lambda - A(t))^{-1} dA(t)^{-1}/dt \quad \text{in } \mathcal{L}(X). \tag{4.16}$$

We use in this case the following two integral equations:

$$\begin{aligned} U_n(t,s) &= \exp(-(t-s)A_n(s)) + \int_s^t P_n(t,\tau)\, U_n(\tau,s)\, d\tau, \\ W_n(t,s) &= R_n(t,s) + \int_s^t Q_n(t,\tau)\, W_n(\tau,s)\, d\tau, \end{aligned} \tag{4.17}$$

for $U_n(t,s)$ and $W_n(t,s) = A_n(t)\,U_n(t,s) - A_n(s)\exp(-(t-s)A_n(s))$, $0 \leqslant s \leqslant t \leqslant T$. The integral kernels are respectively:

$$P_n(t,s) = (\partial/\partial t + \partial/\partial s)\exp(-(t-s)A_n(s)), \qquad 0 \leqslant s \leqslant t \leqslant T,$$
$$Q_n(t,s) = P_n(t,s) - A_n(s)\exp(-(t-s)A_n(s))\, dA_n(s)^{-1}/ds, \qquad 0 \leqslant s \leqslant t \leqslant T,$$
$$R_n(t,s) = \int_s^t Q_n(t,\tau)A_n(s)\exp(-(\tau-s)A_n(s))\, d\tau, \qquad 0 \leqslant s \leqslant t \leqslant T.$$

For the derivation of these equations, compare [20]. From (4.7) and (4.15) it is verified that all the kernels have integrable weak singularities at $t = s$ which are uniform in n, and that they converge strongly in $\mathcal{L}(X)$ as $n \to \infty$ for each $0 \leqslant s < t \leqslant T$.

Therefore the solutions $U_n(t,s)$ and $W_n(t,s)$ also converge strongly in $\mathcal{L}(X)$ to limits $U(t,s)$ and $W(t,s)$ respectively for $0 \leqslant s < t \leqslant T$. In the same way as above $U(t,s)$ is shown to have the properties *a)-d)*. The convergence of $U_n(t,s)J_n(s)$ to $U(t,s)$ is also verified by operating $J_n(s)$ to (4.17) from the right and by using Theorem 3.1.

Let us now prove that the function u given by (4.2) is under (4.1) a strict solution of (E). Let $u_n(t)$, $n \geqslant n_0$, be the sequence defined by (4.13), which converges to $u(t)$ on

$[0, T]$ pointwisely. We write $A_n(t)u_n(t)$ in the form

$$A_n(t)u_n(t) = A_n(t)U_n(t, 0)u_0 + \int_0^t A_n(t)U_n(t, \tau)\{f(\tau) - f(t)\}\,d\tau + \int_0^t W_n(t, \tau)f(\tau)\,d\tau +$$

$$+ \int_0^t \{A_n(\tau)\exp(-(t-\tau)A_n(\tau)) - A_n(t)\exp(-(t-\tau)A_n(t))\}\,f(t)\,d\tau +$$

$$+\{1 - \exp(-tA_n(t))\}\,f(t).$$

$A_n(t)u_n(t)$ is then shown to converge to a continuous function $g(t)$ on $(0, T]$ pointwisely, which is $f(t) - du/dt = g(t) \in A(t)u(t)$ for $0 < t \leq T$; hence $u(t)$ satisfies the equation of (E). Initial condition is verified in the same way as before from

$$\|U(t, 0) - \exp(-tA(0))\|_{\mathcal{L}(X)} \leq Ct^{(\alpha+2\beta+\nu-3)/\alpha} \to 0 \quad \text{as } t \to 0, \tag{4.18}$$

(which follows from (4.17)) and from $\exp(-\tau A(0)) \to 1$ on X as $\tau \to 0$ in the seminorm $p_{A(0)}(\cdot)$.

Conversely, under (4.4), let u be any strict solution of (E) with (4.6). Using as before the fact that $U_n(t, s)J_n(s) \to U(t, s)$ as $n \to \infty$, we easily obtain that

$$u(t) - U(t, \varepsilon)u(\varepsilon) = \int_\varepsilon^t U(t, \tau)f(\tau)\,d\tau, \qquad \varepsilon \leq t \leq T, \ \text{with any } \varepsilon > 0.$$

Therefore, the problem is to verify $u(\varepsilon) - U(\varepsilon, 0)u_0 \to 0$ in X as $\varepsilon \to 0$. We insert $\exp(-\varepsilon A(0))u_0$. Then $\exp(-\varepsilon A(0)) - U(\varepsilon, 0)$ is estimated by (4.18). On the other hand, writing with any $0 < \delta < \varepsilon$:

$$u(\varepsilon) - \exp(-(\varepsilon-\delta)A_n(\delta))u(\delta) = \int_\delta^\varepsilon \frac{\partial \exp(-(\varepsilon-\tau)A_n(\tau))}{\partial\tau}u(\tau)\,d\tau +$$

$$+ \int_\delta^\varepsilon \exp(-(\varepsilon-\tau)A_n(\tau))\frac{du(\tau)}{d\tau}\,d\tau = \int_\delta^\varepsilon P_n(\varepsilon, \tau)u(\tau)\,d\tau +$$

$$+ \int_\delta^\varepsilon \exp(-(\varepsilon-\tau)A_n(\tau))\{J_n(\tau) - 1\}\,g(\tau)\,d\tau + \int_\delta^\varepsilon \exp(-(\varepsilon-\tau)A_n(\tau))\,f(\tau)\,d\tau,$$

where $g(t) = f(t) - du/dt \in A(t)u(t)$, $0 < t \leq T$, we let n tend to ∞ and then δ to 0. Then the initial condition of (E) jointed with the estimate

$$\|\exp(-(\varepsilon-\delta)A(\delta)) - \exp(-(\varepsilon-\delta)A(0))\|_{\mathcal{L}(X)} \leq C\delta(\varepsilon-\delta)^{(\beta+\nu-2)/\alpha}$$

(which follows from (C.1)) and with (4.6) yields that

$$u(\varepsilon) - \exp(-\varepsilon A(0))\,u_0 = \int_0^\varepsilon P(\varepsilon, \tau)\,u(\tau)\,d\tau + \int_0^\varepsilon \exp(-(\varepsilon - \tau)A(\tau))\,f(\tau)\,d\tau,$$

where $P(t, s) = (\partial/t + \partial/s)\exp(-(t - s)A(s))$, $0 \leq s < t \leq T$. Since $P(t, s)$ is shown by (C.1) to satisfy:

$$\|P(t, s)\|_{\mathcal{L}(X)} \leq C(t - s)^{(\beta + \nu - 2)/\alpha},$$

the desired result follows now from the condition (4.6).

REMARK. – From (4.14) and (4.18) in both the cases studied above we verify that, if $u_0 \in \mathcal{D}(A(0))$, then $U(t, 0)\,u_0 \to u_0$ as $t \to 0$ in the norm of X (remember (3.2)). Therefore, if the initial value $u_0 \in \mathcal{D}(A(0))$, then the strict solution given by (4.2) is continuous at $t = 0$ in the norm of X.

REMARK. – In the case of temperately variable domains $\mathcal{D}(A(t))$, the case of constant domains is included. In fact, if $A(t)$, $0 \leq t \leq T$, with constant domain satisfy the condition (P), then everywhere defined multivalued operators $A(t)A(s)^{-1}$ are defined for $0 \leq s,\ t \leq T$. If, for example, there exists a linear section $K(t, s) \in \mathcal{L}(X)$ of $A(t)A(s)^{-1}$ for each $0 \leq s,\ t \leq T$, and if $K(t, s)$ satisfies:

$$\|K(t, s) - 1\|_{\mathcal{L}(X)} \leq K|t - s|^\mu, \qquad 0 \leq s,\ t \leq T,$$

then, since $A(t)^\circ(\lambda - A(t))^{-1}A(s)^{-1} = (\lambda - A(t))^{-1}K(t, s)$ from Theorem 2.7, it follows that

$$\|A(t)^\circ(\lambda - A(t))^{-1}\{A(t)^{-1} - A(s)^{-1}\}\|_{\mathcal{L}(X)} \leq MK|t - s|^\mu/(|\lambda| + 1)^\beta,$$

i.e. (T.1) is valid with $\nu = \beta$.

REMARK. – Contrary to the above remark, there is a substantial difference between the temperately variable domains and the completely variable domains. Example 6.5 described in Section 6 satisfies in fact the Condition (C.1) but not the Condition (T.1).

5. – Degenerate equations.

In this section we apply the results obtained in the preceding sections for studying the parabolic evolution equations in a Banach space X of degenerate type with respect to the time derivative.

Let us first consider

$$\text{(D.E)} \qquad \begin{cases} dM(t)\,v/dt + L(t)\,v = f(t)\,, & 0 < t \leq T\,, \\ \lim_{t \to 0+} M(t)\,v(t) = u_0\,. \end{cases}$$

Here, $M(t)$ and $L(t)$, $0 \leq t \leq T$, are densely defined univalent closed linear operators in X such that $\mathcal{D}(M(t)) \supset \mathcal{D}(L(t))$ and $L(t)^{-1} \in \mathcal{L}(X)$; $T(t)$, $0 \leq t \leq T$, denote the bounded operators $T(t) = M(t)\,L(t)^{-1}$. $f\colon [0, T] \to X$ is a given continuous function, $u_0 \in X$ is an initial value, and $v\colon (0, T] \to X$ is the unknown function.

If we put $u(t) = M(t)\,v(t)$, then (D.E) is reduced to

$$\text{(5.1)} \qquad \begin{cases} du/dt + A(t)\,u \ni f(t)\,, & 0 < t \leq T\,, \\ u(0) = u_0\,, \end{cases}$$

in which the unknown function is $u(t)$ and the operators $A(t) = L(t)\,M(t)^{-1} = T(t)^{-1}$ (cf. Theorem 2.5), $0 \leq t \leq T$, are multivalued linear operators in X. It is then possible to apply Theorem 4.1 and 2 to obtain existence and uniqueness of the strict solution of (5.1). From the Definition of the strict solution in Section 4 we are naturally led to:

DEFINITION. – A function $v\colon (0, T] \to X$ is called *a strict solution* of (D.E) if $v(t) \in \mathcal{D}(L(t))$ $(\subset \mathcal{D}(M(t)))$ for $0 < t \leq T$ with $Mv \in \mathcal{C}^1((0, T]; X)$ and $Lv \in \mathcal{C}((0, T]; X)$, v satisfies the equation in (D.E) for every $0 < t \leq T$, and the initial condition: $M(t)\,v(t) \to u_0$, as $t \to 0$, is satisfied in the sense of the seminorm $\|T(0)\cdot\|_X$.

LEMMA 5.1. – *Let $T \in \mathcal{L}(X)$ be a bounded operator on X, and let $A = T^{-1}$ be its inverse. Then, $(\lambda - A)^{-1} = T(\lambda T - 1)^{-1} = (\lambda T - 1)^{-1}\,T$ for all $\lambda \in \boldsymbol{C}$ in the sense of multivalued operator.*

PROOF. – Let $u \in \mathcal{D}((\lambda T - 1)\,T^{-1}) = \mathcal{D}(A)$; then, $(\lambda T - 1)\,T^{-1}u = \{(\lambda T - 1)f;\ f \in T^{-1}u\} = \{\lambda u - f;\ f \in Au\} = (\lambda - A)\,u$; therefore, $(\lambda T - 1)\,T^{-1} = \lambda - A$; then the first assertion follows from Theorem 2.5.

Let now $u \in (\lambda T - 1)^{-1}\,Tf$; then, $u = T(\lambda u - f)$; so that, $\lambda u - f \in Au$ or $u \in$ $\in (\lambda - A)^{-1}f$; hence $(\lambda T - 1)^{-1}\,T \subset (\lambda - A)^{-1}$. The converse is also true.

This lemma shows that, in order that $A(t)$ satisfy the Condition (P), it suffices to assume on $T(t)$ that

(D.P) There exists a region $\Sigma = \{\lambda \in \boldsymbol{C};\ \operatorname{Re}\lambda < c(|\operatorname{Im}\lambda| + 1)^\alpha\}$ such that, for $\lambda \in \Sigma$, $(\lambda T(t) - 1)^{-1}$, $0 \leq t \leq T$, are univalent bounded operators on X and there an estimate

$$\|T(t)(\lambda T(t) - 1)^{-1}\|_{\mathcal{L}(X)} \leq M/(|\lambda| + 1)^\beta\,, \qquad \lambda \in \Sigma,\ \ 0 \leq t \leq T$$

holds with some exponents $0 < \beta \leq \alpha \leq 1$ and constant c, $M > 0$.

Moreover, since it is also verified from the lemma that

$$A(t)^\circ(\lambda - A(t))^{-1} = \lambda(\lambda - A(t))^{-1} - 1 = (\lambda T(t) - 1)^{-1}, \qquad \lambda \in \Sigma, \ 0 \leq t \leq T,$$

it is sufficient for (T.1) or (C.1) to assume respectively that

(D.T.1) $\|(\lambda T(t) - 1)^{-1}\{T(t) - T(s)\}\|_{\mathcal{L}(X)} \leq K|t-s|^\mu/(|\lambda| + 1)^\nu$, $\lambda \in \Sigma$, $0 \leq s, t \leq T$;

(D.C.1) $T(t)$ is strongly continuously differentiable in t, and the derivative satisfies:

$$\|(\lambda T(t) - 1)^{-1} dT(t)/dt\|_{\mathcal{L}(X)} \leq N/(|\lambda| + 1)^\nu, \qquad \lambda \in \Sigma, \ 0 \leq s, t \leq T.$$

THEOREM 5.2. – *Under* (D.P), *assume* (D.T.1) *and* (T.2), *or assume* (D.C.1) *and* (C.2). *Then, for any Hölder continuous function* $f \in \mathcal{C}^\sigma([0, T]; X)$, $\sigma > (2 - \alpha - \beta)/\alpha$, *and any initial value* $u_0 \in X$, *there exists a strict solution of* (D.E). *Conversely, for* $f \in \mathcal{C}([0, T]; X)$ *and* $u_0 \in X$, *any strict solution* v *of* (D.E) *with* $t^\gamma dMv/dt \in L^\infty(0, T; X)$, $\gamma < (\alpha + \nu - 1)/\alpha + \mu$, *(resp.* $t^\gamma Mv \in L^\infty(0, T; X)$, $\gamma < (\alpha + \beta + \nu - 2)/\alpha$), *when we assumed* (D.T.1) *and* (T.2), *(resp.* (D.C.1) *and* (C.2)), *is unique.*

PROOF. – The unique thing we have to verify now is that $v(t)$ is a strict solution of (D.E) if and only if $u(t) = M(t)v(t)$ is a strict solution of (5.1). Let $u(t)$ be a strict solution of (5.1); then, there exists a function $v(t) \in \mathcal{D}(L(t)) \cap M(t)^{-1}u(t)$, $0 < t \leq T$, such that $du/dt + L(t)v(t) = f(t)$; since $M(t)v(t) = u(t)$, $v(t)$ satisfies the equation of (D.E). The convergence at $t = 0$ is given by the same relation between u and v. Conversely, if $v(t)$ is a strict solution of (D.E), we put $u(t) = M(t)v(t)$ for $0 < t \leq T$ and $u(0) = u_0$; then, the proof is immediate.

Let us next consider a degenerate equation of the dual form of (D.E)

$$\text{(D.E)}' \qquad \begin{cases} M(t)\,du/dt + L(t)u = M(t)g(t), & 0 < t \leq T, \\ u(0) = u_0 \end{cases}$$

in X. Here, $M(t)$ and $L(t)$, $0 \leq t \leq T$, are densely defined univalent closed linear operators in X such that $L(t)^{-1} \in \mathcal{L}(X)$ and that $L(t)^{-1}M(t)$ has a bounded extension on X for each $0 \leq t \leq T$, the bounded extension being denoted by $S(t) = \overline{L(t)^{-1}M(t)}$. g is a given function, u_0 is an initial value, and u is the unknown function.

We operate $M(t)^{-1}$ to the equation; then, since $M(t)^{-1}L(t) \subset S(t)^{-1}$ from Theorem 2.5, an extended equation

$$(5.2) \qquad \begin{cases} du/dt + B(t)u \ni g(t), & 0 < t \leq T, \\ u(0) = u_0 \end{cases}$$

is obtained, where $B(t) = S(t)^{-1}$, $0 \leq t \leq T$, are multivalued linear operators in X. From Lemma 5.1 we have: $(\lambda - B(t))^{-1} = S(t)(\lambda S(t) - 1)^{-1}$ and $B(t)^\circ(\lambda - B(t))^{-1} =$

$= (\lambda S(t) - 1)^{-1}$. Therefore the Conditions (P), (T.1) and (C.1) for $B(t)$ are implied by the conditions which will be obtained by replacing $T(t)$ by $S(t)$ in (D.P), (D.T.1) and (D.C.1) respectively. Let us call these conditions on $S(t)$ (D.P)′, (D.T.1)′ and (D.C.1)′ respectively. We then obtain easily the following theorem.

THEOREM 5.3. – *Under* (D.P)′ *assume* (D.T.1)′ *and* (T.2), *or assume* (D.C.1)′ *and* (C.2). *Then, for any Hölder continuous function* $g \in \mathcal{C}^{\sigma}([0, T]; X)$, $\sigma > (2 - \alpha - \beta)/\alpha$, *and any initial value* $u_0 \in X$, *there exists a strict solution of the extended equation* (5.2). *Conversely, for* $f \in \mathcal{C}([0, T]; X)$ *and* $u_0 \in X$, *any strict solution* u *of* (5.2) *with* $t^{\gamma} du/dt \in L^{\infty}(0, T; X)$, $\gamma < (\alpha + \nu - 1)/\alpha + \mu$, (*resp.* $t^{\gamma} u \in L^{\infty}(0, T; X)$, $\gamma < (\alpha + \beta + \nu - 2)/\alpha$), *when we assumed* (D.T.1)′ *and* (T.2), (*resp.* (D.C.1)′ *and* (C.2)), *is unique.*

REMARK. – As was remarked in Section 4, if the initial value u_0 in (5.1) $\in \mathcal{D}(A(0)) = \mathcal{R}(T(0))$, then the strict solution u is continuous at the $t = 0$ in the norm of X. This means that, if $u_0 \in \mathcal{R}(T(0)) = M(0)\mathcal{D}(L(0))$, then the strict solution v of (D.E) satisfies: $M(t)v(t) \to u_0$, as $t \to 0$, in the norm X.

REMARK. – Similarly to the above remark, if $u_0 \in \mathcal{D}(B(0)) = \mathcal{R}(S(0))$ in (5.2), then the strict solution u is continuous at $t = 0$ in the norm of X. Since we assumed that $L(0)^{-1}M(0) = S(0) \in \mathcal{L}(X)$, $u_0 \in \mathcal{R}(S(0))$ if and only if there exist sequences $u_n \in \mathcal{D}(L(0))$ and $f_n \in \mathcal{D}(M(0))$ such that $L(0)u_n = M(0)f_n$, u_n converges to u_0 in X and f_n is also convergent in X.

REMARK. – As may be easily seen, the strict solution of the extended equation (5.2) is not necessarily that of the original equation (D.E)′. But this is the case if $M(t)$ are bounded operators, since then $S(t) = L(t)^{-1}M(t)$, and there are some interesting examples in which $M(t)$ are bounded operators but have no univalent inverses.

6. – Examples.

It is known that many partial differential equations can be written in the form (E), (D.E) or (D.E)′. We shall study in this section elementary ones, and we shall see how our abstract results apply to those examples.

EXAMPLE 6.1.

$$\begin{cases} du/dt + O_{\infty} u \ni f(t), & 0 < t \leq T, \\ u(0) = u_0. \end{cases} \tag{6.1}$$

Here, O_{∞} is the inverse of the 0 operator in a Banach space X which has been already defined in the proof of Theorem 2.6. We know that $\mathcal{D}(O_{\infty}) = \{0\}$, $O_{\infty} 0 = X$ and that $\rho(O_{\infty}) = \boldsymbol{C}$ and the resolvent $(\lambda - A)^{-1}$ vanishes for all $\lambda \in \boldsymbol{C}$. So that, $\exp(-\tau O_{\infty}) =$

$= 0$, $\tau > 0$, and $\exp(-0O_\infty) = 1$. Moreover, for any $f \in \mathcal{C}([0, T]; X)$ and any $u_0 \in X$, the function $u(t) = 0$ for $t > 0$ and $u(0) = u_0$ is the strict solution of (6.1). Of course this example is too artificial and too extreme, but this shows us significantly what happens in the evolution equations with multivalued linear operators. In fact, if we regard (6.1) as a diffusion equation, then O_∞ has «infinite diffusion» and every solution vanishes at once for $t > 0$.

EXAMPLE 6.2 (Stokes equation). - Let

$$\begin{cases} \partial u/\partial t - \nu \Delta u + \nabla p = f(t, x) & \text{in } (0, T] \times \Omega, \\ \operatorname{div} u = 0 & \text{in } (0, T] \times \Omega, \\ u = 0 & \text{on } (0, T] \times \partial\Omega, \\ u(0, x) = u_0(x) & \text{in } \Omega, \end{cases} \tag{6.2}$$

be the Stokes equation in an open set $\Omega \subset \mathbf{R}^n$ with a smooth boundary $\partial\Omega$. $u = (u_1(t, x), \ldots, u_n(t, x))$ and $p = p(t, x)$ are unknown functions, $f(t, x)$ is a given function and $u_0(x)$ is an initial function. Our problem is to formulate this equation in the $L^2(\Omega)$ space. Set $X = (L^2(\Omega))^n$ and

$$\begin{cases} X_\sigma = \text{the closure of } \{u \in (\mathcal{C}_0^\infty(\Omega))^n;\ \operatorname{div} u = 0 \text{ in } \Omega\} \text{ in } X, \\ X_{\mathcal{G}} = \{\nabla p;\ p \in H^1(\Omega)\}. \end{cases}$$

It is known that X is an orthogonal direct sum of X_σ and $X_{\mathcal{G}}$. Let P denote the orthogonal projection $P: X \to X_\sigma$. We define a multivalued linear operator A in X by

$$\begin{cases} \mathcal{D}(A) = (H^2(\Omega))^n \cap (H_0^1(\Omega))^n \cap X_\sigma, \\ Au = -\Delta u + X_{\mathcal{G}}. \end{cases}$$

Then it is possible to write the Stokes equation in the form

$$\begin{cases} du/dt + \nu A u \ni f(t), & 0 < t \leq T, \\ u(0) = u_0, \end{cases}$$

with $f: [0, T] \to X$ and $u_0 \in X_\sigma$.

This formulation is essentially equivalent to the classical one. As it is well known, the Stokes equation is also written in the form

$$\begin{cases} du/dt + \nu A_s u = Pf(t), & 0 < t \leq T, \\ u(0) = u_0, \end{cases}$$

as an abstract equation in X_σ, using the operator A_s defined by

$$\begin{cases} \mathcal{D}(A_s) = \mathcal{D}(A), \\ A_s = -P\Delta u. \end{cases}$$

Since $(\lambda - \nu A)u \ni f$ is equivalent to $(\lambda - \nu A_s)u = Pf$ and $(1-P)(\nu \Delta u - f) \in X_{\mathcal{S}}$ for $u \in \mathcal{D}(A)$, $(\lambda - \nu A)u \ni f$ if and only if $(\lambda - \nu A_s)u = Pf$. Therefore we have:

$$\rho(A) = \rho(A_s) \quad \text{and} \quad (\lambda - \nu A)^{-1} = (\lambda - \nu A_s)^{-1}P \quad \text{for } \lambda \in \rho(A).$$

Consequently, $\exp(-\tau\nu A) = \exp(-\tau\nu A_s)P$ for $\tau > 0$. In particular, it is verified that

$$\exp(-\tau\nu A)g \to g, \quad \text{as } \tau \to 0, \text{ for the function } g \in \overline{\mathcal{D}(A)} = X_\sigma,$$

$$\exp(-\tau\nu A)h = 0, \quad \text{for } \tau > 0, \text{ for the function } h \in A0 = X_{\mathcal{S}}.$$

EXAMPLE 6.3.

$$\text{(6.3)} \qquad \begin{cases} \partial(m(x)v)/\partial t - \Delta v = f(t,x) & \text{in } (0,T] \times \Omega, \\ v = 0 & \text{on } (0,T] \times \partial\Omega, \\ \lim_{t\to 0} m(x)v(t,x) = u_0(x) & \text{in } \Omega. \end{cases}$$

Here, Ω is a bounded open set in $\boldsymbol{R}^n$ with a smooth boundary $\partial\Omega$, $m(x)$ is a given non negative function defined in Ω such that

$$\text{(6.4)} \qquad m \in L^\infty(\Omega), \qquad m(x) \geq 0.$$

$f(t,x)$ is also a given function, $u_0(x)$ is an initial function, and $v = v(t,x)$ is the unknown function. We consider this problem in the spaces $H^{-1}(\Omega)$ and $L^2(\Omega)$. $\|\cdot\|_{-1}$, $\|\cdot\|$ and $\|\cdot\|_1$ denote respectively the norms of $H^{-1}(\Omega)$, $L^2(\Omega)$ and $H_0^1(\Omega)$. $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ denote respectively the scalar products in $L^2(\Omega)$ and in $H_0^1(\Omega) \times H^{-1}(\Omega)$.

Let us first consider (6.3) in $H^{-1}(\Omega)$. As we have already seen in Section 5, (6.3) can be written as an equation with multivalued operator

$$\begin{cases} du/dt + Au \ni f(t), & 0 < t \leq T, \\ u(0) = u_0, \end{cases}$$

in $H^{-1}(\Omega)$. Here, $A = Lm^{-1}$ is determined by

$$\text{(6.5)} \qquad \begin{cases} \mathcal{D}(A) = \{mv;\ v \in H_0^1(\Omega)\} \\ Au = \{Lv;\ v \in H_0^1(\Omega) \text{ such that } mv = u\}, \end{cases}$$

where L is $-\Delta$ with the Dirichlet boundary condition in $H^{-1}(\Omega)$ and m^{-1} denotes the inverse of the multiplication of $m(x)$ which is a bounded operator from $H_0^1(\Omega)$ to $L^2(\Omega)$.

In order to apply Theorem 3.2, let us verify that our A satisfies Condition (H.P). Since $\mathcal{D}(L) = H_0^1(\Omega)$ and $L^{-1} \in \mathcal{L}(H^{-1}(\Omega), H_0^1(\Omega))$, $A^{-1} = T = mL^{-1} \in \mathcal{L}(H^{-1}(\Omega))$, so that from Lemma 5.1 $(\lambda - A)^{-1} = T(\lambda T - 1)^{-1} = m(\lambda m - L)^{-1}$. We then consider the sesquilinear form $a_\lambda(u, v)$ on $H_0^1(\Omega)$

$$a_\lambda(u, v) = \lambda \int_\Omega m(x)\, u(x)\, \overline{v}(x)\, dx - \int_\Omega \nabla u\, \nabla \overline{v}\, dx, \qquad u, v \in H_0^1(\Omega)$$

for each $\lambda \in \boldsymbol{C}$. It is verified that, for any $0 < \omega < \pi/2$ and some $c > 0$, there exists $\delta > 0$ such that

$$|a_\lambda(u, u)| \geqslant \delta(\|u\|_1^2 + |\lambda| \|\sqrt{m}u\|^2), \qquad u \in H_0^1(\Omega),$$

holds uniformly for every $\lambda \in \Sigma = \{\lambda \in \boldsymbol{C};\ |\arg \lambda| \geqslant \omega \text{ or } |\lambda| \leqslant c\}$. Then the Lax-Milgram Theorem yields $(a_\lambda(u, v) = \langle (\lambda m - L) u, v \rangle)$ that $\lambda m - L$ has a bounded inverse from $H^{-1}(\Omega)$ to $H_0^1(\Omega)$ for $\lambda \in \Sigma$ as well an estimate

$$(6.6) \qquad |\lambda| \|\sqrt{m}(\lambda m - L)^{-1} f\|^2 + \|(\lambda m - L)^{-1} f\|_1^2 \leqslant C\|f\|_{-1}^2, \qquad f \in H^{-1}(\Omega),\ \lambda \in \Sigma.$$

Therefore, noting that $L(\lambda m - L)^{-1} + 1 = \lambda m(\lambda m - L)^{-1}$, we conclude that, under (6.4), $\rho(A) \supset \Sigma$ and the resolvent satisfies:

$$(6.7) \qquad \|(\lambda - A)^{-1} f\|_{-1} = \|m(\lambda m - L)^{-1} f\|_{-1} \leqslant C\|f\|_{-1}/|\lambda|, \qquad f \in H^{-1}(\Omega),\ \lambda \in \Sigma,$$

$$(6.8) \qquad \|(\lambda - A)^{-1} f\| = \|\sqrt{m}\sqrt{m}(\lambda m - L)^{-1} f\| \leqslant C\|f\|_{-1}/|\lambda|^{1/2}, \qquad f \in H^{-1}(\Omega),\ \lambda \in \Sigma.$$

In addition, if we define the part of A in the space $L^2(\Omega)$ by

$$A_0 u = Au \cap L^2(\Omega) \quad \text{with } \mathcal{D}(A_0) = \{u \in \mathcal{D}(A);\ Au \cap L^2(\Omega) \neq \emptyset\},$$

then A_0, in view of (6.8), satisfies similarly $\rho(A_0) \supset \Sigma$ and

$$(6.9) \qquad \|(\lambda - A_0)^{-1}\|_{\mathcal{L}(L^2(\Omega))} \leqslant C/|\lambda|^{1/2}, \qquad \lambda \in \Sigma.$$

To obtain a better estimate for A_0 than (6.9) we have to assume more regularity on $m(x)$. Let $m \in \mathcal{C}^1(\overline{\Omega})$, $m(x) \geqslant 0$, and let for some $0 \leqslant \rho < 1$

$$(6.10) \qquad |\nabla m(x)| \leqslant C m(x)^\rho \quad \text{on } x \in \overline{\Omega}.$$

Then from $a_\lambda(u, mu) = (f, mu)$ with $u = (\lambda m - L)^{-1} f$, $f \in L^2(\Omega)$, we have:

$$\lambda \|mu\|^2 - \|\sqrt{m}\, \nabla u\|^2 = (f, mu) + (\nabla u, u \nabla m).$$

Since (6.10) implies that $|\nabla m(x)| = a(x)\, m(x)^\rho$ with some $a(x) \in L^\infty(\Omega)$, it follows that

$$|\lambda \|mu\|^2 - \|\sqrt{m}\, \nabla u\|^2| \leqslant \|f\|\, \|mu\| + C\|mu\|^\rho \|u\|^{1-\rho} \|\nabla u\|;$$

so that from (6.6), if $|\arg \lambda| \geqslant \omega$, then

$$|\lambda| \|mu\|^{2-\rho} \leqslant \|f\|\, \|mu\|^{1-\rho} + C\|f\|^{2-\rho};$$

and hence it follows together with (6.9) that

$$\|(\lambda - A_0)^{-1} f\| = \|mu\| \leqslant C\|f\|/|\lambda|^{1/(2-\rho)}, \qquad f \in L^2(\Omega),\ \lambda \in \Sigma.$$

EXAMPLE 6.4. – Let us next consider a dual equation of (6.3):

$$\begin{cases} m(x)\,\partial u/\partial t - \Delta u = m(x)\,g(t, x) & \text{in } (0, T] \times \Omega, \\ u = 0 & \text{on } (0, T] \times \partial\Omega, \\ u(0, x) = u_0(x) & \text{in } \Omega. \end{cases} \tag{6.11}$$

Here, $m(x)$ is the same function as in (6.3) and is assumed to satisfy (6.4). $g(t, x)$ is a given function and $u = u(t, x)$ is the unknown function. We shall consider this problem in the spaces $H_0^1(\Omega) = \{H^{-1}(\Omega)\}'$ and $L^2(\Omega)$.

In $H_0^1(\Omega)$ we can write (6.11) in the form

$$\begin{cases} du/dt + Bu \ni g(t), & 0 < t \leq T, \\ u(0) = u_0, \end{cases}$$

where $B = m^{-1}L$ is a multivalued operator given more precisely by

$$\begin{cases} \mathcal{D}(B) = \{u \in H_0^1(\Omega);\ \text{there exists some } v \in H_0^1(\Omega) \text{ such that } Lu = mv\}, \\ Bu = \{v \in H_0^1(\Omega);\ Lu = mv\}. \end{cases} \tag{6.12}$$

Here, L denotes as above $-\Delta$ with Dirichlet boundary conditions in $H^{-1}(\Omega)$ and m^{-1} denotes the inverse of the multiplication by $m(x)$, which is bounded from $H_0^1(\Omega)$ to $L^2(\Omega)$. Since $B^{-1} = S = L^{-1}m \in \mathcal{L}(H_0^1(\Omega))$, we have from Lemma 5.1: $(\lambda - B)^{-1} = (\lambda S - 1)^{-1}S = (\lambda m - L)^{-1}m$. Further, we already observed that, if $\lambda \in \Sigma$, $\lambda m - L$ has a bounded inverse from $H^{-1}(\Omega)$ to $H_0^1(\Omega)$. Hence if we consider, for $u \in H_0^1(\Omega)$ and $\phi \in H^{-1}(\Omega)$, the scalar product

$$\langle(\lambda m - L)^{-1}mu, \phi\rangle = (mu, (\bar{\lambda} m - L)^{-1}\phi) = \langle u, m(\bar{\lambda} m - L)^{-1}\phi\rangle,$$

then, under (6.4), (6.7) implies that

$$\|(\lambda - B)^{-1}u\|_1 = \|(\lambda m - L)^{-1}mu\|_1 \leq C\|u\|_1/|\lambda|, \qquad u \in H_0^1(\Omega), \qquad \text{for } \lambda \in \Sigma.$$

We next extend B to a multivalued operator B_0 in $L^2(\Omega)$ by

$$\begin{cases} \mathcal{D}(B_0) = \{u \in H_0^1(\Omega);\ \text{there exists some } f \in L^2(\Omega) \text{ such that } Lu = mf\}, \\ B_0 u = m^{-1}Lu = \{f \in L^2(\Omega);\ Lu = mf\}; \end{cases} \tag{6.13}$$

here m^{-1} is the inverse of the multiplication operator $m\colon L^2(\Omega) \to L^2(\Omega)$. Since $B_0^{-1} \in \mathcal{L}(L^2(\Omega))$, it is seen by the same duality argument that (6.9) implies

$$\|(\lambda - B_0)^{-1}\|_{\mathcal{L}(L^2(\Omega))} \leq C/|\lambda|^{1/2}, \qquad \lambda \in \Sigma.$$

It is also possible as in Example 6.4 to prove that under the conditions $m \in \mathcal{C}^1(\bar{\Omega})$ and (6.10) that the resolvent $(\lambda - B_0)^{-1}$ satisfies the better estimate

$$\|(\lambda - B_0)^{-1}\|_{\mathcal{L}(L^2(\Omega))} \leq C/|\lambda|^{1/(2-\rho)}, \qquad \lambda \in \Sigma.$$

EXAMPLE 6.5.

$$\begin{cases} d(m(t)v)/dt + Lv = f(t), & 0 < t \leqslant T, \\ \lim_{t\to 0} m(t)v(t) = u_0. \end{cases} \tag{6.14}$$

Here, L is a univalent linear operator in a Banach space X satisfying Condition (H.P). $m(t) \geqslant 0$ is a given non negative function on the interval $[0, T]$, $f(t)$ is a given function with values in X, u_0 is an initial value, $v = v(t)$ is the unknown function.

Putting $u = m(t)v$, we rewrite the equation in the form

$$\begin{cases} du/dt + A(t)u \ni f(t), & 0 < t \leqslant T, \\ u(0) = u_0, \end{cases}$$

where $A(t) = Lm(t)^{-1}$, $0 \leqslant t \leqslant T$, are multivalue operators in X. More precisely, $A(t)$ is given by

$$\begin{cases} A(t) = Lm(t)^{-1} & \text{with } \mathcal{D}(A(t)) = \mathcal{D}(L) \text{ if } m(t) \neq 0, \\ A(t) = O_\infty & \text{with } \mathcal{D}(A(t)) = \{0\} \text{ if } m(t) = 0. \end{cases}$$

In order to apply Theorem 5.2, let us verify the Conditions (D.P) and (D.C.1). We assume that

$$\begin{cases} m \in \mathcal{C}^1([0, T]; \boldsymbol{R}), \\ |m_t(t)| \leqslant Cm(t)^\nu \quad \text{for all } 0 \leqslant t \leqslant T, \end{cases} \tag{6.15}$$

with some $0 < \nu \leqslant 1$; here $m_t(t)$ denotes the derivative of $m(t)$. Take a number d such that $d \geqslant m(t)$ for all $0 \leqslant t \leqslant T$, and choose a sufficiently small number $\tilde{c} > 0$ such that, for a region $\tilde{\Sigma} = \{\lambda \in \boldsymbol{C}; \operatorname{Re}\lambda \leqslant \tilde{c}(|\operatorname{Im}\lambda| + 1)^\alpha\}$, $\bigcup_{0 \leqslant d' \leqslant d} d'\tilde{\Sigma} \subset \Sigma \subset \rho(L)$ holds, where Σ is the region in the Condition (H.P). Then, since

$$(\lambda - A(t))^{-1} = m(t)L^{-1}(\lambda m(t)L^{-1} - 1)^{-1} = m(t)(\lambda m(t) - L)^{-1} \qquad \text{for } \lambda \in \tilde{\Sigma},$$

it is verified that

$$\|(\lambda - A(t))^{-1}\|_{\mathcal{L}(X)} \leqslant Mm(t)^{1-\beta}\left(\frac{m(t)|\lambda|}{m(t)|\lambda| + 1}\right)^\beta |\lambda|^{-\beta} \leqslant \frac{\tilde{M}}{(|\lambda| + 1)^\beta}, \qquad \lambda \in \tilde{\Sigma},$$

hence the Condition (D.P). On the other hand, clearly (6.15) implies $T(t) = A(t)^{-1} = m(t)L^{-1} \in \mathcal{C}^1([0, T]; \mathcal{L}(X))$ with $dT(t)/dt = m_t(t)L^{-1}$. Moreover, it follows that

$$\|(\lambda T(t) - 1)^{-1} dT(t)/dt\|_{\mathcal{L}(X)} = \|m_t(t)(\lambda m(t) - L)^{-1}\|_{\mathcal{L}(X)} \leqslant \frac{Cm(t)^\nu}{(m(t)|\lambda| + 1)^\beta} \leqslant \frac{\tilde{N}}{(|\lambda| + 1)^{\tilde{\nu}}},$$

where $\tilde{\nu} = \min\{\beta, \nu\}$.

EXAMPLE 6.6.

$$\begin{cases} \partial(m(t, x)v)/\partial t - \Delta v = f(t, x) & \text{in } (0, T] \times \Omega, \\ v = 0 & \text{on } (0, T] \times \partial\Omega, \\ \lim_{t \to 0} m(t, x)v(t, x) = u_0(x) & \text{in } \Omega. \end{cases} \tag{6.16}$$

$$\begin{cases} m(t, x)\partial u/\partial t - \Delta u = m(t, x)g(t, x) & \text{in } (0, T] \times \Omega, \\ u = 0 & \text{on } (0, T] \times \partial\Omega, \\ u(0, x) = u_0(x) & \text{in } \Omega. \end{cases} \tag{6.17}$$

Here, $\Omega \subset \boldsymbol{R}^n$ is a bounded open set with smooth boundary $\partial\Omega$. $m(t, x) \geq 0$ is a given non negative function such that

$$m(t) = m(t, \cdot) \in L^\infty(\Omega) \quad \text{for all } 0 \leq t \leq T, \text{ with } \|m(t)\|_{L^\infty(\Omega)} \leq C. \tag{6.18}$$

$v = v(t, x)$ (resp. $u = u(t, x)$) is the unknown function of the problem (6.16) (resp. (6.17)). We consider (6.16) (resp. (6.17)) in the space $H^{-1}(\Omega)$ (resp. $H_0^1(\Omega)$). These problems are written respectively in the form

$$\begin{cases} du/dt + A(t)u \ni f(t), & 0 < t \leq T, \\ u(0) = u_0, \end{cases}$$

$$\begin{cases} du/dt + B(t)u \ni g(t), & 0 < t \leq T, \\ u(0) = u_0. \end{cases}$$

Here, $A(t)$ (resp. $B(t)$), $0 \leq t \leq T$, are multivalued operators in $H^{-1}(\Omega)$ (resp. $H_0^1(\Omega)$) defined analogously to (6.5) (resp. (6.12)). As has been proved in Example 6.3 (resp. Example 6.4), $\rho(A(t))$ (resp. $\rho(B(t))$) contain a region $\Sigma = \{\lambda \in \boldsymbol{C}; |\arg \lambda| \geq \omega$ or $|\lambda| \leq \leq c\}$ with any $0 < \omega < \pi/2$ and with some $c > 0$, and there the estimate

$$\|(\lambda - A(t))^{-1}\|_{\mathcal{L}(H^{-1}(\Omega))} \leq C/|\lambda|, \quad \lambda \in \Sigma, \qquad (\text{resp. } \|(\lambda - B(t))^{-1}\|_{\mathcal{L}(H_0^1(\Omega))} \leq C/|\lambda|, \quad \lambda \in \Sigma),$$

holds, i.e. under (6.18), (D.P) (resp. (D.P)′) in Section 5 is satisfied.

In order to apply Theorem 5.2 (resp. Theorem 5.3), therefore, it suffices to verify that $T(t) = A(t)^{-1} = m(t)L^{-1}$ (resp. $S(t) = B(t)^{-1} = L^{-1}m(t)$) satisfies (D.C.1) (resp. (D.C.1)′). Let us assume in addition to (6.18) that

$$\begin{cases} m \in \mathcal{C}^1([0, T]; L^1(\Omega)), \\ |m_t(t, x)| \leq Cm(t, x)^\nu \quad \text{for all } 0 \leq t \leq T \text{ and a.e. } x \in \Omega. \end{cases} \tag{6.19}$$

with some exponent $0 < \nu \leq 1$; here m_t denotes the derivative of $m(t)$. Using Sobolev embedding theorem $H^1(\Omega) \subset L^{p_n}(\Omega)$, where $p_n = \infty$ if $n = 1$, p_n is any finite number

≥ 1 if $n = 2$, $p_n = 2n/(n-2)$ if $n \geq 3$; we observe that

$$|\langle T(t)\phi, u\rangle| \leq \|m(t)\|_{L^{q_n}(\Omega)} \|L^{-1}\phi\|_{L^{p_n}(\Omega)} \|u\|_{L^{p_n}(\Omega)} \leq$$

$$\leq C\|m(t)\|_{L^{q_n}(\Omega)} \|\phi\|_{-1} \|u\|_1, \qquad \text{for } \phi \in H^{-1}(\Omega) \text{ and } u \in H_0^1(\Omega),$$

where $q_n = 1$ if $n = 1$, q_n is any number > 1 if $n = 2$, $q_n = n/2$ if $n > 3$ (note that always $2/p_n + 1/q_n = 1$). This shows that $\|T(t)\|_{\mathcal{L}(H^{-1}(\Omega))} \leq C\|m(t)\|_{L^{q_n}(\Omega)}$ for $0 \leq t \leq T$. Similarly, we observe that $\|S(t)\|_{\mathcal{L}(H_0^1(\Omega))} \leq C\|m(t)\|_{L^{q_n}(\Omega)}$. On the other hand, since $m_t \in \mathcal{C}([0, T]; L^q(\Omega))$ for any $1 < q < \infty$ from (6.19), $m \in \mathcal{C}^1([0, T]; L^q(\Omega))$, $1 < q < \infty$; we conclude that $T \in \mathcal{C}^1([0, T]; \mathcal{L}(H^{-1}(\Omega)))$ with $dT/dt = m_t(t)L^{-1}$ and $S \in \mathcal{C}^1([0, T]; \mathcal{L}(H_0^1(\Omega)))$ with $dS/dt = L^{-1}m_t(t)$ respectively. To show (D.C.1) let us consider, recalling that $(\lambda T(t) - 1)^{-1} = L(\lambda m(t) - L)^{-1}$ for $\lambda \in \Sigma$, the scalar product

$$|\langle(\lambda T(t) - 1)^{-1} dT/dt\phi, u\rangle| =$$

$$= |(L^{-1}\phi, m_t(t)(\bar{\lambda} m(t) - L)^{-1}Lu)| \leq C\|L^{-1}\phi\| \|m(t)^{\nu}(\bar{\lambda} m(t) - L)^{-1}Lu\|.$$

Using (6.8), we obtain that it is bounded by

$$\|\phi\|_{-1} \|m(t)(\bar{\lambda} m(t) - L)^{-1}Lu\|^{\nu} \|(\bar{\lambda} m(t) - L)^{-1}Lu\|^{1-\nu} \leq C|\lambda|^{-\nu/2} \|\phi\|_{-1} \|u\|_1.$$

Hence

$$\|(\lambda T(t) - 1)^{-1} dT(t)/dt\phi\|_{-1} \leq C|\lambda|^{-\nu/2} \|\phi\|_{-1}, \qquad \phi \in H^{-1}(\Omega),\ \lambda \in \Sigma.$$

The Condition (D.C.1)′ on $S(t)$ is also verified in a similar way. In fact, noting that

$$(\lambda S(t) - 1)^{-1} dS(t)/dt = (\lambda L^{-1}m(t) - 1)^{-1}L^{-1}m_t(t) = (\lambda m(t) - L)^{-1}m_t(t),$$

we observe that

$$\|(\lambda S(t) - 1)^{-1} dS(t)u/dt\|_1 \leq C|\lambda|^{-\nu/2} \|u\|_1, \qquad u \in H_0^1(\Omega),\ \lambda \in \Sigma.$$

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