Annali di Matematica pura ed applicata
(IV), Vol. CLXIII (1993), pp. 181-198

# Resonance at Two Consecutive Eigenvalues for Semilinear Elliptic Equations (*).

PIERPAOLO OMARI - FABIO ZANOLIN

**Abstract.** – *The solvability of the Dirichlet problem for a semilinear elliptic equation is studied in some situations where the classical resonance conditions of Landesman and Lazer may fail.*

## 1. – Introduction.

Let $\Omega$ be a bounded domain in $\boldsymbol{R}^n$, with a smooth boundary, and let $g: \Omega \times \boldsymbol{R} \to \boldsymbol{R}$ be a nonlinear function satisfying the Carathéodory conditions. We consider the Dirichlet problem

$$(1.1) \qquad -\Delta u = g(x, u) + h(x), \qquad \text{in } \Omega,$$

$$(1.2) \qquad u = 0, \qquad \text{on bdry}\Omega,$$

where $h \in L^p(\Omega)$, for some suitable $p \geq 2$, is given.

It is well known that, when $g$ grows at most linearly with respect to its second variable, the solvability of (1.1)-(1.2) depends on the interaction of the ratio $g(x, s)/s$ with the spectrum $\sigma = \{\lambda_N: N = 1, 2, \ldots\}$ of $-\Delta$ in $H_0^1(\Omega)$. The conditions imposed on $g$ are usually classified as nonresonant or resonant, according as they yield the solvability of (1.1)-(1.2) for every $h$ or not. Of course, in the linear case $g(x, s) = \lambda s$, such conditions reduce to $\lambda \notin \sigma$ or to $\lambda = \lambda_N \in \sigma$; accordingly, by the Fredholm alternative, (1.1)-(1.2) has a solution for every $h$, or has a solution if and only if $h \in E_N^\perp$, where $E_N$ denotes the eigenspace corresponding to the eigenvalue $\lambda_N$. Many papers have been devoted to the obtention of nonresonance conditions (see e.g. [Do], [B-N], [A-M], [B-DF], [M-Wa], [DF-G], [Gos], [C-O]), as well as of resonance conditions (see e.g. [L-L], [A-L-P], [B-B-F], [F-F]). Here, we are concerned with the latter situation, that is with the resonant case. Namely, for a pair of consecutive eigenvalues $\lambda_N < \lambda_{N+1}$, we assume

(*) Entrata in Redazione il 25 maggio 1990.

Indirizzo degli AA.: P. OMARI: Dipartimento di Scienze Matematiche, Università di Trieste, Piazzale Europa 1, 34127 Trieste, Italia; F. ZANOLIN: Dipartimento di Matematica e Informatica, Università di Udine, Via Zanon 6, 33100 Udine, Italia.

$(\mathrm{i}_1)\ \lambda_N s^2 \leqslant sg(x, s) \leqslant \lambda_{N+1} s^2, \quad$ *for* $|s| \geqslant r > 0$ *and a.e.* $x \in \Omega$,

and

$(\mathrm{i}_2)\ h \in E_N^{\perp} \cap E_{N+1}^{\perp}$.

Comparing with the linear case, one could expect that $(\mathrm{i}_1)$ and $(\mathrm{i}_2)$ imply the existence of solutions for (1.1)-(1.2). Yet, generally speaking, this is not true, even if $(\mathrm{i}_1)$ is assumed to hold for every $s$ and a.e. $x \in \Omega$, as is shown by a counterexample in [I-N]. Therefore, some further conditions on the behaviour of $g$ must be imposed. Here, to conclude solvability, in addition to $(\mathrm{i}_1)$ and $(\mathrm{i}_2)$, we assume, if $N > 1$,

$(\mathrm{i}_3)\ \operatorname*{ess\,inf}_{x \in \Omega'} sg(x, s) - \lambda_N s^2 \to +\infty, \quad$ *as* $s \to +\infty$ *or* $s \to -\infty$,

and

$(\mathrm{i}_4)\ \lambda_{N+1} s^2 - \operatorname*{ess\,sup}_{x \in \Omega''} sg(x, s) \to +\infty, \quad$ *as* $s \to +\infty$ *or* $s \to -\infty$,

where $\Omega'$ and $\Omega''$ are sufficiently large subsets of $\Omega$. While, if $N = 1$, we assume either

$(\mathrm{i}_3')\ \operatorname*{ess\,inf}_{x \in \Omega'} sg(x, s) - \lambda_1 s^2 \to +\infty, \quad$ *as* $|s| \to +\infty$,

where $\Omega'$ is a sufficiently large subset of $\Omega$, or

$(\mathrm{i}_3'')\ sg(x, s) - \lambda_1 s^2 \geqslant 0, \quad$ *for* $s \in \mathbf{R}$ *and a.e.* $x \in \Omega$.

The special assumptions considered with respect to the first eigenvalue $\lambda_1$ are due to the fact that each nonzero eigenfunction in $E_1$ has a definite sign in $\Omega$, while in $E_N$, with $N > 1$, each nonzero eigenfunction changes sign on subsets of $\Omega$ of positive measure. We recall that $(\mathrm{i}_3'')$ was already assumed in [DF-N], [I-N-W], [Gu], [D-T], but it was always coupled with a nonresonance condition with respect to the second eigenvalue $\lambda_2$.

We point out that, under $(\mathrm{i}_1)$, $(\mathrm{i}_3)$, (respectively $(\mathrm{i}_3')$, or $(\mathrm{i}_3'')$) and $(\mathrm{i}_4)$, it may happen that

$$g_-(x) = \limsup_{s \to -\infty} (g(x, s) - \lambda_N s) = 0, \qquad \text{a.e. in } \Omega,$$

and

$$g_+(x) = \liminf_{s \to +\infty} (g(x, s) - \lambda_N s) = 0, \qquad \text{a.e. in } \Omega,$$

as well as

$$\gamma_-(x) = \limsup_{s \to -\infty} (\lambda_{N+1} s - g(x, s)) = 0, \qquad \text{a.e. in } \Omega,$$

and

$$\gamma_+(x) = \liminf_{s \to +\infty} (\lambda_{N+1} s - g(x, s)) = 0, \qquad \text{a.e. in } \Omega.$$

So that the classical Landesman-Lazer conditions [L-L] may fail, both at the eigenvalue $\lambda_N$ and at the eigenvalue $\lambda_{N+1}$, since they require respectively

$$\int_\Omega g_+ v^+ - \int_\Omega g_- v^- > \int_\Omega hv \quad (=0, \text{ by } (i_2)),$$

for all $(0 \neq)\ v \in E_N$, and

$$\int_\Omega \gamma_+ w^+ - \int_\Omega \gamma_- w^- > \int_\Omega hw \quad (=0, \text{ by } (i_2)),$$

for all $(0 \neq)\ w \in E_{N+1}$, where $u^+ = \max\{u, 0\}$ and $u^- = max\{-u, 0\}$ (the element of volume $dx$ is omitted in the above integrals).

On the other hand, conditions $(i_1)$, $(i_3)$, (respectively $(i_3')$) and $(i_4)$ imply the validity of the Ahmad-Lazer-Paul conditions [A-L-P], with respect to $\lambda_N$ and $\lambda_{N+1}$. Namely, if $G(x, s)$ denotes the primitive $\int_{[0,s]} g(x, t)\, dt$, we have

$$\lim_{\|v\| \to +\infty} \int_\Omega (G(x, v(x)) - (\lambda_N/2)|v(x)|^2)\, dx = +\infty \qquad (\text{by } (i_3)),$$

where $v \in E_N$, and

$$\lim_{\|w\| \to +\infty} \int_\Omega ((\lambda_{N+1}/2)|w(x)|^2 - G(x, w(x)))\, dx = +\infty \qquad (\text{by } (i_4)),$$

where $w \in E_{N+1}$ (here, $\|\cdot\|$ denotes any norm in $E_N$ and $E_{N+1}$). Yet, to the best of our knowledge, it remains an open problem to establish the existence of solutions to (1.2)-(1.2), under $(i_1)$, $(i_2)$ and the Ahmad-Lazer-Paul conditions, both at $\lambda_N$ and at $\lambda_{N+1}$. So that our results can be regarded as (partial) contributions in this direction. Other papers related to these questions are [T], [Gon], [Ca], [Co].

Actually, our results are more in the spirit of those obtained in [F-K] (see also [He], [F-H], [Dr], [W], [Hi]). But, in all these papers resonance occurs only at the eigenvalue $\lambda_N$, since the function $g$ is required to satisfy a nonresonance condition with respect to the eigenvalue $\lambda_{N+1}$, namely, it is always assumed that, for some $\delta > 0$,

$$sg(x, s) \leq (\lambda_{N+1} - \delta) s^2, \qquad \text{for } |s| \geq r > 0 \text{ and a.e. } x \in \Omega .$$

On the contrary, under our assumptions double resonance (at $\lambda_N$ and at $\lambda_{N+1}$) may arise.

Moreover, as a consequence of our main theorems, in the case $g(x, s) = g(s)$, $g(0) = 0$ and $g \in C^1(\boldsymbol{R})$, we can replace conditions $(i_1)$, $(i_3)$ and $(i_4)$ with the following

$(i_5)$ $\lambda_N \leq g'(s) \leq \lambda_{N+1}$, *for* $s \in \boldsymbol{R}$,

and again conclude solvability, for any $h$ satisfying $(i_2)$. In this way, we complete some known results, which relate the solvability of (1.1)-(1.2) to the location of the range $R(g')$ of $g'$ with respect to the spectrum $\sigma$. Indeed, it is well-known ([Do], [$M_1$]) that if $\mathrm{cl}\, R(g') \cap \sigma = \emptyset$, then (1.1)-(1.2) is uniquely solvable for every $h$, whereas if $\mathrm{int}\, R(g') \cap \sigma \neq \emptyset$, then there exist $h$ such that (1.1)-(1.2) has more than one solution [Da]. Now we can say that if $\mathrm{int}\, R(g') \cap \sigma = \emptyset$, then (1.1)-(1.2) has at least one solution for any $h$ satisfying ($i_2$), even if $R(g') \cap \sigma \neq \emptyset$. In this way we also extend to a more general framework a result in [Di], obtained in the study of periodic solutions of a second order ordinary differential equation, by a technique which strongly relies upon the one dimensional character of the problem. It is also worthy to mention that, even if, under ($i_5$), $G$ is a convex function, our result is independent from those obtained by methods based on convex analysis, like e.g. in [M-Wi].

Finally, we observe that using the theorems stated in the next section one can easily obtain some nonresonance results for problem (1.1)-(1.2), which recover previous ones contained in [O-$Z_1$], [O-$Z_3$].

## 2. – The existence results.

*The case of 2m-th order elliptic equations.*

Let $\Omega$ be a bounded domain in $\boldsymbol{R}^n$, with a boundary of class $C^{2m}$, with $m \geq 1$, and let

$$\mathcal{L}u = \sum_{0 \leq |i|,\, |j| \leq m} (-1)^{|j|} D^j (a_{ij}(x) D^i u),$$

be a symmetric uniformly strongly elliptic differential operator of order $2m$, acting on functions $u$ defined on $\Omega$. The coefficients $a_{ij}$ are real value functions defined on $\mathrm{cl}\,\Omega$, with $a_{ij} \in C^{|j|}(\mathrm{cl}\,\Omega)$, for $0 \leq |i|,\ |j| \leq m$. We are concerned with the weak solvability of the Dirichlet problem

$$\mathcal{L}u = f(x, u) + h(x), \qquad \text{in } \Omega,$$

$$\partial^i u / \partial n^i = 0, \qquad \text{for } 0 \leq i \leq m-1, \qquad \text{on bdry}\,\Omega,$$

where $\partial/\partial n$ denotes the differentiation with respect to the outward normal to the boundary. We assume that $h \in L^2(\Omega)$ and $f\colon \Omega \times \boldsymbol{R} \to \boldsymbol{R}$ satisfies the Carathéodory conditions and

($h_1$) *there exist $a > 0$ and $b \in L^2(\Omega)$ such that $|f(x, s)| \leq a|s| + b(x)$, for $s \in \boldsymbol{R}$ and a.e. $x \in \Omega$.*

In order to study the above problem, we consider the operator $L\colon D(L) \subset L^2(\Omega) \to L^2(\Omega)$, with domain $D(L) = H^{2m}(\Omega) \cap H_0^m(\Omega)$, induced by $\mathcal{L}$. It is known (see e.g. [F], [DF]) that $L$ is a (densely defined) self-adjoint linear operator, having a closed range $R(L)$ and a spectrum $\sigma(L)$ made up of an increasing sequence $\{\lambda_N\}$ of real eigenvalues, with $\lambda_N \to +\infty$, as $N \to +\infty$. Moreover, the corresponding eigenspaces $N(L - \lambda_N)$ are finite dimensional.

From now on, we suppose that 0 is an eigenvalue and we denote by $\mu_1$ the minimal positive eigenvalue (this is a model situation which one can always reduce to in the study of resonance problems between two consecutive eigenvalues $\lambda_N < \lambda_{N+1}$). We also indicate by $K: R(L) \to R(L)$ the right inverse of $L$, which is compact. For each real valued function $u$ defined on $\Omega$, we define $\Omega^-(u) = \{x \in \Omega: u(x) < 0\}$ and $\Omega^+(u) = \{x \in \Omega: u(x) > 0\}$. The following assumptions are then considered:

(h$_2$) *for every* $v \in N(L) - \{0\}$, *the sets* $\Omega^{\pm}(v)$ *have both positive measure*

and

(h$_3$) *for every* $w \in N(L - \mu_1) - \{0\}$, *the sets* $\Omega^{\pm}(w)$ *have both positive measure.*

THEOREM 1. – *Let* (h$_1$), (h$_2$), (h$_3$) *and*

(h$_4$) $0 \leqslant sf(x, s) \leqslant \mu_1 s^2$, *for* $|s| \geqslant r > 0$ *and a.e.* $x \in \Omega$,

*hold. Moreover, assume that there exist subsets* $\Omega'$ *and* $\Omega''$ *of* $\Omega$, *with*

(h$_5$) $\operatorname{meas}(\Omega' \cap \Omega^-(v)) > 0$ *and* $\operatorname{meas}(\Omega' \cap \Omega^-(v)) > 0$, *for every* $v \in N(L) - \{0\}$

*and*

(h$_6$) $\operatorname{meas}(\Omega'' \cap \Omega^-(w)) > 0$ *and* $\operatorname{meas}(\Omega'' \cap \Omega^-(w)) > 0$, *for every* $w \in N(L - \mu_1) - \{0\}$,

*such that*

(h$_7$) $\operatorname*{ess\,inf}_{x \in \Omega'} sf(x, s) \to +\infty$, *as* $s \to +\infty$ *or* $s \to -\infty$,

*and*

(h$_8$) $\mu_1 s^2 - \operatorname*{ess\,sup}_{x \in \Omega''} sf(x, s) \to +\infty$, *as* $s \to +\infty$ *or* $s \to -\infty$.

*Then equation*

$$Lu = f(x, u) + h, \tag{2.1}$$

*has at least one solution* $u \in H^{2m}(\Omega) \cap H_0^m(\Omega)$, *for each* $h \in L^2(\Omega)$ *such that*

(h$_9$) $h \in N(L)^{\perp} \cap N(L - \mu_1)^{\perp}$.

REMARK 1. – A preliminary version of Theorem 1 has been presented in [O]. Its proof combines topological degree methods with some technical arguments introduced in [O-Z$_1$] and refined in [O-Z$_3$]. The main effort is devoted to the obtention of a priori bounds for the $L^2$-norm of the term $f(x, u; \lambda)$ (cf. relation (2.15)). We also point out that, even if here we confine ourselves to selfadjoint problems, nevertheless our technique can be adapted to the treatment of some classes of nonselfadjoint problems as well. Some results in this direction are contained in [O-$Z_2$] and [A-O-Z].

PROOF. – We use the Leray-Schauder continuation theorem as stated in [$M_2$, Th. IV.5]. We set $A: L^2(\Omega) \to L^2(\Omega)$, $A = \nu I$, with $\nu \in ]0, \mu_1[$, and $N: L^2(\Omega) \to L^2(\Omega)$, $Nu = f(\cdot, u) + h$, $u \in L^2(\Omega)$. Clearly, $L$ is a Fredholm mapping of index zero, $A$ and $N$ are $L$-completely continuous and the kernel $N(L - A) = \{0\}$. Then equation (2.1) will have a solution $u \in D(L)$, if a constant $R > 0$ can be found such that, for every $u \in D(L)$ satisfying, for some $\lambda \in ]0, 1[$, the equation

$$Lu = (1 - \lambda) Au + \lambda Nu, \tag{2.1$_\lambda$}$$

it results

$$|u|_{L^2} < R. \tag{2.2}$$

Henceforth, in the process of this proof, the $L^2$-scalar product $(\cdot, \cdot)_{L^2}$ and the $L^2$-norm $|\cdot|_{L^2}$ will be simply denoted by $(\cdot,\cdot)$ and $|\cdot|$, respectively. Setting

$$f(x, s; \lambda) = (1 - \lambda) \nu s + \lambda f(x, s),$$

equation (2.1$_\lambda$) can be rewritten in the form

$$Lu = f(x, u; \lambda) + \lambda h. \tag{2.1$_\lambda$}$$

Using ($h_4$), ($h_7$) and ($h_8$), we can easily construct (see e.g. [O-$Z_1$]) two continuous, nondecreasing and sublinear functions $\alpha^{\pm}: \mathbf{R} \to [0, +\infty[$ such that

$$\alpha^-(s) \leq sf(x, s), \qquad \text{for } |s| \geq r \text{ and a.e. } x \in \Omega',$$

and

$$sf(x, s) \leq \mu_1 s^2 - \alpha^+(s), \qquad \text{for } |s| \geq r \text{ and a.e. } x \in \Omega'',$$

with

$$\alpha^{\pm}(s) \to +\infty, \qquad \text{as } s \to +\infty, \qquad \text{and} \qquad \alpha^{\pm}(s) = 0, \qquad \text{for } s \leq 0,$$

if ($h_7$) and ($h_8$) hold at $+\infty$ (the other cases being treated similarly). Accordingly, conditions ($h_4$), ($h_7$) and ($h_8$) can be formulated as

$$\alpha^-(\chi_{\Omega'}(x)s) \leq sf(x, s) \leq \mu_1 s^2 - \alpha^+(\chi_{\Omega''}(x)s), \qquad \text{for } |s| \geq r \text{ and a.e. } x \in \Omega,$$

where $\chi_{\Omega'}$ and $\chi_{\Omega''}$ stand for the characteristic functions of $\Omega'$ and $\Omega''$, respectively. Hence, it follows

$$\alpha^-(\chi_{\Omega'}(x)s) \leq sf(x, s; \lambda) \leq \mu_1 s^2 - \alpha^+(\chi_{\Omega''}(x)s), \tag{2.3}$$

for $|s| \geq r$, a.e. $x \in \Omega$ and $\lambda \in [0, 1]$.

We claim that (2.3) and ($h_1$) yield, for some function $c \in L^1(\Omega)$,

$$sf(x, s; \lambda) \geq (1/\mu_1)|f(x, s; \lambda)|^2 + \beta^+(\chi_{\Omega''}(x)f(x, s; \lambda)) - c(x), \tag{2.4}$$

for $s \in \boldsymbol{R}$, a.e. $x \in \Omega$ and $\lambda \in [0,1]$, with $\beta^+ : \boldsymbol{R} \to [0, +\infty[$ a continuous sublinear function such that

$$\beta^+(s) \to +\infty, \qquad \text{as } s \to +\infty, \qquad \text{and} \qquad \beta^+(s) = 0, \qquad \text{for } s \leq 0.$$

In fact, by (2.3), we have

$$\operatorname{sign}(s) f(x, s; \lambda) = |f(x, s; \lambda)| \leq \mu_1 |s|,$$

for $|s| \geq r$, a.e. $x \in \Omega$ and $\lambda \in [0,1]$. Multiplying (2.3) by $(1/\mu_1) f(x, s; \lambda)/s = (1/\mu_1)|f(x, s; \lambda)|/|s|$, we get

$$\text{(2.5)} \qquad sf(x, s; \lambda) \geq (1/\mu_1)|f(x, s; \lambda)|^2 + \alpha^+(\chi_{\Omega''}(x)s)|f(x, s; \lambda)|/(\mu_1|s|),$$

for $|s| \geq r$, a.e. $x \in \Omega$ and $\lambda \in [0,1]$. Suppose $s \geq r$ and $x \in \Omega''$. We prove that, for some constant $c_1 > 0$,

$$\text{(2.6)} \qquad sf(x, s; \lambda) \geq (1/\mu_1)|f(x, s; \lambda)|^2 + (1/2)\alpha^+(|f(x, s; \lambda)|/\mu_1) - c_1,$$

for $s \geq r$, a.e. $x \in \Omega''$ and $\lambda \in [0,1]$. Indeed, if $(x, s, \lambda)$ is such that

$$(0 \leq) f(x, s; \lambda) < (1/2)\mu_1 s,$$

it is

$$sf(x, s; \lambda) \geq (2/\mu_1)|f(x, s; \lambda)|^2 \geq (1/\mu_1)|f(x, s; \lambda)|^2 + (1/2)\alpha^+(f(x, s; \lambda)/\mu_1) - c_1,$$

recalling that, since $\alpha^+$ is sublinear, there exists a constant $c_1 > 0$ such that $(1/2)\alpha^+(\xi/\mu_1) \leq (1/\mu_1)\xi^2 + c_1$, for every $\xi$. Whereas, if $(x, s, \lambda)$ is such that

$$\mu_1 s \geq f(x, s; \lambda) \geq (1/2)\mu_1 s \ (\geq 0),$$

by (2.5) and $\alpha^+$ non-decreasing, we get

$$sf(x, s; \lambda) \geq (1/\mu_1)|f(x, s; \lambda)|^2 + (1/2)\alpha^+(f(x, s; \lambda)/\mu_1).$$

Hence, (2.6) follows, setting $\beta^+(\xi) = (1/2)\alpha^+(\xi/\mu_1)$. Note that, for $s \leq -r$ and a.e. $x \in \Omega$, or $s \geq r$ and a.e. $x \in \Omega - \Omega''$, (2.5) simply reads as

$$sf(x, s; \lambda) \geq (1/\mu_1)|f(x, s; \lambda)|^2,$$

by the properties of $\alpha^+$. Hence (2.6) holds for $|s| \geq r$, a.e. $x \in \Omega$. Finally, (2.4) is proved, using $(h_1)$.

By (2.4), we also derive, for every $u \in D(L)$ and $\lambda \in [0,1]$,

$$\text{(2.7)} \quad u(x)f(x, u(x); \lambda) \geq (1/\mu_1)|f(x, u(x); \lambda)|^2 + \beta^+(\chi_{\Omega''}(x)f(x, u(x); \lambda)) - c(x),$$

for a.e. $x \in \Omega$. Integrating (2.7) on $\Omega$ and using the properties of $\beta^+$, we find

$$\text{(2.8)} \qquad (u, f(\cdot, u; \lambda)) \geq (1/\mu_1)|f(\cdot, u; \lambda)|^2 + \int_{\Omega''} \beta^+(f(x, u(x); \lambda))\,dx - \int_{\Omega} c(x)\,dx,$$

for every $u \in D(L)$ and $\lambda \in [0,1]$.

Let $u \in D(L)$ be a possible solution to $(2.1_\lambda)$, for some $\lambda \in ]0,1[$. Denote by

$$P,\ Q\colon L^2(\Omega) \to L^2(\Omega)\,,$$

the orthogonal projections onto $N(L)$ and $N(L-\mu_1)$, respectively, and set, for simplicity,

$$\varphi = f(\cdot,\, u;\, \lambda)\,.$$

Clearly, it is

$$P\varphi = 0\,.$$

Applying the operator $K$ to both sides of $(2.1_\lambda)$, we get, as $\varphi$, $h \in R(L)$ and $KL = I - P$,

$$u - Pu = K\varphi + \lambda Kh\,. \tag{2.9}$$

We want to bound $|\varphi|$. Multiplying (2.9) by $\varphi$, and using $\varphi \in R(L) = N(L)^\perp$, $K$ symmetric and $(h_9)$, we find

$$(u - Pu,\, \varphi) = (u,\, \varphi) = (K\varphi,\, \varphi) + \lambda(Kh,\, \varphi) =$$

$$= (KQ\varphi + K(I-Q)\varphi,\, Q\varphi + (I-Q)\varphi) + \lambda(Kh,\, Q\varphi + (I-Q)\varphi) =$$

$$= (KQ\varphi,\, Q\varphi) + 2(KQ\varphi, (I-Q)\varphi) + (K(I-Q)\varphi, (I-Q)\varphi) + \lambda(h,\, KQ\varphi) + \lambda(Kh, (I-Q)\varphi) =$$

$$= (1/\mu_1)|Q\varphi|^2 + (2/\mu_1)(Q\varphi, (I-Q)\varphi) + (K(I-Q)\varphi, (I-Q)\varphi) +$$

$$+ (\lambda/\mu_1)(h,\, Q\varphi) + \lambda(Kh, (I-Q)\varphi) =$$

$$= (1/\mu_1)|Q\varphi|^2 + (K(I-Q)\varphi, (I-Q)\varphi) + \lambda(Kh, (I-Q)\varphi)\,.$$

Denoting by $\mu_2$ the smallest (positive) eigenvalue of $L$ greater than $\mu_1$, we obtain, using the Cauchy-Schwarz inequality,

$$(u,\, \varphi) \leqslant (1/\mu_1)|Q\varphi|^2 + (1/\mu_2)|(I-Q)\varphi|^2 + |Kh|\ |(I-Q)\varphi|\,. \tag{2.10}$$

On the other hand, by (2.8), as $\beta^+ \geqslant 0$, we get

$$(1/\mu_1)|Q\varphi|^2 + (1/\mu_1)|(I-Q)\varphi|^2 - \int_\Omega c(x)\,dx = (1/\mu_1)|\varphi|^2 - \int_\Omega c(x)\,dx \leqslant (u,\, \varphi)\,. \tag{2.11}$$

A comparison between (2.10) and (2.11) yields

$$(\mu_1^{-1} - \mu_2^{-1})|(I-Q)\varphi|^2 - |Kh|\ |(I-Q)\varphi| - \int_\Omega c(x)\,dx \leqslant 0\,,$$

and then, for some constant $c_2 > 0$,

$$|(I-Q)\varphi| \leqslant c_2\,. \tag{2.12}$$

Hence, using (2.8) and (2.10) again, we have

$$\int_{\Omega''} \beta^+ (f(x, u(x); \lambda))\, dx \leq (\mu_2^{-1} - \mu_1^{-1}) |(I - Q)\varphi|^2 + |Kh|\, |(I - Q)\varphi| + \tag{2.13}$$

$$+ \int_{\Omega} c(x)\, dx \leq 0 + |Kh|\, c_2 + \int_{\Omega} |c(x)|\, dx = c_3 .$$

Now, we assume the existence of a sequence $\{u_n\}$ of solutions to $(2.1_\lambda)$, with $\lambda = \lambda_n$, such that

$$|Q\varphi_n| \to +\infty ,$$

where $\varphi_n = f(\cdot, u_n; \lambda_n)$. Then, since $N(L - \mu_1)$ is finite dimensional, possibly passing to a subsequence, we get, as $n \to +\infty$,

$$Q\varphi_n / |Q\varphi_n| \to w \in N(L - \mu_1),$$

and hence, by (2.12),

$$\varphi_n / |Q\varphi_n| = Q\varphi_n / |Q\varphi_n| + (I - Q)\varphi_n / |Q\varphi_n| \to w .$$

Moreover, possibly passing to a further subsequence,

$$\varphi_n(x) / |Q\varphi_n| \to w(x), \qquad \text{as } n \to +\infty, \text{ a.e. in } \Omega .$$

Further, by $(h_3)$ and $|w| = 1$, the sets $\Omega^{\pm}(w)$ have both positive measure. Thus, we have, in particular,

$$\varphi_n(x) \to +\infty , \qquad \text{as } n \to +\infty ,$$

in $\Omega^+(w)$ and then

$$\beta^+ (\varphi_n(x)) \to +\infty , \qquad \text{as } n \to +\infty , \tag{2.14}$$

in $\Omega'' \cap \Omega^+(w)$, with $\Omega'' \cap \Omega^+(w)$ having positive measure by $(h_6)$. On the other hand, as $\beta^+ \geq 0$, it is by (2.13)

$$\int_{\Omega'' \cap \Omega^+(w)} \beta^+ (\varphi_n(x))\, dx \leq \int_{\Omega''} \beta^+ (\varphi_n(x))\, dx \leq c_3 ;$$

so that, by Fatou's lemma, we could conclude that the function

$$\liminf_{n \to +\infty} \beta^+ (\varphi_n(\cdot)),$$

is integrable on $\Omega'' \cap \Omega^+(w)$: thus contradicting (2.14). This implies the existence of a constant $c_4 > 0$, independent of $u$ and $\lambda$, such that

$$|\varphi|_{L^2} = |f(x, u; \lambda)|_{L^2} \leq c_4 . \tag{2.15}$$

Since $K: R(L) \to R(L)$ is continuous, from (2.9) and (2.15), we derive

$$|u - Pu| \leqslant (1/\mu_1)|\varphi| + \lambda|Kh| \leqslant (1/\mu_1)c_4 + |Kh| = c_5. \tag{2.16}$$

Now, in order to bound $|Pu|$, assume, by contradiction, that there exists a sequence $\{u_n\}$ of solutions to $(2.1_\lambda)$, with $\lambda = \lambda_n$, such that

$$|Pu_n| \to +\infty, \qquad \text{as } n \to +\infty.$$

Since $N(L)$ is finite dimensional, arguing as above, we get, possibly for a subsequence,

$$u_n(x)/|Pu_n| \to v(x), \qquad \text{as } n \to +\infty, \text{ a.e. in } \Omega,$$

for some $v \in N(L)$, with $|v| = 1$, Since, by $(h_2)$ and $|v| = 1$, the sets $\Omega^{\pm}(v)$ have both positive measure, we obtain, in particular,

$$u_n(x) \to +\infty, \qquad \text{as } n \to +\infty,$$

in $\Omega^+(v)$ and then

$$\alpha^-(u_n(x)) \to +\infty, \qquad \text{as } n \to +\infty, \tag{2.17}$$

in $\Omega' \cap \Omega^+(v)$, with $\Omega' \cap \Omega^+(v)$ having positive measure by $(h_5)$. From the right hand side of (2.3), using $(h_1)$, we derive, for some function $d \in L^1(\Omega)$,

$$\alpha^-(\chi_{\Omega'}(x)s) - d(x) \leqslant sf(x, s; \lambda),$$

for $s \in \boldsymbol{R}$, a.e. $x \in \Omega$ and $\lambda \in [0,1]$. Accordingly, using $P\varphi = 0$, we obtain

$$\int_{\Omega'} \alpha^-(u_n(x))\,dx - \int_{\Omega} d(x)\,dx \leqslant (u_n, f(\cdot, u_n; \lambda_n)) = (u_n, \varphi_n) =$$

$$= (Pu_n, \varphi_n) + ((I-P)u_n, \varphi_n) = ((I-P)u_n, \varphi_n) \leqslant |(I-P)u_n|\,|\varphi_n|.$$

This implies, by (2.15), (2.16) and $\alpha^- \geqslant 0$,

$$\int_{\Omega' \cap \Omega^+(v)} \alpha^-(u_n(x))\,dx \leqslant \int_{\Omega'} \alpha^-(u_n(x))\,dx \leqslant c_6,$$

for some constant $c_6 > 0$: thus contradicting (2.17), by Fatou's lemma. Finally, we conclude the existence of a constant $c_7 > 0$, independent of $u$ and $\lambda$, such that

$$|Pu| \leqslant c_7.$$

Hence, (2.2) follows, for any $R > c_5 + c_7$. Q.E.D.

REMARK 2. – If $(h_2)$ fails, then condition $(h_5)$ and $(h_7)$ can be replaced for instance by

$$\operatorname*{ess\,inf}_{x \in \Omega} sf(x, s) \to +\infty, \qquad \text{as } |s| \to +\infty.$$

A similar condition can be assumed, in place of $(h_6)$ and $(h_8)$, if $(h_3)$ fails.

Theorem 1 takes a particularly simple form when $f(x, s) = f(s)$ does not depend on the $x$-variable. In such a case one can choose $\Omega' = \Omega'' = \Omega$, so that ($h_5$) and ($h_6$) are fulfilled by ($h_2$) and ($h_3$), and ($h_6$) and ($h_8$) read respectively

($h_7'$) $sf(s) \to +\infty$, *as* $s \to +\infty$ *or* $s \to -\infty$,

and

($h_8'$) $\mu_1 s^2 - sf(s) \to +\infty$, *as* $s \to +\infty$ *or* $s \to -\infty$.

Accordingly, we can state the following result.

COROLLARY 1. – *Assume* ($h_2$) *and* ($h_3$). *Suppose that* $f(x, s) = f(s)$ *and let*

($h_4'$) $0 \leq sf(s) \leq \mu_1 s^2$, *for* $|s| \geq r > 0$,

($h_7'$) *and* ($h_8'$) *be satisfied. Then equation* (2.1) *has at least one solution* $u \in H^{2m}(\Omega) \cap H_0^m(\Omega)$, *for each* $h \in L^2(\Omega)$ *satisfying* ($h_9$).

A further consequence of Theorem 1 is the following Corollary 2, which extends to elliptic equations a similar result obtained in [D], for the periodic problem for a second order ordinary differential equation by a phase-plane analysis argument which of course cannot be transferred to the present situation.

COROLLARY 2. – *Let* ($h_2$) *and* ($h_3$) *hold. Assume that* $f(x, s) = f(s)$, $f(0) = 0$, $f \in C^1(\boldsymbol{R})$ *and*

($k_1$) $0 \leq f'(s) \leq \mu_1$, *for* $s \in \boldsymbol{R}$.

*Then equation* (2.1) *has at least one solution* $u \in H^{2m}(\Omega) \cap H_0^m(\Omega)$, *for each* $h \in L^2(\Omega)$ *satisfying* ($h_9$).

PROOF. – At first we notice that ($k_1$) implies ($h_1$). Moreover, from ($k_1$), it follows that $f'(s) = 0$, for all $s$, or $f'(s) = \mu_1$, for all $s$, or there are points $s_1$, $s_2$ such that $0 < f'(s_1)$, $f'(s_2) < \mu_1$. Since the first two situations are trivial, let us consider the third one. By continuity, there exist $s^* \in \boldsymbol{R}$, $\varepsilon > 0$, $\delta > 0$ such that

$$\varepsilon \leq f'(s) \leq \mu_1 - \varepsilon,$$

for every $s \in [s^* - \delta, s^* + \delta]$. Let $s^* \geq 0$ be and take $s > s^* + \delta$; we have

$$\varepsilon\delta \leq f(s) = \int_{[0, s^*]} f'(t)\,dt + \int_{[s^*, s^* + \delta]} f'(t)\,dt + \int_{[s^* + \delta, s]} f'(t)\,dt \leq \mu_1 s - \varepsilon\delta.$$

Hence, we easily conclude that ($h_7'$) and ($h_8'$) hold. Similarly one works, if $s^* < 0$. Q.E.D.

REMARK 3. – Dealing with second order elliptic operators, the above stated results, where ($h_2$) and ($h_3$) are assumed, model more efficiently the case where 0 and $\mu_1$

are higher order eigenvalues. In the following subsection we will restrict to this kind of operators and study the situation where 0 is the first eigenvalue, so that ($h_2$) fails.

*The case of second order elliptic equations.*

Let $\Omega$ be a bounded domain in $\mathbf{R}^n$ ($n \geq 2$), with a boundary of class $C^2$, and let

$$\mathcal{L}u = - \sum_{i, j = 1, \ldots, n} \partial / \partial x_j (a_{ij}(x) \partial u / \partial x_i) + a_0(x) u ,$$

be a symmetric uniformly strongly elliptic second order differential operator, acting on real valued functions $u$ defined on $\Omega$. The coefficients $a_{ij}$ are real valued functions defined on $\mathrm{cl}\,\Omega$, with $a_{ij} \in C^1(\mathrm{cl}\,\Omega)$, for $i, j = 1, \ldots, n$, and $a_0 \in C^0(\mathrm{cl}\,\Omega)$, $a_0(x) \geq 0$ on $\mathrm{cl}\,\Omega$. Under these assumptions, it is well-known that the first eigenvalue $\lambda_1$ of $\mathcal{L}$ in $H_0^1(\Omega)$ is simple and that there exists a corresponding smooth eigenfunction $\phi$, with $\phi > 0$ in $\Omega$ and $\partial \phi / \partial n < 0$ on $\mathrm{bdry}\,\Omega$. Let us consider the Dirichlet problem

$$\mathcal{L}u - \lambda_1 u = f(x, u) + h(x) , \qquad \text{in } \Omega ,$$

$$u = 0 , \qquad \text{on } \mathrm{bdry}\,\Omega .$$

We assume that $h \in L^p(\Omega)$, with $p > n$, and $f: \Omega \times \mathbf{R} \to \mathbf{R}$ satisfies the Carathéodory conditions and

($h_1'$) *there exist* $a > 0$ *and* $b \in L^p(\Omega)$, *with* $p > n$, *such that* $|f(x, s)| \leq a|s| + b(x)$, *for* $s \in \mathbf{R}$ *and a.e.* $x \in \Omega$.

As well-known, $\mathcal{L} - \lambda_1$ induces a (densely defined) self-adjoint linear operator $D(L) \subset L^2(\Omega) \to L^2(\Omega)$, with domain $D(L) = H^2(\Omega) \cap H_0^1(\Omega)$. Since the kernel $N(L)$ of $L$ is one dimensional and is spanned by the function $\phi$, all nonzero eigenfunctions in $N(L)$ have a definite sign on $\Omega$, so that ($h_2$) fails. While, setting $\mu_1 = \lambda_2 - \lambda_1 > 0$, $\lambda_2$ being the second eigenvalue of $\mathcal{L}$, all nonzero eigenfunctions in $N(L - \mu_1)$ change sign on subsets of $\Omega$ of positive measure, and hence ($h_3$) holds. Of course, all the other structural assumptions previously considered are satisfied. The following theorems complete, for second order elliptic operators, the results stated in the previous subsection.

THEOREM 2. – *Let* ($h_1'$),

($h_4''$) $0 \leq sf(x, s)$, *for* $s \in \mathbf{R}$ *and a. e.* $x \in \Omega$,

*and*

($h_4'''$) $sf(x, s) \leq \mu_1 s^2$, *for* $|s| \geq r > 0$ *and a. e.* $x \in \Omega$,

*hold. Moreover, assume that there exists a subset* $\Omega''$ *of* $\Omega$, *such that* ($h_6$) *and* ($h_8$) *are* *fulfilled. Then equation*

$$Lu = f(x, u) + h, \tag{2.18}$$

*has at least one solution* $u \in W^{2,p}(\Omega) \cap H_0^1(\Omega)$, *for each* $h \in L^p(\Omega)$, *with* $p > n$, *satisfying* ($h_9$).

PROOF. – We start observing that, using ($h_1'$) and $h \in L^p(\Omega)$, with $p > n$, by the $L^p$-theory for the Dirichlet problem [A] and a standard bootstrap argument, each solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$ to

$$Lu = f(x, u; \lambda) + \lambda h, \tag{2.19$_\lambda$}$$

for some $\lambda \in [0,1]$, belongs to $W^{2,p}(\Omega)$ and hence to $C^1(\mathrm{cl}\,\Omega)$. Here, as in the proof of Theorem 1, we set, for some $\nu \in ]0, \mu_1[$,

$$f(x, u; \lambda) = (1 - \lambda)\nu s + \lambda f(x, s).$$

Then, we proceed as in that proof up of to the point where

$$|u - Pu|_{L^2} \leq c_5, \tag{2.20}$$

is obtained. In order to get (2.2), we assume by contradiction that there exists a sequence $\{u_n\}$ in $W^{2,p}(\Omega) \cap H_0^1(\Omega)$ of solutions to (2.19$_\lambda$), with $\lambda = \lambda_n \in [0, 1[$, such that, as $n \to +\infty$,

$$|u_n|_{L^2} \to +\infty,$$

and then

$$|u_n|_{C^1} \to +\infty.$$

Using again ($h_1'$) and $h \in L^p(\Omega)$, with $p > n$, by the $L^p$-theory and the compact imbedding of $W^{2,p}(\Omega)$ in $C^1(\mathrm{cl}\,\Omega)$, we obtain that, possibly passing to a subsequence,

$$u_n / |u_n|_{C^1} \to v, \qquad \text{in } C^1(\mathrm{cl}\,\Omega), \tag{2.21}$$

with $|v|_{C^1} = 1$. From (2.20), we also deduce that

$$(u_n - Pu_n)/|u_n|_{C^1} \to 0, \qquad \text{in } L^2(\Omega),$$

and then $v - Pv = 0$, i.e. $v \in N(L)$. Since $|v|_{C^1} = 1$, we have that either $v > 0$ in $\Omega$ and $\partial v/\partial n < 0$ on bdry $\Omega$, or $v < 0$ in $\Omega$ and $\partial v/\partial n > 0$ on bdry $\Omega$. Assuming, for instance, that the first eventuality holds, we deduce, from (2.21), that $u_n > 0$ in $\Omega$, for all large

$n$. Now, taking the $L^2$-scalar product of $(2.19_\lambda)$ by $v$, we obtain, recalling that $\lambda_n \in [0, 1[$,

$$0 = \int_\Omega f(x, u_n; \lambda_n) v(x)\, dx = (1 - \lambda_n) \nu \int_\Omega u_n(x) v(x)\, dx +$$

$$+ \lambda_n \int_\Omega f(x, u_n(x)) v(x)\, dx > \lambda_n \int_\Omega f(x, u_n(x)) v(x)\, dx,$$

thus contradicting $(h_4'')$, when $n$ is large. Q.E.D.

We recall that the sign condition $(h_4'')$ was already considered in [DF-N], [I-N-W], [Gu], [D-T], but it was always coupled with some stronger nonresonance condition at the second eigenvalue $\mu_1$. The next Corollary 3 illustrates better such a difference in the case of an autonomous nonlinearity $f$.

COROLLARY 3. – *Suppose that* $f(x, s) = f(s)$ *and let*

$(h_4^{IV})$ $0 \leq sf(s)$, *for* $s \in \boldsymbol{R}$,

$(h_4^{V})$ $sf(s) \leq \mu_1 s^2$, *for* $|s| \geq r > 0$,

*and* $(h_8')$ *be satisfied. Then equation* (2.18) *has at least one solution* $u \in W^{2,p}(\Omega) \cap H_0^1(\Omega)$, *for each* $h \in L^p(\Omega)$, *with* $p > n$, *satisfying* $(h_9)$.

COROLLARY 4. – *Assume that* $f(x, s) = f(s), f(0) = 0, f \in C^1(\boldsymbol{R})$ *and* $(k_1)$ *holds, Then equation* (2.18) *has at least one solution* $u \in W^{2,p}(\Omega) \cap H_0^1(\Omega)$, *for each* $h \in L^p(\Omega)$, *with* $p > n$, *satisfying* $(h_9)$.

PROOF. – It sufficies to observe that $(h_4^{IV})$ and $(h_4^{V})$ are fulfilled and then to work as in the proof of Corollary 2, in order to apply Corollary 3. Q.E.D.

*The one-dimensional case.*

In order to explain better the meaning of conditions $(h_5)$, $(h_6)$, $(h_7)$, $(h_8)$ and the role of the sets $\Omega^\pm$, we present now the simplified stements which can be obtained in the one-dimensional case. Let us consider the two-point boundary value problem

$$-u'' - \lambda_N u = f(x, u) + h(x), \qquad x \in ]0, \pi[, \tag{2.21}$$

$$u(0) = u(\pi) = 0. \tag{2.22}$$

We recall that, here, $\lambda_N = N^2$, for $N = 1, 2, \ldots$, and the corresponding eigenspace is spanned by the function $\sin(Nx)$. We assume that $f: ]0, \pi[ \times \boldsymbol{R} \to \boldsymbol{R}$ satisfies the Carathéodory conditions. Then a careful reading of the proof of the preceding results (in particular observing that $h \in L^1(0, \pi)$ sufficies) shows that the following theorems hold.

PROPOSITION 1. – *Let* $N > 1$ *be. Suppose that* ($h_1$) *and*

$$0 \leq sf(x, s) \leq (2N+1)s^2, \quad \text{for } |s| \geq r > 0 \text{ and a.e. } x \in ]0, \pi[,$$

*hold. Moreover, assume that there exist subintervals* $J'$ *and* $J''$ *of* $]0, \pi[$, *with* $\text{meas}(J') > \pi/N$ *and* $\text{meas}(J'') > \pi/(N+1)$, *such that*

$$\operatorname*{ess\,inf}_{x \in J'} sf(x, s) \to +\infty, \quad \text{as } s \to +\infty \text{ or } s \to -\infty,$$

*and*

$$(2N+1)s^2 - \operatorname*{ess\,sup}_{x \in J''} sf(x, s) \to +\infty, \quad \text{as } s \to +\infty \text{ or } s \to -\infty.$$

*Then problem* (2.21)-(2.22) *has at least one solution* $u \in W^{2,1}(0, \pi)$, *for each* $h \in L^1(0, \pi)$ *such that*

$$\int_{]0,\pi[} h(x) \sin(Nx)\, dx = 0 = \int_{]0,\pi[} h(x) \sin((N+1)x)\, dx.$$

PROPOSITION 2. – *Let* ($h_1$),

$$0 \leq sf(x, s), \quad \text{for } s \in \mathbf{R} \text{ and a.e. } x \in ]0, \pi[,$$

*and*

$$sf(x, s) \leq 3s^2, \quad \text{for } |s| \geq r > 0 \text{ and a.e. } x \in ]0, \pi[,$$

*hold. Moreover, assume that there exist a subinterval* $J''$ *of* $]0, \pi[$, *with* $\text{meas}(J'') > \pi/2$, *such that*

$$3s^2 - \operatorname*{ess\,sup}_{x \in J''} sf(x, s) \to +\infty, \quad \text{as } s \to +\infty \text{ or } s \to -\infty.$$

*Then problem* (2.21)-(2.22) *has at least one solution* $u \in W^{2,1}(0, \pi)$, *for each* $h \in L^1(0, \pi)$ *such that*

$$\int_{]0,\pi[} h(x) \sin(x)\, dx = 0 = \int_{]0,\pi[} h(x) \sin(2x)\, dx.$$

*A counterexample.*

We conclude this paper showing that the assumptions considered in the above theorems are, in some sense, sharp. Let us consider the two point boundary value problem

$$-u'' - 4u = f(x, u) + h(x), \quad x \in ]0, \pi[, \tag{2.23}$$

$$u(0) = u(\pi) = 0, \tag{2.24}$$

where $h(x) = c \sin x$, with $c < 0$, and $f(x, s)$ is defined as follows

$$f(x, s) = \sin(2x)\sqrt{s}, \qquad \text{for } x \in ]0, \pi/2[, \quad s \geq 0,$$
$$= 0, \qquad \text{otherwise}.$$

Observe that the functions $f$ and $h$ satisfy the following conditions:

$$0 \leq sf(x, s) \leq 5s^2, \qquad \text{for } |s| \geq 1 \text{ and } x \in ]0, \pi[,$$

$$5s^2 - \operatorname*{ess\,sup}_{x \in ]0, \pi[} sf(x, s) = 5s^2 - s\sqrt{s} \to +\infty, \qquad \text{as } s \to +\infty,$$

and

$$\int_{]0, \pi[} h(x) \sin(2x)\, dx = 0 = \int_{]0, \pi[} h(x) \sin(3x)\, dx.$$

Concerning the condition

$$\operatorname*{ess\,inf}_{x \in J'} sf(x, s) \to +\infty, \qquad \text{as } s \to +\infty,$$

we note that it can be satisfied only on subsets $J'$ of $]0, \pi[$, with meas $(J') < \pi/2$. Thus all the assumptions of Proposition 1 are fulfilled, for $N = 2$ and $J'' = ]0, \pi[$, with the only exception of the condition: meas $(J') > \pi/N$. On the other hand, it can be easily checked that problem (2.23)-(2.24) possesses no solution. Indeed, any possible solution $u(x)$ of (2.23)-(2.24) must satisfy

$$\int_{]0, \pi[} f(x, u(x)) \sin(2x)\, dx = 0$$

and therefore, by definition of $f$, $u(x) \leq 0$, on $]0, \pi/2[$, which in turns implies that

$$-u'' - 4u = c \sin x, \qquad \text{on } ]0, \pi/2[.$$

This yields a contradiction by a direct inspection of the linear equation.

## REFERENCES

[A-O-Z] AFUWAPE A. U. - OMARI P. - ZANOLIN F., *Nonlinear perturbations of differential operators with nontrivial kernel and applications to third order periodic boundary value problems,* J. Math. Anal. Appl., **143** (1989), pp. 35-56.

[A] AGMON S., *The $L^p$ approach to the Dirichlet problem,* Ann. Scuola Norm. Sup. Pisa, **13** (1959), pp. 405- 448.

[A-M] AMANN H. - MANCINI G., *Some applications of monotone operator theory to resonance problems,* Nonlinear Analysis T.M.A., **3** (1979), pp. 815-830.

[A-L-P] AHMAD S. - LAZER A. C. - PAUL J. L., *Elementary critical point theory and perturbations of elliptic boundary value problems at resonance,* Indiana Univ. Math. J., **25** (1976), pp. 933-944.

[B-B-F] BARTOLO P. - BENCI V. - FORTUNATO D., *Abstract critical point theorems and applications to some nonlinear problems with «strong» resonance at infinity,* Nonlinear Analysis T. M. A., 7 (1983), pp. 981-1012.

[B-DF] BERESTYCKI H. - DE FIGUEIREDO D.G., *Double resonance in semilinear elliptic problems,* Comm. Partial Diff. Equations, **9** (1981), pp. 91-120.

[B-N] BREZIS H. - NIRENBERG L., *Characterizations of the ranges of some nonlinear operators and applications to boundary value problems,* Ann. Sc. Norm. Sup. Pisa, **5** (1978), pp. 225-326.

[Ca] CAC N. P., *On an elliptic boundary value problem at double resonance,* J. Math. Anal. Appl., **132** (1988), pp. 473-483.

[Co] COSTA D. G., *A note on unbounded perturbations of linear resonant problems,* preprint (1989).

[C-O] COSTA D. G. - OLIVEIRA A. S., *Existence of solutions for a class of semilinear elliptic problems at double resonance,* Bol. Soc. Bras. Mat., **19** (1988), pp. 21-37.

[Da] DANCER E. N., *Non-uniqueness for nonlinear boundary value problems,* Rocky Mountain J. Math., **13** (1983), pp. 401-412.

[DF] DE FIGUEIREDO D. G., *The Dirichlet problem for nonlinear elliptic equations: A Hilbert space approach,* Lecture Notes in Mathematics, **446**, Springer-Verlag, Berlin (1975), pp. 144-165.

[DF-G] DE FIGUEIREDO D. G. - GOSSEZ J. P., *Conditions de non-résonance pour certain problèmes elliptiques semi-linéaires,* C. R. Acad. Sci. Paris, **302** (1986), pp. 543-545.

[DF-N] DE FIGUEIREDO D. G. - NI W. M., *Perturbations of second order linear elliptic problems by nonlinearities without Landesman-Lazer condition,* Nonlinear Analysis T.M.A., **5** (1979), pp. 629-634.

[Di] DING T., *Nonlinear oscillations at a point of resonance,* Scientia Sinica (Serie A), **25** (1982), pp. 918- 931.

[Do] DOLPH C. L., *Nonlinear integral equations of the Hammerstein type,* Trans. Amer. Mat. Soc., **66** (1949), pp. 289-307.

[Dr] DRABEK P., *Solvability of nonlinear problems at resonance,* Comment. Math. Univ. Carolinae, **23** (1982), pp. 359-368.

[D-T] DRABEK P. - TOMICZEK P., *Remark on the structure of the range of second order nonlinear elliptic operator,* Comment. Math. Univ. Carolinae, **30** (1989), pp. 455-464.

[F-F] FABRY C. - FONDA A., *Periodic solutions of nonlinear differential equations with double resonance,* Ann. Mat. Pura Appl., **157** (1990), pp. 99-116.

[F] FRIEDMAN A., *Partial Differential Equations,* Holt, Rinehart & Winston, New York (1969).

[F-H] FUCIK S. - HESS P., *Nonlinear perturbations of linear operators having nullspace with strong unique continuation property,* Nonlinear Analysis T.M.A., 3 (1979), pp. 271-277.

[F-K] FUCIK S. - KRBEC M., *Boundary value problems with bounded nonlinearity and general nullspace of the linear part,* Math. Z., **155** (1977), pp. 129-138.

[Gon] GONÇALVES J. V. A., *Existence of saddle points for functional on Hilbert spaces: Applications to Hammerstein equations,* J. Integral Equations, 8 (1985), pp. 229-238.

[Gos] GOSSEZ J. P., *Nonresonance near the first eigenvalue of a second order elliptic* *problem,* Lecture Notes in Mathematics, **1324**, Springer-Verlag, Berlin (1986), pp. 97- 104.

[Gu] GUPTA C. P., *Solvability of a boundary value problem with the nonlinearity satisfying a sign condition,* J. Math. Anal. Appl., **129** (1988), pp. 482-492.

[He] HESS P., *A remark on the preceding paper of Fucik and Krbec,* Math. Z., **155** (1977), pp. 138-141.

[Hi] HIRANO N., *Unbounded nonlinear perturbations of linear elliptic problems at resonance,* J. Math. Anal. Appl., **132** (1988), pp. 434-446.

[I-N] IANNACCI R. - NKASHAMA M. N., *Nonlinear boundary value problems at resonance,* Nonlinear Analysis T.M.A., **11** (1987), pp. 455-473.

[I-N-W] IANNACCI R. - NKASHAMA M. N. - WARD J. R. jr., *Nonlinear second order elliptic partial differential equations at resonance,* Trans. Amer. Math. Soc., **311**(1989), pp. 711-726.

[L-L] LANDESMAN E. M. - LAZER A. C., *Nonlinear perturbations of linear elliptic boundary value problems at resonance,* J. Math. Mech., **19** (1970), pp. 609-623.

[$M_1$] MAWHIN J., *Contractive mappings and periodically perturbed conservative systems,* Arch. Math. Scripta Fac. Sci. Mat. UJEP Brunensis, **12** (1976), pp. 67-74.

[$M_2$] MAWHIN J., *Topological Degree Methods in Nonlinear Boundary Value Problems, , C.B.M.S. Regional Conference Series in Mathematics,* **40**, A.M.S., Providence (1979).

[M-Wi] MAWHIN J. - WILLEM M., *Critical points of convex perturbations of some indefinite quadratic forms and semilinear boundary value problems at resonance,* Ann. Inst. Henri Poincaré, **6** (1986), pp. 431-453.

[M-Wa] MAWHIN J. - WARD J. R. jr., *Nonresonance and existence for nonlinear elliptic boundary value problems,* Nonlinear Analysis T.M.A., **6** (1981), pp. 677- 684.

[O] OMARI P., *Resonance at two consecutive eigenvalues for nonlinear differential equations,* unpublished internal report (1988).

[O-$Z_1$] OMARI P. - ZANOLIN F., *Existence results for forced nonlinear periodic BVPs at resonance,* Ann. Mat. Pura Appl., **141** (1985), pp. 127-157.

[O-$Z_2$] OMARI P. - ZANOLIN F., *Boundary value problems for forced nonlinear equations at resonance,* Lecture Notes in Mathematics, **1151**, Springer-Verlag, Berlin (1985), pp. 285- 294.

[O-$Z_3$] OMARI P. - ZANOLIN F., *A note on nonlinear oscillations at resonance,* Acta Math. Sinica (New Series), **3** (1987), pp. 351-361.

[T] THEWS K., *A reduction method for some nonlinear Dirichlet problems,* Nonlinear Analysis T.M.A., **3** (1979), pp. 795-813.

[W] WARD J. R. jr., *Applications of critical point theory to weakly nonlinear boundary value problems at resonance,* Houston J. Math., **10** (1984), pp. 291-305.