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On Improved Regularity of Weak Solutions of Some Degenerate, Anisotropic Elliptic Systems (*).

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Summary. – *We consider a (possibly) vector-valued function* $u\colon \Omega \to \mathbb{R}^N$, $\Omega \subset \mathbb{R}^n$, *minimizing the integral* $\int_\Omega (|D_1 u|^2 + \ldots + |D_{n-1}|^2 + |D_n u|^p)\,dx$, $2-2/(n+1)<p<2$, *where* $D_i u = \partial u/\partial x_i$, *or some more general functional retaining the same behaviour; we prove higher integrability for* Du: $D_1 u, \ldots, D_{n-1} u \in L^{p/(p-1)}$ *and* $D_n u \in L^2$; *this result allows us to get existence of second weak derivatives:* $D(D_1 u), \ldots, D(D_{n-1} u) \in L^2$ *and* $D(D_n u) \in L^p$.

0. – Introduction.

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$; u be such that $u\colon \Omega \to \mathbb{R}^N$, $N \geq 1$. We consider an integral functional of the type

$$I(u) = \int_\Omega F(Du(x))\,dx . \tag{0.1}$$

Here F satisfies an anisotropic growth condition, namely,

$$a \sum_{i=1}^n |\xi_i|^{q_i} - b \leq F(\xi) \leq c \sum_{i=1}^n |\xi_i|^{q_i} + d, \qquad \forall \xi \in \mathbb{R}^{nN}, \tag{0.2}$$

where a, b, c and d are positive constants and $1 < q_i$, $i = 1, \ldots, n$. The isotropic case, that is $q_i = q$ $\forall i$, has been deeply studied [12]. In the last few years the anisotropic case, in which at least one of the q_i's differs from the others, has been attracting some attention: in [13], [15] it is shown that minimizers of (0.1) may be singular, if no restriction is assumed on the q_i's. On the other hand, if the exponents q_i are not too far apart, some regularity results for minimizers of (0.1) have been proven in [10], [11] and [16]. Let us point out that [10], [11] and [16] deal with scalar min-

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imizers $u\colon \Omega \to \mathbb{R}$, that is, $N = 1$. Vector-valued mappings $u\colon \Omega \to \mathbb{R}^N$ have been considered in [14], where $q_i \geq 2$. In the present paper we take $1 < q_i \leq 2$ and, under additional restrictions on F, we prove higher regularity results for local minimizers of (0.1). A typical example of a functional, in this class, is

$$I(u) = \int_\Omega \frac{1}{2} \sum_{i=1}^{n-1} |D_i u|^2 + \frac{1}{p} (\alpha + |D_n u|^2)^{p/2}, \qquad 0 \leq \alpha \leq 1. \tag{0.3}$$

Here $Du = (D_1 u, \dots, D_n u)$, $1 < p < 2$. The main effort of this work is to obtain results when, in (0.3), $\alpha = 0$, namely, the degenerate case. In a previous paper [4] we studied (0.3) when $\alpha \neq 0$, and deduced higher integrability and higher differentiability results for minimizers. However, the results of the current work do not follow from this earlier paper. Please see the Remark 4 at the end of Theorem 4 in section 1.

We introduce notations and the main results in section 1; section 2 contains some preliminary lemmas necessary for our work. The proofs of the theorems appear in section 3, 4 and 5.

1. - Notation and main results.

Let Ω be a bounded open set of $\mathbb{R}^n$, $n \geq 2$, u be a (possibly) vector-valued function, $u\colon \Omega \to \mathbb{R}^N$, $N \geq 1$; we consider integrals

$$I(u) = \int_\Omega F(Du(x))\,dx, \tag{1.1}$$

where $F\colon \mathbb{R}^{nN} \to \mathbb{R}$ is in $C^1(\mathbb{R}^{nN})$ and satisfies, for some positive constants c, m,

$$|F(\xi)| \leq c\left(1 + \sum_{i=1}^{n-1} |\xi_i|^2 + |\xi_n|^p\right), \tag{1.2}$$

$$\left|\frac{\partial F}{\partial \xi_i^\alpha}(\xi)\right| \leq c(1 + \xi_i|) \quad \text{if } i = 1, \dots, n-1, \tag{1.3}$$

$$\left|\frac{\partial F}{\partial \xi_n^\alpha}(\xi)\right| \leq c(1 + \xi_n|^{p-1}) \tag{1.4}$$

and

$$\sum_{j=1}^{n} \sum_{\beta=1}^{N} \left(\frac{\partial F}{\partial \xi_j^\beta}(\nu) - \frac{\partial F}{\partial \xi_j^\beta}(\lambda)\right)(\nu_j^\beta - \lambda_j^\beta) \geq \tag{1.5}$$

$$\geq m \sum_{j=1}^{n-1} |\nu_j - \lambda_j|^2 + m(1 + |\nu_n|^2 + |\lambda_n|^2)^{(p-2)/2} |\nu_n - \lambda_n|^2,$$

for every $\lambda, \nu, \xi \in \mathbb{R}^{nN}$, $\alpha = 1, \dots, N$. Here, $\lambda = \{\lambda_i^\alpha\}$, $\xi = \{\xi_i^\alpha\}$, $|\lambda_i|^2 = \sum_{\alpha=1}^{N} |\lambda_i^\alpha|^2$, etc.

About p, we assume that

$$1 < p < 2. \tag{1.6}$$

We say that u minimizes the integral (1.1) if $u\colon \Omega \to \mathbb{R}^N$, $u \in W^{1,p}(\Omega)$ with $D_i u \in L^2(\Omega)$, $i = 1, \dots, n-1$, and

$$I(u) \leqslant I(u + \phi). \tag{1.7}$$

for every $\phi\colon \Omega \to \mathbb{R}^N$ with $\phi \in W_0^{1,p}(\Omega)$ and $D_i \phi \in L^2(\Omega)$, $i = 1, \dots, n-1$. We will prove the following higher integrability result for $D_n u$:

THEOREM 1. – *Let $u\colon \Omega \to \mathbb{R}^N$ satisfy $u \in W^{1,p}(\Omega)$ with $D_i u \in L^2(\Omega)$, $i = 1, \dots, n-1$, where*

$$2 - 2/n < p < 2. \tag{1.8}$$

If F satisfies (1.2), ..., (1. 5) *and u minimizes the integral* (1.1), *then*

$$D_n u \in L^2_{\mathrm{loc}}(\Omega). \tag{1.9}$$

The higher integrability result (1.9) for $D_n u$ allows us to improve on the integrability of Du in the following way:

THEOREM 2. – *Under the assumptions of Theorem* 1 *we have*

$$D_i u \in L^r_{\mathrm{loc}}(\Omega), \qquad i = 1, \dots, n-1 \quad \forall r < \frac{2n}{n-1}, \tag{1.10}$$

$$D_n u \in L^t_{\mathrm{loc}}(\Omega), \qquad \forall t < \frac{pn}{n-1}. \tag{1.11}$$

Let us explicitly remark that (1.8) implies $2 < pn/(n-1) < 2n/(n-1)$; moreover, when $n = 2$, (1.8) is just $1 < p < 2$ and we have the following

COROLLARY 1. – *Under the assumptions of Theorem* 1, *we get*

$$\text{if} \quad n = 2 \quad \text{then} \quad u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega) \quad \text{for some } \alpha > 0.$$

The higher integrability result (1.10) contained in Theorem 2 allows us to get the existence of second weak derivatives:

THEOREM 3. – *Under the assumptions of Theorem* 1, *if p verifies the additional restriction*

$$2 - 2/(n+1) < p < 2, \tag{1.12}$$

then

$$D(D_i u) \in L^2_{\mathrm{loc}}(\Omega), \qquad i = 1, \dots, n-1, \tag{1.13}$$

$$D((1 + |D_n u|^2)^{(p-2)/4} D_n u) \in L^2_{\mathrm{loc}}(\Omega), \tag{1.14}$$

$$D(D_n u) \in L^p_{\mathrm{loc}}(\Omega). \tag{1.15}$$

REMARK 1. – Condition (1.12) is stronger than (1.8).

REMARK 2. – A straightforward application of Sobolev imbedding theorem gives us hölder continuity of u also in dimension three, more precisely, we have

COROLLARY 2. – *Under the assumptions of Theorem* 1, *if* p *verifies the additional restriction* (1.12), *then*

$$\text{when} \quad n = 2 \quad \text{we have } u \in C^{0,\beta}_{\mathrm{loc}}(\Omega), \qquad \forall \beta < 1;$$

$$\text{when} \quad n = 3 \quad \text{we have } u \in C^{0,1-1/p}_{\mathrm{loc}}(\Omega).$$

REMARK 3. – For $1 < p < 2$, let us consider the integrals

$$I(u) = \int_\Omega \left(\frac{1}{2} \sum_{i=1}^{n-1} |D_i u(x)|^2 + \frac{1}{p} |D_n u(x)|^p \right) dx, \tag{1.16}$$

$$I(u) = \int_\Omega \left(\frac{1}{2} \sum_{i=1}^{n-1} |D_i u(x)|^2 + \frac{1}{p}(1 + |D_n u(x)|^2)^{p/2} \right) dx: \tag{1.17}$$

they verify (1.2), ... ,(1. 5). In [4] the following regularity result has been proven:

THEOREM 4. – *Let* $F \in C^2(\mathbb{R}^{nN})$ *and* (1.2), ... ,(1. 4) *hold; in addition, let us assume that, for some positive constants* M_1, M_2,

$$M_1\left(\sum_{i=1}^{n-1} |\lambda_i|^2 + (1 + |\xi_n|^2)^{(p-2)/2} |\lambda_n|^2\right) \leq \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} \frac{\partial^2 F}{\partial \xi_j^\beta \partial \xi_i^\alpha}(\xi) \lambda_i^\alpha \lambda_j^\beta \leq \tag{1.5'}$$

$$\leq M_2\left(\sum_{i=1}^{n-1} |\lambda_i|^2 + (1 + |\xi_n|^2)^{(p-2)/2} |\lambda_n|^2\right),$$

for every λ, $\xi \in \mathbb{R}^{nN}$. *About* p, *we assume that*

$$\begin{cases} 1 < p < 2 & \text{if } n = 2, 3, \\ 98/97 < p < 2 & \text{if } n = 4; \\ 2 - 4/n < p < 2 & \text{if } n \geq 5. \end{cases} \tag{1.18}$$

Then, for a vectorvalued function $u \in W^{1,p}(\Omega)$ *with* $D_i u \in L^2(\Omega)$, $i = 1, \ldots, n-1$, *minimizing the integral* (1.1), *we get*

$$D_n \in L^2_{\mathrm{loc}}(\Omega), \tag{1.19}$$

$$D(D_i u) \in L^2_{\mathrm{loc}}(\Omega), \qquad i = 1, \ldots, n-1 \qquad \text{and} \qquad D(D_n u) \in L^p_{\mathrm{loc}}(\Omega). \tag{1.20}$$

REMARK 4. – Clearly, the left-hand side of (1.5') implies (1.5). The functional (1.17) satisfies (1.5'); however, the integral (1.16) which satisfies (1.5) does not satisfy (1.5'). Also note that (1.8) implies (1.18). We may summarize as follows: Theorem 4 requires good integrands such as (1.17), but a less restrictive range for p. Theorem 1, on the other hand, allows for lesser restrictions on the integrands (e.g., degeneracies are allowed), but requires more restrictions on p.

REMARK 5. – Higher integrability properties contained in Theorem 1 and 2 are proven by a technique of [7]: we gain a fractional order derivative of $\widehat{V}(Du)$, a suitable function of Du, thereby improving its integrability; also see [3], [4], [6], [14].

2. – Preliminaries.

For a vector-valued function $f(x)$, define the difference

$$\tau_{s,h} f(x) = f(x + he_s) - f(x),$$

where $h \in \mathbb{R}$, is the unit vector in the x_s direction, and $s = 1, 2, \dots, n$. For $x_0 \in \mathbb{R}^n$, let $B_R(x_0)$ be the ball centered at x_0 with radius R. We will often suppress x_0 whenever there is no danger of confusion. We now state several lemmas that are crucial to our work. In the following $f\colon \Omega \to \mathbb{R}^k$, $k \geq 1$; B_ϱ, B_R, $B_{2\varrho}$ and B_{2R} are concentric balls.

LEMMA 2.1. – *If* $0 < \varrho < R$, $|h| < R - \varrho$, $1 \leq t < \infty$, $s \in \{1, \dots, n\}$, $f, D_s f \in L^t(B_R)$, *then*

$$\int_{B_\varrho} |\tau_{s,h} f(x)|^t dx \leq |h|^t \int_{B_R} |D_s f(x)|^t dx.$$

(*See* [12, page 45], [5, page 28].)

LEMMA 2.2. – *Let* $f \in L^t(B_{2\varrho})$, $1 < t < \infty$, $s \in \{1, \dots, n\}$; *if there exists a positive constant* C *such that*

$$\int_{B_\varrho} |\tau_{s,h} f(x)|^t dx \leq C|h|^t,$$

for every h *with* $|h| < \varrho$, *then there exists* $D_s f \in L^t(B_\varrho)$. (*See* [12, page 45], [5, page 26].)

LEMMA 2.3. – *If* $f \in L^2(B_{3\varrho})$ *and for some* $d \in (0, 1)$ *and* $C > 0$

$$\sum_{s=1}^{n} \int_{B_\varrho} |\tau_{s,h} f(x)|^2 dx \leq C|h|^{2d},$$

for every h *with* $|h| < \varrho$, *then* $f \in L^r(B_{\varrho/4})$ *for every* $r < 2n/(n - 2d)$.

PROOF. – The previous inequality tells us that $f \in W^{b,2}(B_{\varrho/2})$ for every $b < d$, so we can apply the imbedding theorem for fractional Sobolev spaces. [2, chapter VII].

LEMMA 2.4. – *For every t with $1 \leq t < \infty$ there exists a positive constant C such that*

$$\int_{B_R} |\tau_{s,h} f(x)|^t dx \leq C \int_{B_{2R}} |f(x)|^t dx,$$

for every $f \in L^t(B_{2R})$, for every h with $|h| < R$, for every $s = 1, 2, \ldots, n$.

LEMMA 2.5. – *For every $\gamma \in (-1/2, 0)$ we have*

$$(2\gamma + 1)|a - b| \leq \frac{|(1 + |a|^2)^\gamma a - (1 + |b|^2)^\gamma b|}{(1 + |a|^2 + |b|^2)^\gamma} \leq \frac{c(k)}{2\gamma + 1} |a - b|,$$

for all $a, b \in \mathbb{R}^k$. (See [1].)

3. - Proof of Theorem 1.

Since u minimizes the integral (1.1) with growth conditions as in (1.2), ...,(1. 4), u solves the Euler equation,

$$\text{(3.1)} \qquad \int_\Omega \sum_{i=1}^n \sum_{\alpha=1}^N \frac{\partial F}{\partial \xi_i^\alpha} (Du(x)) D_i \phi^\alpha(x)\, dx = 0,$$

for all functions $\phi: \Omega \to \mathbb{R}^N$, with $\phi \in W_0^{1,p}(\Omega)$ and $D_1\phi, \ldots, D_{n-1}\phi \in L^2(\Omega)$. Let $R > 0$ be such that $\overline{B_{4R}} \subset \Omega$ and let B_ϱ and B_R be concentric balls, $0 < \varrho < R \leq 1$. Fix s take $0 < |h| < R$ and let $\eta: \mathbb{R}^n \to \mathbb{R}$ be a «cut off» function in $C_0^2(B_R)$ with

$$\eta \equiv 1 \text{ on } B_\varrho, \quad 0 \leq \eta \leq 1, \quad |D\eta| \leq C_1/(R - \varrho) \quad \text{and} \quad |DD\eta| \leq C_1/(R - \varrho)^2.$$

Using $\phi = \tau_{s,-h}(\eta^2 \tau_{s,h} u)$ in (3.1) we get, as usual

$$0 = \sum_{i=1}^n \sum_{\alpha=1}^N \int \frac{\partial F}{\partial \xi_i^\alpha}(Du)\, \tau_{s,-h}(D_i(\eta^2 \tau_{s,h} u^\alpha))\, dx =$$

$$= \sum_{i=1}^n \sum_{\alpha=1}^N \int \tau_{s,h}\left(\frac{\partial F}{\partial \xi_i^\alpha}(Du)\right)(2\eta D_i \eta \tau_{s,h} u^\alpha + \eta^2 \tau_{s,h} D_i u^\alpha)\, dx,$$

so that

$$\text{(3.2)} \qquad (I) = \int_{B_R} \sum_{i=1}^n \sum_{\alpha=1}^N \tau_{s,h}\left(\frac{\partial F}{\partial \xi_i^\alpha}(Du)\right) \tau_{s,h} D_i u^\alpha \eta^2\, dx =$$

$$= -\int_{B_R} \sum_{i=1}^n \sum_{\alpha=1}^N \tau_{s,h}\left(\frac{\partial F}{\partial \xi_i^\alpha}(Du)\right) 2\eta D_i \eta \tau_{s,h} u^\alpha\, dx = (II).$$

We apply (1.5) so that

$$(3.3)\qquad m\int_{B_R}\sum_{i=1}^{n-1}|\tau_{s,h}D_iu(x)|^2\eta^2(x)\,dx+$$

$$+m\int_{B_R}(1+|D_nu(x)|^2+|D_nu(x+he_s)|^2)^{(p-2)/2}|\tau_{s,h}D_nu(x)|^2\eta^2(x)\,dx\leqslant(I).$$

Set

$$(3.4)\quad \widehat{V}(\xi)=|V(\xi_n)|+\sum_{i=1}^{n-1}|\xi_i|,\qquad V(\xi_n)=(1+|\xi_n|^2)^{(p-2)/4}\xi_n,\qquad \forall\xi\in\mathbb{R}^{nN}.$$

Clearly,

$$(3.5)\qquad |\tau_{s,h}\widehat{V}(Du)|\leqslant|\tau_{s,h}V(D_nu)|+\sum_{i=1}^{n-1}|\tau_{s,h}D_iu|$$

and

$$(3.6)\qquad \widehat{V}(Du)\in L^r\quad\text{if and only if}\quad\begin{cases}D_iu\in L^r, & i=1,\dots,n-1,\\ D_nu\in L^{rp/2}.\end{cases}$$

Using Lemma 2.5 we find

$$(3.7)\quad C_2|\tau_{s,h}D_nu(x)|\leqslant\frac{|\tau_{s,h}V(D_nu(x))|}{(1+|D_nu(x)|^2+|D_nu(x+he_s)|^2)^{(p-2)/4}}\leqslant C_3|\tau_{s,h}D_nu(x)|,$$

for some positive constants C_2, C_3 depending only on N and p. Then, since $\eta=1$ on B_ϱ,

$$(3.8)\qquad m\int_{B_\varrho}|\tau_{s,h}\widehat{V}(Du)|^2dx\leqslant$$

$$\leqslant mC_4\int_{B_R}|\tau_{s,h}V(D_nu)|^2\eta^2dx+mC_4\int_{B_R}\sum_{i=1}^{n-1}|\tau_{s,h}D_iu|^2\eta^2dx\leqslant(1+C_3^2)C_4(I),$$

for some positive constant C_4, depending only on n. Now, let us estimate (II) in (3.2): using growth conditions (1.3), (1.4) and the properties of the «cut off» function η, we have

$$(3.10)\qquad (II)\leqslant\frac{4\tilde{n}cC_1}{R-\varrho}\int_{B_R}\Bigg(\sum_{i=1}^{n-1}|D_iu(x+he_s)|+|D_nu(x+he_s)|^{p-1}+1+$$

$$+\sum_{i=1}^{n-1}|D_iu(x)|+|D_nu(x)|^{p-1}\Bigg)|\tau_{s,h}u(x)|\,dx.$$

Now, changing variables and recalling $|h| < R$, we get

$$\int_{B_R} |D_i u(x+he_s)|^2 dx = \int_{B_R + he_s} |D_i u(y)|^2 dy \leq \int_{B_{2R}} |D_i u(y)|^2 dy \tag{3.11}$$

and

$$\int_{B_R} |D_n u(x+he_s)|^{2(p-1)} dx = \int_{B_R + he_s} |D_n u(y)|^{2(p-1)} dy \leq \int_{B_{2R}} |D_n u(y)|^{2(p-1)} dy\,; \tag{3.12}$$

let us remark that $0 < 2(p-1) < p$, so the integrals in (3.12) are finite. We use Hölder's inequality in (3.10) and we apply (3.11), (3.12) in order to get

$$(II) \leq C_5 \left(\left(|B_R| + \sum_{i=1}^{n-1} \int_{B_{2R}} |D_i u|^2 dx + \int_{B_{2R}} |D_n u|^{2(p-1)} dx \right) \int_{B_R} |\tau_{s,h} u|^2 dx \right)^{1/2}, \tag{3.13}$$

for some positive constant C_5 independent of h. Let us treat the last integral in (3.13); recalling that $D_s u \in L^2$ for $s = 1, \ldots, n-1$, we may use Lemma 2.1 in order to get

$$\int_{B_R} |\tau_{s,h} u|^2 dx \leq |h|^2 \int_{B_{2R}} |D_s u|^2 dx\,, \qquad \forall s = 1, \ldots, n-1\,. \tag{3.14}$$

Since $D_n u \in L^p$ and $p < 2$, the last integral in (3.13), corresponding to $s = n$, is dealt with as follows. We write

$$\int_{B_R} |\tau_{n,h} u|^2 dx = \int_{B_R} |\tau_{n,h} u|^a |\tau_{n,h} u|^{2-a} dx\,, \tag{3.15}$$

where $0 < a < 2$ is to be choosen later. Let us assume that, for some $\sigma \in [p, 2)$

$$D_n u \in L^{\sigma}_{\mathrm{loc}}(\Omega)\,. \tag{3.16}$$

Now we use Hölder's inequality in (3.15) with exponents σ/a and $\sigma/(\sigma - a)$, provided $a < \sigma$:

$$\int_{B_R} |\tau_{n,h} u|^2 dx \leq \left(\int_{B_R} |\tau_{n,h} u|^{\sigma} dx \right)^{a/\sigma} \left(\int_{B_R} |\tau_{n,h} u|^{(2-a)\sigma/(\sigma-a)} dx \right)^{(\sigma-a)/\sigma}. \tag{3.17}$$

Because of (3.16), we may apply Lemma 2.1 in order to get

$$\left(\int_{B_R} |\tau_{n,h} u|^{\sigma} dx \right)^{a/\sigma} \leq |h|^a \left(\int_{B_{2R}} |D_n u|^{\sigma} dx \right)^{a/\sigma}. \tag{3.18}$$

If

$$(2-\sigma)\sigma/(\sigma - a) \leq \sigma^* = \sigma n/(n-\sigma)\,, \tag{3.19}$$

then, (3.16), Sobolev imbedding theorem and Lemma 2.4 allows us to write

$$\left(\int_{B_R} |\tau_{n,h} u|^{(2-a)\sigma/(\sigma-a)} dx\right)^{(\sigma-a)/\sigma} \leq C_6 \left(\int_{B_{2R}} |u|^{(2-a)\sigma/(\sigma-a)} dx\right)^{(\sigma-a)/\sigma}, \tag{3.20}$$

for some positive constant C_6 independent of h. Collecting the previous inequalities, we get

$$\int_{B_R} |\tau_{n,h} u|^2 dx \leq |h|^a \left(\int_{B_{2R}} |D_n u|^\sigma dx\right)^{a/\sigma} C_6 \left(\int_{B_{2R}} |u|^{(2-a)\sigma/(\sigma-a)} dx\right)^{(\sigma-a)/\sigma}. \tag{3.21}$$

Thus, noting that $a < 2$ and $|h| < R \leq 1$, (3.14) and (3.21) give us

$$\int_{B_R} |\tau_{s,h} u|^2 dx \leq C_7 |h|^a, \qquad \forall s = 1, \dots, n-1, n, \tag{3.22}$$

for some positive constant C_7 independent of h. Eventually, inserting (3.22) into (3.13), we get

$$(II) \leq C_8 |h|^{a/2} \tag{3.23}$$

for some positive constant C_8 independent of h. Let us recall the restrictions on a: $0 < a < \sigma$ and (3.19); when exploiting (3.19), we get

$$a \leq \frac{n+2}{\sigma}\left(\sigma - \frac{2n}{n+2}\right) = a_0 . \tag{3.24}$$

Since we assumed (1.8), we have $2 > \sigma \geq p > 2 - 2/n \geq 2n/(n+2)$, so $0 < a_0 < \sigma$ and we can take

$$a = a_0 = \frac{n+2}{\sigma}\left(\sigma - \frac{2n}{n+2}\right) \tag{3.25}$$

in the previous calculations. Let us put together (3.8), (3.2) and (3.23):

$$m \int_{B_\varrho} |\tau_{s,h} \hat{V}(Du)|^2 dx \leq C_8 |h|^{a/2} \tag{3.26}$$

for every $s = 1, \dots, n$, for every $h : |h| < R$; this inequality allows us to apply Lemma 2.3 and we get

$$\hat{V}(Du) \in L^r(B_{\varrho/4}), \qquad \forall r < \frac{2n}{n - a/2} . \tag{3.27}$$

Since $B_{\varrho/4}$ has only to verify $\varrho < R \leq 1$ and $\overline{B_{4R}} \subset \Omega$, we also have $L^r_{\text{loc}}(\Omega)$ in (3.27); moreover, because of (3.6), we arrive at

$$D_n u \in L^t_{\text{loc}}(\Omega), \qquad \forall t < \frac{pn}{n - a/2} = \hat{t}(\sigma). \tag{3.28}$$

Let us summarize as follows: we have proved that, if for some $\sigma \in [p, 2)$

$$D_n u \in L^{\sigma}_{\mathrm{loc}}(\Omega), \tag{3.16}$$

then

$$D_n u \in L^{t}_{\mathrm{loc}}(\Omega), \qquad \forall t < \hat{t}(\sigma) = \frac{2pn\sigma}{(n-2)\sigma + 2n}.$$

Now, we want to estimate how much we gave gained, that is, $\hat{t}(\sigma) - \sigma$:

$$\hat{t}(\sigma) - \sigma = \sigma \frac{2n(p-1) - (n-2)\sigma}{(n-2)\sigma + 2n} \geq p \frac{2n(p-1) - (n-2)2}{(n-2)2 + 2n} = p \frac{2n(p - (2 - 2/n))}{(n-2)2 + 2n} = \delta(n, p); \tag{3.29}$$

since we assumed (1.8), we have $\delta(n, p) > 0$. Eventually, we have proved that, under (1.8), there exists $\delta = \delta(n, p) > 0$ such that

$$D_n u \in L^{\sigma}_{\mathrm{loc}}(\Omega), \qquad p \leq \sigma < 2, \Rightarrow D_n u \in L^{\sigma + \delta/2}_{\mathrm{loc}}(\Omega). \tag{3.30}$$

Thus (3.30) allows us to start a bootstrap argument that, after a finite number of steps, gives us

$$D_n u \in L^{2}_{\mathrm{loc}}(\Omega); \tag{1.9}$$

this ends the proof. ∎

4. - Proof of Theorem 2.

We argue as in the proof of Theorem 1: starting from (3.1) we arrive at (3.13); now we have the result of Theorem 1:

$$D_n u \in L^{2}_{\mathrm{loc}}(\Omega), \tag{1.9}$$

so we get (3.14) also for $s = n$:

$$\int_{B_R} |\tau_{s,h} u|^2 dx \leq |h|^2 \int_{B_{2R}} |D_s u|^2 dx, \qquad \forall s = 1, \dots, n-1, n. \tag{4.1}$$

We put together (3.8), (3.2), (3.13) and (4.1): for some positive constant C_9, independent of h, we have

$$m \int_{B_\varrho} |\tau_{s,h} \hat{V}(Du)|^2 dx \leq C_9 |h|, \tag{4.2}$$

for every $s = 1, \dots, n$, for every $h: |h| < R$. Now, a straightforward application of Lemma 2.3 yields

$$\widehat{V}(Du) \in L^r(B_{\varrho/4}), \qquad \forall r < \frac{2n}{n-1}. \tag{4.3}$$

Since $B_{\varrho/4}$ has only to verify $\varrho < R \leqslant 1$ and $\overline{B_{4R}} \subset \Omega$, we also have $L^r_{\mathrm{loc}}(\Omega)$ in (4.3). Looking back to (3.6), we get

$$D_i u \in L^r_{\mathrm{loc}}(\Omega), \quad i = 1, \dots, n-1, \quad \forall r < \frac{2n}{n-1} \quad \text{and } D_n u \in L^t_{\mathrm{loc}}(\Omega), \quad \forall t < \frac{pn}{n-1}. \tag{4.4}$$

This ends the proof. ∎

PROOF OF COROLLARY 1. – We explicitly remark that (1.8) implies $2 < pn/(n-1) < 2n/(n-1)$ so that (4.4) tells us that $Du \in L^{2+\varepsilon}_{\mathrm{loc}}$ for some $\varepsilon > 0$ and, when $n = 2$, the Sobolev imbedding theorem ends the proof. ∎

5. – Proof of Theorem 3.

First of all, we have $p/(p-1) < 2n/(n-1)$ if and only if $2 - 2/(n+1) < p$, so that (4.4) and (1.12) yield

$$D_i u \in L^{p/(p-1)}_{\mathrm{loc}}(\Omega), \qquad \forall i = 1, \dots, n-1. \tag{5.0}$$

Now, we argue as in the proof of Theorem 1: starting from (3.1), we arrive at (3.8); in order to get differentiability for $D_n u$, that is (1.15), we have to estimate $\tau_{s,h} D_n u$. We use the left-hand side of (3.7), Hölder's inequality with $2/(2-p)$ and $2/p$ in order to get

$$\begin{aligned}
&\int_{B_R} |\tau_{s,h} D_n u(x)|^p \eta^p(x)\, dx \leqslant \\
&\leqslant C_2^{-p} \int_{B_R} (1 + |D_n u(x)|^2 + |D_n u(x + he_s)|^2)^{p(2-p)/4} |\tau_{s,h} V(D_n u(x))|^p \eta^p(x)\, dx \leqslant \\
&\leqslant C_2^{-p} \left(\int_{B_R} (1 + |D_n u(x)|^2 + |D_n u(x + he_s)|^2)^{p/2} dx \right)^{(2-p)/2} \times \\
&\qquad \times \left(\int_{B_R} |\tau_{s,h} V(D_n u(x))|^2 \eta^2(x)\, dx \right)^{p/2}.
\end{aligned} \tag{5.1}$$

Now, splitting the integral and changing variables yield

$$C_2^{-p}\left(\int_{B_R}(1+|D_n u(x)|^2+|D_n u(x+he_s)|^2)^{p/2}dx\right)^{(2-p)/2}\leqslant$$

$$\leqslant C_{10}\left(\int_{B_{2R}}(1+|D_n u(y)|^p)\,dy\right)^{(2-p)/2}=C_{11}\,,$$

for some positive constants C_{10}, C_{11} independent of h, so that

$$C_{11}^{-2/p}\left(\int_{B_R}|\tau_{s,h}D_n u|^p\eta^p\,dx\right)^{2/p}\leqslant\int_{B_R}|\tau_{s,h}V(D_n u)|^2\eta^2\,dx\,, \tag{5.2}$$

then, using (5.2), (3.8) and (3.2) we arrive at

$$\frac{mC_4}{2}C_{11}^{-2/p}\left(\int_{B_R}|\tau_{s,h}D_n u|^p\eta^p\,dx\right)^{2/p}+\frac{mC_4}{2}\int_{B_R}|\tau_{s,h}V(D_n u)|^2\eta^2\,dx+ \tag{5.3}$$

$$+mC_4\int_{B_R}\sum_{i=1}^{n-1}|\tau_{s,h}D_i u|^2\eta^2\,dx\leqslant(1+C_3^2)\,C_4(I)=(1+C_3^2)\,C_4(II)\,.$$

We recall that, from (3.2)

$$(II)=-\int\sum_{i=1}^{n}\sum_{\alpha=1}^{N}\tau_{s,h}\left(\frac{\partial F}{\partial\xi_i^\alpha}(Du)\right)2\eta D_i\eta\tau_{s,h}u^\alpha\,dx\,;$$

this integral is now handled in a different way: in the proof of Theorem 1 we estimate the difference $\tau_{s,h}((\partial F/\partial\xi_i^\alpha)(Du))$; now we shift the difference operator $\tau_{s,h}$ from $(\partial F/\partial\xi_i^\alpha)(Du)$ to $2\eta D_i\eta\tau_{s,h}u^\alpha$:

$$(II)=-\int\sum_{i=1}^{n}\sum_{\alpha=1}^{N}\tau_{s,h}\left(\frac{\partial F}{\partial\xi_i^\alpha}(Du)\right)2\eta D_i\eta\tau_{s,h}u^\alpha\,dx= \tag{5.4}$$

$$=-\int\sum_{i=1}^{n}\sum_{\alpha=1}^{N}\frac{\partial F}{\partial\xi_i^\alpha}(Du)\,\tau_{s,-h}(2\eta D_i\eta\tau_{s,h}u^\alpha)\,dx\,.$$

We use the growth conditions (1.3), (1.4) and Hölder's inequality with $p/(p-1)$, p in (5.4) in order to get

$$(1+C_3^2)\,C_4(II)\leqslant C_{12}\left(\int_{B_{2R}}\left(1+\sum_{i=1}^{n-1}|D_i u|^{p/(p-1)}+|D_n u|^p\right)dx\right)^{(p-1)/p}\cdot \tag{5.5}$$

$$\cdot\left(\int_{B_{2R}}|\tau_{s,-h}(2\eta D\eta\tau_{s,h}u)|^p\,dx\right)^{1/p},$$

for some positive constant C_{12} independent of h. Now we use the higher integrability result of Theorem 2 as stated in (5.0):

$$\left(\int_{B_{2R}} \left(1 + \sum_{i=1}^{n-1} |D_i u|^{p/(p-1)} + |D_n u|^p\right) dx\right)^{(p-1)/p} = C_{13} < \infty . \tag{5.6}$$

Let us apply Lemma 2.1:

$$\left(\int_{B_{2R}} |\tau_{s,-h}(2\eta D\eta \tau_{s,h} u)|^p dx\right)^{1/p} \leqslant |h| \left(\int_{B_{3R}} |D_s(2\eta D\eta \tau_{s,h} u)|^p dx\right)^{1/p} = |h| \left(\int_{B_R} |D_s(2\eta D\eta \tau_{s,h} u)|^p dx\right)^{1/p}, \tag{5.7}$$

since $\eta = 0$ outside B_R. Taking into account (5.3), (5.5), (5.6) and (5.7), we arrive at

$$\left(\int_{B_R} |\tau_{s,h} D_n u|^p \eta^p dx\right)^{2/p} + \int_{B_R} |\tau_{s,h} V(D_n u)|^2 \eta^2 dx + \int_{B_R} \sum_{i=1}^{n-1} |\tau_{s,h} D_i u|^2 \eta^2 dx \leqslant \leqslant C_{14} |h| \left(\int_{B_R} |D_s(2\eta D\eta \tau_{s,h} u)|^p dx\right)^{1/p} = (III), \tag{5.8}$$

for some positive constant C_{14}, independent of h. Now, using the inequality $2ab \leqslant \leqslant \varepsilon a^2 + b^2/\varepsilon$, that holds true for every $\varepsilon > 0$, we have

$$(III) \leqslant \frac{C_{14}^2 |h|^2}{\varepsilon} + \varepsilon \left(\int_{B_R} |D_s(2\eta D\eta \tau_{s,h} u)|^p dx\right)^{2/p}. \tag{5.9}$$

The integral in the previous inequality is dealt with as follows:

$$\left(\int_{B_R} |D_s(2\eta D\eta \tau_{s,h} u)|^p dx\right)^{1/p} \leqslant \left(\int_{B_R} |D_s(2\eta D\eta) \tau_{s,h} u|^p dx\right)^{1/p} + + \left(\int_{B_R} |2\eta D\eta \tau_{s,h} D_s u|^p dx\right)^{1/p} = (A) + (B). \tag{5.10}$$

Now we keep in mind the properties of the «cut off» function η and we use Lemma 2.1 in order to get

$$(A) \leqslant C_{15} \left(\int_{B_{2R}} |D_s u|^p dx\right)^{1/p} |h| = C_{16} |h|, \tag{5.11}$$

for some positive constants C_{15}, C_{16} independent of h. On the other hand, recalling the properties of η and using Hölder's inequality, we have

$$(5.12)\qquad (B) \leq C_{17}\left(\int_{B_R} |\tau_{s,h} D_s u|^p \eta^p\, dx\right)^{1/p} \leq C_{17}\left(\sum_{i=1}^{n}\int_{B_R} |\tau_{s,h} D_i u|^p \eta^p\, dx\right)^{1/p} \leq$$

$$\leq C_{18}\left(\sum_{i=1}^{n-1}\int_{B_R} |\tau_{s,h} D_i u|^p \eta^p\, dx\right)^{1/p} + C_{18}\left(\int_{B_R} |\tau_{s,h} D_n u|^p \eta^p\, dx\right)^{1/p} \leq$$

$$\leq C_{19}\left(\sum_{i=1}^{n-1}\int_{B_R} |\tau_{s,h} D_i u|^2 \eta^2\, dx\right)^{1/2} + C_{18}\left(\int_{B_R} |\tau_{s,h} D_n u|^p \eta^p\, dx\right)^{1/p},$$

for some positive constants C_{17}, C_{18}, C_{19}, independent of h. We insert (5.11) and (5.12) into (5.10), we use the resulting inequality in (5.9) and we keep in mind (5.8): we get

$$\left(\int_{B_R} |\tau_{s,h} D_n u|^p \eta^p\, dx\right)^{2/p} + \int_{B_R} |\tau_{s,h} V(D_n u)|^2 \eta^2\, dx + \int_{B_R} \sum_{i=1}^{n-1} |\tau_{s,h} D_i u|^2 \eta^2\, dx \leq$$

$$\leq \frac{C_{20}\,|h|^2}{\varepsilon} + \varepsilon C_{20}\left(|h|^2 + \int_{B_R} \sum_{i=1}^{n-1} |\tau_{s,h} D_i u|^2 \eta^2\, dx + \left(\int_{B_R} |\tau_{s,h} D_n u|^p \eta^p\, dx\right)^{2/p}\right),$$

for some positive constant C_{20}, independent of $h$and ε, so taking $\varepsilon = 1/(2C_{20})$, we finally get

$$(5.13)\qquad \int_{B_R} |\tau_{s,h} V(D_n u)|^2 \eta^2\, dx + \int_{B_R} \sum_{i=1}^{n-1} |\tau_{s,h} D_i u|^2 \eta^2\, dx \leq C_{21}\,|h|^2,$$

$$(5.14)\qquad \int_{B_R} |\tau_{s,h} D_n u|^p \eta^p\, dx \leq C_{21}^{p/2}\,|h|^p,$$

for some positive constant C_{21}, independent of h. Since $\eta = 1$ on $B_\varrho \subset B_R$, we can apply Lemma 2.2 and, after recalling (3.4) for the definition of $V(D_n u)$, we get (1.13), (1.14), (1.15), thus ending the proof. ∎

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