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Random time change and an integral representation for marked stopping times*

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Summary. Consider the set $\mathscr{C}$ of all possible distributions of triples (τ, κ, η), such that τ is a finite stopping time with associated mark κ in some fixed Polish space, while η is the compensator random measure of (τ, κ). We prove that $\mathscr{C}$ is convex, and that the extreme points of $\mathscr{C}$ are the distributions obtained when the underlying filtration is the one induced by (τ, κ). Moreover, every element of $\mathscr{C}$ has a corresponding unique integral representation. The proof is based on the peculiar fact that $\mathsf{E} V_{\tau,\kappa} = 0$ for every predictable process V which satisfies a certain moment condition. From this it also follows that $T_{\tau,\kappa}$ is $U(0, 1)$ whenever T is a predictable mapping into $[0, 1]$ such that the image of ζ, a suitably discounted version of η, is a.s. bounded by Lebesgue measure. Iterating this, one gets a time change reduction of any simple point process to Poisson, without the usual condition of quasi-leftcontinuity. The paper also contains a very general version of the Knight–Meyer multivariate time change theorem.

1. Introduction

This paper deals primarily with *marked stopping times* of the form (τ, κ), where τ is a random time in $(0, \infty)$ while κ is an associated random mark in some fixed Polish space K, and where the process $\xi_t(A) = 1\{\tau \leqq t, \kappa \in A\}$ is assumed to be adapted to the underlying filtration $\mathscr{F} = (\mathscr{F}_t)$ for every set A in the Borel σ-field $\mathscr{B}(K)$. We shall always assume that $\mathscr{F}$ satisfies the usual conditions of right-continuity and completeness. Note also that all stopping times in this paper are assumed to be a.s. finite, unless otherwise specified. By the *compensator* of (τ, κ) we shall mean the a.s. unique random measure η on $(0, \infty) \times K$, which is such that the process $\eta_t(A) = \eta((0, t] \times A)$ is predictable while the difference $\xi_t(A) - \eta_t(A)$ is a martingale for every $A \in \mathscr{B}(K)$. Marked stopping times and their compensated versions are

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obviously fundamental, as they form the basic building blocks in marked point processes and in purely discontinuous local martingales, respectively.

In the special case when $\mathscr{F}$ is *induced* by (τ, κ), i.e., when $\mathscr{F}$ is the smallest filtration that makes the process ξ_t adapted, one may compute the compensator explicitly through the formula

$$\eta_t(A) = \int_0^{(t \wedge \tau)+} \frac{\mu(ds \times A)}{\mu([s, \infty) \times K)}, \qquad A \in \mathscr{B}(K), \tag{1}$$

in terms of the distribution μ of (τ, κ) (cf. Jacod (1979), p. 86, or Elliott (1982), p. 203). Hence in this case, the distribution of the triple (τ, κ, η) is uniquely determined by μ, and we shall denote it by P_μ. We shall also refer to η in this case as the *natural compensator.*

In general there is no formula like (1), and the interplay between the filtration $\mathscr{F}$ and the random pair (τ, κ) may often be quite subtle. Nevertheless, it will be shown in Theorem 6.1 below that the distribution of an arbitrary triple (τ, κ, η) has a unique integral representation as a mixture of measures -P_μ, in the sense that

$$\mathsf{P}(\tau, \kappa, \eta)^{-1} = \int P_\mu \nu(d\mu), \tag{2}$$

for some probability distribution ν over the space of all possible measures μ. It will further be seen that every such measure ν may occur in (2), for a suitable choice of filtered probability space $(\Omega, \mathscr{F}, \mathsf{P})$ and of pair (τ, κ). This means in particular that the class $\mathscr{C}$ of all possible distributions of triples (τ, κ, η) (defined on arbitrary filtered probability spaces) is convex, and that the extreme points of $\mathscr{C}$ are exactly the distributions P_μ.

We pause to remark that our notion of extremal measure differs from the usual one in the martingale literature, where one considers a fixed adapted process X (or more generally a set of such processes) on some filtered space $(\Omega, \mathscr{F})$, and studies extremality within the convex set of all probability measures on Ω, such that (each) X becomes a martingale. Though the two notions are obviously closely related, the results one obtains in the context of fixed processes are rather different (cf. Jacod and Yor 1977; Lépingle et al. 1981).

In the natural case, the distribution μ of (τ, κ) can be recovered up to time τ from the compensator η, via formula (1). The same construction applies in the general case, but yields instead of μ a *random* subprobability measure ζ supported by the set $(0, \tau] \times K$, which will play a basic role in the sequel. We shall refer to ζ as the *discounted compensator* of (τ, κ). To get an explicit expression for ζ, write $\bar{\eta}_t = \eta_t(K)$ and $\bar{\zeta}_t = \zeta_t(K) = \zeta((0, t] \times K)$, and note that the process $Z_t = 1 - \bar{\zeta}_t$ must satisfy the Doléans differential equation

$$dZ_t = -Z_{t-} d\bar{\eta}_t, \qquad Z_0 = 1, \tag{3}$$

whose unique solution is given by

$$Z_t = \exp(-\bar{\eta}_t^c) \prod_{s \leq t} (1 - \Delta\bar{\eta}_s), \qquad t \geq 0, \tag{4}$$

(cf. Brémaud (1981), p. 338; or Rogers and Williams (1987), p. 29). Here $\Delta\bar{\eta}_s = \bar{\eta}_s - \bar{\eta}_{s-}$, while $\bar{\eta}^c$ denotes the continuous component of $\bar{\eta}$. The measure ζ may now be obtained in terms of η and Z as

$$\zeta(A) = \iint_A Z_{t-} \eta(dt\,dx), \qquad A \in \mathscr{B}(\mathbb{R}_+ \times K). \tag{5}$$

We may now give a probabilistic interpretation of (2). As we shall see in Theorem 6.2, our discounted compensator ζ can always be extended to a random probability measure $\hat{\zeta}$ on $(0, \infty) \times K$, such that

$$\zeta = \pi_\tau \hat{\zeta} \quad \text{and} \quad \mathsf{P}[(\tau, \kappa) \in \cdot \mid \hat{\zeta}] = \hat{\zeta} \quad \text{a.s.}, \tag{6}$$

where π_τ denotes restriction to $(0, \tau] \times K$. The distribution of $\hat{\zeta}$ is then unique and agrees with the mixing measure ν in (2). It is suggestive to think of (6) (hence also of (2)) in terms of first choosing a probability measure $\hat{\zeta}$ at random with distribution ν, and then picking a random pair (τ, κ) in accordance with the distribution $\hat{\zeta}$. To get the desired compensator η for (τ, κ), we may finally take $\mathscr{F}$ to be the smallest filtration that makes the process $(\hat{\zeta}, \xi_t)$ adapted.

The discounted compensator ζ of (τ, κ) plays an important role even in the context of random time change. Thus if T is a predictable mapping from $\mathbb{R}_+ \times K$ into $[0, 1]$, such that the image random measure ζT^{-1} is a.s. bounded by Lebesgue measure λ, then $T_{\tau,\kappa}$ turns out (see Theorem 4.3) to be $U(0, 1)$ (uniformly distributed on $[0, 1]$). This might be surprising, since the total mass of ζ is typically strictly less than one. In the special case when τ is $U(0, 1)$ with natural compensator η, the result is essentially contained in Section 5 of Kallenberg (1988). The general statement is somewhat deeper, mainly because τ is not assumed to be totally inaccessible.

By iterating the mentioned type of transformations for individual stopping times, one may reduce an arbitrary simple point process ξ to homogeneous Poisson with respect to a suitably transformed filtration, through a random time change which only depends on the compensator η of ξ, and on certain randomizations needed to resolve the possible discontinuities of η at the jumps of ξ. The resulting Theorem 5.1 extends the classical result of Papangelou (1972) and Meyer (1971) in the quasi-leftcontinuous case, where η itself defines the new time scale. In general the appropriate time change is different, which explains the previously noted deviation from Poisson when η has jumps (cf. Brown and Nair 1988b).

All results mentioned so far are based on a peculiar but extremely useful moment identity in Theorem 4.1, which states that $\mathsf{E} V_{\tau,\kappa} = 0$ for every predictable process V on $\mathbb{R}_+ \times K$ that satisfies a certain moment condition. More precisely, we define

$$U_{t,x} = V_{t,x} + \frac{1}{Z_t} \int_0^{t+} \int V \, d\zeta, \qquad t \geqq 0, \quad x \in K, \tag{7}$$

where V is assumed to be such that the right-hand side makes sense, and we prove that

$$\mathsf{E}|U_{\tau,\kappa}| < \infty \quad \text{implies} \quad \mathsf{E} V_{\tau,\kappa} = 0. \tag{8}$$

Various primitive versions of this rather incredible fact have been noted earlier in the context of exchangeable processes (cf. Theorem 4.1 in Kallenberg (1989)). Note incidentally that a martingale component may be added to the process V in (8), to yield a similar statement for suitable semimartingales.

In addition to the mentioned results which are all closely related, we include in Sect. 2 a general multivariate time change theorem, even though the result uses different methods for its proof. More specifically, we consider predictable transformations which reduce a family of continuous local martingales and quasi-leftcontinuous point processes to a pair of a centered Gaussian process X and an independent Poisson process η, both defined on abstract spaces. The result contains Knight's classical (1970, 1971) reduction of orthogonal continuous martingales to independent Brownian motions (where the previous one-dimensional version is due to Dambis (1965) and Dubins and Schwarz (1965)), as well as the corresponding point process reduction to Poisson, due to Meyer (1971). However, our present version is more general, since the continuous martingales are not assumed to be orthogonal, nor are the predictable transformations assumed to be monotone. The result is rather closely related to certain invariance theorems in Kallenberg (1988, 1989).

Our proof uses a simplified version of the method in Cocozza and Yor (1980) based on exponential martingales, which in turn is much easier than the usual textbook proof (cf. Ikeda and Watanabe (1981), p. 86; or Karatzas and Shreve (1988), p. 187). Other approaches are suggested by Kurtz (1980) and, in the point process case, by Aalen and Hoem (1978) and Brown and Nair (1988a). Pitman and Yor (1986) contains an approximation theorem related to Knight's result, while Grigelionis (1971) and Karoui and Lepeltier (1977) prove a representation of certain marked point processes in terms of a Poisson process. One might also mention the partially successful attempts by Merzbach and Nualart (1986) and others to transform a two-dimensional point process to Poisson by means of suitable 'stopping lines'.

The mentioned results will essentially appear in reversed order. Thus we begin in Sect. 2 with the general reduction theorem. Our moment identity (8) and the related predictable mappings of marked stopping times will appear in Sect. 4, and in Sect. 5 we consider monotone transformations of (sequences of) stopping times, along with their associated filtrations. Finally, Sect. 6 contains a proof of our integral representation (2) and of the related existence of a random distribution $\hat{\zeta}$. Some technical prerequisites for Sect. 4–6 have been relegated to a special Sect. 3, in order not to distract the reader's attention from the main ideas.

Throughout the paper, we shall often use standard terminology, notation and results from stochastic calculus without explicit references, and in most cases the reader may consult the texts by Dellacherie and Meyer (1975/80), Jacod (1979), or Elliott (1982) for details. All random objects are assumed to be defined on filtered probability spaces $(\Omega, \mathscr{F}, \mathsf{P})$ satisfying the usual conditions, and E will then denote integration with respect to P. To avoid the constant nuisance of extending the probability space (in applications of Lemma 3.1), we shall always assume Ω to be rich enough to support any randomization variables we may need. Some further conventions for this paper are to let $1\{\dots\}$ denote the indicator function of the set within brackets, to use I as the identity mapping on any space, and to write $a \lesssim b$ as synonymous to $a = 0(b)$. Finally note that, given any Polish space K, we use $\mathscr{B}(K)$ to denote the Borel σ-field in K, write $\mathscr{M}(K)$ for the class of locally finite measures on $\mathscr{B}(K)$, and let $\mathscr{M}_1(K)$ be the subclass of probability measures.

2. Reduction to Poisson and Gaussian processes

Our aim in this section is to state and prove the general reduction theorem for continuous local martingales and quasi-leftcontinuous marked point processes, mentioned in the introduction. Thus we fix a filtered probability space $(\Omega, \mathscr{F}, \mathsf{P})$, where the filtration $\mathscr{F} = (\mathscr{F}_t)$ is assumed to satisfy the usual conditions of right-continuity and completeness. We shall also fix a Polish space K endowed with its Borel σ-field $\mathscr{B}(K)$.

By a *K-marked point process* on $(0, \infty)$ we shall mean a locally finite, integer valued random measure ξ on $(0, \infty) \times K$ satisfying $\xi(\{t\} \times K) \leqq 1$ for each t, and such that the process $\xi_t(B) = \xi((0, t] \times B)$, $t \geqq 0$, is adapted to $\mathscr{F}$ for every bounded $B \in \mathscr{B}(K)$. Note that there exists an a.s. unique random measure $\hat{\xi}$ on $\mathbb{R}_+ \times K$, the so called *compensator* of ξ, such that the process $\hat{\xi}_t(B) = \hat{\xi}((0, t] \times B)$, $t \geqq 0$, is a compensator of $\xi_t(B)$ in the usual sense for every bounded $B \in \mathscr{B}(K)$. Further recall that ξ is *quasi-leftcontinuous* iff $\hat{\xi}_t(B)$ is a.s. continuous for every bounded $B \in \mathscr{B}(K)$, i.e. iff $\hat{\xi}(\{t\} \times K) = 0$ for all t a.s. For the definition of Poisson random measures on abstract measurable spaces, we may refer to Kallenberg (1986), p. 15.

Theorem 2.1. *Let $M_1, \ldots, M_d$ be continuous local martingales, and let ξ be a quasi-leftcontinuous K-marked point process on $(0, \infty)$ with compensator $\hat{\xi}$. Fix a σ-finite measure space $(S, \mathscr{S}, \mu)$, along with an abstract space T equipped with a non-negative definite function $\rho: T^2 \to \mathbb{R}$. Add a point ∂ to S, and consider a family of predictable processes $U_{jt}: \Omega \times \mathbb{R}_+ \to \mathbb{R}$, $j = 1, \ldots, d$, $t \in T$, and $V: \Omega \times \mathbb{R}_+ \times K \to S \cup \{\partial\}$, satisfying*

$$\int_0^\infty U_{jt}^2 \mathrm{d}[M_j, M_j] < \infty \quad \textit{a.s.}, \qquad j = 1, \ldots, d, \quad t \in T,$$

$$\sum_{j=1}^d \sum_{k=1}^d \int_0^\infty U_{js} U_{kt} \mathrm{d}[M_j, M_k] = \rho_{st} \quad \textit{a.s.}, \qquad s, t \in T,$$

$$\hat{\xi} V^{-1} = \mu \quad \textit{a.s. on } S .$$

Then $\eta = \xi V^{-1}$ is a Poisson random measure on S with intensity μ, while

$$X_t := \sum_{j=1}^d \int_0^\infty U_{jt} \mathrm{d}M_j , \qquad t \in T,$$

is an independent centered Gaussian process on T with covariance function ρ.

Proof. First we conclude from Itô's formula that, if M is a continuous local martingale while J is a quasi-leftcontinuous simple point process with compensator $\hat{J}$, and if $u \geqq 0$ is a constant, then the processes

$$Z_t = \exp(\mathrm{i}M_t + \tfrac{1}{2}[M, M]_t), \qquad t \geqq 0,$$

$$Y_t = \exp(-uJ_t + (1 - \mathrm{e}^{-u})\hat{J}_t), \qquad t \geqq 0,$$

are local martingales. Applying this to the processes

$$M(t) = \sum_{k=1}^m c_k \sum_{j=1}^d \int_0^t U_{j, t_k} \mathrm{d}M_j, \qquad t \geqq 0,$$

$$J_j(t) = \int_K \int_0^{t+} 1\{V \in A_j\} \mathrm{d}\xi, \qquad t \geqq 0, \qquad j = 1, \ldots, n,$$

where $c_1, \ldots, c_m \in \mathbb{R}$ and $u_1, \ldots, u_n \in \mathbb{R}_+$ are constants while $A_1, \ldots, A_n \in \mathcal{S}$ are arbitrary disjoint sets with finite μ-measure, and noting that the resulting processes Z and $Y_1, \ldots, Y_n$ are bounded and orthogonal, it follows that the process

$$N_t = \exp\{iM_t + \tfrac{1}{2}[M, M]_t - \sum_{j=1}^{n} (u_j J_j(t) - (1 - e^{-u_j})\hat{J}_j(t)\}, \qquad t \geqq 0,$$

is a bounded martingale. Hence $\mathsf{E}N_\infty = \mathsf{E}N_0 = 1$. Since

$$M_\infty = \sum_{k=1}^{m} c_k X_{t_k}; \qquad J_j(\infty) = \eta(A_j), \qquad j = 1, \ldots, n,$$

$$[M, M]_\infty = \sum_h \sum_k c_h c_k \sum_i \sum_j \int U_{i,t_h} U_{j,t_k} \mathrm{d}[M_i, M_j] = \sum_h \sum_k c_h c_k \rho_{t_h, t_k},$$

$$\hat{J}_j(\infty) = \iint 1\{V \in A_j\} \mathrm{d}\hat{\xi} = \mu(A_j), \qquad j = 1, \ldots, n,$$

we obtain

$$\mathsf{E}\exp\left\{i \sum_{k=1}^{m} c_k X_{t_k} - \sum_{j=1}^{n} u_j \eta(A_j)\right\} = \exp\left\{-\frac{1}{2}\sum_h \sum_k c_h c_k \rho_{t_h, t_k} - \sum_{j=1}^{n} (1 - e^{-u_j})\mu(A_j)\right\},$$

and the result follows. □

We remark that an even simpler proof is available when the space S is sufficiently nice (Polish will do) while the measure μ is diffuse, since we may then replace the processes $Y_1, \ldots, Y_n$ above by one single bounded martingale

$$Y_t = e^{\hat{J}_t} 1\{J_t = 0\}, \qquad t \geqq 0,$$

where

$$J_t = \int_K \int_0^{t+} 1\{V \in A\} \mathrm{d}\xi, \qquad t \geqq 0,$$

and conclude as before that

$$\mathsf{E}\left[\exp\left\{i \sum_k c_k X_{t_k}\right\};\ \eta(A) = 0\right] = \exp\left\{-\frac{1}{2}\sum_h \sum_k c_h c_k \rho_{t_h, t_k} - \mu(A)\right\}.$$

Since obviously $\mathsf{E}\eta = \mu$, the assertion now follows as in Theorem 3.3 of Kallenberg (1986).

We mention this because of recent efforts to get a simple proof of the multivariate time change theorem for orthogonal point processes (cf. Brown and Nair 1988a), and also because the same method gives a very simple proof of Theorem 4.3 below, in the special case when the stopping time τ is totally inaccessible.

We conclude this section by showing how the classical time change theorems of Knight and Meyer can be deduced from Theorem 2.1. First we take $M_1, \ldots, M_d$ to be mutually orthogonal continuous local martingales starting at 0, and such that $[M_j, M_j]_\infty = \infty$ a.s. for each j. Here we choose $T = \{1, \ldots, d\} \times \mathbb{R}_+$, and define

$$U_{ijt}(s) = \delta_{ij} 1\{[M_i, M_i]_s \leqq t\}, \qquad i, j = 1, \ldots, d, \quad s, t \geqq 0,$$

$$\tau_j(t) = \inf\{s \geqq 0; [M_j, M_j]_s > t\}, \qquad j = 1, \ldots, d, \quad t \geqq 0.$$

Then

$$X_j(t) := \sum_{i=1}^{d} \int_0^{\infty} U_{ijt} \mathrm{d}M_i = \int_0^{\infty} 1\{[M_j, M_j]_s \leqq t\} \mathrm{d}M_j(s) = M_j \circ \tau_j(t) ,$$

while

$$\sum\sum_{i,j=1}^{d} \int_0^{\infty} U_{ihs} U_{jkt} \mathrm{d}[M_i, M_j] = \delta_{hk} \int \{[M_k, M_k]_r \leqq s \wedge t\} \mathrm{d}[M_k, M_k]_r = \delta_{hk}(s \wedge t) ,$$

so Theorem 2.1 shows that the time changed processes $X_j = M_j \circ \tau_j$ have the same finite-dimensional distributions as d independent Brownian motions. Moreover, the X_j inherit the property of right-continuity from the τ_j, so they must in fact be a.s. continuous. Thus $X_1, \ldots, X_d$ are independent Brownian motions, as noted by Knight (1970, 1971).

Next we assume that $\xi_1, \ldots, \xi_d$ are mutually orthogonal quasi-leftcontinuous simple point processes with compensators $\hat{\xi}_1, \ldots, \hat{\xi}_d$, and such that $\hat{\xi}_j(\infty) = \hat{\xi}_j(0, \infty) = \infty$ a.s. for each j. In this context, orthogonality means that $\xi_i(\{t\})\xi_j(\{t\}) = 0$ for all $t > 0$ and $i \neq j$. To make this situation fit into our general framework, we take $K = \{1, \ldots, d\}$, and put $\xi = (\xi_1, \ldots, \xi_d)$ and $\hat{\xi} = (\hat{\xi}_1, \ldots, \hat{\xi}_d)$. Now choose $S = \{1, \ldots, d\} \times \mathbb{R}_+$ and define

$$V_j(t) = (j, \hat{\xi}_j(t)), \qquad j = 1, \ldots, d, \quad t \geqq 0 ,$$

$$\tau_j(t) = \inf\{s \geqq 0; \hat{\xi}_j(s) > t\}, \qquad j = 1, \ldots, d, \quad t \geqq 0 ,$$

Then

$$\eta_j(t) := \xi_j\{s \geqq 0; \hat{\xi}_j(s) \leqq t\} = \xi_j \circ \tau_j(t), \qquad j = 1, \ldots, d, \quad t \geqq 0 ,$$

while

$$\hat{\xi}_j\{s \geqq 0; \hat{\xi}_j(s) \leqq t\} = t, \qquad j = 1, \ldots, d, \quad t \geqq 0 ,$$

so Theorem 2.1 shows that $\eta_1, \ldots, \eta_d$ are independent Poisson processes on $\mathbb{R}_+$ with intensity λ. This is the result of Meyer (1971). Our theorem even allows us to conclude that the two sets of processes $X_1, \ldots, X_d$ and $\eta_1, \ldots, \eta_d$ are independent, whenever the martingales $M_1, \ldots, M_d$ and the point processes $\xi_1, \ldots, \xi_d$ are defined on the same filtered probability space.

Applications of this kind are all very simple from our present general point of view. With somewhat greater effort, we could also deduce from Theorem 2.1 some non-trivial representation theorems for stochastic integrals with respect to stable Lévy processes, but that would bring us too far afield . . .

3. Some preliminaries

In this section we have gathered some auxiliary results which will be needed in subsequent sections. The idea is that the reader may skip this section on a first reading, and return to the specific results when need arises. Only Lemmas 3.2 and 3.3 may be of some independent interest, while the others are more technical.

Our first result is the simple coupling lemma from Kallenberg (1988), which we restate here for the reader's convenience.

Lemma 3.1. *Let ξ and η be random elements in some Polish spaces S and T, such that $\xi \stackrel{d}{=} f(\eta)$ for some measurable mapping $f\colon T \to S$. Further assume that γ is $U(0, 1)$ and independent of ξ. Then there exists some measurable function η' of ξ and γ, such that $\eta' \stackrel{d}{=} \eta$ and $\xi = f(\eta')$ a.s.*

Note that, by enlarging the probability space if necessary, we may always assume the existence of a randomization variable γ with the stated properties. No further comments will be made on this point.

For the next few results, recall that the notions of stopping time, martingale, etc., are with respect to a fixed right-continuous and complete filtration $\mathscr{F} = (\mathscr{F}_t)$. Note also that K denotes a fixed Polish space endowed with the Borel σ-field $\mathscr{B}(K)$. We shall say that a marked stopping time (τ, κ) is *pure*, if its $\mathscr{F}$-compensator remains predictable with respect to the induced filtration and therefore agrees with the *natural* compensator in (1.1). The next lemma plays a crucial role in the construction of the random probability measure $\hat{\zeta}$ of (1.6).

Lemma 3.2. *Let (τ, κ) be a pure marked stopping time in $(0, 1) \times K$ with* ess sup $\tau = 1$, *and let U be an adapted left-continuous K-valued process on $[0, 1)$. Then there exists some left-continuous K-valued process V on $[0, 1)$, such that V is independent of (τ, κ) and satisfies $V = U$ a.s. on $[0, \tau]$.*

Proof. Define $(\pi_t f)_s = f(s \wedge t)$ for any function f on $[0, 1)$ and constants $s, t \in [0, 1)$. Letting $t \in [0, 1)$ and $A \in \mathscr{B}((t, 1] \times K)$, we get

$$\mathsf{P}[(\tau, \kappa) \in A | \mathscr{F}_t] = \mathsf{E}[\eta A | \mathscr{F}_t] = \mathsf{E}[\eta A | \tau > t]\, 1\{\tau > t\} \quad \text{a.s.,}$$

so

$$\begin{aligned} \mathsf{P}\{(\tau, \kappa) \in A, \pi_t U \in \cdot\} &= \mathsf{E}[\mathsf{P}[(\tau, \kappa) \in A | \mathscr{F}_t];\ \ \pi_t U \in \cdot] \\ &= \mathsf{P}[(\tau, \kappa) \in A | \tau > t]\, \mathsf{P}\{\tau > t, \pi_t U \in \cdot\} \\ &= \mathsf{P}\{(\tau, \kappa) \in A\}\, \mathsf{P}[\pi_t U \in \cdot | \tau > t]\,. \end{aligned} \tag{1}$$

Taking $A = (s, 1) \times K$ with $s \in (t, 1)$, we get in particular

$$\mathsf{P}[\pi_t U \in \cdot | \tau > s] = \mathsf{P}[\pi_t U \in \cdot | \tau > t], \qquad 0 \leqq t < s < 1\,,$$

so by the Daniell–Kolmogorov theorem, there exists some left-continuous process V on $[0, 1)$, such that

$$\mathsf{P}\{\pi_t V \in \cdot\} = \mathsf{P}[\pi_t U \in \cdot | \tau > t], \qquad t \in [0, 1)\,. \tag{2}$$

Taking V to be independent of (τ, κ), we get from (1) and (2)

$$\mathsf{P}\{(\tau, \kappa) \in A, \pi_t U \in \cdot\} = \mathsf{P}\{(\tau, \kappa) \in A\}\, \mathsf{P}\{\pi_t V \in \cdot\} = \mathsf{P}\{(\tau, \kappa) \in A, \pi_t V \in \cdot\}\,. \tag{3}$$

Let us now write

$$\tau_n = \max\{k2^{-n} < \tau, k \in \mathbb{Z}_+\}, \qquad n \in \mathbb{N}\,.$$

Letting $A \in \mathscr{B}((0, 1) \times K)$, we get from (3) with $t = k2^{-n}$,

$$\mathsf{P}\{(\tau_n, \kappa) \in A, \tau_n = k2^{-n}, \pi_{\tau_n} U \in \cdot\} = \mathsf{P}\{(\tau_n, \kappa) \in A, \tau_n = k2^{-n}, \pi_{\tau_n} V \in \cdot\}\,.$$

Summing over k yields

$$(\tau_n, \kappa, \pi_{\tau_n} U) \stackrel{d}{=} (\tau_n, \kappa, \pi_{\tau_n} V) ,$$

and since U and V are left-continuous, it follows that

$$(\tau, \kappa, \pi_\tau U) \stackrel{d}{=} (\tau, \kappa, \pi_\tau V) .$$

By Lemma 3.1, there will then exist some random elements τ', κ' and V', such that

$$(\tau', \kappa', V') \stackrel{d}{=} (\tau, \kappa, V); \qquad (\tau, \kappa, \pi_\tau U) = (\tau', \kappa', \pi_{\tau'} V') \quad \text{a.s.}$$

It follows in particular that $(\tau', \kappa') = (\tau, \kappa)$ a.s., so even V' is independent of (τ, κ), and we have $\pi_\tau U = \pi_\tau V'$ a.s. □

The following uniqueness assertion is part of Theorem 6.1 below and will be needed for its proof. Note that the corresponding statement for general point processes is false.

Lemma 3.3. *For marked stopping times (τ, κ) in $(0, \infty) \times K$ with compensators η, the distribution of (τ, κ, η) is uniquely determined by that of η.*

Proof. The stochastic intervals $\{t \geqq 0; \eta_t K \geqq s\}$, $s \geqq 0$, are right-closed and predictable, so their left endpoints

$$\tau_s = \inf\{t \geqq 0; \eta_t K \geqq s\}, \qquad s \geqq 0,$$

are predictable stopping times. Since for each s the random measure $\pi_{\tau_s}\eta$ is $\mathscr{F}_{\tau_s -}$-measurable, the restriction of τ_s to sets of the form $\{\pi_{\tau_s}\eta \in B\}$ with $B \in \mathscr{B}(\mathscr{M}((0, \infty) \times K))$ are again predictable stopping times. By the compensating property of η, we thus obtain for any $A \in \mathscr{B}((0, \infty) \times K)$,

$$\mathsf{P}\{(\tau, \kappa) \in A, \tau \geqq \tau_s, \pi_{\tau_s}\eta \in B\} = \mathsf{E}[\eta\{(t, x) \in A; t \geqq \tau_s\}; \pi_{\tau_s}\eta \in B] . \tag{4}$$

Since the τ_s are measurable functions of η, it follows that the probabilities on the left of (4) are uniquely determined by the distribution $\mathsf{P}\eta^{-1}$. Now the class of events occurring on the left of (4) is closed under finite intersections, so by a monotone class argument, every set in the generated σ-field has a unique probability. Thus $\mathsf{P}\eta^{-1}$ determines the joint distribution of all random measures

$$1\{(\tau, \kappa) \in A, \tau \geqq \tau_s\}\pi_{\tau_s}\eta, \qquad A \in \mathscr{B}(\mathbb{R}_+ \times K),\ s \geqq 0 . \tag{5}$$

Now

$$1\{\tau \geqq \tau_s\}\pi_{\tau_s}\eta \uparrow \eta \quad \text{as} \quad s \uparrow \eta_\tau K ,$$

so by taking the supremum in (5) over rational s, we may conclude that even the random measures $1_A(\tau, \kappa)\eta$ have uniquely determined distributions. The asserted uniqueness of $\mathsf{P}(\tau, \kappa, \eta)^{-1}$ now follows by another monotone class argument. □

The idea of attaching additional marks to a marked stopping time (τ, κ) by means of predictable mappings will be useful in Section 6. The next lemma shows how this affects the compensator.

Lemma 3.4. *Let (τ, κ) be a marked stopping time in $(0, \infty) \times K$ with compensator η, let V be a predictable process on $\mathbb{R}_+ \times K$, and let W denote the random mapping $(t, x) \to (t, x, V_{t,x})$, $t \geqq 0, x \in K$. Then the triple $(\tau, \kappa, V_{\tau,\kappa})$ is a marked stopping time in $\mathbb{R}_+ \times K \times \mathbb{R}$ with compensator ηW^{-1}.*

Proof. The triple $(\tau, \kappa, V_{\tau,\kappa})$ is a marked stopping time, since $V_{\tau,\kappa}$ is $\mathscr{F}_\tau$-measurable. To see that the associated compensator is given by ηW^{-1}, we fix a set $B \in \mathscr{B}(K \times \mathbb{R})$, and note that

$$1\{(\tau, \kappa, V_{\tau,\kappa}) \in [0, t] \times B\} = \int_0^{t+} \int_K 1_B(x, V_{s,x}) \delta_{\tau,\kappa}(\mathrm{d}s\,\mathrm{d}x), \qquad t \geqq 0 . \tag{6}$$

Here the integrand on the right is a predictable process in the pair (s, x), so the process in (1) is compensated by the process

$$\int_0^{t+} \int_K 1_B(x, V_{s,x}) \eta(\mathrm{d}s\,\mathrm{d}x) = \eta W^{-1}([0, t] \times B), \qquad t \geqq 0 ,$$

and the assertion follows. □

The next result is a simple identity for conditional probabilities which will be used repeatedly in subsequent sections.

Lemma 3.5. *Fix a probability space $(\Omega, \mathscr{A}, \mathsf{P})$, let $\mathscr{G}$ and $\mathscr{H}$ be sub-σ-fields of $\mathscr{A}$, and consider an atom G of $\mathscr{G}$ and a set $A \in \mathscr{A}$ with $A \subset G$. Then*

$$\mathsf{P}[A \mid \mathscr{G} \vee \mathscr{H}] = \frac{\mathsf{P}[A \mid \mathscr{H}]}{\mathsf{P}[G \mid \mathscr{H}]} 1_G 1\{\mathsf{P}[G \mid \mathscr{H}] > 0\} \quad a.s. , \tag{7}$$

(*with the convention* $0/0 = 0$).

Proof. Since

$$\mathsf{E}[\mathsf{P}[A \mid \mathscr{G} \vee \mathscr{H}]; G^c] = \mathsf{P}(A \backslash G) = 0 ,$$

both sides of (7) vanish a.s. on G^c. Next we write $H_0 = \{\mathsf{P}[G \mid \mathscr{H}] = 0\}$, and note that

$$\mathsf{E}[\mathsf{P}[A \mid \mathscr{G} \vee \mathscr{H}]; H_0] = \mathsf{P}(A \cap H_0) \leqq \mathsf{P}(G \cap H_0) = \mathsf{E}[\mathsf{P}[G \mid \mathscr{H}]; H_0] = 0 .$$

Thus $\mathsf{P}[A \mid \mathscr{G} \vee \mathscr{H}] = 0$ a.s. on H_0. It remains to prove (7) on $G \backslash H_0$. Now any $\mathscr{G} \vee \mathscr{H}$-measurable subset of $G \backslash H_0$ has the form $G \cap H$, where $H \in \mathscr{H}$ with $H \subset H_0^c$. It is hence enough to show that both sides of (7) have the same integral over $G \cap H$, which is seen from

$$\mathsf{E}\left[\frac{\mathsf{P}[A \mid \mathscr{H}]}{\mathsf{P}[G \mid \mathscr{H}]}; G \cap H\right] = \mathsf{E}\left[\frac{\mathsf{P}[A \mid \mathscr{H}]}{\mathsf{P}[G \mid \mathscr{H}]} \mathsf{P}[G \mid \mathscr{H}]; H\right] = \mathsf{E}[\mathsf{P}[A \mid \mathscr{H}]; H]$$

$$= \mathsf{P}(A \cap H) = \mathsf{P}(A \cap G \cap H) = \mathsf{E}[\mathsf{P}[A \mid \mathscr{G} \vee \mathscr{H}]; G \cap H].$$

□

Next we state without proofs a couple of simple technical results needed for the proof of Theorem 4.6, for which no reference could be found in the literature. Both statements may be proved by straightforward monotone class arguments.

Lemma 3.6. *Let V be a predictable process on $\mathbb{R}_+ \times K$, and let σ be a finite predictable stopping time. Then the process $V(\sigma, \cdot)$ on K is $\mathscr{F}_{\sigma-} \times \mathscr{B}(K)$-measurable.*

Lemma 3.7. *Fix a probability space with a sub-σ-field $\mathscr{G}$, let ξ be a locally finite random measure on K, and let X be an $\mathbb{R}_+$-valued and $\mathscr{G} \times \mathscr{B}(K)$-measurable process on K. Then*

$$\mathsf{E}[\textstyle\int X \mathrm{d}\xi \mid \mathscr{G}] = \int X \mathrm{d}\mathsf{E}[\xi \mid \mathscr{G}] \quad a.s. ,$$

where $\mathsf{E}[\xi|\mathscr{G}]$ *denotes the a.s. unique measure valued version of the process* $\mathsf{E}[\xi A|\mathscr{G}]$, $A \in \mathscr{B}(K)$.

We conclude this section with two technical lemmas, where certain sets considered in the proof of Theorem 6.2 below are shown to be measurable. As before, K is an arbitrary Polish space endowed with its Borel σ-field, and $\mathscr{M}_1(K)$ denotes the class of all probability measures on K. For the first result, write $S(a, b) = (a, b) \times [0, 1] \times \mathbb{R}_+ \times K$.

Lemma 3.8. *For each* $s \in (0, 1]$, *let* B_s *denote the class of all measures* μ *on* $S(0, 1)$, *such that whenever* $a, b, y \in [0, 1]$, $t \geqq 0$, *and* $\nu \in \mathscr{M}_1(K)$ *are such that* $a < b \leqq s$, *and* $\mu = \lambda \times \delta_y \times \delta_t \times \nu$ *on* $S(a, b)$, *then* μ *has the same form on* $S(a, (b \vee y) \wedge s)$. *Those sets* B_s *are measurable.*

Proof. Fix $s \in (0, 1]$, and note that $B_s = \bigcap B_{a,b}$, where $B_{a,b}$ is the set of all measures with the stated property for fixed a and b. Here the intersection extends over all a and b with $0 \leqq a < b \leqq s$, but the formula remains true if we restrict the intersection to rational a and b. This is because every non-empty interval (a, b) contains a similar interval with rational endpoints. It is thus enough to prove that $B_{a,b}$ is measurable for fixed a and b.

Let us then introduce the classes A_r of measures μ with the stated product form on $S_{a,r}$, where $r \in (a, 1]$ is arbitrary, and write $A_r = A_r' \cap A_r'' \cap A_r'''$, where A_r'' and A_r''' are the sets of measures with degenerate projections from $S_{a,r}$ onto coordinate spaces 2 and 3, respectively, while A_r' is the set of measures with projection of the form $\lambda \times \nu$ onto $(a, r) \times K$. Then A_r'' and A_r''' are expressible in terms of suitable moments of order 0, 1 and 2, and are therefore measurable. To see that even A_r' is measurable, we may write it as the intersection of the sets $\{\nu_t = \nu_r\}$ over rational $t \in (a, r)$, where ν_t denotes the normalized projection on μ from $S_{a,t}$ onto K. This shows that each set A_r is measurable.

For any $\mu \in A_b$, the degenerate value of y in the product formula may be expressed in terms of moments of order 0 and 1, which shows that $y = y_\mu$ is a measurable function of μ on A_b. By putting $y_\mu = 0$ for $\mu \in A_b^c$, we may extend y_μ to a measurable function on the entire measure space. Writing $b_\mu = (y_\mu \vee b) \wedge s$, it is easy to verify that

$$B_{a,b}^c = A_b \cap \bigcup_{r>b} (\{r \leqq b_\mu\} \cap A_r^c) ,$$

and this clearly remains true if the union on the right is restricted to rational $r > b$. The desired measurability of $B_{a,b}$ now follows from the measurability of the sets A_r and the functions b_μ. □

Lemma 3.9. *For any bounded measure* μ *on* $[0, 1]^2 \times \mathbb{R}_+ \times K$, *define*

$$y_\mu(s) = \sup\{y \in [0, 1]; \mu([0, s] \times [y, 1] \times \mathbb{R}_+ \times K) > 0\}, \qquad s \in [0, 1] ,$$

$$t_\mu(s) = \sup\{t \in \mathbb{R}_+; \mu([0, s] \times [0, 1] \times [t, \infty) \times K) > 0\}, \qquad s \in [0, 1] .$$

For each $s \in (0, 1]$, *let* C_s *be the class of measures* μ *as above, such that*

$$\mu((y_\mu(r+), 1] \times [0, 1] \times [0, t_\mu(r+)] \times K) = 0, \qquad r \in [0, s) , \tag{8}$$

Those sets C_s *are measurable.*

Proof. First we show for fixed s that $y_\mu(s)$ and $t_\mu(s)$ are measurable functions of μ. It is then enough to show for measures m on $\mathbb{R}$ that sup(supp m) is a measurable function of m, which is obvious from the relation

$$\{m \in \mathcal{M}(\mathbb{R}); \sup(\operatorname{supp} m) \leqq x\} = \{m \in \mathcal{M}(\mathbb{R}); m(x, \infty) = 0\}, \qquad x \in \mathbb{R} .$$

Approximating from the right by step functions, it follows that even $y_\mu(r+)$ and $t_\mu(r+)$ are measurable in μ for fixed $r \in (0, 1)$. Keeping r fixed, we next approximate $y_\mu(r+)$ and $t_\mu(r+)$ from above by functions y_μ^n and t_μ^n taking values in the set $\{k2^{-n}; k \in \mathbb{N}\}$. Then $y_\mu^n \downarrow y_\mu(r+)$ and $t_\mu^n \downarrow t_\mu(r+)$, so by dominated convergence,

$$\mu((y_\mu^n, 1] \times [0, 1] \times [0, t_\mu^n] \times K) \to \mu((y_\mu(r+), 1] \times [0, 1] \times [0, t_\mu(r+)] \times K) . \tag{9}$$

Since the y_μ^n and t_μ^n are again measurable and take only countably many values, the left-hand side of (9) is measurable in μ, so the same thing is true for the expression on the right. This proves the measurability in (8) for *fixed r*.

From this follows the measurability of the sets C_s', defined by (8) but with r restricted to the rational numbers. To complete the proof, it remains to show that $C_s' = C_s$. Let us then take $\mu \in C_s'$ and fix $r \in [0, s)$. Choose rational numbers $r_1, r_2, \ldots$ with $s > r_n \downarrow r$. Since $y_\mu(r+)$ and $t_\mu(r+)$ are non-decreasing and right-continuous, we get $y_\mu(r_n+) \downarrow y_\mu(r+)$ and $t_\mu(r_n+) \downarrow t_\mu(r+)$, so by dominated convergence,

$$0 = \mu((y_\mu(r_n+), 1] \times [0, 1] \times [0, t_\mu(r_n+)] \times K)$$

$$\to \mu((y_\mu(r+), 1] \times [0, 1] \times [0, t_\mu(r+)] \times K) .$$

which shows that the relation in (8) remains true for r. Hence $\mu \in C_s$. □

4. Moment identities and predictable transformations

Our plan in this section is first to prove in Theorem 4.1 the basic moment identity (1.8), from which the predictable reduction of a marked stopping time to $U(0, 1)$ will be deduced in Theorem 4.3. The latter result yields in particular some simple but extremely useful tail and moment estimates for ζ and η. The concluding Theorem 4.5 is a multivariate extension of Theorem 4.3.

All random objects in this section are defined on a fixed probability space equipped with a right-continuous and complete filtration $\mathcal{F} = (\mathcal{F}_t)$. Until further notice, we fix a Polish space K with associated Borel σ-field $\mathcal{B}(K)$, and consider a marked stopping time (τ, κ) in $(0, \infty) \times K$ with compensator η and discounted compensator ζ. Recall how ζ was constructed from η in (1.5) by means of the Doléans exponential process Z of (1.4), which in turn arose as the unique solution to the differential equation in (1.3). As before, we shall use $\bar{\eta}$ and $\bar{\zeta}$ to denote the projections of η and ζ onto $(0, \infty)$. Recall also how a measure valued process (η_t) was associated with the random measure η via the formula $\eta_t(A) = \eta((0, t] \times A)$, and similarly for $(\bar{\eta}_t)$, (ζ_t) and $(\bar{\zeta}_t)$.

Theorem 4.1. *Let V be a predictable and a.s. ζ-integrable process on $\mathbb{R}_+ \times K$, such that $\iint V d\zeta = 0$ a.s. on $\{Z_\tau = 0\}$, and let U be given by* (1.7) (*with the convention* $0/0 = 0$). *Then the process*

$$M_t = U_{\tau,\kappa} 1\{\tau \leqq t\} - \int_0^{t+} \int U d\eta, \qquad t \geqq 0, \tag{1}$$

exists and satisfies $M_\infty = V_{\tau,\kappa}$. If we further assume that $\mathsf{E}|U_{\tau,\kappa}| < \infty$, then M becomes a uniformly integrable martingale, and we get $\mathsf{E}V_{\tau,\kappa} = 0$. In that case also $\mathsf{E}M^{*p} \lesssim \mathsf{E}|V_{\tau,\kappa}|^p$ for any $p > 1$.

Here the main assertion is the fact that $\mathsf{E}|U_{\tau,\kappa}| < \infty$ implies $\mathsf{E}V_{\tau,\kappa} = 0$, but the remaining statements are useful to obtain multivariate extensions of this result. In fact, assume that $(\tau_1, \kappa_1), \ldots, (\tau_d, \kappa_d)$ are marked stopping times with compensators $\eta_1, \ldots, \eta_d$, such that the martingales $1\{\tau_j \leqq t\} - \bar{\eta}_j(t)$ are orthogonal, and let $V_1, \ldots, V_d$ be predictable processes on $\mathbb{R}_+ \times K$ with associated processes $U_1, \ldots, U_d$ as defined in (1.7), such that

$$\mathsf{E}|U_j(\tau_j, \kappa_j)| < \infty \quad \text{and} \quad \mathsf{E}|V_j(\tau_j, \kappa_j)|^{p_j} < \infty, \qquad j = 1, \ldots, d, \tag{2}$$

for some constants $p_1, \ldots, p_d > 1$ satisfying $p_1^{-1} + \ldots + p_d^{-1} \leqq 1$. Then the corresponding martingales $M_1, \ldots, M_d$ from (1) are again orthogonal and bounded in $L_{p_1}, \ldots, L_{p_d}$, so even their product is a uniformly integrable martingale, and we get

$$\mathsf{E} \prod_{j=1}^{d} V_j(\tau_j, \kappa_j) = 0. \tag{3}$$

Note that the orthogonality condition is automatically fulfilled when $\tau_1 < \ldots < \tau_d$. A special case of the mentioned result is contained in Theorem 4.1 of Kallenberg (1989), where the τ_j are independent and $U(0, 1)$, while $\mathscr{F}$ is essentially the generated filtration.

For the proof of Theorem 4.1, and in fact already for the definitions of ζ and U, we need some basic properties of Z.

Lemma 4.2. *The process Z is a.s. non-increasing with*

$$Z \geqq 0 \quad \textit{and} \quad Z_{\tau-} > 0 \quad \textit{a.s.}, \tag{4}$$

while the reciprocal process $Y = 1/Z$ satisfies

$$dY_t = Y_t d\bar{\eta}_t \quad \textit{as long as} \quad Z_t > 0. \tag{5}$$

Proof. From (1.4) it is clear that the statements in (4) are equivalent to

$$\Delta\bar{\eta} \leqq 1 \quad \text{and} \quad \tau \leqq \inf\{t \geqq 0; \Delta\bar{\eta}_t = 1\} \quad \text{a.s.}, \tag{6}$$

and (6) follows easily from the definition of η by means of the predictable section theorem (cf. Jacod (1979), p. 76, or Liptser and Shiryayev (1978), p. 240).

Next we may integrate by parts to see that, as long as $Z_t > 0$,

$$0 = d(Z_t Y_t) = Z_{t-} dY_t + Y_t dZ_t.$$

Combining this with (1.3) yields

$$dY_t = -Y_t Y_{t-} dZ_t = Y_t Y_{t-} Z_{t-} d\bar{\eta}_t = Y_t d\bar{\eta}_t,$$

as asserted in (5). □

Proof of Theorem 4.1. From Lemma 4.2 and (1.5) it is clear that $\eta \leqq \zeta/Z_{\tau-}$ a.s., which shows that V is also a.s. η-integrable. From (1.7) and the hypothesis on V, it is further seen that $|U - V| \leqq (Z_{\tau-})^{-1} \iint |V| \mathrm{d}\zeta < \infty$, so even U is a.s. η-integrable. Letting t be such that $Z_t > 0$ and writing $Y = 1/Z$, we get by (1.7) (applied twice), Lemma 4.2, and Fubini's theorem,

$$\int_0^{t+} \int (U - V)\mathrm{d}\eta = \int_0^{t+} Y_r \mathrm{d}\bar{\eta}_r \int_0^{r+} \int V \mathrm{d}\zeta = \int_0^{t+} \mathrm{d}Y_r \int_0^{r+} \int V \mathrm{d}\zeta$$
$$= \int_0^{t+} \int V_{s,y}(Y_t - Y_{s-})\mathrm{d}\zeta_{s,y} = U_{t,x} - V_{t,x} - \int_0^{t+} \int V \mathrm{d}\eta \, ,$$

which shows that

$$V_{t,x} = U_{t,x} - \int_0^{t+} \int U \mathrm{d}\eta \, , \tag{7}$$

Adding this to (1.7) and letting $t \uparrow \tau$, we get in particular

$$\int_0^{\tau-} \int U \mathrm{d}\eta = Y_{\tau-} \int_0^{\tau-} \int V \mathrm{d}\zeta \, . \tag{8}$$

Let us next assume that $Z_\tau = 0$. By (1.7) and the hypothesis on V, we get

$$\int_0^{\tau+} \int V \mathrm{d}\zeta = 0 = U_{\tau,x} - V_{\tau,x} \, , \tag{9}$$

and from (8) and (9),

$$\int_0^{\tau+} \int U \mathrm{d}\eta = \int_0^{\tau-} \int U \mathrm{d}\eta + \int_{[\tau]} \int U \mathrm{d}\eta$$
$$= Y_{\tau-} \int_0^{\tau-} \int V \mathrm{d}\zeta + \int_{[\tau]} \int V \mathrm{d}\eta = Y_{\tau-} \int_0^{\tau+} \int V \mathrm{d}\zeta = 0 = U_{\tau,x} - V_{\tau,x} \, .$$

This shows that (7) holds for $t = \tau$, even if $Z_\tau = 0$. From (1.7) it is then clear that (7) is generally true.

Putting $t = \tau$ and $x = \kappa$ in (7), we obtain

$$V_{\tau,\kappa} = U_{\tau,\kappa} - \int_0^{\tau+} \int U \mathrm{d}\eta = M_\tau = M_\infty \, .$$

Noting that U is predictable and assuming $\mathsf{E}|U_{\tau,\kappa}| < \infty$, we get $\mathsf{E}\iint |U| \mathrm{d}\eta < \infty$, so in this case M is a uniformly integrable martingale, and hence

$$\mathsf{E}V_{\tau,\kappa} = \mathsf{E}M_\infty = \mathsf{E}M_0 = 0 \, .$$

By Doob's inequality, we further get for any $p > 1$,

$$\mathsf{E}|M^*|^p \lesssim \mathsf{E}|M_\infty|^p = \mathsf{E}|V_{\tau,\kappa}|^p \, . \qquad \square$$

We now have the tools to prove the first main result of this section.

Theorem 4.3. *Let T be a predictable mapping of $\mathbb{R}_+ \times K$ into some probability space $(S, \mathscr{S}, m)$, such that $\zeta T^{-1} \leqq m$. Then $\mathsf{P}\{T_{\tau,\kappa} \in \cdot\} = m$.*

Proof. Fix $B \in \mathscr{S}$, and define

$$V_{t,x} = 1_B(T_{t,x}) - mB, \qquad t \geqq 0, \; x \in K \, ,$$

and note that V is bounded and hence ζ-integrable. By the hypothesis on T, we get

$$\int_0^{t+}\int V\mathrm{d}\zeta = \int_0^{t+}\int 1_B(T)\mathrm{d}\zeta - mB\int_0^{t+}\mathrm{d}\bar{\zeta} \leq \zeta T^{-1}B - mB(1-Z_t) \leqq Z_t mB\,, \quad (10)$$

and repeating the argument with B replaced by B^c (which only changes the sign of V), we get instead

$$-\int_0^{t+}\int V\mathrm{d}\zeta \leqq Z_t mB^c\,. \quad (11)$$

If $Z_t = 0$, it follows from (10) and (11) that the integral on the left if zero, so V satisfies the conditions in Proposition 4.1. Defining U by (1.7), we obtain from (10) and (11),

$$-1 \leqq 1_B(T_{t,x}) - 1 = V_{t,x} - mB^c \leqq U_{t,x} \leqq V_{t,x} + mB = 1_B(T_{t,x}) \leqq 1,$$

so $|U| \leqq 1$. Hence Proposition 4.1 yields

$$\mathsf{P}\{T_{\tau,\kappa}\in B\} = \mathsf{E}V_{\tau,\kappa} + mB = mB,$$

and since B was arbitrary, this shows that $T_{\tau,\kappa}$ has distribution m. □

The last theorem can be used to obtain bounds for the distributions of Z_τ, $Z_{\tau-}$ and $\bar{\eta}_\tau$. Such estimates could also be obtained from the integral representation (1.2).

Corollary 4.4.

(a) $\mathsf{P}\{Z_{\tau-} \leqq s\} \leqq s \leqq \mathsf{P}\{Z_\tau < s\}$, $\quad s\in[0,1]$,

(b) $1 - t \leqq \mathsf{P}\{\bar{\eta}_\tau \geqq t\} \leqq e^{1-t}$, $\quad t \geqq 0$.

(c) $\mathsf{E}|\bar{\eta}_\tau|^p$ *is bounded by some absolute constant* c_p *for every* $p > 0$.

Proof. Let γ be $U(0,1)$ and independent of $\mathscr{F}$, and enlarge the filtration so as to make the process $1\{\tau \leqq t, \gamma\in\cdot\}$ adapted. Then (τ,γ) becomes a marked stopping time in $\mathbb{R}_+\times[0,1]$ with compensator $\bar{\eta}\times\lambda$. Define a predictable mapping T from $\mathbb{R}_+\times[0,1]$ to $[0,1]$ by

$$T_{t,x} = 1 - Z_{t-} - x\Delta Z_t, \qquad t\in\mathbb{R}_+,\ x\in[0,1]\,, \quad (12)$$

and note that $(\bar{\zeta}\times\lambda)T^{-1} = \lambda$ on $[0, 1 - Z_\tau]$. Hence Theorem 4.3 shows that $T_{\tau,\gamma}$ is $U(0,1)$, and part (a) follows from the fact that, by (12),

$$Z_\tau \leqq 1 - T_{\tau,\gamma} \leqq Z_{\tau-}\,. \quad (13)$$

Next we get from (1.4) for any $t \geqq 0$,

$$-\log Z_t = \bar{\eta}_t^c - \sum_{s\leqq t}\log(1-\Delta\bar{\eta}_s) \geqq \bar{\eta}_t^c + \sum_{s\leqq t}\Delta\bar{\eta}_s = \bar{\eta}_t \geqq 1 - Z_t\,,$$

so using (13) and the fact that $\Delta\bar{\eta} \leqq 1$, we obtain

$$T_{\tau,\gamma} \leqq \bar{\eta}_\tau \leqq \bar{\eta}_{\tau-} + 1 \leqq 1 - \log Z_{\tau-} \leqq 1 - \log(1 - T_{\tau,\gamma})\,.$$

We thus get (b) by writing

$$1 - t \leqq \mathsf{P}\{\bar{\eta}_\tau \geqq t\} \leqq \mathsf{P}\{1 - \log(1 - T_{\tau,\gamma}) \geqq t\} = \mathsf{P}\{1 - T_{\tau,\gamma} \leqq \mathrm{e}^{1-t}\} \leqq \mathrm{e}^{1-t}\,.$$

Finally, we may obtain (c) from (b) by writing

$$\mathsf{E}|\bar{\eta}_\tau|^p = \int_0^\infty \mathsf{P}\{|\bar{\eta}_\tau|^p \geqq s\}\mathrm{d}s \leqq \int_0^\infty \exp(1 - s^{1/p})\mathrm{d}s = p\mathrm{e}\int_0^\infty \mathrm{e}^{-x}x^{p-1}\mathrm{d}x < \infty\,. \quad \square$$

We remark that part (c) of the last corollary may also be obtained by the following direct argument, suggested by a referee. First note that it is equivalent to prove the assertion with M_τ in place of $\bar{\eta}_\tau$, where M denotes the martingale $\bar{\xi} - \bar{\eta}$. Then write

$$\mathsf{E}M_\tau^2 = \mathsf{E}[M, M]_\tau \leqq 1 + \mathsf{E}\sum(\Delta\bar{\eta}_t)^2 \leqq 1 + \mathsf{E}\eta_\tau = 2\,, \tag{14}$$

and proceed by induction over $p = 2^n$, $n \in \mathbb{N}$, using the Burkholder–Davis–Gundy inequality to obtain

$$\begin{aligned}\mathsf{E}M_\tau^{2^{n+1}} &\lesssim \mathsf{E}[M, M]_\tau^{2^n} \leqq \mathsf{E}(1 + \textstyle\sum(\Delta\bar{\eta}_t)^2)^{2^n} \leqq \mathsf{E}(1 + \bar{\eta}_\tau)^{2^n} \\ &= \mathsf{E}(2 - M_\tau)^{2^n} \leqq 2^{2^n - 1}(2^{2^n} + \mathsf{E}M_\tau^{2^n})\,.\end{aligned}$$

Let us illustrate the usefulness of the last corollary by mentioning a few applications. First we note how the result can be used to prove a law of the iterated logarithm for simple point processes. Define $\phi(s) = (2s\log\log s)^{1/2}$ for $s \geqq \mathrm{e}$, and write $a_t \lesssim b_t$ for positive a_t and b_t to mean that $\limsup(a_t/b_t) \leqq 1$. Let ξ be a simple point process on $(0, \infty)$, and denote its compensator by η. Then, *with probability one*, $\xi_t \to \infty$ *iff* $\eta_t \to \infty$ *as* $t \to \infty$, *and in that case*

$$|\xi_t - \eta_t| \lesssim c\phi(\xi_t) \sim c\phi(\eta_t), \qquad t \to \infty\,, \tag{15}$$

where c is an absolute constant. (One may e.g. take $c = 2$.) This follows by a rather straightforward application of the Skorohod embedding theorem and the law of large numbers of martingales, as stated in Hall and Heyde (1980), pp. 269 and 36, respectively, where the required uniform conditional moment estimates may be obtained from Corollary 4.4 (or from (14) for the second moment to get $c = 2$).

Though results like (15) might be available in the more general context of martingales with bounded jumps, the present version of the statement seems to be less known. Note in particular that the equivalence of $\xi_t \to \infty$ and $\eta_t \to \infty$ generalizes Lévy's conditional version of the Borel-Cantelli lemma. Note also that (15) implies $\xi_t \sim \eta_t$, which is known in discrete time (cf. Neveu (1972), p. 152). Finally, (15) holds trivially with $c = 1$ when ξ is quasi-leftcontinuous, since in that case $\xi \circ \eta^{-1}$ is a unit rate Poisson process.

As another application of Corollary 4.4, we note that our integrability conditions $\mathsf{E}|U_{\tau,\kappa}| < \infty$ and $\mathsf{E}|V_{\tau,\kappa}|^p < \infty$ in Theorem 4.1 may be replaced by similar conditions involving ζ. More specifically, let V be any predictable mapping from $\mathbb{R}_+ \times K$ to $\mathbb{R}_+$. *Then* $\mathsf{E}V_{\tau,\kappa} < \infty$, *provided that* $\mathsf{E}\iint V^p \mathrm{d}\zeta < \infty$ *for some* $p > 2$. The observation is particularly useful when the filtration is the natural one, since ζ is then bounded by the distribution μ of (τ, κ), and the condition becomes $\iint \mathsf{E}V^p \mathrm{d}\mu < \infty$. The resulting version of Theorem 4.1 should be compared with Theorem 4.1 in Kallenberg (1989).

To prove our claim, note that, by Hölder's inequality with $p^{-1} + q^{-1} = 1$,

$$\mathsf{E}V_{\tau,\kappa} \leqq (\mathsf{E}Z_{\tau-}^{1-q})^{1/q}(\mathsf{E}Z_{\tau-}V_{\tau,\kappa}^p)^{1/p}\,. \tag{16}$$

Under the stated condition on V we obtain

$$\mathsf{E}Z_{\tau-}V_{\tau,\kappa}^p = \mathsf{E}\iint Z_{t-}V_{t,x}^p \mathrm{d}\eta_{t,x} = \mathsf{E}\iint V^p \mathrm{d}\zeta < \infty\,. \tag{17}$$

By Corollary 4.4(a), it is further seen that

$$\mathsf{E}Z_{\tau-}^{1-q} \leqq \int_0^1 s^{1-q}\mathrm{d}s = \frac{1}{2-q} < \infty\,. \tag{18}$$

Thus the right-hand side of (16) is finite, and the assertion follows.

As a final application of Corollary 4.4, we discuss the proof of the existence of a compensator associated with an arbitrary stopping time (and then, by iteration, of a general point process). Here the easiest approach, due to K.M. Rao, involves as the only technical difficulty a proof that a certain family of random sums is uniformly integrable (cf. Ikeda and Watanabe (1981), p. 37; or Karatzas and Shreve (1988), p. 25). Now those sums turn out to be of the form $\eta_n(\tau_n)$ for certain approximating stopping times τ_n with associated compensators η_n, so their uniform integrability is immediate from Corollary 4.4. Realizing this makes Rao's already simple proof even more transparent. Note that no circular reasoning is involved, since the discrete version of our corollary is elementary and doesn't require the general Doob–Meyer decomposition.

We conclude this section with a multivariate version of Theorem 4.3. For motivation, note that if the marked stopping times (τ_j, κ_j) are such that their associated martingales in Theorem 4.1 are (strongly) orthogonal, and if $T_1, T_2, \ldots$ are predictable mappings into $[0, 1]$ satisfying $\zeta_j T_j^{-1} \leqq \lambda$ a.s., where ζ_j denotes the discounted compensator of (τ_j, κ_j), then the random variables $T_j(\tau_j, \kappa_j)$ are independent and $U(0, 1)$. Without orthogonality the statement fails, but we show how it can be saved by a suitable modification of the compensators, as long as the τ_j remain a.s. distinct.

Theorem 4.5. *Let $(\tau_1, \kappa_1), \ldots, (\tau_d, \kappa_d)$ be marked stopping times in $(0, \infty) \times K$ with compensators $\eta_1, \ldots, \eta_d$, and assume that $\tau_1, \ldots, \tau_d$ are a.s. distinct. Put $\bar{\eta}_k = \eta_k(\cdot \times K)$ for each k, and define the random processes and measures*

$$\rho_k(t) = \prod_{j=1}^{k-1} 1\{\tau_j \neq t\}, \qquad \hat{\rho}_k(t) = 1 - \sum_{j=1}^{k-1} \Delta\bar{\eta}_j(t)\,,$$

$$\eta'_k(\mathrm{d}t\,\mathrm{d}x) = \frac{\rho_k(t)}{\hat{\rho}_k(t)}\,\eta_k(\mathrm{d}t\,\mathrm{d}x), \qquad \bar{\eta}'_k(\mathrm{d}t) = \eta'_k(\mathrm{d}t \times K)\,, \tag{19}$$

$$Z'_k(t) = \exp(-\bar{\eta}'^c_k(t)) \prod_{s \leqq t} (1 - \Delta\bar{\eta}'_k(s))\,,$$

$$\zeta'_k(\mathrm{d}t\,\mathrm{d}x) = Z'_k(t-)\eta'_k(\mathrm{d}t\,\mathrm{d}x)\,,$$

(with the convention $0/0 = 0$ in (19)). Further assume that $T_1, \ldots, T_d$ are predictable mappings from $\mathbb{R}_+ \times K$ to $[0, 1]$, such that $\zeta'_k T_k^{-1} \leqq \lambda$ a.s. for each k. Then the random variables $T_k(\tau_k, \kappa_k)$ are independent and $U(0, 1)$.

Before proceeding to the proof, we note that $\eta'_1, \ldots, \eta'_d$ are a.s. well-defined, since $\hat{\rho}_k(t) = 0$ implies $\rho_k(t) = 0$ for all k and t, outside a fixed P-nullset. In fact, each $\xi_k = \delta_{\tau_1} + \ldots + \delta_{\tau_{k-1}}$ is a simple point process with compensator $\hat{\xi}_k = \bar{\eta}_1 + \ldots + \bar{\eta}_{k-1}$, and the relation $\hat{\rho}_k(t) = 0$ is equivalent to $\hat{\xi}_k\{t\} = 1$, which implies $\xi_k\{t\} = 1$. Note also that the processes Z'_k are well-defined and non-increasing, since $\Delta\bar{\eta}'_k \leqq 1$.

Proof. Fix Borel sets $B_1, \ldots, B_d \subset [0, 1]$, and define for $t \geqq 0$, $x \in K$, and $k = 1, \ldots, d$,

$$V_k(t, x) = 1_{B_k} \circ T_k(t, x) - \lambda B_k\,,$$

$$U_k(t, x) = V_k(t, x) + \frac{1}{Z'_k(t)} \int_0^{t+} \int V_k \mathrm{d}\zeta'_k\,, \tag{20}$$

$$M_k(t) = U_k(\tau_k, \kappa_k) 1\{\tau_k \leqq t\} - \int_0^{t+} \int U_k \mathrm{d}\eta'_k\,. \tag{21}$$

As in the proof of Theorem 4.3, we note that $|U_k| \leqq 1$ for each k, and that the integral in (20) equals 0 when $Z'_k(t) = 0$. It follows as before that $V_k(\tau_k, \kappa_k) = M_k(\infty)$.

We need to show that the processes M_k are uniformly integrable martingales. To see this, we note that the random set $\{t \geqq 0; \Delta(\bar{\eta}_1 + \ldots + \bar{\eta}_d)_t > 0\}$ is predictable, and hence is covered by the graphs of some predictable stopping times $\sigma_1, \sigma_2, \ldots$, where the latter may be taken to be finite and distinct. Write $D = \bigcup [\sigma_j]$, and note that $\eta'_k = \eta_k$ on D^c. Furthermore, each U_k coincides on D^c with a predictable process

$$U'_k(t, x) = V_k(t, x) + (U_k - V_k)(t-), \qquad t \geqq 0,\ x \in K .$$

Thus the processes

$$M'_k(t) = \int_0^{t+} 1_{D^c} \mathrm{d}M_k = U'_k(\tau_k, \kappa_k) 1\{\tau_k \in [0, t] \setminus D\} - \int_0^{t+} \int 1_{D^c} U'_k \mathrm{d}\eta_k, \qquad t \geqq 0 ,$$

are uniformly integrable martingales.

For fixed $k \in \{1, \ldots, d\}$ and a finite predictable stopping time σ, we next define $\mathscr{F}'_{\sigma-} = \mathscr{F}_{\sigma-} \vee \sigma\{\xi_1, \ldots, \xi_{k-1}\}$, where $\xi_j \equiv 1\{\tau_j = \sigma, \kappa_j \in \cdot\}$, and show that

$$\mathsf{E}[\Delta M_k(\sigma) | \mathscr{F}'_{\sigma-}] = 0 \quad \text{a.s.} \tag{22}$$

Let us then write

$$\Delta M_k(\sigma) = \int U_k(\sigma, \cdot) \mathrm{d}\xi_k - \int U_k(\sigma, \cdot)\, \mathrm{d}\eta'_k(\{\sigma\} \times \cdot) . \tag{23}$$

From Corollary 1.45 in Jacod (1979) it is clear that $\mathsf{E}[\xi_k | \mathscr{F}_{\sigma-}] = \eta_k(\{\sigma\} \times \cdot)$ a.s., so by our Lemma 3.5, we get

$$\mathsf{P}[\xi_k | \mathscr{F}'_{\sigma-}] = \eta_k(\{\sigma\} \times \cdot) \frac{1\left\{\sigma \neq \tau_1, \ldots, \tau_{k-1};\ \sum_{j=1}^{k-1} \bar{\eta}_j\{\sigma\} < 1\right\}}{1 - \sum_{j=1}^{k-1} \bar{\eta}_j\{\sigma\}} = \eta'_k(\{\sigma\} \times \cdot) . \tag{24}$$

Thus (22) will follow from (23) by Lemma 3.7, if we can only show that the process $U_k(\sigma, \cdot)$ on K is $\mathscr{F}'_{\sigma-} \times \mathscr{B}(K)$-measurable.

To see this, we conclude from Lemma 3.6 that the process $V_k(\sigma, \cdot)$ is $\mathscr{F}_{\sigma-} \times \mathscr{B}(K)$-measurable. Moreover, the processes Z'_k and $\int\int 1_{[0, \cdot]} V_k \mathrm{d}\zeta'_k$ are adapted and of bounded variation, so their left-continuous versions are predictable, and hence their left-hand limits at σ are $\mathscr{F}_{\sigma-}$-measurable. It remains to show that both processes have $\mathscr{F}'_{\sigma-}$-measurable jumps at σ. But this follows easily from the mentioned measurability of $V_k(\sigma, \cdot)$, plus the fact that $\eta'_k(\{\sigma\} \times \cdot)$ is $\mathscr{F}'_{\sigma-}$-measurable by (24). This completes the proof of (22).

By the definition of D, we may write

$$\int_0^{t+} 1_D \mathrm{d}M_k = \sum_{j=1}^{\infty} \Delta M_k(\sigma_j) 1\{\sigma_j \leqq t\}, \qquad t \geqq 0 , \tag{25}$$

and using Corollary 1.45 in Jacod (1979), it is seen from (22) that each term on the right is a martingale. Since $|U_k| \leqq 1$, we further obtain from (21) and (24),

$$\mathsf{E} \sum_{j=1}^{\infty} |\Delta M_k(\sigma_j)| \leqq \mathsf{E} \sum_{j=1}^{\infty} (1\{\tau_k = \sigma_j\} + \bar{\eta}'_k\{\sigma_j\}) = 2 \sum_{j=1}^{\infty} \mathsf{P}\{\tau_k = \sigma_j\} \leqq 2 , \tag{26}$$

so the series in (25) converges in L_1 for each t, and the sum is uniformly integrable in t. Thus the integral on the left is a uniformly integrable martingale. Since the same thing was shown to be true for the integral M_k' over D^c, even the sum M_k must be a uniformly integrable martingale. Since $|M_k(\infty)| = |V_k(\tau_k, \kappa_k)| \leqq 1$, it follows in particular that $M_k^* \leqq 1$ a.s.

We proceed to show that the product $M_J = \Pi_J M_j$ is a martingale for every non-empty index set $J \subset \{1, \ldots, d\}$. Through repeated integration by parts (cf. Lemma 5.6 in Kallenberg (1989)), we get

$$M_J(t) = \sum_{j \in J} \int_0^{t+} M_{J\setminus\{j\}}(s-)\mathrm{d}M_j(s) + \sum_I \sum_{s \leqq t} M_{J\setminus I}(s-)\prod_{i \in I} \Delta M_i(s), \qquad t \geqq 0 ,$$

where the outer summation in the last term extends over all subsets $I \subset J$ of size $\geqq 2$. Here the integrals in the first sum on the right are clearly martingales. Since the τ_j are a.s. distinct, the inner sum in the second term equals

$$\sum_j 1\{\sigma_j \leqq t\} M_{J\setminus I}(\sigma_j -)\prod_{i \in I} \Delta M_i(\sigma_j) , \tag{27}$$

and to see that each term in (27) is a martingale, it suffices as before to show that

$$\mathsf{E}\left[M_{J\setminus I}(\sigma -)\prod_{i \in I} \Delta M_i(\sigma) \middle| \mathscr{F}_{\sigma -} \right] = 0 \quad \text{a.s.} ,$$

for every finite predictable stopping time σ. Here the first factor on the left is $\mathscr{F}_{\sigma-}$-measurable and may be ignored. Letting k be the largest index in I and defining $\mathscr{F}'_{\sigma-}$ as before, it is then enough to show that

$$\mathsf{E}\left[\prod_{i \in I} \Delta M_i(\sigma) \middle| \mathscr{F}'_{\sigma -} \right] = 0 \quad \text{a.s.}$$

But this follows from (22), since $\Delta M_i(\sigma)$ is clearly $\mathscr{F}'_{\sigma-}$-measurable for every $i < k$. To obtain the martingale property of M_J, it remains to notice that, by (26) and the fact that $|M_k| \leqq 1$ for each k,

$$\mathsf{E} \sum_{j=1}^{\infty} \left| M_{J\setminus I}(\sigma_j -)\prod_{i \in I} \Delta M_i(\sigma_j) \right| \leqq 2^{d-1}\mathsf{E} \sum_{j=1}^{\infty} \sum_{i=1}^{d} |\Delta M_i(\sigma_j)| \leqq d2^d < \infty .$$

The martingale property of M_J yields

$$\mathsf{E} \prod_{j \in J} (1_{B_j} \circ T_j(\tau_j, \kappa_j) - \lambda B_j) = \mathsf{E} \prod_{j \in J} V_j(\tau_j, \kappa_j) = \mathsf{E} M_J(\infty) = \mathsf{E} M_J(0) = 0 ,$$

and it follows by induction over $|J|$ that

$$\mathsf{P} \bigcap_{j \in J} \{T_j(\tau_j, \kappa_j) \in B_j\} = \prod_{j \in J} \lambda B_j, \qquad J \subset \{1, \ldots, d\} .$$

In particular, the relation for $J = \{1, \ldots, d\}$ expresses the fact that $T_1(\tau_1, \kappa_1), \ldots, T_d(\tau_d, \kappa_d)$ are independent and $U(0, 1)$. □

5. Random time change

The main result of this section is our extension in Theorem 5.1 of the classical time change reduction of quasi-leftcontinuous simple point processes to homogeneous Poisson. For motivation, let $\xi = \sum \delta_{\tau_j}$ be an a.s. unbounded simple point process adapted to some filtration $\mathscr{F}$, and let η denote the compensator of ξ. Assuming that $0 < \tau_1 < \tau_2 < \dots$, we may apply the previous time change results to $\tau_1, \tau_2, \dots$, to obtain a sequence of independent exponentially distributed random variables, which may then be combined to form a Poisson process. Unless ξ is quasi-leftcontinuous, we need in addition a sequence of independent $U(0, 1)$ random variables $\kappa_1, \kappa_2, \dots$ independent of $\mathscr{F}$. The resulting time change process T then becomes

$$T_t = \eta_t^c - \sum_{s \leqq t,\, \xi\{s\} = 0} \log(1 - \Delta\eta_s) - \sum_{j \leqq \xi[0,t]} \log(1 - \kappa_j \Delta\eta_{\tau_j}), \qquad t \geqq 0 . \tag{1}$$

Note that T_t reduces to η_t when the latter process is continuous.

A refined version of the classical result states that the image process $\tilde{\xi}$ is Poisson with respect to a suitably time changed filtration $\tilde{\mathscr{F}}$, in the sense that $\tilde{\xi}$ becomes $\tilde{\mathscr{F}}$-adapted with compensator λ (cf. Theorem 10.33 in Jacod (1979)). This is the version of the classical result which we shall extend below to arbitrary point processes. Let us then define the left- and rightcontinuous inverses of T by

$$L_s = \inf\{t \geqq 0; T_t \geqq s\}, \qquad R_s = \inf\{t \geqq 0; T_t > s\}, \qquad s \geqq 0 . \tag{2}$$

and recall that the classical choice is to take $\tilde{\mathscr{F}}_s = \mathscr{F}_{R_s}$ for each s. In the general case, we introduce for each $s \geqq 0$ the extended filtration

$$\mathscr{F}_t^{(s)} = \mathscr{F}_t \vee \sigma\{\pi_s(\xi T^{-1})\}, \qquad t \geqq 0 , \tag{3}$$

where the operator π_s denotes restriction to $[0, s]$, and define

$$\mathscr{G}_s = \mathscr{F}_{L_s-}^{(s)}, \qquad \tilde{\mathscr{F}}_s = \mathscr{G}_{s+}, \qquad s \geqq 0 . \tag{4}$$

Theorem 5.1. *Let ξ be an a.s. unbounded $\mathscr{F}$-adapted simple point process on* $(0, \infty)$ *with compensator η, let $\kappa_1, \kappa_2, \dots$ be* i.i.d. $U(0, 1)$ *and independent of $\mathscr{F}$, and define $T, L, R, \mathscr{F}^{(s)}$ and $\tilde{\mathscr{F}}$ by* (1)–(4). *Then ξT^{-1} is $\tilde{\mathscr{F}}$-Poisson, and (to motivate* (4)) *L_s is an $\mathscr{F}^{(s)}$-predictable stopping time for each s. In the quasi-leftcontinuous case, T_t and $\tilde{\mathscr{F}}_s$ reduce to the classical choices η_t and $\mathscr{F}_{R_s}$.*

A lemma will be needed for the proof.

Lemma 5.2. *Let (τ, κ) be a marked stopping time in* $(0, \infty) \times (0, 1)$ *with discounted compensator of the form $\zeta = \bar{\zeta} \times \lambda$, and define $Z_t = 1 - \bar{\zeta}_t$. Further consider a stopping time $\vartheta < \tau$ and an $\mathscr{F}_\vartheta$-measurable random variable $\alpha \geqq 0$, and write*

$$Y_t = -\log Z_t, \qquad T_{t,x} = -\log(Z_{t-} + x\Delta Z_t), \qquad t \geqq 0,\ x \in (0, 1) ,$$

$$L_s = \inf\{t \geqq 0; \alpha + Y_t \geqq s\} + \infty \cdot 1\{\alpha \geqq s\}, \qquad s \geqq 0 .$$

Then each L_s is a predictable stopping time, and the random variable $\sigma = \alpha + T_{\tau,\kappa}$ satisfies

$$\mathrm{P}[\sigma > s + h \mid \mathscr{F}_{L_s-}, \sigma > s] = e^{-h} \quad \text{a.s. on } \{\alpha < s\}, \qquad s, h \geqq 0 . \tag{5}$$

Proof. Write μ for the exponential distribution on $\mathbb{R}_+$ with density e^{-x}, and define for fixed $s \geqq 0$ and $B \in \mathscr{B}(\mathbb{R}_+)$,

$$V_{t,x} = (1_B(T_{t,x} - Y_{L_s-}) - \mu B)1\{L_s \leqq t \wedge \tau\}, \qquad t \geqq 0, x \in (0,1)\,.$$

Since $\zeta T^{-1} \leqq \mu$, we get for $t \wedge \tau \geqq L_s$,

$$\int_0^{t+}\int V \mathrm{d}\zeta = \int_{L_s-}^{t+}\int (1_B(T - Y_{L_s-}) - \mu B)\mathrm{d}\zeta \leqq \zeta T^{-1}(B + Y_{L_s-}) - (Z_{L_s-} - Z_t)\mu B$$

$$\leqq \exp(-Y_{L_s-})\mu B - (Z_{L_s-} - Z_t)\mu B = Z_t \mu B\,.$$

Hence the process U in (1.8) satisfies $U \leqq 1 - \mu B + \mu B = 1$. The same argument with B replaced by B^c yields $-U \leqq 1 - \mu B^c + \mu B^c = 1$, so in fact $|U| \leqq 1$. Note also that the integral in (1.8) equals zero when $Z_t = 0$, as required.

Since $\alpha > s$ implies $\alpha + Y_{L_s-} \leqq s$, we may choose $B = (s + h - \alpha - Y_{L_s-}, \infty)$, so that

$$V_{t,x} = (1\{T_{t,x} > s + h\} - \exp(\alpha + Y_{L_s-} - s - h))1\{L_s \leqq t \wedge \tau\}, \quad t \geqq 0, x \in (0,1)\,. \tag{6}$$

We shall prove below that L_s is a predictable stopping time and that the process V is predictable on $\mathbb{R}_+ \times (0,1)$. Theorem 4.1 then shows that the process M in (4.1) is a uniformly integrable martingale with $M_\infty = V_{\tau,\kappa}$. Choosing an announcing sequence of stopping times $\tau_n \uparrow L_s$ and noting that $M_t = 0$ for $t < L_s$, we obtain by optional sampling and martingale convergence,

$$0 = \mathsf{E}M_{\tau_n} = \mathsf{E}[V_{\tau,\kappa}|\mathscr{F}_{\tau_n}] \to \mathsf{E}[V_{\tau,\kappa}|\mathscr{F}_{L_s-}] \quad \text{a.s.}$$

Hence the limiting variable vanishes a.s., and we get

$$\mathsf{P}[\sigma > s + h|\mathscr{F}_{L_s-}] = \exp(\alpha + Y_{L_s-} - s - h) \quad \text{a.s.} \quad \text{on } \{L_s \leqq \tau\}, \qquad h \geqq 0\,.$$

Thus (5) follows by Lemma 3.5 plus the fact that $\alpha < s < \sigma$ implies $L_s \leqq \tau$.

To see that each L_s is a predictable stopping time, we note that the process

$$Y'_t = (Y_t + \alpha)1\{\vartheta < t, \alpha < s\}, \qquad t \geqq 0\,,$$

is predictable, since Y inherits its predictability from Z, while the processes $1\{\vartheta < t, \alpha < s\}$ and $\alpha 1\{\vartheta < t, \alpha < s\}$ are both adapted and left-continuous. From the fact that $Y_t = 0$ for $t \leqq \vartheta$, we get

$$L_s = \inf\{t \geqq 0;\ Y'_t \geqq s\}, \qquad s \geqq 0\,,$$

so L_s is the left endpoint of the predictable interval $I = \{t \geqq 0;\ Y'_t \geqq s\}$. It remains to notice that I is closed, since Y' is right-continuous on (ϑ, ∞), while $L_s > \vartheta$ because $Y'_{\vartheta+} = \alpha$.

To see that the process V in (6) is predictable, we note first that T has this property. By the predictability of the stopping time L_s, it remains to show that the event $\{L_s \leqq \tau\}$ and the random variables α and Y_{L_s-} are $\mathscr{F}_{L_s-}$-measurable. But this follows from the predictability of Y and L_s, the stopping time properties of ϑ and τ, and the fact that $\vartheta < L_s$. □

By a similar argument, or by applying Lemma 5.2 with $\alpha = 0$ and transforming the time scale, we obtain the following result which will be needed later.

Lemma 5.3. *Let* (τ, κ), ζ *and* Z *be such as in Lemma 5.2, but define instead*

$$T_{t,x} = 1 - Z_{t-} - x\Delta Z_t, \qquad t \geqq 0,\ x \in (0, 1)\,,$$
$$L_s = \inf\{t \geqq 0;\ 1 - Z_t \geqq s\}, \qquad s \in [0, 1)\,,$$

and put $\sigma = T_{\tau,\kappa}$. *Then*

$$\mathsf{P}[\sigma \in B \mid \mathscr{F}_{L_s-}, \sigma > s] = \frac{\lambda B}{1-s} \quad a.s., \quad B \in \mathscr{B}((s, 1]),\ s \in [0, 1)\,.$$

Proof of Theorem 5.1. Put $\sigma_0 = \tau_0 = 0$ and $\sigma_n = T \circ \tau_n$, $n \in \mathbb{N}$, and define for each $n \in \mathbb{Z}_+$ the process

$$T_t^{(n)} = \sigma_n 1\{t > \tau_n\} - \int_{t \wedge \tau_n +}^{t \wedge \tau_{n+1}+} \log(1 - \mathrm{d}\eta_s), \qquad t \geqq 0\,, \tag{7}$$

where

$$\int_A \log(1 - \mathrm{d}\eta_s) = -\int_A \mathrm{d}\eta_s^c + \sum_{s \in A} \log(1 - \Delta\eta_s)\,.$$

Writing $\nu = \xi T^{-1}[0, s]$, it is easily verified that the process $T' = T^{(\nu)}$ is $\mathscr{F}^{(s)}$-predictable. From the fact that ξT^{-1} is unit rate Poisson, as explained at the beginning of this section, it is further clear that $\sigma_\nu < s < \sigma_{\nu+1}$ a.s., so $T' = T$ on the interval $(\tau_\nu, \tau_{\nu+1})$, while $T'_{\tau_{\nu+1}} \geqq T_{\tau_{\nu+1}}$. Writing $L'_s = \inf\{t \geqq 0;\ T'_t > s\}$, it follows easily that $L'_s = L_s > \tau_\nu$ a.s., and since T' is right-continuous on (τ_ν, ∞), L_s is a.s. the left endpoint of an $\mathscr{F}^{(s)}$-predictable closed interval. Hence L_s is an $\mathscr{F}^{(s)}$-predictable stopping time.

To prove the last assertion, assume that ξ is quasi-leftcontinuous, i.e. that η is a.s. continuous. Then $R_s < L_{s+h}$ for all $s \geqq 0$ and $h > 0$, so

$$\mathscr{F}_{R_s} \subset \mathscr{F}_{R_s}^{(s+h)} \subset \mathscr{F}_{L_{s+h}-}^{(s+h)} = \mathscr{G}_{s+h}\,,$$

and therefore $\mathscr{F}_{R_s} \subset \mathscr{G}_{s+} = \tilde{\mathscr{F}}_s$. To get the reverse relation, we note that

$$\pi_s(\xi T^{-1}) = (\pi_{R_s}\xi) T^{-1} \in \mathscr{F}_{R_s}\,,$$

and further that for all $t \geqq 0$,

$$A \cap \{t < L_s\} \in \mathscr{F}_{L_s-} \subset \mathscr{F}_{R_s}\,, \qquad A \in \mathscr{F}_t\,.$$

Thus

$$A \cap \{t < L_s\} \in \mathscr{F}_{R_s}, \qquad A \in \mathscr{F}_t^{(s)}\,,$$

which shows that

$$\mathscr{G}_s = \mathscr{F}_{L_s-}^{(s)} \subset \mathscr{F}_{R_s}\,.$$

Hence we get

$$\tilde{\mathscr{F}}_s = \mathscr{G}_{s+} \subset \mathscr{F}_{R_{s+}} = \mathscr{F}_{R_s}\,,$$

as desired.

To prove that ξT^{-1} is $\tilde{\mathscr{F}}$-Poisson amounts to showing that the process $M_s = (\xi T^{-1})_s - s$ is an $\tilde{\mathscr{F}}$-martingale. By right-continuity of M and reverse martingale convergence, it is enough to show that M is a $\mathscr{G}$-martingale on $(0, \infty)$, and since M is $\mathscr{G}$-adapted, we may clearly replace $\mathscr{G}$ by any larger filtration $\mathscr{G}'$. In particular, we may replace $\mathscr{F}_s$ by $\mathscr{F}'_s = \mathscr{F}_s \vee \sigma\{\pi_s \Xi\}$, $s \geqq 0$, where $\Xi = \sum \delta_{(\tau_j, \kappa_j)}$, and define $\mathscr{G}'$ accordingly as in (3) and (4). Note that Ξ is then $\mathscr{F}'$-adapted with compensator $\eta \times \lambda$. Dropping the primes, we may assume instead that Ξ is adapted to $\mathscr{F}$ with compensator $\eta \times \lambda$, and prove that M is a $\mathscr{G}$-martingale.

We shall prove below that

$$\mathsf{P}[\sigma_{v+1} > s + h | \mathscr{G}_s] = e^{-h} \quad \text{a.s.,} \qquad s, h > 0 , \tag{8}$$

where $\sigma_n = T \circ \tau_n$ and $v = \xi T^{-1}[0, s]$ as before. Considering any finite union U of disjoint intervals $I_j = (s_j, s_j + h_j]$, $j = 1, \ldots, n$, to the right of s, we get from (8) by successive conditioning on $\mathscr{G}_{s_n}, \ldots, \mathscr{G}_{s_1}$,

$$\mathsf{P}[\xi T^{-1} U = 0 | \mathscr{G}_s] = \mathsf{P}\left[\bigcap_{j=1}^{n} \{\xi T^{-1} I_j = 0\} \middle| \mathscr{G}_s \right] = \prod_{j=1}^{n} e^{-h_j} = e^{-\lambda U} \quad \text{a.s.}$$

Fixing $s > 0$, this will then hold simultaneously outside a fixed nullset for all U determined by rational endpoints, and since ξT^{-1} is a.s. simple, we may conclude by Theorem 3.3 in Kallenberg (1986) that ξT^{-1} is conditionally unit rate Poisson on (s, ∞), given $\mathscr{G}_s$. This yields in particular the desired martingale property of M.

To prove (8), we note that v is $\mathscr{G}_s$-measurable, and further that a.s. $v = n$ iff $\sigma_n < s < \sigma_{n+1}$. Thus it suffices to show for any fixed $n \in \mathbb{Z}_+$ and $s, h > 0$ that

$$\mathsf{P}[\sigma_{n+1} > s + h | \mathscr{G}_s] = e^{-h} \quad \text{a.s. on} \quad \{\sigma_n < s < \sigma_{n+1}\} . \tag{9}$$

Defining $T^{(n)}$ as in (7) and putting

$$L'_s = \inf\{t \geqq 0;\ T_t^{(n)} \geqq s\} + \infty \cdot 1\{\sigma_n \geqq s\}, \tag{10}$$

we note as before that $L_s = L'_s$ on $\{\sigma_n < s < \sigma_{n+1}\}$. Since even L'_s is an $\mathscr{F}^{(s)}$-predictable stopping time, it is hence equivalent to prove (9) with $\mathscr{G}_s$ replaced by $\mathscr{F}^{(s)}_{L'_s-}$, or rather to take (10) as our new definition of L_s. Writing $\alpha = \sigma_n$, $\sigma = \sigma_{n+1}$, $\tau = \tau_{n+1}$ and $\kappa = \kappa_{n+1}$, we may then conclude from Lemma 5.2 that

$$\mathsf{P}[\sigma > s + h | \mathscr{F}_{L_s-}, \sigma > s] = e^{-h} \quad \text{a.s. on} \quad \{\alpha < s\} .$$

In the same notation, (9) becomes

$$\mathsf{P}[\sigma > s + h | \mathscr{G}_s] = e^{-h} \quad \text{a.s. on} \quad \{\alpha < s < \sigma\} ,$$

so we need only show that the restriction of $\mathscr{G}_s$ to $\{\alpha < s < \sigma\}$ is contained in $\mathscr{F}_{L_s-} \cap \{\sigma > s\}$. (The restrictions of the two σ-fields are in fact equal.)

To see this, recall that $\mathscr{G}_s$ is generated by the sets $A \cap \{t < L_s\}$ with $A \in \mathscr{F}_t^{(s)}$ and $t \geqq 0$. Hence its restriction to $\{\alpha < s < \sigma\}$ is generated by the sets

$$A \cap \{t < L_s\} \cap \{\alpha < s < \sigma\}, \qquad A \in \mathscr{F}_t,\ t \geqq 0 , \tag{11}$$

and

$$\{\pi_s(\xi T^{-1}) \in \cdot\} \cap \{t < L_s\} \cap \{\alpha < s < \sigma\}, \qquad t \geqq 0 , \tag{12}$$

Here the former class generates the σ-field

$$\mathscr{F}_{L_s-} \cap \{\alpha < s < \sigma\} \subset \mathscr{F}_{L_s-} \cap \{\sigma > s\},$$

so it is enough to consider the class in (12). Comparing with (11), it is clear that we may omit the set $\{t < L_s\}$ in (12), and prove instead that

$$\{\pi_s(\xi T^{-1}) \in \cdot\} \cap \{\alpha < s < \sigma\} \in \mathscr{F}_{L_s-} \cap \{\alpha < s < \sigma\} . \tag{13}$$

But on $\{\alpha < s < \sigma\}$, our present L_s from (10) agrees with the original version from (2), and moreover $\xi T^{-1}[s] = 0$, so we get

$$\pi_s(\xi T^{-1}) = \pi_{s-}(\xi T^{-1}) = (\pi_{L_s-}\xi) T^{-1} \in \mathscr{F}_{L_s-} ,$$

and (13) follows. □

The remainder of this section is devoted to an extension of Lemma 5.3 to the context of marked stopping times, a result needed for the proof of the integral representation in Sect. 6. The assertion is easy to believe but surprisingly hard to prove, so the reader might skip to the next section on a first reading.

We return to the notation and conventions of Sects. 1 and 4. Thus (τ, κ) is a marked stopping time in $(0, \infty) \times K$ with discounted compensator ζ, as given by (1.4) and (1.5) in terms of the ordinary compensator η, and we have $Z_t = 1 - \bar{\zeta}_t$, where $\bar{\zeta}_t = \zeta([0, t] \times K)$. Moreover, L is the left-continuous inverse of $1 - Z$, as defined in Lemma 5.3. Recall that π_s denotes restriction to $[0, s]$ or $[0, s] \times K$, and write π for projection onto $\mathbb{R}_+$. Given a process T on $\mathbb{R}_+ \times [0, 1]$, we define the process $T \times I$ on $\mathbb{R}_+ \times K \times [0, 1]$ by $(T \times I)_{t,x,y} = (T_{t,y}, x)$.

Proposition 5.4. *Let (τ, κ) be a marked stopping time in $(0, \infty) \times K$ with discounted compensator ζ, define Z and L as before, let γ be $U(0, 1)$ and independent of $\mathscr{F}$, and put $\sigma = T_{\tau,\gamma}$, where*

$$T_{t,y} = 1 - Z_t + y \Delta Z_t, \qquad t \geqq 0, \quad y \in [0, 1] .$$

Let $\mathscr{G}$ be the right-continuous filtration generated by the σ-fields $\mathscr{F}_{L_s-}$ and the process $1\{\sigma \leqq s, \kappa \in \cdot\}$. Then the pair (σ, κ) has discounted $\mathscr{G}$-compensator

$$\zeta' = \pi_\sigma((\zeta \times \lambda)(T \times I)^{-1}). \tag{14}$$

To appreciate this result, note that the discounted compensator of σ equals

$$\bar{\zeta}' := \zeta' \pi^{-1} = \pi_\sigma((\zeta \times \lambda) T^{-1}) = \pi_\sigma \lambda ,$$

as predicted by Lemma 5.3. Thus the discounted (but not the ordinary) compensator is transformed by the same time change T as the stopping time itself. Proposition 5.4 merely states that this remains true in the presence of marks.

We shall base our proof of Proposition 5.4 on two rather technical lemmas.

Lemma 5.5. *Let τ, κ, ζ, Z and L be as before, and put $\sigma = T_\tau$ where $T = 1 - Z$. Denote by $\mathscr{G}$ the right-continuous filtration on $[0, 1]$ generated by the σ-fields $\mathscr{F}_{L_s-}$. Then (σ, κ) is a marked stopping time in $[0, 1] \times K$ with discounted compensator $\zeta' = \zeta(T \times I)^{-1}$. Writing $\bar{\zeta}' = \zeta'(\cdot \times K)$ and $L'_s = \inf\{r \geqq 0; \bar{\zeta}'_r \geqq s\}$, we further have the identities*

$$\bar{\zeta}'_\sigma = \sigma, \qquad \zeta'_{L'_s} = \zeta_{L_s}, \qquad \bar{\zeta}'\{s \geqq 0; \bar{\zeta}'_s \neq s\} = 0 .$$

Proof. Let η denote the compensator of $\xi = \delta_{(\tau,\kappa)}$, and put

$$\xi' = \xi(T \times I)^{-1} = \delta_{(\sigma,\kappa)}, \quad \eta' = \eta(T \times I)^{-1} .$$

We shall show that ξ' is adapted to $\mathscr{G}$ with compensator η'. To this aim, we first note that $T_t < s$ iff $t < L_s$, so that for any $s \in [0, 1]$,

$$\xi'([0, s) \times \cdot) = \xi([0, L_s) \times \cdot), \qquad \eta'([0, s) \times \cdot) = \eta([0, L_s) \times \cdot) . \tag{15}$$

Here the random measures on the right are measurable with respect to $\mathscr{F}_{L_s-} \subset \mathscr{G}_s$, since the processes $\xi([0, t) \times \cdot)$ and $\eta([0, t) \times \cdot)$ are $\mathscr{F}$-predictable, while L_s is an $\mathscr{F}$-predictable stopping time. By the right-continuity of $\mathscr{G}$, it follows that even $\xi'([0, s] \times \cdot)$ and $\eta'([0, s] \times \cdot)$ are $\mathscr{G}_s$-measurable, which means that ξ' and η' are adapted to $\mathscr{G}$.

Next we fix $0 < r < s < t$, and let $\tau_1, \tau_2, \ldots$ be a sequence of $\mathscr{F}$ stopping times announcing the $\mathscr{F}$-predictable stopping time L_r. Since $\xi - \eta$ is a uniformly integrable $\mathscr{F}$ martingale measure, while L_s and L_t are $\mathscr{F}$-predictable stopping times, we get from (15) by optional sampling,

$$\mathsf{E}[(\xi' - \eta')([s, t) \times \cdot)|\mathscr{F}_{\tau_n}] = \mathsf{E}[(\xi - \eta)([L_s, L_t) \times \cdot)|\mathscr{F}_{\tau_n}] = 0 \quad \text{a.s.}$$

By martingale convergence, it follows that

$$\mathsf{E}[(\xi' - \eta')([s, t) \times \cdot)|\mathscr{F}_{L_r-}] = 0 \quad \text{a.s.},$$

and replacing r by $r' \in (r, s)$, we get

$$\mathsf{E}[(\xi' - \eta')([s, t) \times \cdot)|\mathscr{G}_r] = 0 \quad \text{a.s.}.$$

Since $\mathsf{E}(\xi' + \eta')([0, 1] \times K) = 2 < \infty$, we finally obtain by dominated convergence,

$$\mathsf{E}[(\xi' - \eta')((r, s] \times \cdot)|\mathscr{G}_r] = 0 \quad \text{a.s.},$$

which means that $\xi' - \eta'$ is a (measure valued) $\mathscr{G}$-martingale.

To see that η' is $\mathscr{G}$-predictable, conclude from the definitions of η', T and L that, for any $s \in [0, 1]$ and $A \in \mathscr{B}(K)$,

$$\Delta\eta'_s(A) = \eta\{(t, x) \in \mathbb{R}_+ \times A; T_t = s\} = \Delta\eta_{L_s}(A) \cdot 1\{T \circ L_s = s\} . \qquad (16)$$

If $\Delta\bar{\eta}_{L_s} > 0$ for some s with $T \circ L_s = s$, there must exist some rational $s' \in (0, s)$ with $L_{s'} = L_s$, so $\Delta\bar{\eta}'$ is supported by the countable set

$$\{T \circ L_s; s \in \mathbb{Q} \cap (0, 1), T \circ L_s > s\} .$$

From (16) it is further seen that

$$\Delta\eta'_{T \circ L_s} = \Delta\eta_{L_s}, \qquad s \in (0, 1) . \qquad (16')$$

Since the continuous component of η' is automatically $\mathscr{G}$-predictable, it thus remains to show that, for every fixed $s \in (0, 1)$, the process

$$\Delta\eta_{L_s} 1\{s < T \circ L_s \leqq r\}, \qquad r \in [0, 1] ,$$

is $\mathscr{G}$-predictable. To see this, we note that $T \circ L_s \in \mathscr{F}_{L_s-} \subset \mathscr{G}_s$ by the $\mathscr{F}$-predictability of T. The restriction σ_s of $T \circ L_s$ to the set $\{T \circ L_s > s\}$ is then a $\mathscr{G}$-predictable stopping time $> s$, and it remains to notice that

$$\Delta\eta_{L_s} \in \mathscr{F}_{L_s-} \subset \mathscr{G}_s \subset \mathscr{G}_{\sigma_s-} .$$

To see that the discounted compensator ζ' of (σ, κ) equals $\zeta(T \times I)^{-1}$, conclude from (16) that each atom of $\bar{\eta}'$ is the image under T of a unique atom of $\bar{\eta}$. This shows in particular that $\bar{\eta}'^c = \bar{\eta}^c T^{-1}$. Since moreover $T^{-1}[0, s) = [0, L_s)$, we get

$$Z'_{s-} := \exp(-\bar{\eta}'^c_s) \prod_{r<s} (1 - \Delta\bar{\eta}'_r) = \exp(-\bar{\eta}^c_{L_s}) \prod_{t<L_s} (1 - \Delta\bar{\eta}_t) = Z_{L_s-} .$$

Writing $\bar{T} = T \times I$, we thus obtain

$$\zeta'(\mathrm{d}s\mathrm{d}x) = Z'_{s-}\eta'(\mathrm{d}s\mathrm{d}x) = Z_{L_s-}\eta\bar{T}^{-1}(\mathrm{d}s\mathrm{d}x) .$$

Since clearly

$$L \circ T_t = t, \qquad t \geqq 0 \quad \text{a.e.} \quad \bar{\eta} , \qquad (17)$$

it follows that for any $A \in \mathscr{B}([0,1] \times K)$,

$$\zeta'(A) = \int_A Z_{L_s-}\eta\bar{T}^{-1}(\mathrm{d}s\mathrm{d}x) = \int_{\bar{T}^{-1}A} Z_{L(T_t)-}\eta(\mathrm{d}t\mathrm{d}x) = \int_{\bar{T}^{-1}A} Z_{t-}\eta(\mathrm{d}t\mathrm{d}x)$$

$$= \int_{\bar{T}^{-1}A} \zeta(\mathrm{d}t\mathrm{d}x) = \int_A \zeta\bar{T}^{-1}(\mathrm{d}s\mathrm{d}x) = \zeta\bar{T}^{-1}(A)\,,$$

as asserted.

To prove the final identities, note first that

$$\bar{\zeta}'_\sigma = \bar{\zeta}T^{-1}[0,\sigma] = \bar{\zeta}\{t \geqq 0;\, T_t \leqq \sigma\} = \bar{\zeta}_\tau = T_\tau = \sigma\,.$$

Next we note that, for any $s \geqq 0$,

$$\bar{\zeta}'_s = \bar{\zeta}T^{-1}[0,s] = \bar{\zeta}T^{-1}\bigcap_{u>s}[0,u) = \bar{\zeta}\bigcap_{u>s}[0,L_u) = \inf_{u>s} T(L_u-)\,. \tag{18}$$

Now we have for s in the range of T,

$$s = T(L_s) \leqq T(L_u-) \leqq u, \quad u > s\,,$$

since in this case $L_u > L_s$ for $u > s$. Hence by (18), $\bar{\zeta}' \circ T_t = T_t$ for all $t \geqq 0$, and it follows that

$$\bar{\zeta}'\{s \geqq 0;\, \bar{\zeta}'_s \neq s\} = \bar{\zeta}\{t \geqq 0;\, \bar{\zeta}' \circ T_t \neq T_t\} = \bar{\zeta}(\emptyset) = 0\,.$$

To prove the second identity, we note that

$$L'_s \wedge T_\infty = T \circ L_s, \qquad s \geqq 0\,. \tag{19}$$

In fact, we get for $r < T \circ L_s$,

$$\bar{\zeta}'[0,r) = \bar{\zeta}[0,L_r) = T(L_r-) \leqq r < T \circ L_s \leqq s\,,$$

so $L'_s \geqq r$, and as $r \uparrow T \circ L_s$ we get $L'_s \geqq T \circ L_s$. Conversely, assuming that $L_s < \infty$ and letting $r > T \circ L_s$, we get $L_r > L(T \circ L_s) \geqq L_s$, so

$$\bar{\zeta}'[0,r] \geqq \bar{\zeta}'[0,r) = \bar{\zeta}[0,L_r) \geqq \bar{\zeta}[0,L_s] = T \circ L_s \geqq s\,,$$

whence $L'_s \leqq r$, and as $r \downarrow T \circ L_s$ we get $L'_s \leqq T \circ L_s$. It remains to notice that (19) is trivially true when $L_s = \infty$. From (19) we conclude by the definition of L that

$$L \circ L'_s \leqq L_s, \qquad s \geqq 0\,. \tag{20}$$

It is now easy to show that

$$T_t \leqq L'_s \quad \text{iff} \quad t \leqq L_s, \quad t \geqq 0 \quad \text{a.e.} \quad \bar{\zeta}\,. \tag{21}$$

In fact, $t \leqq L_s$ implies $T_t \leqq T \circ L_s \leqq L'_s$ in view of (19). Conversely, using (18) and (20), we get from $T_t \leqq L'_s$,

$$t = L \circ T_t \leqq L \circ L'_s = L_s, \qquad t \geqq 0 \quad \text{a.e.} \quad \bar{\zeta}\,.$$

By (21) we obtain for any $A \in \mathscr{B}(K)$,

$$\zeta'_{L'_s}(A) = \zeta'([0,L'_s] \times A) = \zeta\{(t,x) \in \mathbb{R}_+ \times A;\, T_t \leqq L'_s\}$$

$$= \zeta\{(t,x) \in \mathbb{R}_+ \times A;\, t \leqq L_s\} = \zeta_{L_s}(A)\,.$$

□

Lemma 5.6. *Let* (τ, κ) *be a marked stopping time in* $(0, \infty) \times K$ *with discounted compensator* ζ *satisfying* $\bar{\zeta}_\tau \equiv \tau$ *and* $\zeta\{t \geqq 0; \bar{\zeta}_t \neq t\} = 0$, *define* Z *and* L *as before, and assume that the process* ζ_{L_t} *is* $\mathscr{F}$*-adapted. Write*

$$T_{t,y} = t + y\Delta Z_t, \quad t \geqq 0, \; y \in [0, 1] ,$$

let γ *be* $U(0, 1)$ *and independent of* $\mathscr{F}$, *and put* $\sigma = T_{\tau,\gamma}$. *Denote by* $\mathscr{G}$ *the right-continuous filtration generated by* $\mathscr{F}$ *and by the process* $1\{\sigma \leqq t, \kappa \in \cdot\}$. *Then the pair* (σ, κ) *has discounted* $\mathscr{G}$*-compensator* $\zeta' := \pi_\sigma((\zeta \times \lambda)(T \times I)^{-1})$.

Proof. Since $\bar{\zeta}\{t \geqq 0; \bar{\zeta}_t \neq t\} = 0$, we have $T_{t,y} = 1 - Z_t + y\Delta Z_t$ for $\bar{\zeta}$-a.e. $t \geqq 0$ and for all $y \in [0, 1]$, so $\bar{\zeta}' := \zeta' \pi^{-1} = \pi_\sigma((\bar{\zeta} \times \lambda)T^{-1}) = \pi_\sigma \lambda$. Moreover, $\sigma = \tau + \gamma \Delta Z_\tau = 1 - Z_\tau + \gamma \Delta Z_\tau$, so Lemma 5.3 shows that σ has discounted $\mathscr{G}'$-compensator $\pi_\sigma \lambda = \bar{\zeta}'$, where $\mathscr{G}'$ is the right-continuous filtration generated by the σ-fields $\mathscr{F}_{L_s-}$ and by the process $1\{\sigma \leqq s, \kappa \in \cdot\}$. Now $s \leqq L_s$ in the present case, so $\mathscr{G}_s \subset \mathscr{G}'_s$, and therefore even the discounted $\mathscr{G}$-compensator of σ equals $\bar{\zeta}'$. It remains to show that the $\mathscr{G}$-compensator η' of the pair (σ, κ) equals

$$d\eta'_{t,x} = Y_t d\zeta'_{t,x}, \qquad t \geqq 0, \; x \in K , \tag{22}$$

for some process Y on $\mathbb{R}_+$, since the discounted $\mathscr{G}$-compensator must then be of the same form but with another process Y'_t, and projection onto $\mathbb{R}_+$ yields $Y' = 1$ a.e. $\bar{\zeta}'$.

To see this, we first consider the restrictions of τ and σ to the $\mathscr{F}$-predictable set $S = \{t \geqq 0; Z_{t-} = t\}$. For $t \in S$ we have $Z_t = t$ and $\Delta Z_t = 0$, so $\tau \in S$ implies $\sigma = \tau$. Note also that $\tau \neq \sigma \in S$ may occur only if $\gamma = 1$, which is a.s. excluded. Thus $\{\tau \in S\} = \{\sigma \in S\} = \{\sigma = \tau\}$ a.s., so it is enough to show that the $\mathscr{F}$- and $\mathscr{G}$-compensators of (τ, κ) agree a.s. on the set $S \times K$. But this is immediate from Lemma 3.5.

The remaining part of $(0, \tau]$ may be decomposed into disjoint intervals $I_j = (\alpha_j, \beta_j]$ of length $\beta_j - \alpha_j = Z_{\alpha_j} - Z_{\beta_j} = -\Delta Z_{\beta_j}$. Since the α_j with $I_j \neq \varnothing$ are exactly the jump times of the $\mathscr{F}$-adapted process $1 - Z \circ L$, we may choose the α_j to be a.s. distinct stopping times. Writing

$$\beta_j = \alpha_j + \Delta(Z \circ L)_{\alpha_j}, \qquad \rho_j := \zeta(\{\beta_j\} \times \cdot) = \Delta(\zeta_L)_{\alpha_j}, \qquad j \in \mathbb{N} ,$$

it is further seen that both β_j and ρ_j are $\mathscr{F}_{\alpha_j}$-measurable for each j. In particular, even the β_j are stopping times, and the intervals I_j are $\mathscr{F}$-predictable. Note also that a.s. $\{\tau = \beta_j\} = \{\tau \in I_j\} = \{\sigma \in I_j\}$, since $\bar{\zeta}_\tau = \tau$ while $\gamma \neq 1$ a.s. It remains to prove (22) on each inteval I_j.

Dropping the subscripts, we thus consider a stochastic interval $(\alpha. \beta]$ with an $\mathscr{F}_\alpha$-measurable β. Redefining α and β to be ∞ when $\alpha = \beta$, it is seen that β becomes predictable, and we may write

$$\rho := \mathsf{P}[\tau = \beta, \kappa \in \cdot | \mathscr{F}_{\beta-}] = \mathsf{P}[\tau = \beta, \kappa \in \cdot | \mathscr{F}_\alpha] \quad \text{a.s.} \tag{23}$$

Recall that a.s. $\{\tau = \beta\} = \{\tau \in (\alpha, \beta]\} = \{\sigma \in (\alpha, \beta]\}$, and that $\sigma = \beta - \gamma(\beta - \alpha)$ on $\{\tau = \beta\}$. Write $\tilde{\sigma}$ for the restriction of σ to the set $\{\tau = \beta\}$. Since the latter is $\mathscr{G}_\sigma$-measurable, $(\tilde{\sigma}, \kappa)$ is another marked $\mathscr{G}$-stopping time, and $\tilde{\sigma}$ is a.s. restricted to $(\alpha, \beta] \cup \{\infty\}$, so the compensator $\tilde{\eta}$ of $(\tilde{\sigma}, \kappa)$ is a.s. supported by the predictable set $(\alpha, \beta] \times K$. It remains to show that $\tilde{\eta} = \chi \times \rho$ a.s. on $(\alpha, \beta] \times K$ for some random measure χ on $(\alpha, \beta]$.

To see this, we are going to prove for any fixed $s < t$ that the random measure $\mathsf{P}[\tilde{\sigma} \leqq t, \kappa \in \cdot | \mathscr{G}_s]$ is a.s. proportional to ρ on the set $\{\alpha \leqq s < \beta \wedge \sigma\}$. Using

Doleans' L_1-approximation theorem for continuous compensators (cf. Rogers and Williams (1987), p. 373), which applies since σ and hence also $\tilde{\sigma}$ is totally inaccessible, we may conclude that $\tilde{\eta}$ has the stated form on every interval $(\alpha_n, \beta]$, where $\alpha_n = \inf\{k2^{-n} > \alpha; k \in \mathbb{N}\}$. It only remains to let $n \to \infty$, in order to get the result on $(\alpha, \beta]$.

To prove the statement about $\mathsf{P}[\tilde{\sigma} \leqq t, \kappa \in \cdot \,|\mathscr{G}_s]$, we note that by Lemma 3.5,

$$\frac{\mathsf{P}[\tilde{\sigma} \leqq t, \kappa \in \cdot\,|\mathscr{G}_s]}{\mathsf{P}[\tilde{\sigma} \leqq t|\mathscr{G}_s]} = \frac{\mathsf{P}[\tilde{\sigma} \leqq t, \kappa \in \cdot\,|\mathscr{F}_s]}{\mathsf{P}[\tilde{\sigma} \leqq t|\mathscr{F}_s]} \quad \text{a.s. on} \quad \{s \in [\alpha, \beta \wedge \sigma)\} .$$

Expressing the event $\{\tilde{\sigma} \leqq t\}$ in terms of τ, α, β and γ, and conditioning on $\mathscr{F}_s \vee \sigma\{\tau, \alpha, \beta\}$, it is clear from the independence of γ and $\mathscr{F}$ that

$$\frac{\mathsf{P}[\tilde{\sigma} \leqq t, \kappa \in \cdot\,|\mathscr{F}_s]}{\mathsf{P}[\tilde{\sigma} \leqq t|\mathscr{F}_s]} = \frac{\mathsf{P}[\tau = \beta, \kappa \in \cdot\,|\mathscr{F}_s]}{\mathsf{P}[\tau = \beta|\mathscr{F}_s]} \quad \text{a.s. on} \quad \{s \in [\alpha, \beta \wedge \tau)\} ,$$

so it suffices to show that

$$\mathsf{P}[\tau = \beta, \kappa \in \cdot\,|\mathscr{F}_s] = \rho \quad \text{a.s. on} \quad \{s \in [\alpha, \beta \wedge \tau)\} . \tag{24}$$

Let us then choose an announcing sequence of stopping times β_n for β, put $\beta_n' = \beta_n \vee \alpha$, and define $\alpha_n = (s \vee \alpha) \wedge \beta_n'$. Then $\mathscr{F}_s$ and $\mathscr{F}_{\alpha_n}$ agree on the set $\{s = \alpha_n\} = \{s \in [\alpha, \beta_n']\}$, so by (23),

$$\mathsf{P}[\tau = \beta, \kappa \in \cdot\,|\mathscr{F}_s] = \mathsf{P}[\tau = \beta, \kappa \in \cdot\,|\mathscr{F}_{\alpha_n}] = \rho \quad \text{a.s. on} \quad \{s \in [\alpha, \beta_n']\} ,$$

and (24) follows as we let $n \to \infty$. □

Proof of Proposition 5.4. Define $\tilde{\tau} = Y_\tau$ where $Y = 1 - Z$, and let $\tilde{\mathscr{F}}$ denote the right-continuous filtration generated by the σ-fields $\mathscr{F}_{L_s-}$. Then Lemma 5.5 shows that the pair $(\tilde{\tau}, \kappa)$ is a marked stopping time in $(0, 1] \times K$ with discounted compensator $\chi = \zeta(Y \times I)^{-1}$. Writing $\bar{\chi} = \chi(\cdot \times K)$ and $L_s = \inf\{r \geqq 0; \bar{\chi}_r \geqq s\}$, it is further seen that

$$\bar{\chi}_{\tilde{\tau}} = \tilde{\tau}, \qquad \bar{\chi}_{L_s} = \zeta_{L_s}, \qquad \bar{\chi}\{s \geqq 0; \bar{\chi}_s \neq s\} = 0 .$$

In particular, the second of these relations shows that $\bar{\chi}_{L_s}$ is adapted to $\tilde{\mathscr{F}}$.

Next we write $\tilde{Z}_s = 1 - \bar{\chi}_s$, and define

$$\tilde{T}_{t,y} = t + y\Delta\tilde{Z}_t, \qquad t \geqq 0,\ y \in [0, 1] .$$

Noting that $L \circ Y_t = t$ for t outside the $\mathscr{F}$-predictable set

$$A = \bigcup_{r \in \mathbb{Q}_+} \{t > r; Z_t = Z_r\} ,$$

we get in view of (16′),

$$\tilde{T}_{Y_t, y} = Y_t + y\Delta\tilde{Z}_{Y_t} = Y_t + y\Delta Z_{L \circ Y_t} = Y_t + y\Delta Z_t = T_{t,y}, \quad t \in A^c,\ y \in [0, 1] . \tag{25}$$

Since $\bar{\zeta}(A) = 0$ and therefore $\tau \in A^c$ a.s., it follows that $\sigma = \tilde{T}_{\tilde{\tau}, \gamma}$ a.s. Note also that $\mathscr{G}$ agrees with the right-continuous filtration generated by $\tilde{\mathscr{F}}$ and by the process $1\{\sigma \leqq s, \kappa \in \cdot\}$. Thus Lemma 5.6 shows that the pair (σ, κ) has discounted $\mathscr{G}$-compensator

$$\zeta' = \pi_\sigma((\chi \times \lambda)(\tilde{T} \times I)^{-1}) = \pi_\sigma((\zeta \times \lambda)(\tilde{T} \circ (Y \times I) \times I)^{-1}) ,$$

which by (25) agrees with the expression in (14). □

As we have seen, much of the technical difficulties in this section arose from the fact that we insist on working with right-continuous transformations of our processes and left-continuous mappings of the associated filtrations. Reversing this (as in Jacod (1979), p. 321) gives a smoother theory, which unfortunately doesn't seem to apply here, for various reasons.

6. Integral representations and random distributions

In this section we retain the framework and conventions of Sects. 1 and 4. Thus we consider arbitrary marked stopping times (τ, κ) in $(0, \infty) \times K$ with compensators η and their discounted versions ζ. Our aim is to establish the previously announced integral representation for the distribution of (τ, κ, η), as well as the equivalent extension of ζ to a random probability distribution for (τ, κ).

As in Sect. 1, we shall write P_μ for the distribution of (τ, κ, η) when (τ, κ) has distribution μ, while η is the compensator of (τ, κ) with respect to the induced filtration. Here μ is an arbitrary element of $\mathcal{M}_1((0, \infty) \times K)$, the space of all probability measures on $(0, \infty) \times K$. Recall that $\mathcal{M}_1((0, \infty) \times K)$ is endowed with the σ-field generated by all projections $m \mapsto m(B)$ for arbitrary $B \in \mathcal{B}((0, \infty) \times K)$. We may now state the main result of the paper.

Theorem 6.1. *Let (τ, κ) be a marked stopping time in $(0, \infty) \times K$ with compensator η. Then*

$$\mathsf{P}(\tau, \kappa, \eta)^{-1} = \int P_\mu \nu(\mathrm{d}\mu) \tag{1}$$

for some probability measure ν on $\mathcal{M}_1((0, \infty) \times K)$, and ν is uniquely determined by $\mathsf{P}\eta^{-1}$. Moreover, any measure ν as above may occur in (1).

For the last statement, we emphasize that the underlying filtered probability space $(\Omega, \mathcal{F}, \mathsf{P})$ is *not* regarded as fixed. To state the equivalent extension theorem for discounted compensators, recall that a random probability measure on $(0, \infty) \times K$ is a random element in the space $\mathcal{M}_1((0, \infty) \times K)$. As before, we write π_t for restriction of such measures to the subset $(0, t] \times K$.

Theorem 6.2. *Let (τ, κ) be a marked stopping time in $(0, \infty) \times K$ with discounted compensator ζ. Then there exists (on a suitable extension of the probability space) some random probability measure $\hat{\zeta}$ on $(0, \infty) \times K$, such that a.s.*

$$\zeta = \pi_\tau \hat{\zeta} \quad \textit{and} \quad \mathsf{P}[(\tau, \kappa) \in \cdot \mid \hat{\zeta}] = \hat{\zeta}\,. \tag{2}$$

Moreover, $\mathsf{P}\hat{\zeta}^{-1}$ is uniquely determined by $\mathsf{P}\zeta^{-1}$, and it agrees with the measure ν in Theorem 6.1.

To provide motivation, we shall first establish the equivalence between (1) and (2). The existence of the random measure $\hat{\zeta}$ in Theorem 6.2 will then be established via some further lemmas.

Lemma 6.3. *Let (τ, κ, ζ) be a random element in $\mathbb{R}_+ \times K \times \mathcal{M}(\mathbb{R}_+ \times K)$. Then*

$$\mathsf{P}(\tau, \kappa, \zeta)^{-1} = \int \mu\{(t, x) \in \mathbb{R}_+ \times K; (t, x, \pi_t \mu) \in \cdot\} \nu(\mathrm{d}\mu) \tag{3}$$

for some probability measure ν on $\mathcal{M}_1(\mathbb{R}_+ \times K)$, iff

$$\zeta = \pi_\tau \hat{\zeta} \quad \textit{and} \quad \mathsf{P}[(\tau, \kappa) \in \cdot \mid \hat{\zeta}] = \hat{\zeta} \qquad \textit{a.s.} \tag{4}$$

for some random probability measure $\hat{\zeta}$ on $\mathbb{R}_+ \times K$. In that case $\nu = \mathsf{P}\hat{\zeta}^{-1}$, and ν is uniquely determined by $\mathsf{P}\zeta^{-1}$.

Proof. Assume that (4) holds for some random probability measure $\hat{\zeta}$ on $\mathbb{R}_+ \times K$, and define $\nu = \mathsf{P}\hat{\zeta}^{-1}$. We then obtain (3) by writing

$$\begin{aligned}\mathsf{P}\{(\tau, \kappa, \zeta) \in \cdot\} &= \mathsf{P}\{(\tau, \kappa, \pi_\tau \hat{\zeta}) \in \cdot\} = \mathsf{E}\mathsf{P}[(\tau, \kappa, \pi_\tau \hat{\zeta}) \in \cdot \mid \hat{\zeta}] \\ &= \mathsf{E}\hat{\zeta}\{(t, x) \in \mathbb{R}_+ \times K; (t, x, \pi_t \hat{\zeta}) \in \cdot\} \\ &= \int \mu\{(t, x) \in \mathbb{R}_+ \times K; (t, x, \pi_t \mu) \in \cdot\} \nu(d\mu) .\end{aligned} \tag{5}$$

Assume conversely that (3) holds for some probability measure ν on $\mathcal{M}_1(\mathbb{R}_+ \times K)$. Choose random elements ζ' in $\mathcal{M}_1(\mathbb{R}_+ \times K)$ and (τ', κ') in $\mathbb{R}_+ \times K$, such that $\mathsf{P}\zeta'^{-1} = \nu$ and $\mathsf{P}[(\tau', \kappa') \in \cdot \mid \zeta'] = \zeta'$ a.s. Proceeding as in (5), it is seen that the distribution of the triple $(\tau', \kappa', \pi_{\tau'} \zeta')$ is given by the right-hand side of (3), so $(\tau', \kappa', \pi_{\tau'} \zeta') \stackrel{d}{=} (\tau, \kappa, \zeta)$. By Lemma 3.1, we may then choose another triple $(\hat{\tau}, \hat{\kappa}, \hat{\zeta}) \stackrel{d}{=} (\tau', \kappa', \zeta')$, such that $(\hat{\tau}, \hat{\kappa}, \pi_{\hat{\tau}} \hat{\zeta}) = (\tau, \kappa, \zeta)$ a.s. In particular we get $(\tau, \kappa, \hat{\zeta}) \stackrel{d}{=} (\tau', \kappa', \zeta')$ and $\pi_\tau \hat{\zeta} = \zeta$ a.s., so (4) follows. Note also that $\mathsf{P}\hat{\zeta}^{-1} = \mathsf{P}\zeta'^{-1} = \nu$.

To prove the uniqueness of ν, write $Y_t = \zeta_t(K)$ and $\hat{Y}_t = \hat{\xi}_t(K)$, and define

$$L_s = \inf\{t \geqq 0;\ Y_t \geqq s\}, \qquad \hat{L}_s = \inf\{t \geqq 0;\ \hat{Y}_t \geqq s\}, \qquad s \in [0, 1) .$$

Next choose the random variable σ to be $U(0, 1)$ and independent of $\hat{\zeta}$, and put $\hat{\tau} = \hat{L}_\sigma$. Then (4) shows that $(\hat{\tau}, \hat{\zeta}) \stackrel{d}{=} (\tau, \hat{\zeta})$, and since clearly $\hat{\tau} = g(\sigma, \hat{\zeta})$ for some measurable function g: $[0, 1) \times \mathcal{M}_1(\mathbb{R}_+ \times K) \to \mathbb{R}_+$, there exists by Lemma 3.1 some random pair $(\sigma', \zeta') \stackrel{d}{=} (\sigma, \hat{\zeta})$, such that $(g(\sigma', \zeta'), \zeta') = (\tau, \hat{\zeta})$ a.s. In particular, $(\sigma', \hat{\zeta}') \stackrel{d}{=} (\sigma, \hat{\zeta})$ and $g(\sigma', \hat{\zeta}) = \tau$ a.s. Thus σ may be chosen such that in addition $\tau = \hat{L}_\sigma$ a.s. Then a.s.

$$\begin{aligned}\mathsf{P}[\sigma \geqq s, \tau \in B \mid \hat{\zeta}] &= \lambda\{x \in [s, 1]; \hat{L}_x \in B\} \\ &= \int_B \left(1\{s \leqq \hat{Y}_{t-}\} + \frac{\hat{Y}_t - s}{\Delta \hat{Y}_t} 1\{\hat{Y}_{t-} < s \leqq \hat{Y}_t\} \right) \lambda\{x \in [0, 1]; \hat{L}_x \in dt\} \\ &= \mathsf{E}\left[\left(1\{s \leqq \hat{Y}_{\tau-}\} + \frac{\hat{Y}_\tau - s}{\Delta \hat{Y}_\tau} 1\{\hat{Y}_{\tau-} < s \leqq \hat{Y}_\tau\} \right); \tau \in B \,\middle|\, \hat{\zeta} \right],\end{aligned}$$

and since $\hat{Y} = Y$ a.s. on $[0, \tau]$, we get

$$\mathsf{P}[\sigma \geqq s \mid \tau, \hat{\zeta}] = 1\{s \leqq Y_{\tau-}\} + \frac{Y_\tau - s}{\Delta Y_\tau} 1\{Y_{\tau-} < s \leqq Y_\tau\} \quad \text{a.s.} \tag{6}$$

Note also that $\pi_{\hat{L}_s} \hat{\zeta} = \pi_{L_s} \zeta$ a.s. on $\{s \leqq Y_\tau\}$. Using these facts and the independence of σ and $\hat{\zeta}$, we get for any $s \in [0, 1)$,

$$\begin{aligned}(1 - s)\mathsf{P}\{\pi_{\hat{L}_s} \hat{\zeta} \in \cdot\} &= \mathsf{P}\{\sigma \geqq s, \pi_{\hat{L}_s} \hat{\zeta} \in \cdot\} \mathsf{E}[\mathsf{P}[\sigma \geqq s \mid \hat{\zeta}, \tau]; \pi_{\hat{L}_s} \hat{\zeta} \in \cdot] \\ &= \mathsf{P}\{s \leqq Y_{\tau-}, \pi_{L_s} \zeta \in \cdot\} + \mathsf{E}[(Y_\tau - s)/\Delta Y_\tau;\ s \in (Y_{\tau-}, Y_\tau], \pi_{L_s} \zeta \in \cdot] .\end{aligned}$$

Since τ is a.s. the last point of increase of Y, the right-hand side is uniquely determined by $\mathsf{P}\zeta^{-1}$, and hence so is the distribution of $\pi_{\hat{L}_s} \hat{\zeta}$ for each $s \in [0, 1)$. The same thing is then true for $\nu = \mathsf{P}\hat{\zeta}^{-1}$, since $\pi_{\hat{L}_s} \hat{\zeta} \uparrow \hat{\zeta}$ as $s \uparrow 1$. □

Weaker versions of Theorem 6.2 are established in the next two lemmas.

Lemma 6.4. *Let (τ, κ) be a marked stopping time in $(0, \infty) \times K$ with discounted compensator ζ. Then there exists some random probability measure $\hat{\zeta}$ on $\mathbb{R}_+ \times K$, such that*

$$\zeta = \pi_\tau \hat{\zeta} \quad and \quad \mathsf{P}[\tau \in \cdot \,|\, \hat{\zeta}] = \hat{\zeta}\pi^{-1} \quad a.s. \tag{7}$$

Proof. Put $Y_t = 1 - Z_t = \bar{\zeta}_t = \zeta_t(K) = \zeta([0, t] \times K)$ as before, and define

$$L_s = \inf\{t \geqq 0;\ Y_t \geqq s\}, \qquad D_s = (Y \circ L)_{s-}, \qquad s \in [0, 1)\,,$$

$$T_{t,y} = Y_{t-} + y\Delta Y_t, \qquad (T \times I)_{t,x,y} = (T_{t,y}, x), \qquad t \geqq 0,\quad x \in K,\ y \in [0, 1]\,.$$

Introduce a $U(0, 1)$ random variable γ independent of $\mathscr{F}$, and define

$$\sigma = T_{\tau,\gamma}, \qquad \chi = (\zeta \times \lambda)(T \times I)^{-1}\,, \tag{8}$$

so that χ becomes a random measure on $[0, \sigma] \times K$ with $\chi\pi^{-1} = \lambda$ on $[0, \sigma]$. Letting $\mathscr{G}$ denote the filtration generated by the σ-fields $\mathscr{F}_{L_s-}$ and the process $1\{\sigma \leqq s\}$, it is further seen from Lemma 5.3 that σ is $U(0, 1)$ and a pure $\mathscr{G}$ stopping time, in the sense of Sect. 3.

Next we note that the processes L_s, D_s and χ_s are non-decreasing and left-continuous, in case of χ with respect to the weak topology for measures on K. Note also that L_s and D_s are $\mathscr{F}_{L_s-}$-measurable, the latter because of the predictability of Y. Even χ_s is in fact $\mathscr{F}_{L_s-}$-measurable, as may be seen from the formula

$$\chi_s(A) = \zeta_{L_s}(A) - \frac{Y_{L_s} - s}{Y_{L_s} - Y_{L_s-}}\, \Delta\zeta_{L_s}(A) 1\{L_s < \infty\}, \qquad A \in \mathscr{B}(K)\,.$$

Thus the processes L, D and χ are adapted to $\mathscr{G}$. By Lemma 3.2, there exist some left-continuous processes $\hat{L}$, $\hat{D}$ and $\hat{\chi}$ independent of σ, such that a.s.

$$\hat{L}_s = L_s, \qquad \hat{D}_s = D_s, \qquad \hat{\chi}_s = \chi_s, \qquad s \in [0, \sigma]\,. \tag{9}$$

Conditioning on the event $\{\sigma \geqq s\}$ for arbitrary $s \in [0, 1)$, it is seen that the processes $\hat{L}_s$, $\hat{D}_s$ and $\hat{\chi}_s$ are a.s. non-decreasing, and further that $\hat{\chi}_s$ is a.s. continuous. By a standard extension argument, we may then define a random measure $\hat{\chi}$ on $[0, 1) \times K$, such that a.s. $\hat{\chi}_s(A) = \hat{\chi}([0, s] \times A)$ for all $s \in [0, 1)$ and $A \in \mathscr{B}(K)$. From the conditioning on $\{\sigma \geqq s\}$, it is also seen that $\hat{\chi}\pi^{-1} = \lambda$ a.s., and in particular that $\hat{\chi}([0, 1) \times K) = 1$ a.s.

Our next aim is to establish the a.s. relations

$$L_s = \hat{L}_s = \tau, \qquad \chi_s = \hat{\chi}_s\,, \qquad s \in [\sigma, Y_\tau]\,, \tag{10}$$

$$L_s \wedge \hat{L}_s > \tau, \qquad\qquad s > Y_\tau\,. \tag{11}$$

We then note that τ is a.s. the last point of increase of Y, and that $\gamma > 0$ a.s. Using the definition of L_s, we thus obtain $L_s = \tau$ a.s. on $\{s \in [\sigma, Y_\tau]\}$ and $L_s = \infty$ a.s. on $\{s > Y_\tau\}$. To prove the remaining relations in (10) and (11), we may first assume that $\Delta Y_\tau = 0$. Then (10) follows from (9), and to prove (11) we need to show only that $\hat{L}_s > \tau$ for $s > Y_\tau$. Now $\Delta Y_\tau = 0$ implies $L_s = \hat{L}_s < L_\sigma = \tau$ a.s. for all $s < \sigma$, so if $\hat{L}_s = \tau$ for some $s > \sigma$, then σ must be the left endpoint of an interval where $\hat{L}$ is constant. But this is a.s. excluded by the independence of σ and $\hat{L}$, since there are only countably many such points for a fixed $\hat{L}$.

Turning to the case when $\Delta Y_\tau > 0$, we note that

$$D_{s+} = \inf\{t \geqq s; L_t > L_{s+}\}, \qquad s \in [0, \sigma) .$$

From (9) it follows by conditioning on σ that also

$$D_{s+} = \inf\{t \geqq s; \hat{L}_t > \hat{L}_{s+}\}, \qquad s \in [0, 1) .$$

Comparing these relations and using (9), we get

$$\inf\{t \geqq s; \hat{L}_t > L_{s+}\} = \inf\{t \geqq s; L_t > L_{s+}\}, \qquad s \in [0, \sigma) .$$

Now $\Delta Y_\tau > 0$ implies a.s. that $L_s = \tau$ for all sufficiently large $s < \sigma$, so for any such s,

$$\inf\{t \geqq s; \hat{L}_t > \tau\} = \inf\{t \geqq s; L_t > \tau\} = Y_\tau \quad \text{a.s.} ,$$

which shows that $\hat{L}$ satisfies (10) and (11) a.s.

To prove the second part of (10) when $\Delta Y_\tau > 0$, we note that χ_s increases linearly on every interval where L is constant. By conditioning on σ, we get the same property for the processes $\hat{\chi}$ and $\hat{L}$. Since L is constant on $(Y_{\tau-}, Y_\tau]$, and since $\hat{L} = L$ a.s. on the same interval by the relevant parts of (9) and (10), it follows that both χ_s and $\hat{\chi}_s$ increase linearly on the mentioned interval. Moreover, χ and $\hat{\chi}$ agree a.s. on the a.s. non-empty subinterval $(Y_{\tau-}, \sigma]$. Thus $\chi = \hat{\chi}$ a.s. on the entire interval. This completes the proof of (10) and (11).

We may now define the random probability measure $\hat{\zeta}$ on $\mathbb{R}_+ \times K$ by

$$\hat{\zeta} = \hat{\chi}(\hat{L} \times I)^{-1} , \tag{12}$$

where $(\hat{L} \times I)_{s,x} = (\hat{L}_s, x)$. Noting that

$$L \circ T_{t,y} = t, \qquad (t, y) \in \mathbb{R}_+ \times [0, 1] \quad \text{a.e.} \quad \zeta\pi^{-1} \times \lambda ,$$

and using (8)–(12), we get for any $t \in [0, \tau]$ and $A \in \mathscr{B}(K)$,

$$\begin{aligned} \hat{\zeta}_t(A) &= \hat{\chi}\{(s, x) \in [0, 1) \times A; \hat{L}_s \leqq t\} \\ &= \chi\{(s, x) \in [0, 1) \times A; L_s \leqq t\} \\ &= (\zeta \times \lambda)\{(u, x, y) \in \mathbb{R}_+ \times A \times [0, 1]; L \circ T_{u,y} \leqq t\} = \zeta_t(A), \end{aligned}$$

which shows that $\hat{\zeta} = \zeta$ a.s. on $[0, \tau] \times K$. Since $\zeta = 0$ on $(\tau, \infty) \times K$, this proves the first relation in (7). To prove the second one, we write

$$\begin{aligned} \mathsf{P}[\tau \in \cdot \,|\hat{\zeta}] &= \mathsf{P}[\hat{L}_\sigma \in \cdot\,|\hat{\zeta}] = \mathsf{E}[\mathsf{P}[\hat{L}_\sigma \in \cdot\,|\hat{\zeta}, \hat{L}]|\hat{\zeta}] = \mathsf{E}[\lambda\hat{L}^{-1}|\hat{\zeta}] \\ &= \mathsf{E}[(\hat{\chi}\pi^{-1})\hat{L}^{-1}|\hat{\zeta}] = \mathsf{E}[\hat{\zeta}\pi^{-1}|\hat{\zeta}] = \hat{\zeta}\pi^{-1}. \end{aligned}$$

□

Lemma 6.5. *The random measure $\hat{\zeta}$ of Theorem* 6.2 *exists when τ is totally inaccessible.*

Proof. Choose $\hat{\zeta}$ as in Lemma 6.4, and introduce some random elements τ' in $(0, \infty)$ and κ' in K, such that

$$\mathsf{P}[(\tau', \kappa') \in \cdot\,|\hat{\zeta}] = \hat{\zeta} \quad \text{a.s.} \tag{13}$$

Then $(\tau', \hat{\zeta}) \overset{d}{=} (\tau, \hat{\zeta})$, so by Lemma 3.1 there exists some triple $(\tau'', \kappa'', \zeta'') \overset{d}{=} (\tau', \kappa', \hat{\zeta})$ with $(\tau'', \zeta'') = (\tau, \hat{\zeta})$ a.s. But then $(\tau, \kappa'', \hat{\zeta}) \overset{d}{=} (\tau', \kappa', \hat{\zeta})$, so (13) remains true for the pair (τ, κ''), which shows that we may take $\tau' = \tau$ in (13). We shall prove below that the pair (τ, κ') has $\mathscr{G}$-compensator η, where $\mathscr{G}$ is the right-continuous and complete filtration generated by η and (τ, κ'). Then Lemma 3.3 yields $(\tau, \kappa', \eta) \stackrel{d}{=} (\tau, \kappa, \eta)$, so there exists by Lemma 3.1 some triple $(\tilde{\tau}, \tilde{\kappa}, \tilde{\zeta}) \stackrel{d}{=} (\tau, \kappa', \hat{\zeta})$ with $(\tilde{\tau}, \tilde{\kappa}, \pi_{\tilde{\tau}}\tilde{\zeta}) = (\tau, \kappa, \zeta)$ a.s. In particular, $\pi_\tau \tilde{\zeta} = \zeta$ a.s. and $(\tau, \kappa, \tilde{\zeta}) \stackrel{d}{=} (\tau, \kappa, \hat{\zeta})$, so (13) remains true for the triple $(\tau, \kappa, \tilde{\zeta})$. Thus (2) holds with $\hat{\zeta}$ replaced by $\tilde{\zeta}$.

To see that the $\mathscr{G}$-compensator of (τ, κ') equals η, let us fix a $t \geqq 0$ and a measurable function $f: (t, \infty) \times K \to \mathbb{R}_+$, and write $\mathscr{G}_t^o = \sigma\{\pi_t \zeta, \tau > t\}$. Using (13) and Lemmas 3.5 and 4.2, we get a.s. on $\{\tau > t\}$,

$$\begin{aligned} \mathsf{E}[\textstyle\int f d\eta \mid \mathscr{G}_t^o] &= \mathsf{E}[\textstyle\iint f_{s,x} Z_{s-}^{-1} \zeta(dsdx) \mid \mathscr{G}_t^o] \\ &= \mathsf{E}[\textstyle\iint f_{s,x} \hat{Z}_{s-}^{-1} 1\{s \leqq \tau\} \hat{\zeta}(dsdx) \mid \mathscr{G}_t^o] \\ &= \mathsf{E}[\textstyle\iint f_{s,x} \hat{Z}_{s-}^{-1} \mathsf{P}[s \leqq \tau \mid \hat{\zeta}, \tau > t] \hat{\zeta}(dsdx) \mid \mathscr{G}_t^o] \\ &= \mathsf{E}[\textstyle\iint f_{s,x} \hat{Z}_t^{-1} \hat{\zeta}(dsdx) \mid \mathscr{G}_t^o] = Z_t^{-1} \mathsf{E}[\textstyle\int f d\hat{\zeta} \mid \mathscr{G}_t^o], \end{aligned}$$

where $\hat{Z}_t = \hat{\zeta}((t, \infty) \times K)$. Hence by (13) and Lemma 3.5,

$$\mathsf{E}[f(\tau, \kappa') \mid \mathscr{G}_t^o] = \mathsf{E}[\mathsf{E}[f(\tau, \kappa') \mid \hat{\zeta}, \tau > t] \mid \mathscr{G}_t^o] = \mathsf{E}[\hat{Z}_t^{-1} \textstyle\int f d\hat{\zeta} \mid \mathscr{G}_t^o] = \mathsf{E}[\textstyle\int f d\eta \mid \mathscr{G}_t^o],$$

a.s. on $\{\tau > t\}$, which shows that the measure valued process $1\{\tau \leqq t, \kappa \in \cdot\} - \eta_t$ is a martingale with respect to the induced filtration. By right-continuity, the martingale property extends to the filtration $\mathscr{G}$, and it remains to notice that η is adapted continuous and hence $\mathscr{G}$-predictable. □

Proof of Theorem 6.2. In view of Lemma 6.3, it suffices to prove the existence of $\hat{\zeta}$. Let us then put $Y = 1 - Z$, and note by Lemma 3.4 that the discounted compensator ζ' of the marked stopping time $(\tau, \kappa') := (\tau, Y_\tau, \tau, \kappa)$ equals the image of ζ under the mapping $(t, x) \mapsto (t, Y_t, t, x)$. Assuming the assertion to be true for (τ, κ'), so that there exists some random probability measure $\hat{\zeta}'$ on $\mathbb{R}_+ \times [0, 1] \times \mathbb{R}_+ \times K$ satisfying

$$\zeta' = \pi_\tau \hat{\zeta}' \quad \text{and} \quad \mathsf{P}[(\tau, \kappa') \in \cdot \mid \hat{\zeta}'] = \hat{\zeta}' \quad \text{a.s.},$$

it is clear that the projection of $\hat{\zeta}'$ onto $\mathbb{R}_+ \times K$ satisfies (2). It is thus enough to prove the assertion with κ replaced by κ' and K by $K' := [0, 1] \times \mathbb{R}_+ \times K$.

Let us then define

$$T_{t,y} = Y_t - y\Delta Y_t, \qquad t \geqq 0, \; y \in [0, 1], \tag{14}$$

let γ be a $U(0, 1)$ random variable independent of $\mathscr{F}$, and put $\sigma = T_{\tau,\gamma}$. Then Proposition 5.4 shows that, under a suitable choice of filtration $\mathscr{G}$, the pair (σ, κ') becomes a marked stopping time with discounted compensator

$$\zeta' := \pi_\sigma((\zeta \times \lambda)(T \times I)^{-1}), \tag{15}$$

where $(T \times I)_{t,x,y} = (T_{t,y}, x)$ as before. In particular, the discounted compensator of σ equals

$$\zeta' \pi^{-1} = \pi_\sigma((\bar{\zeta} \times \lambda) T^{-1}) = \pi_\sigma \lambda \quad \text{a.s.}, \tag{16}$$

so σ is $U(0, 1)$, pure, and totally inaccessible. Hence there exists by Lemma 6.5 some random probability measure $\hat{\zeta}'$ on $\mathbb{R}_+ \times K'$ with

$$\zeta' = \pi_\sigma \hat{\zeta}' \quad \text{and} \quad \mathsf{P}[(\sigma, \kappa') \in \cdot \mid \hat{\zeta}'] = \hat{\zeta}' \quad \text{a.s.} \tag{17}$$

Projecting this onto the time scale and noting that by (16) also

$$\zeta'\pi^{-1} = \pi_\sigma \lambda \quad \text{and} \quad \mathsf{P}[\sigma \in \cdot \,|\lambda] = \lambda \quad \text{a.s.}\,,$$

we may conclude by the uniqueness assertion in Lemma 6.3 that

$$\hat{\zeta}'\pi^{-1} = \lambda \quad \text{a.s.} \tag{18}$$

Combining this with (17) yields $\mathsf{P}[\sigma \in \cdot \,|\hat{\zeta}'] = \lambda$ a.s., which shows that σ and $\hat{\zeta}$ are independent.

From (14) and (15) it is clear that, if $\zeta' = \zeta'(\omega)$ is of the form $\lambda \times \delta_y \times \delta_t \times \nu$ on some non-empty rectangle $(a, b) \times K'$, then $Y_{t-} \leqq a < b \leqq Y_t = y$ and $\zeta(\{t\} \times \cdot) = \nu$, so ζ' must have the same form on $(a, y \wedge \sigma) \times K'$. This shows that $\zeta' \in B_\sigma$, where the sets B_s are defined as in Lemma 3.8, and from (17) we then obtain $\hat{\zeta}' \in B_s$ a.s. on $\{\sigma \geqq s\}$ for every $s \in (0, 1)$. By the measurability of the B_s and the independence between σ and $\hat{\zeta}'$, it follows that $\hat{\zeta}' \in B_s$ a.s. for every $s \in (0, 1)$, and then also for $s = 1$. Now it is clear from (14) that, for a suitable random measure ν on K,

$$(\zeta \times \lambda)(T \times I)^{-1} = \lambda \times \delta_{Y_\tau} \times \delta_\tau \times \nu \quad \text{on } (Y_{\tau-}, Y_\tau] \times K'\,,$$

and combining (15) and (17), we get

$$\pi_\sigma \hat{\zeta}' = \pi_\sigma((\zeta \times \lambda)(T \times I)^{-1}) \quad \text{a.s.}\,, \tag{19}$$

so $\hat{\zeta}'$ has a.s. the same representation on $(Y_{\tau-}, \sigma] \times K'$. Since $\hat{\zeta}' \in B_1$ a.s., the latter representation extends a.s. to $(Y_{\tau-}, Y_\tau] \times K'$, so (19) extends to

$$\pi_{Y_\tau} \hat{\zeta}' = (\zeta \times \lambda)(T \times I)^{-1} \quad \text{a.s.} \tag{20}$$

Let us next define the functions t_μ and y_μ and the sets C_s as in Lemma 3.9, and note that for $r < \sigma$, the function $t_{\zeta'}(r+)$ agrees with the right-continuous inverse R_r of Y, while $y_{\zeta'}(r+) = Y \circ t_{\zeta'}(r+) = Y \circ R_r$. It follows that for any $t \leqq t_{\zeta'}(r+)$ and $x \in [0, 1]$,

$$T_{t,x} \leqq Y_t \leqq Y \circ t_{\zeta'}(r+) = y_{\zeta'}(r+)\,,$$

so

$$(T \times I)^{-1}((y_{\zeta'}(r+), 1] \times [0, 1] \times [0, t_{\zeta'}(r+)] \times K) = \varnothing\,,$$

and therefore $\zeta' \in C_\sigma$ by (15). Arguing as for the sets B_s, we may conclude that $\hat{\zeta}' \in C_1$ a.s. In particular,

$$\hat{\zeta}'((y_{\hat{\zeta}'}(\sigma+), 1] \times [0, 1] \times [0, t_{\hat{\zeta}'}(\sigma+)] \times K) = 0 \quad \text{a.s.}$$

Now the independence of $\hat{\zeta}'$ and σ implies that $y_{\hat{\zeta}'}$ and $t_{\hat{\zeta}'}$ are a.s. continuous at σ, and moreover τ is a.s. the last point of increase of Y, so we get a.s.

$$t_{\hat{\zeta}'}(\sigma+) = t_{\hat{\zeta}'}(\sigma) = t_{\zeta'}(\sigma) = \tau, \qquad y_{\hat{\zeta}'}(\sigma+) = y_{\hat{\zeta}'}(\sigma) = y_{\zeta'}(\sigma) = Y_\tau\,.$$

Hence

$$\hat{\zeta}'((Y_\tau, 1] \times [0, 1] \times [0, \tau] \times K) = 0 \quad \text{a.s.} \tag{21}$$

Let us now define $\hat{\zeta} = \hat{\zeta}' h^{-1}$ where $h(s, y, t, x) := (t, y, t, x)$, and conclude from (17) that

$$\mathsf{P}[(\tau, \kappa') \in \cdot \,|\hat{\zeta}] = \mathsf{E}[\mathsf{P}[h(\sigma, \kappa') \in \cdot \,|\hat{\zeta}']|\hat{\zeta}] = \mathsf{E}[\hat{\zeta}' h^{-1}|\hat{\zeta}] = \hat{\zeta} \qquad \text{a.s.}$$

Since $h \circ (T \times I)$ is merely projection of $\mathbb{R}_+ \times K' \times [0, 1]$ onto $\mathbb{R}_+ \times K'$, it is further seen from (20) that

$$(\pi_{Y_\tau} \hat{\zeta}')h^{-1} = (\zeta \times \lambda)(T \times I)^{-1} h^{-1} = \zeta \quad \text{a.s.}\,,$$

so by (21) we get a.s.

$$\begin{aligned}\pi_\tau \hat{\zeta} &= \hat{\zeta}' h^{-1}(\cdot \cap ([0, \tau] \times K')) = \hat{\zeta}'(h^{-1}(\cdot) \cap ([0, 1]^2 \times [0, \tau] \times K)) \\ &= \hat{\zeta}'(h^{-1}(\cdot) \cap ([0, Y_\tau] \times [0, 1] \times [0, \tau] \times K)) \\ &= (\pi_{Y_t} \hat{\zeta}')h^{-1}(\cdot \cap ([0, \tau] \times K')) = \pi_\tau \zeta = \zeta\,.\end{aligned}$$

Thus $\hat{\zeta}$ has the required properties. □

Proof of Theorem 6.1. In view of Theorem 6.2 and Lemma 6.3, it remains to show that any probability measure on $\mathcal{M}_1((0, \infty) \times K)$ may occur as the distribution of $\hat{\zeta}$ in (2). But given any such measure ν, we may take $\hat{\zeta}$ to be a random probability measure on $(0, \infty) \times K$ with distribution ν, and let τ and κ be such that $\mathsf{P}[(\tau, \kappa) \in \cdot \,|\, \hat{\zeta}] = \hat{\zeta}$. Choosing $\mathcal{F}$ to be the filtration generated by the process $(\hat{\zeta}, 1\{\tau \leqq t, \kappa \in \cdot\})$, $t \geqq 0$, it is clear that (τ, κ) becomes a marked stopping time in $(0, \infty) \times K$, whose compensator is given by (1.1) with μ replaced by $\hat{\zeta}$. Hence the discounted compensator of (τ, κ) equals $\pi_\tau \hat{\zeta}$, and (2) follows. □

Let us finally remark that, although Lemma 6.4 was stated and proved for arbitrary marked stopping times (τ, κ), it was only used in the special case when τ is totally inaccessible. Likewise, Lemma 3.2 was proved for marked stopping times, but it was only applied in a markless situation. Finally, we proved the uniqueness assertion in Lemma 6.3 without taking advantage of Lemma 3.3. The resulting redundancy in those three cases was intentional. Indeed, we feel that there ought to be a simpler way of proving Theorem 6.1, where the rather awkward time change arguments of Sect. 5 are avoided. The redundant arguments might be helpful to a reader who wants to search for one. In this connection, note that Lemma 6.4 (in its present generality) gives a direct proof of Theorem 6.1 for stopping times without marks. The same lemma could be used to prove the general result in discrete time, by a method mimicking the proof of Lemma 6.5.

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