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The Extension of Regular and Rational Embeddings

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Introduction

In the paper [1] Abhyankar proves the so called "The Geometric Epimorphism Theorem": If $\varphi: \mathbb{C} \times 0 \to \mathbb{C}^2$ is an embedding then there exists $\Phi \in \mathrm{Iso}(\mathbb{C}^2)$ such that

$$\mathrm{res}_{\mathbb{C} \times 0} \Phi = \varphi ,$$

Abhyankar states there the following conjecture: Conjecture 1 in [1]: "For any $n \geqq 3$ the above theorem in $\mathbb{C}^n$ is false" (we get here an equivalent formulation). He formulates also the Conjectures 2 and 3, which are particular cases of Conjecture 1.

Our paper is concerned with seeking the sufficient conditions for an embedding $\varphi: X \to \mathbb{C}^n$ (where X is a closed algebraic subset of $\mathbb{C}^n$) to have an extension $\Phi \in \mathrm{Iso}(\mathbb{C}^n)$. In particular we prove:

Theorem 1.1. *Let $\varphi: \mathbb{C}^k \times 0 \to \mathbb{C}^n$ be an embedding and $n \geqq 3k+1$. Then there exists $\Phi \in \mathrm{Iso}(\mathbb{C}^n)$ such that*

$$\mathrm{res}_{\mathbb{C}^k \times 0} \Phi = \varphi ,$$

and

Theorem 1.2. *Let X be a closed smooth algebraic subset of $\mathbb{C}^n$ (not necessarily irreducible) of dimension (not necessarily pure) k. Let $\varphi: X \to \mathbb{C}^n$ be an embedding. If $n \geqq 4k+2$ then there exists $\Phi \in \mathrm{Iso}(\mathbb{C}^n)$ such that*

$$\mathrm{res}_X \Phi = \varphi .$$

From our Theorem 1 it follows that all three conjectures in [1] are false. It also gets the partial answers for the questions 1, 3, 4 which are contained in [1]. The paper is divided into two sections. In the first part of this paper we investigate extensions of regular embeddings and in the second part we consider extensions of rational embeddings.

Notations and Conventions

Our notations are generally the same as in [3]. Additionally we denote by $\mathrm{cl}_Z(X)$ the closure of X in the Zariski topology of $\mathbb{C}^n$. Let X, Y be closed algebraic subsets of $\mathbb{C}^n$ and let $\varphi: X \to Y$ be a regular mapping. We call φ a regular embedding (or embedding for short) if: 1) $\varphi(X)$ is closed in the Zariski topology 2) φ^{-1} exists and is a regular mapping $\varphi(X) \to X$. $\varphi: X \to Y$ is called an algebraic isomorphism if φ satisfies 1) and 2) and $\varphi(X) = Y$. If $X = Y = \mathbb{C}^n$ then we denote by $\mathrm{Iso}(\mathbb{C}^n)$ the set of all algebraic isomorphisms from X onto Y.

Let X, Y be algebraic varieties. By a rational mapping $\varphi: X \to Y$ we mean a mapping which is defined on non-empty and open subset $\mathrm{dom}(\varphi)$ of X and such that $\varphi: \mathrm{dom}(\varphi) \to Y$ is a regular mapping. By $\varphi(X)$ we mean the set $\varphi(\mathrm{dom}(\varphi))$. If $\varphi, \psi: X \to Y$ are rational mappings we say that $\varphi = \psi$ if there exists an open dense subset U of X such that φ, ψ are defined and regular on U and $\mathrm{res}_U \varphi = \mathrm{res}_U \psi$. A rational mapping $\varphi: X \to Y$ will be called a rational embedding if φ^{-1} is a rational mapping from $\mathrm{cl}_Z \varphi(X)$ onto X. We say that $\varphi: X \to Y$ is a birational isomorphism if φ is a rational embedding and $\mathrm{cl}_Z \varphi(X) = Y$.

1. The Extension of Regular Embeddings

The following proposition is well-known.

Proposition 1.1. *Let X, Y be closed algebraic subsets of $\mathbb{C}^n$ and let $\varphi: X \to Y$ be a regular mapping and $\tilde{X} = \mathrm{cl}_Z(\varphi(X))$. The following conditions are equivalent:*

1) *φ is a regular embedding*
2) *$\varphi_*: \mathbb{C}[\tilde{X}] \ni h \to h \circ \varphi \in \mathbb{C}[X]$ is an isomorphism.*

If in addition X and Y are smooth then these two conditions are equivalent to

3) *φ is an embedding in the sense of differential geometry i.e. φ is closed injective and $d_a\varphi: T_aX \to T_aY$ is injective for all $a \in X$.*

Lemma A1. *Let $V \subset \mathbb{C}^k \times \mathbb{C}^n$ be a closed algebraic set and let $\pi: \mathbb{C}^k \times \mathbb{C}^n \ni (x, y) \to (0, y) \in 0 \times \mathbb{C}^n$ denote the projection. Assume that $\varphi = \mathrm{res}_V \pi$ is an embedding. Then there exists an algebraic isomorphism $\Phi: \mathbb{C}^k \times \mathbb{C}^n \to \mathbb{C}^k \times \mathbb{C}^n$ such that*

$$\mathrm{res}_V \Phi = \varphi .$$

Proof. Let $\tilde{V} = \varphi(V)$. From Proposition 1.1 $\varphi_*: \mathbb{C}[\tilde{V}] \to \mathbb{C}[V]$ is an isomorphism. Let $\bar{x}_i = \mathrm{res}_V x_i$, $i = 1, \ldots, k$. There exist $\bar{h}_i \in \mathbb{C}[\tilde{V}]$ such that $\bar{x}_i = \bar{h}_i \circ \varphi$. In other words there exist polynomials $h_i \in \mathbb{C}[y_1, \ldots, y_n]$ such that

$$x_i = h_i(y) \bmod I(V) .$$

Let us recall that $I(V)$ denotes the ideal of a set V in $\mathbb{C}[x_1, \ldots, x_k, y_1, \ldots, y_n]$.

Let

$$\Phi: \mathbb{C}^k \times \mathbb{C}^n \ni (x, y) \to (x_1 - h_1(y), \ldots, x_k - h_k(y), y) \in \mathbb{C}^k \times \mathbb{C}^n .$$

It is clear that Φ is an algebraic isomorphism and

$$\mathrm{res}_V \Phi = \varphi ,$$

because the $x_i - h_i(y)$ vanish on V.

Lemma B1. *Let L^s be a linear subspace of $\mathbb{C}^n$, $\dim L^s = s$ and let $X \subset L^s \subset \mathbb{C}^n$ be a closed algebraic set. If $\varphi: X \to \mathbb{C}^n$ is an embedding and $\varphi(X) \subset H^{n-s}$, where H^{n-s} denotes an $(n-s)$-dimensional linear subspace of $\mathbb{C}^n$, then φ has an extension, i.e. there exists an algebraic isomorphism $\Phi: \mathbb{C}^n \to \mathbb{C}^n$, such that*

$$\operatorname{res}_X \Phi = \varphi .$$

Proof. We may assume that $L^s = \mathbb{C}^s \times 0$, $H^{n-s} = 0 \times \mathbb{C}^{n-s}$. Let us denote an extension of φ to L^s by $\tilde{\varphi}$ and define the mapping Ψ by the formula

$$\Psi: \mathbb{C}^s \times \mathbb{C}^{n-s} \ni (x, y) \to (x, \tilde{\varphi}(x) - y) \in \mathbb{C}^s \times \mathbb{C}^{n-s} .$$

We have $\Psi(X) = \operatorname{graph}(\varphi) := \Gamma$. Let π be the projection

$$\pi: \mathbb{C}^s \times \mathbb{C}^{n-s} \to 0 \times \mathbb{C}^{n-s} \quad \text{and} \quad \sigma = \operatorname{res}_\Gamma \pi .$$

The following diagram

$$\begin{tikzcd} X \arrow[r, "\psi"] \arrow[dr, "\varphi"'] & \Gamma \arrow[d, "\sigma"] \\ & \varphi(X) \end{tikzcd} ,$$

where $\psi = \operatorname{res}_X \Psi: X \to \Gamma$, commutes and ψ is an isomorphism. Therefore

$$\varphi_* = \psi_* \circ \sigma_* ,$$

and φ_*, ψ_* are isomorphisms, hence σ_* is also an isomorphism. This implies that σ is an embedding. From Lemma A1 it follows that there exists $\Sigma: \mathbb{C}^n \to \mathbb{C}^n$, $\Sigma \in \operatorname{Iso}(\mathbb{C}^n)$, satisfying the equality

$$\operatorname{res}_\Gamma \Sigma = \sigma .$$

If we put $\Phi = \Sigma \circ \Psi$, then $\Phi \in \operatorname{Iso}(\mathbb{C}^n)$ and

$$\operatorname{res}_X \Phi = (\operatorname{res}_\Gamma \Sigma) \circ (\operatorname{res}_X \Psi) = \sigma \circ \psi = \varphi ,$$

which completes the proof.

Lemma C1. *Let $X \subset \mathbb{C}^n$ be a smooth (not necessarily irreducible) closed algebraic set of dimension (not necessarily pure) k. If $n > 2k+1$ then we can change coordinates in such a way that*

$$\varphi: X \ni (x_1, \ldots, x_n) \to (0, x_2, \ldots, x_n) \in 0 \times \mathbb{C}^{n-1} ,$$

is an embedding.

Proof. Let us recall the notion of multiplicity of an algebraic set X at a point x. It is the number

$$\operatorname{mult}_x(X) = \min i(x, X \cap L^{n-r}) ,$$

where L^{n-r} is an $(n-r)$-dimensional linear subspace of $\mathbb{C}^n$ and x is a component of $X \cap L^{n-r}$ here $r = \dim X$. (see [3, p. 75, Definition 5.9]).

Let Y be an irreducible algebraic set. The smooth points y of Y are characterised by the property

$$y \in Y \quad \text{is smooth} \Leftrightarrow \operatorname{mult}_y(Y) = 1 .$$

Mult behaves in the following way with respect to projections:

Theorem [3, pp. 77–78, 5.14]. *If $x \in Y \subset \mathbb{P}^n$ and y satisfies*

1) $y \notin Y$
2) $y \notin E_x^* Y = \{$*the tangent cone at a point x to* $Y\}$
3) $\overline{yx} \cap Y = \{x\}$,

then

$$\operatorname{mult}_x(Y) = \operatorname{mult}_{x'}(Y') \deg(\operatorname{res}_Y p_y),$$

where $Y' = p_y(Y) \subset \mathbb{P}^{n-1}$, $x' = p_y(x) \in Y'$.

Corollary. *If x, Y, x', Y' are as in the above theorem and x is a smooth point of Y then x' is also a smooth point of Y'.*

Proof. The result is clear from the equality $1 = \operatorname{mult}_{x'}(Y') \deg(\operatorname{res}_Y p_y)$ because mult, deg are natural numbers.

For a further application, note that if Y is a smooth set then $E_x^* Y = T_x Y$ for every $x \in Y$: this follows from the definition of the tangent cone (see also [3, p. 76]).

Let us go back to the proof of Lemma C1. First we shall assume that X is irreducible. Let $S = X \times X \times \mathbb{C}$, then $\dim S = 2k+1$ and S is an affine variety. We define the regular mapping

$$\psi : S \ni (a, b, t) \to ta + (1-t)b \in \mathbb{C}^n .$$

Let $\operatorname{Tan}(X) = \bigcup_{x \in X} T_x X$. $\operatorname{Tan}(X)$ is locally the image of $X \times \mathbb{C}^k$ by a regular mapping. Since X is pseudocompact (in the Zariski topology) then there exists an algebraic set Y such that $\operatorname{Tan}(X) \subset Y$ and $\dim Y \leqq 2k$. The set $C = \operatorname{cl}_Z(\psi(S)) \cup Y$ is algebraic and $\dim C \leqq 2k+1$. Let $\tilde{C}$ $(\tilde{X})$ be the closure of C (X) in $\mathbb{P}^n$. If $H_\infty = \{x \in \mathbb{P}^n : x_0 = 0\}$ is a hyperplane at infinity then $H_\infty \setminus \tilde{C} \neq \emptyset$ because each component of $\tilde{C}$ contains points which are outside H_∞ and $\dim H_\infty \geqq \dim \tilde{C}$. Hence also $H_\infty \setminus (\tilde{C} \cup \tilde{X}) \neq \emptyset$ and there exists a point $y \in H_\infty \setminus (\tilde{C} \cup \tilde{X})$. We can change affine coordinates in such a way that $y = (0, 1, 0, \ldots, 0) \in H_\infty$. From the above Corollary (for $Y = \tilde{X}$) it follows that $p_y(X)$ contains only smooth points of $p_y(\tilde{X})$. On the other hand $p_y(\tilde{X} \cap H_\infty)$ is contained in H_∞, hence $\varphi = \operatorname{res}_X p_y : X \to \mathbb{C}^n$ is a closed mapping. Therefore $\operatorname{im} \varphi = \operatorname{cl}_Z(\varphi(X))$ and because φ is injective (by the choice of y) then φ is a bijection from X onto the smooth affine variety $\varphi(X)$. From Zariski's Main Theorem [3, p. 48, 3.26] it follows that φ^{-1} is regular. In homogeneous coordinates

$$p_y((x_0, x_1, \ldots, x_n)) = (x_0, 0, x_2, \ldots, x_n),$$

i.e.

$$\varphi((x_1, \ldots, x_n)) = (0, x_2, \ldots, x_n),$$

in affine coordinates, which ends the proof of this case.

If $X=\bigcup_{i=1}^{r} X_i$, where X_i are irreducible components of X, then the proof is similar. In this case

$$C=\mathrm{cl}_Z\left(\left(\bigcup_{i\geqq j}^{r}\varphi_{ij}(X_i\times X_j\times\mathbb{C})\right)\cup\bigcup_{k=1}^{r}\mathrm{Tan}(X_k)\right),$$

where

$$\varphi_{ij}: X_i\times X_j\times\mathbb{C}\ni(a,b,t)\to ta+(1-t)b\in\mathbb{C}^n,$$

i.e. C is the union of components of dimension not greater than $2k+1$, and $H_\infty\backslash(\tilde{C}\cup\tilde{X})\neq\emptyset$. The rest of the proof is exactly as above.

Corollary. *If $X\subset\mathbb{C}^n$ is a closed algebraic smooth set, $\dim X=k$ and $n>2k+1$ then we can change coordinates in such a way that the projection*

$$\varphi: X\ni(x,y)\to(0,y)\in 0\times\mathbb{C}^{2k+1},$$

is an embedding.

Proof. We apply $s=n-(2k+1)$ times Lemma C1, in each step changing coordinates in an appropriate way, but only in that part of $\mathbb{C}^n$ which is the image of the projection which was used in the previous step.

Theorem 1.1. *Let $\varphi:\mathbb{C}^k\times 0\to\mathbb{C}^n$ be an embedding and $n\geqq 3k+1$. There exists $\Phi\in\mathrm{Iso}(\mathbb{C}^n)$ such that*

$$\mathrm{res}_{\mathbb{C}^k\times 0}\Phi=\varphi.$$

Proof. If $k=0$ then the proof is trivial: it is sufficient to take as Φ an appropriate translation. We may assume that $k>0$. Let us put $X=\varphi(\mathbb{C}^k\times 0)$ and apply the last corollary to X and the second copy of $\mathbb{C}^n$. Then there exist coordinates (in the second copy of $\mathbb{C}^n$) such that

$$\sigma: X\ni(x,y)\to(0,y)\in 0\times\mathbb{C}^{2k+1},$$

is an embedding. Let $\psi=\sigma\circ\varphi$. From Lemma B1 we obtain $\Psi\in\mathrm{Iso}(\mathbb{C}^n)$ such that

$$\mathrm{res}_{\mathbb{C}^k\times 0}\Psi=\psi.$$

From Lemma A1 it follows that there exists $\Sigma\in\mathrm{Iso}(\mathbb{C}^n)$ satisfying the equality

$$\mathrm{res}_X\Sigma=\sigma.$$

If we take $\Phi=\Sigma^{-1}\circ\Psi$ then

$$\mathrm{res}_{\mathbb{C}^k\times 0}\Phi=\sigma^{-1}\circ(\mathrm{res}_{\mathbb{C}^k\times 0}\Psi)=\sigma^{-1}\circ(\sigma\circ\varphi)=\varphi.$$

Theorem 1.2. *Let $X\subset\mathbb{C}^n$ be a closed algebraic set which is smooth and not necessarily irreducible of dimension (not necessarily pure) k. Let $\varphi: X\to\mathbb{C}^n$ be an embedding. If $n\geqq 4k+2$ then there exists $\Phi\in\mathrm{Iso}(\mathbb{C}^n)$ such that*

$$\mathrm{res}_X\Phi=\varphi.$$

Proof. Let us choose the coordinates in the first copy of $\mathbb{C}^n$ in such a way that the projection

$$\sigma: X\ni(x,y)\to(0,y)\in 0\times\mathbb{C}^{2k+1},$$

is an embedding into $\mathbb{C}^{2k+1}$. With the second copy we do the same with respect to $\tilde{X}=\varphi(X)$ and we denote the projection $\tilde{X}\to 0\times\mathbb{C}^{2k+1}$ by λ. From Lemma B1 it follows that the embedding $\lambda\circ\varphi\circ\sigma^{-1}$ has an extension Ψ, i.e. $\Psi\in\operatorname{Iso}(\mathbb{C}^n)$ and

$$\operatorname{res}_{\sigma(X)}\Psi=\lambda\circ\varphi\circ\sigma^{-1}.$$

Let Σ be an extension of σ and Λ be an extension of λ. We obtain that

$$\Phi=\Lambda^{-1}\circ\Psi\circ\Sigma\in\operatorname{Iso}(\mathbb{C}^n)$$

and

$$\operatorname{res}_X\Phi=\lambda^{-1}\circ(\lambda\circ\varphi\circ\sigma^{-1})\circ\sigma=\varphi. \quad \text{Q.E.D.}$$

Corollary 1. *If $n\geqq 2$ and $A,B\subset\mathbb{C}^n$, $\#A=\#B<\infty$ then any bijection $\varphi:A\to B$ has an extension, i.e. there exists $\Phi\in\operatorname{Iso}(\mathbb{C}^n)$ such that*

$$\operatorname{res}_A\Phi=\varphi.$$

Remark. For $n=1$ the last corollary is false. Hence for $\dim X=0$ the estimation in Theorem 1.2 is sharp.

Corollary 2. *For all $n\neq 3$ an embedding $\varphi:\mathbb{C}\to\mathbb{C}^n$ always has an extension.* (see [1] for the case $n=2$).

Corollary 3. *Let X, Y be lines (in the sense of* [1]*). If $n\neq 3$ then X, Y are equivalent. (in the sense of* [1]*).*

Hence Conjectures 1–3 in [1] are false.

Corollary 4. *Let $\mathfrak{a}$ be an ideal of $\mathbb{C}[x_1,\dots,x_n]$. If $\mathbb{C}[x_1,\dots,x_n]/\mathfrak{a}$ is isomorphic to $\mathbb{C}[t_1,\dots,t_k]$ and $3k+1\leqq n$ then there exist polynomials $p_1,\dots,p_k$, $r_1,\dots,r_{n-k}$ such that $\mathfrak{a}=(r_1,\dots,r_{n-k})$ and $\mathbb{C}[x_1,\dots,x_n]=\mathbb{C}[p_1,\dots,p_k,r_1,\dots,r_{n-k}]$.*

Corollary 5. *Let $\mathfrak{a},\mathfrak{b}$ be ideals of $\mathbb{C}[x_1,\dots,x_n]$ and $A=\mathbb{C}[x_1,\dots,x_n]/\mathfrak{a}$, $B=\mathbb{C}[x_1,\dots,x_n]/\mathfrak{b}$. Let us assume that the rings A and B are isomorphic, regular and $n\geqq 4\cdot\dim A+2$. Then there exists an automorphism*

$$\Phi:\mathbb{C}[x_1,\dots,x_n]\to\mathbb{C}[x_1,\dots,x_n],$$

such that $\Phi(\mathfrak{a})=\mathfrak{b}$.

Example. (see [2]) Let $X\subset\mathbb{C}^2$ be the union of the sets

$$\Gamma_1=\{(x,y):xy=1\},\qquad \Gamma_2=\{(x,y):xy=2\},$$

$L=\{(x,y):x=0\}$. Let $\varphi:X\to\mathbb{C}^2$ satisfy the following conditions:

$$\operatorname{res}_{L\cup\Gamma_1}\varphi=\operatorname{res}_{L\cup\Gamma_1}(\text{identity of }\mathbb{C}^2)$$

$$\operatorname{res}_{\Gamma_2}\varphi=\operatorname{res}_{\Gamma_2}T(2),$$

where $T(m)(x,y)=(mx,my)$. X is a smooth algebraic set, φ is an embedding, but φ has no extension.

2. The Extensions of Rational Embeddings

Proposition 2.1. *Let X, Y be algebraic varieties in $\mathbb{C}^n$ and $\varphi: X \to Y$ be a rational mapping. Let us denote $\tilde{X}$ by $\mathrm{cl}_Z(\varphi(X))$. The following conditions are equivalent:*

1) *$\varphi: X \to Y$ is a rational embedding.*
2) *$\varphi_*: \mathbb{C}(\tilde{X}) \ni h \to h \circ \varphi \in \mathbb{C}(X)$ is an isomorphism*
3) *there exists a non-empty, open (in the Zariski topology) set $U \subset X$ such that*

$$\mathrm{res}_U \varphi: U \to \tilde{X},$$

is a regular and injective mapping.

Proof. See e.g. [3].

Lemma A2. *Let $V \subset \mathbb{C}^k \times \mathbb{C}^n$ be an algebraic variety and let $\varphi: V \ni (x, y) \to (0, y) \in 0 \times \mathbb{C}^n$ denote the projection. Let us assume that φ is a rational embedding. Then there exists a birational isomorphism $\Phi: \mathbb{C}^k \times \mathbb{C}^n \to \mathbb{C}^k \times \mathbb{C}^n$ such that*

$$\mathrm{res}_V \Phi = \varphi.$$

Proof. Let us put $\tilde{V} = \mathrm{cl}_Z(\varphi(V))$. The mapping

$$\varphi_*: \mathbb{C}(\tilde{V}) \ni h \to h \circ \varphi \in \mathbb{C}(V),$$

is an isomorphism. In particular there exist polynomials $P_i, Q_i \in \mathbb{C}[y_1, \ldots, y_n]$ such that

$$x_i = \frac{P_i}{Q_i} \quad \text{in} \quad \mathbb{C}(V).$$

The mapping

$$\Phi(x, y) = (x_1 - W_1, \ldots, x_k - W_k, y),$$

where $W_i = P_i/Q_i \in \mathbb{C}(y_1, \ldots, y_n)$ is a birational isomorphism and

$$\mathrm{res}_V \Phi = \varphi. \quad \text{Q.E.D.}$$

Lemma B2. *Let L^s be a linear subspace of $\mathbb{C}^n$, $\dim L^s = s$, and let $X \subset L^s \subset \mathbb{C}^n$ be an algebraic variety. If*

$$\varphi: X \to \mathbb{C}^n,$$

is a rational embedding and $\varphi(X) \subset H^{n-s}$, where H^{n-s} denotes an $(n-s)$-dimensional linear subspace of $\mathbb{C}^n$, then there exists a birational isomorphism $\Phi: \mathbb{C}^n \to \mathbb{C}^n$ such that

$$\mathrm{res}_X \Phi = \varphi.$$

Proof. The proof is analogous to that of Lemma B1. We may assume that $L^s = \mathbb{C}^s \times 0$, $H^{n-s} = 0 \times \mathbb{C}^{n-s}$. We denote $\mathrm{cl}_Z(\varphi(X))$ by $\tilde{X}$ and the extension of φ to L^s by $\tilde{\varphi}$. Let us define the mapping Ψ by the following formula:

$$\psi: \mathbb{C}^s \times \mathbb{C}^{n-s} \ni (x, y) \to (x, \tilde{\varphi}(x) - y) \in \mathbb{C}^s \times \mathbb{C}^{n-s}.$$

It is clear that Ψ is a birational mapping. Let $V = \mathrm{cl}_Z(\Psi(X))$, $\sigma: V \ni (x, y) \to (0, y) \in \tilde{X}$ and $\psi = \mathrm{res}_X \Psi$. σ, ψ, φ are dominating and $\sigma \circ \psi = \varphi$ that is $\psi_* \circ \sigma_* = \varphi_*$ i.e. $\sigma_*: \mathbb{C}(\tilde{X}) \to \mathbb{C}(V)$ is an isomorphism, hence σ is a rational embedding (see Prop. 2.1). Lemma A2 shows that there exists a birational mapping $\Sigma: \mathbb{C}^n \to \mathbb{C}^n$

such that $\operatorname{res}_V \Sigma = \sigma$. If we put $\Phi = \Sigma \circ \Psi$ then Φ is a birational mapping and $\operatorname{res}_X \Phi = \varphi$.

Lemma C2. *Let $X \subset \mathbb{C}^n$ be an algebraic variety of dimension k. We can change coordinates in $\mathbb{C}^n$ in such a way that*

$$\varphi : X \ni (x, y) \to (0, y) \in 0 \times \mathbb{C}^{k+1},$$

is a rational embedding.

Proof. See [3, p. 48].

Theorem 2.1. *Let $\varphi : \mathbb{C}^k \times 0 \to \mathbb{C}^n$ be a birational isomorphism onto its image and $n \geqq 2k+1$. There exists a birational isomorphism $\Phi : \mathbb{C}^n \to \mathbb{C}^n$ such that*

$$\operatorname{res}_{\mathbb{C}^k \times 0} \Phi = \varphi .$$

Proof. See the proof of Theorem 1.1.

Theorem 2.2. *Let $X \subset \mathbb{C}^n$ be an algebraic variety of dimension k. Let $\varphi : X \to \mathbb{C}^n$ be a rational embedding. If $n \geqq 2k+2$ then there exists a birational isomorphism $\Phi : \mathbb{C}^n \to \mathbb{C}^n$ such that*

$$\operatorname{res}_X \Phi = \varphi .$$

Proof. See the proof of Theorem 1.2.

Corollary. *If $\varphi : \mathbb{C} \times 0 \to \mathbb{C}^3$ is an embedding then there exists a birational mapping $\Phi : \mathbb{C}^3 \to \mathbb{C}^3$ such that $\operatorname{res}_{\mathbb{C} \times 0} \Phi = \varphi$.*

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