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Correction to
A Geometric Characterization of Non-Atomic Measure Spaces

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Professors J. B. Conway and J. Szücs have kindly pointed out to the author that there is a gap in the proof of Theorem 1 in the paper [2]. The oversight occurs in the first sentence on p. 58 of [2], where it is asserted that "...the closure is mid-point convex". Due to the fact that the ball $U(L_R^\infty(\mu))$ (all unexplained notation is that of [2]) need not satisfy any countability axioms with respect to the weak* topology, the above assertion only becomes convincing under some additional hypothesis such as "the space $L_R^1(\mu)$ is separable". The purpose of the present note is to indicate how the technique introduced in [2] can be further exploited to yield a valid proof of Theorem 1 as originally formulated.

A persual of the argument given in [2] shows that what is needed for completeness is a proof of the following

Assertion. Let (S, Σ, μ) be a finite non-atomic measure space and let $-1<\lambda<1$. Then there is a sequence $\{f_n\} \subset \operatorname{ext}(U(L_R^\infty(\mu)))$ such that

$$f_n \to \lambda \chi_S \text{ weak*}.$$

Without essential loss of generality we may assume that $\mu(S)=1$. Put $\alpha=\frac{1}{2}(1+\lambda)$. By induction and the Liapounoff convexity theorem we can construct sets

$$A_1^{(n)}, \ldots, A_{2^{n-1}}^{(n)},$$

$$B_1^{(n)}, \ldots, B_{2^{n-1}}^{(n)},$$

for each positive integer n, such that

$$A_k^{(n)} \cup B_k^{(n)} = A_k^{(n-1)}, \qquad k=1, \ldots, 2^{n-2},$$

$$A_k^{(n)} \cup B_k^{(n)} = B_{k-2^{n-2}}^{(n-1)}, \qquad k=2^{n-2}+1, \ldots, 2^{n-1},$$

$$A_k^{(n)} \cap B_k^{(n)} = \phi, \qquad k=1, \ldots, 2^{n-1},$$

$$\mu(A_k^{(n)}) = \begin{cases} \alpha\mu(A_k^{(n-1)}), & k=1, \ldots, 2^{n-2}, \\ \alpha\mu(B_{k-2^{n-2}}^{(n-1)}), & k=2^{n-2}+1, \ldots, 2^{n-1} \end{cases}$$

and

$$A^{(n)} \cup B^{(n)} = S,$$

where

$$A^{(n)} \equiv A_1^{(n)} \cup \cdots \cup A_{2^n-1}^{(n)},$$

$$B^{(n)} \equiv B_1^{(n)} \cup \cdots \cup B_{2^n-1}^{(n)}.$$

Now let $\langle \cdot, \cdot \rangle$ be the inner product on the Hilbert space $L_R^2(\mu)$ and let χ_n be the characteristic function of the set $A^{(n)}$. Then one can verify that

$$\mu(A^{(n)}) = \alpha, \ \forall n,$$

$$\langle \chi_n, \chi_{n+p} \rangle = \alpha^2, \ \forall n, \ \forall p \geqq 1,$$

and consequently that

$$\begin{aligned} &\langle \chi_m - \alpha\chi_S, \chi_n - \alpha\chi_S \rangle \\ &= \langle \chi_m, \chi_n \rangle - \alpha^2 \\ &= \delta_{mn}\alpha(1-\alpha), \end{aligned}$$

where χ_S is the characteristic function of the set S. This equation establishes the weak convergence in $L_R^2(\mu)$ of the sequence $\{\chi_n\}$ to $\alpha\chi_S$.

Finally, we put f_n equal to the difference between χ_n and the characteristic function of the set $B^{(n)}$. Thus

$$f_n = \begin{cases} 1 & \text{on} \quad A^{(n)} \\ -1 & \text{on} \quad B^{(n)}, \end{cases}$$

and we have $f_n \to (2\alpha - 1)\chi_S$ weakly in $L_R^2(\mu)$, and therefore, $f_n \to \lambda\chi_S$ weak* in $L_R^\infty(\mu)$. q.e.d.

In conclusion we note that Conway and Szücs have established a significant generalization of the theorem under discussion in their forthcoming paper [1].

References

1. Conway, J., Szücs, J.: The weak sequential closure of certain sets of extreme points in a von Neumann algebra, Ind. Univ. Math. J., to appear.
2. Holmes, R.: A geometric characterization of non-atomic measure spaces. Math. Ann. **182**, 55—59 (1969).

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