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Probability Theory and Related Fields

The intrinsic local time sheet of Brownian motion

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Summary. McGill showed that the intrinsic local time process $\tilde{L}(t, x)$, $t \geqq 0$, $x \in \mathbb{R}$, of one-dimensional Brownian motion is, for fixed $t > 0$, a supermartingale in the space variable, and derived an expression for its Doob-Meyer decomposition. This expression referred to the derivative of some process which was not obviously differentiable. In this paper, we provide an independent proof of the result, by analysing the local time of Brownian motion on a family of decreasing curves. The ideas involved are best understood in terms of stochastic area integrals with respect to the Brownian local time sheet, and we develop this approach in a companion paper. However, the result mentioned above admits a direct proof, which we give here; one is inevitably drawn to look at the local time process of a Dirichlet process which is not a semimartingale.

1. Introduction

Let $(B_t)_{t \geqq 0}$ be Brownian motion on $\mathbb{R}$, $B_0 = 0$, with jointly continuous local time $\{L(t, x): t \geqq 0, x \in \mathbb{R}\}$. For each $x \in \mathbb{R}$ we define

$$A(t, x) \equiv \int_0^t I_{\{B_s \leqq x\}} \, \mathrm{d}s, \qquad \tau(t, x) \equiv \inf\{u: A(u, x) > t\}$$

$$\tilde{B}(t, x) \equiv B_{\tau(t, x)}, \qquad \mathscr{E}_x \equiv \sigma(\{\tilde{B}(t, x): t \geqq 0\}).$$

The time-change $\tau(\cdot, x)$ wipes out all time spent in (x, ∞) by B, so that $\tilde{B}(\cdot, x)$ is what we would see of B if all excursions above x were deleted and the gaps left in the time axis closed up. Relative to this time scale, the process of local time in x must be

$$\tilde{L}(t, x) \equiv L(\tau(t, x), x),$$

the *intrinsic local time of B at level x*. Various properties of the Brownian excursion filtration $(\mathscr{E}_x)$ have been studied before; see [W1], [Wi], [W2], [M1], [J], [R] for a selection of earlier work.

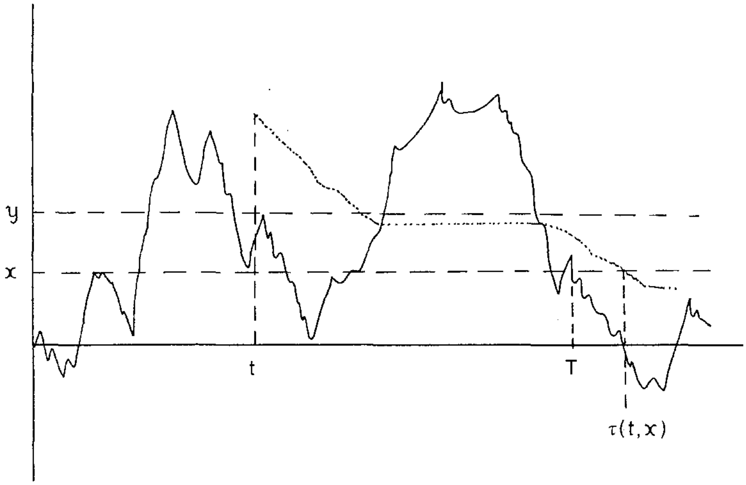

Fig. 1

In a stimulating and inventive paper, McGill [M] has investigated the intrinsic local time process of B with a view to obtaining a stochastic integral representation result for the Brownian local time sheet. The starting point for the current study was a puzzling result in [M]. As a simple consequence of the Ray-Knight theorem, McGill shows that

(1) $$(\tilde{L}(t,x)+2x^-)_{x\in\mathbb{R}} \quad \text{is an } \mathscr{E}_x\text{-supermartingale,}$$

and, after lengthy calculations, concludes that the increasing process of this supermartingale is

(2) $$\frac{1}{2}\frac{d}{dt}\left(\int_a^x \tilde{L}(t,y)^2\,dy\right).$$

(Let us remark straight away that our definition of $\tilde{L}$ is *double* that of McGill, and that the first argument of $\tilde{L}$ for us is t, the variable traditionally associated with the horizontal axis.)

Herein lies the puzzle: if a function is indeed differentiable, one should be able to exhibit its derivative – so what *is* the derivative (2)?

To answer this question, we firstly ask rather, "What is the increasing process of the supermartingale (1)?" To understand this better, let us consider the situation where $x>0$, and $\tilde{B}(t,x)<x$, depicted in Fig. 1. We assume that $t>0$ is fixed for the time being.

The dashed continuous decreasing curve is the graph of $z\mapsto\tau(t,z)$, and $T=\sup\{s<\tau(t,x): B_s=x\}$. By the Ray-Knight theorem, $(L(T,z))_{z\geq x}$ is an $(\mathscr{E}_z)$-martingale, and $L(T,z)=L(\tau(t,z),z)$, at least for $z>x$ sufficiently close to x. Ultimately, as the level z rises, the time $\tau(t,z)$ slips back until $(\tau(t,z),z)$ lies on the Brownian path, and at that level, the processes $L(T,z)$ and $\tilde{L}(t,z)$ begin to differ. Thus by the level y in the diagram, $\tilde{L}(t,y)<L(T,y)$. It seems at least plausible that this discrepancy should be compensated for by *the amount of*

local time spent by B on the dashed curve between $\tau(t, y)$ and $\tau(t, x)$. This turns out to be exactly correct.

In more detail then, defining for $s > \xi > 0$

$$\psi_s(\xi) \equiv \inf\{z: A(s, z) > \xi\},$$

and setting $\psi_s(\xi) = +\infty$ if $s \leqq \xi$, it is not hard to show that ψ is jointly continuous in $\{(s, \xi): s > \xi \geqq 0\}$, and, for fixed ξ, the process $s \mapsto \psi_s(\xi)$ is decreasing in $s > \xi$; the dashed curve in Fig. 1 is just the graph of $\psi.(t)$.

Fixing $t > 0$, if we let λ be the local time at zero of the continuous semimartingale $\{B_s - \psi_s(t): s > t\}$, we have the following result.

Theorem 1. *Fix $a \in \mathbb{R}$. The process*

$$C_{a,x} \equiv \lambda(\tau(t, a)) - \lambda(\tau(t, x)) \qquad (x \geqq a)$$

is continuous, $(\mathscr{E}_x)$-adapted, and increasing, and

$$\{\tilde{L}(t, x) + 2x^- + C_{a,x}: x \geqq a\}$$

is an $(\mathscr{E}_x)$-martingale bounded in L^2.

The proof of this result, using Tanaka's formula, and ideas of McGill and Jeulin, is quite straightforward, and is dealt with in Sect. 2.

Now this does not help us so far to identify the derivative (2). Nonetheless, we begin to see the connections more clearly if we take a few more steps and develop (2):

$$\frac{1}{2}\int_a^x \tilde{L}(t, y)^2 \, dy \equiv \frac{1}{2}\int_a^x L(\tau(t, y), y)^2 \, dy \tag{3}$$

$$= \int_a^x \left[\int_0^{\tau(t,y)} L(s, y)\, L(ds, y)\right] dy$$

$$= \int_{-\infty}^{\infty} dy \int_0^{\infty} L(ds, y)\, L(s, y)\, I_{\{s \leqq \tau(t, y)\}}\, I_{(a,x]}(y)$$

$$= \int_0^{\infty} L(s, B_s)\, I_{\{s \leqq \tau(t, B_s)\}}\, I_{(a,x]}(B_s)\, ds, \tag{4}$$

by the occupation-density property of local time;

$$= \int_0^{\infty} L(s, B_s)\, I_{\{A(s, B_s) \leqq t\}}\, I_{(a,x]}(B_s)\, ds \tag{5}$$

$$= \int_0^{\infty} L(s, B_s)\, I_{\{B_s \leqq \psi_s(t)\}}\, I_{(a,x]}(B_s)\, ds. \tag{6}$$

From this we see that *the differentiability in t of* (3) *is, in view of* (5), *essentially the same thing as the existence of local time for the process $Y_s \equiv A(s, B_s)$.*

The only problem is that Y *is not a semimartingale.* Indeed, if

$$X_t \equiv \int_0^t L(s, B_s)\,\mathrm{d}B_s - A(t, B_t), \tag{7}$$

then we show that *X has zero quadratic variation, yet has infinite order p variation for any* $p<4/3$. We prove this in [RWa 2]. It is not known whether a continuous Dirichlet process (which Y is) already has a local time, though some partial results are known; see Bertoin [B] and the references therein.

Nonetheless, in such a concrete setting, one can establish the existence of local time for Y. Indeed, writing S_t for $\sup_{u \leqq t} B_u$, we prove in Sect. 3 that *there exists a jointly continuous version of*

$$\{\lambda(s, a; t)\colon 0<t<s,\, a\in\mathbb{R},\, a \neq B_t - S_t\},$$

where $\lambda(\cdot, a; t)$ is the local time at a of the semimartingale $B_\cdot - \psi_\cdot(t)$, and we deduce from this that for $f \in C_b(\mathbb{R})$

$$\int_0^u L(s, B_s)\, f(A(s, B_s))\,\mathrm{d}s = \int_0^u f(y)\,\lambda(u, 0; y)\,\mathrm{d}y. \tag{8}$$

From this and (5), it is *now* clear that (3) is differentiable with respect to t, and it is easy to confirm from here that the derivative is indeed the compensator of $\tilde{L}(t, x) + 2x^-$, as identified by Theorem 1.

There is yet another way to characterise the 'local time on a curve' which appears in the compensator of $\tilde{L}(t, x) + 2x^-$, and this is in terms of the stochastic area integral with respect to L, as developed by Walsh [W]. This approach is the one used in the companion paper [RW], where the main ideas of the construction of stochastic area integrals are developed, and an integral representation result is proved; *every L^2 random variable has a stochastic area integral representation.* The consequences of this for the problems discussed here are pursued.

The final section of the paper is a direct and independent derivation of Lemma 4.2 of McGill [M], using the powerful techniques of Itô excursion theory. If you like that kind of thing, you can certainly avoid lots of calculations!

2. The compensator of $\tilde{L}(t, \cdot)$

Recall the definition of the process ψ:

$$\begin{aligned} \psi_s(\xi) &= \inf\{x\colon A(s, x) > \xi\} && \text{if } s>\xi \\ &= +\infty && \text{if } s \leqq \xi. \end{aligned}$$

We record briefly a few properties of ψ. For fixed s, $\psi_s(\cdot)$ is strictly monotone in $[0, s)$, and C^1 in $(0, s)$, with derivative $1/L(s, \psi_s(\xi))$ at ξ. For fixed ξ, $\psi_\cdot(\xi)$ is decreasing in (ξ, ∞), but is not strictly decreasing; in any time interval (u, v) throughout which $B_s > \psi_u(\xi)$, the function $\psi_\cdot(\xi)$ is constant. The last property of ψ which we shall need for the moment is its joint continuity; but if $s_n > \xi_n$,

$s_n \to s > \xi = \lim \xi_n$, if $x_n \equiv \psi_{s_n}(\xi_n)$ and if for some subsequence $x_{n_j} \to \alpha$, we have by the joint continuity of A that

$$\xi_{n_j} = A(s_{n_j}, x_{n_j}) \to A(s, \alpha)$$

so that $\xi = A(s, \alpha)$. Strict monotonicity in x of A implies that $\alpha = \psi_s(\xi)$, from which continuity of ψ in $\{(s, \xi): s > \xi \geqq 0\}$ follows.

For the rest of this section, let us fix $t > 0$, abbreviating $\psi_s(t)$ to ψ_s. Since ψ is continuous, decreasing and adapted, the semimartingale $\{B_s - \psi_s: s > t\}$ has a local time λ at zero. The main result of this section is the following.

Theorem 2. *Fix $a \in \mathbb{R}$. The process*

$$C_{a,x} \equiv \lambda(\tau(t, a)) - \lambda(\tau(t, x)) \qquad (x \geqq a)$$

is continuous, increasing and $(\mathscr{E}_x)$-adapted. The process

$$\{Z_{a,x}: x \geqq a\} \equiv \{\tilde{L}(t, x) + 2x^- + C_{a,x}: x \geqq a\}$$

is an $(\mathscr{E}_x)$-martingale bounded in L^2.

Proof. Monotonicity of C follows by monotonicity of λ and $\tau(t, \cdot)$. It is elementary to show that $\tau(t, \cdot)$ is left-continuous and decreasing, and if for some $b > a$, $\tau(t, b+) = u < \tau(t, b) = v$, then $A(\cdot, b)$ is equal to t throughout (u, v) and hence if follows that $B_s > b$ for all $s \in (u, v)$ a.s. But $\psi_s = b$ for all $s \in (u, v)$ so that $B_s - \psi_s > 0$ throughout (u, v); hence the local time λ at zero for $B - \psi$ cannot grow on (u, v). Thus $\lambda(\tau(t, \cdot))$ has no jump at b, even though τ has.

To see that C is $(\mathscr{E}_x)$-adapted, notice that for $y \leqq x$

$$A(\tau(s, x), y) = \int_0^s I_{\{\tilde{B}(u,x) \leqq y\}} \, du$$

so that $A(\tau(s, x), y)$ is $\mathscr{E}_x$-measurable, and hence for $s > t$, $\psi_{\tau(s,x)}$ is also $\mathscr{E}_x$-measurable. For $s \in (\tau(t, x), \tau(t, a))$, $\psi_s < x$, from which it is not hard to deduce that $\lambda(\tau(t, a)) - \lambda(\tau(t, x))$ is the local time at zero of the $\mathscr{E}_x$-measurable process $\tilde{B}(\cdot, x) - \psi_{\tau(\cdot, x)}$ between the times t and $A(\tau(t, a), x)$, both $\mathscr{E}_x$-measurable.

Now for the main part of the theorem, the statement that $Z_{a,\cdot}$ is an $(\mathscr{E}_x)$-martingale. Let us fix $y > x \geqq a$ and consider $Z_{a,y} - Z_{a,x}$. Because $\tilde{B}(t, x) \leqq x$ for all t, x, Tanaka's formula at the stopping time $\tau(t, x)$ yields

$$-\tfrac{1}{2}\tilde{L}(t, x) = \tilde{B}(t, x) - (x \wedge 0) - \int_0^{\tau(t,x)} I_{\{B_s \leqq x\}} \, dB_s,$$

so that

$$\tfrac{1}{2}\{\tilde{L}(t, y) - \tilde{L}(t, x)\} = \tilde{B}(t, x) - \tilde{B}(t, y) - (x \wedge 0) + (y \wedge 0) + \int_0^{\tau(t,y)} I_{(x,y]}(B_s) \, dB_s - \int_{\tau(t,y)}^{\tau(t,x)} I_{\{B_s \leqq x\}} \, dB_s. \tag{9}$$

Next, using the Tanaka-formula definition of λ between the stopping times $\tau(t, y)$ and $\tau(t, x)$ yields similarly

$$(10)\qquad \tfrac{1}{2}\{\tilde{\lambda}(t,x)-\tilde{\lambda}(t,y)\}=(\tilde{B}(t,y)-\tilde{\psi}(t,y))\wedge 0-(\tilde{B}(t,x)-\tilde{\psi}(t,x))\wedge 0 + \int_{\tau(t,y)}^{\tau(t,x)} I_{\{B_s-\psi_s\leqq 0\}}\,\mathrm{d}(B_s-\psi_s),$$

where $\tilde{\lambda}(t, y)\equiv\lambda(\tau(t, y))$, $\tilde{\psi}(t, y)\equiv\psi_{\tau(t,y)}$ are the customary abbreviations. Notice that $\tilde{\psi}(t, y)=y$, and that $\tilde{B}(t, y)\leqq y$, so adding (9) and (10) gives some cancellations, yielding

$$(11)\qquad \begin{aligned} &\tfrac{1}{2}\{\tilde{L}(t,y)-\tilde{L}(t,x)+\tilde{\lambda}(t,x)-\tilde{\lambda}(t,y)\}\\ &=x-y-(x\wedge 0)+(y\wedge 0)+\int_0^{\tau(t,y)} I_{(x,y]}(B_s)\,\mathrm{d}B_s\\ &\quad+\int_{\tau(t,y)}^{\tau(t,x)}(I_{\{B_s\leqq\psi_s\}}-I_{\{B_s\leqq x\}})\,\mathrm{d}B_s-\int_{\tau(t,y)}^{\tau(t,x)} I_{\{B_s\leqq\psi_s\}}\,\mathrm{d}\psi_s. \end{aligned}$$

Observe that for $\tau(t, y)<s\leqq\tau(t, x)$, ψ_s takes values in $[x, y)$, so that the penultimate term on the right-hand side of (11) simplifies to

$$\int_{\tau(t,y)}^{\tau(t,x)} I_{\{x<B_s\leqq\psi_s\}}\,\mathrm{d}B_s.$$

To simplify the final term on the right-hand side of (11), notice that

$$\int I_{\{B_s>\psi_s\}}\,\mathrm{d}\psi_s=0.$$

Indeed, if we suppose that $B_s>\psi_s$ for all s in some interval (u, v), and also that $\psi_v<\psi_u$, then we have

$$t=A(v,\psi_v)=A(u,\psi_u)>A(u,\psi_v).$$

However, since $B_s>\psi_s\geqq\psi_v$ for all s in (u, v), it must be that $L(v, x)=L(u, x)$ for all $x<\psi_v$, and therefore $A(v, \psi_v)=A(u, \psi_v)$, a contradiction.

Thus the final term on the right-hand side of (11) is

$$-\int_{\tau(t,y)}^{\tau(t,x)}\mathrm{d}\psi_s=\tilde{\psi}(t,y)-\tilde{\psi}(t,x)=y-x,$$

and assembling this gives

$$(12)\qquad \tfrac{1}{2}\{\tilde{L}(t,y)-\tilde{L}(t,x)+\tilde{\lambda}(t,x)-\tilde{\lambda}(t,y)+2y^- -2x^-\} = \int_0^{\tau(t,x)} I_{\{x<B_s\leqq y\wedge\psi_s\}}\,\mathrm{d}B_s.$$

Now the left-hand side of (12) is $Z_{a,y}-Z_{a,x}$, and the right-hand side is bounded in L^2 uniformly in y. To see this, we estimate

$$\begin{aligned}
(13)\qquad & E\left(\int_0^{\tau(t,x)} I_{\{x<B_s\leqq y\wedge\psi_s\}}\,\mathrm{d}B_s\right)^2 \\
&=E\left(\int_0^{\tau(t,x)} I_{\{x<B_s\leqq y\wedge\psi_s\}}\,\mathrm{d}s\right) \\
&=E\int_0^{\infty} I_{\{x<B_s\leqq y\wedge\psi_s\}}\,\mathrm{d}s, \quad \text{because } \psi_s\leqq x \text{ for } s\geqq\tau(t,x); \\
&\leqq E\int_0^{\infty} I_{\{x<B_s\leqq\psi_s\}}\,\mathrm{d}s \\
&=E\int_0^{\infty} I_{\{x<B_s,\,A(s,B_s)\leqq t\}}\,\mathrm{d}s \\
&=E\int_x^{\infty}\mathrm{d}y\int_0^{\infty} L(\mathrm{d}s,y)\,I_{\{A(s,y)\leqq t\}} \\
&=E\int_x^{\infty}\mathrm{d}y\,L(\tau(t,y),y) \\
&\leqq\int_x^0 c\,\mathrm{d}y+\int_0^{\infty} c\,P^0(\text{hit } y \text{ before } t)\,\mathrm{d}y
\end{aligned}$$

where $c=E(L(\tau(t,0),0))$.

Thus we have an L^2 bound on the right-hand side of (12). However, since the right-hand side of (12) is a stochastic integral *involving only increments of B at times when $B>x$*, by Lemma 1.2 of McGill [M], *its conditional expectation given $\mathscr{E}_x$ is zero.* Since the left-hand side of (12) is, as already noted, $Z_{a,y}-Z_{a,x}$, the proof is complete. □

3. The local time process of $A(t, B_t)$

The main aim of this section is to prove the following result.

Theorem 3. *There exists a jointly continuous version of the process*

$$\{\lambda(s,a;t): 0<t<s,\ a\in\mathbb{R},\ a\neq B_t-S_t\},$$

where $\lambda(\cdot,\cdot;t)$ is the local time process of $\{B_s-\psi_s(t): s>t\}$, and $S_t\equiv\sup_{u\leqq t} B_u$.

Remarks. The proof of this result uses Kolmogorov's Lemma, but with some modification of the underlying processes in order to get round a technical difficulty. These modifications prevent one from concluding the joint continuity in (a, s, t) which one wants, but the result we give is sufficient for the applications we have in mind. Before proving Theorem 3, we look at some easy consequences.

Theorem 4. *For bounded measurable* $f\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, *for* $t \geqq 0$,

$$\int_0^t f(s, A(s, B_s))\, L(s, B_s)\, \mathrm{d}s = \int_0^t \mathrm{d}y \int_0^t f(s, y)\, \lambda(\mathrm{d}s, 0; y) \tag{14}$$

Proof. It suffices to prove (14) for the case where f depends only on its second argument, the general case following by familiar techniques. So we aim to prove that

$$\int_0^t f(A(s, B_s))\, L(s, B_s)\, \mathrm{d}s = \int f(y)\, \lambda(t, 0; y)\, \mathrm{d}y, \tag{15}$$

and, since $\int I_{\{0\}}(A(s, B_s))\, \mathrm{d}s = 0$, it suffices to prove (15) for f vanishing in some neighbourhood of 0. Take some $\phi \in C_K^\infty$ which is non-negative, supported in $(0, \infty)$ and such that $\int \phi(x)\, \mathrm{d}x = 1$. Then the right-hand side of (15) is equal to

$$\lim_{\varepsilon \downarrow 0} \int \phi(x)\, \mathrm{d}x \int f(y)\, \lambda(t, \varepsilon x; y)\, \mathrm{d}y \tag{16}$$

using Theorem 3. But (16) is equal to

$$\lim_{\varepsilon \downarrow 0} \int \mathrm{d}y\, f(y) \int_y^t \phi(\varepsilon^{-1}(B_s - \psi_s(y)))\, \mathrm{d}s/\varepsilon$$

$$= \lim_{\varepsilon \downarrow 0} \int_0^t \frac{\mathrm{d}s}{\varepsilon} \int \phi(v)\, f(A(s, B_s - \varepsilon v))\, \varepsilon L(s, B_s - \varepsilon v)\, \mathrm{d}v$$

using the substitution $v = \varepsilon^{-1}(B_s - \psi_s(y))$ and the fact that $\partial \psi_s(y)/\partial y = L(s, \psi_s(y))^{-1}$;

$$= \int_0^t f(A(s, B_s))\, L(s, B_s)\, \mathrm{d}s,$$

as stated. □

From this we can deduce easily McGill's result that the derivative (2) is the compensator of $\tilde{L}(t, x) + 2x^-$, identified in Theorem 1.

Corollary. *For* $x \geqq a$,

$$\tfrac{1}{2} \int_a^x \tilde{L}(t, y)^2\, \mathrm{d}y = \int_0^t \{\lambda(\tau(s, a)) - \lambda(\tau(s, x))\}\, \mathrm{d}s. \tag{17}$$

Proof. By (5) the left-hand side of (17) is

$$\int_0^\infty L(s,B_s)\, I_{[0,t]}(A(s,B_s))\, I_{[a,x)}(B_s)\,\mathrm{d}s$$

$$=\int_0^t \mathrm{d}v \int_0^\infty \lambda(\mathrm{d}s,0;v)\, I_{[a,x)}(B_s) \text{ by (14);}$$

$$=\int_0^t \mathrm{d}v \int_0^\infty \lambda(\mathrm{d}s,0;v)\, I_{[a,x)}(\psi_s(v)),$$

since $\lambda(\cdot, 0; v)$ grows only when $B_s - \psi_s(v)$ is zero;

$$=\int_0^t \mathrm{d}v \int_0^\infty \lambda(\mathrm{d}s,0;v)\, I_{\{A(s,a)\leqq v<A(s,x)\}}$$

$$=\int_0^t \mathrm{d}v \int_0^\infty \lambda(\mathrm{d}s,0;v)\, I_{\{\tau(v,x)<s\leqq\tau(v,a)\}},$$

from which (17) follows immediately. □

Proof (of Theorem 3). Fix $\varepsilon>0$ small, and $N\in\mathbb{N}$ large, define $T\equiv T_N \equiv N\wedge\inf\{u: |B_u|>N\}$, and set

$$A^\varepsilon(s,x)\equiv A(s\wedge T,0)+\int_0^x (L(s\wedge T,y)\vee\varepsilon)\,\mathrm{d}y,$$

$$\psi_s^\varepsilon(t)\equiv\inf\{x: A^\varepsilon(s,x)>t\}.$$

Then throughout $\{(s,x): 0<s\leqq T, \inf_{u\leqq s} B_u<x<\sup_{u\leqq s} B_u\}$, we have $A^\varepsilon(s,x)\to A(s,x)$, *with eventual equality*. Moreover, for $0<t<s\leqq T$, we have $|\psi_s^\varepsilon(t)|\leqq|\psi_s(t)|\leqq N$, and $\psi_s^\varepsilon(t)\to\psi_s(t)$ as $\varepsilon\downarrow 0$, *with eventual equality*. Notice that for $a\neq B_t-S_t$, there is an interval of time to the right of t during which $B_\cdot - a$ does not encounter $\psi_\cdot(t)$ nor, for all sufficiently small ε, $\psi_\cdot^\varepsilon(t)$, since $\psi_{t+}(t)=S_t$. Thus it is sufficient to prove the existence of a jointly continuous local time $\lambda^\varepsilon(s,a;t)$ for $B_\cdot-\psi_\cdot^\varepsilon(t)$, because for $0<t<s\leqq T$ and all $a\in\mathbb{R}$, $a\neq B_t-S_t$, $\lambda^\varepsilon(s,a;t)$ is eventually equal to $\lambda(s,a;t)$. To establish the existence of a jointly continuous version of $\lambda^\varepsilon(\cdot,\cdot;\cdot)$, we analyse Tanaka's formula. For $0<t<s\leqq N$, we have

$$\text{(18)}\qquad \tfrac{1}{2}\lambda^\varepsilon(s,a;t)=(B_{s\wedge T}-\psi_{s\wedge T}^\varepsilon(t)-a)^+-(B_{t\wedge T}-\psi_{t\wedge T}^\varepsilon(t\wedge T)-a)^+ - \int_{t\wedge T}^{s\wedge T} I_{\{B_u-\psi_u^\varepsilon(t)-a>0\}}\,\mathrm{d}(B_u-\psi_u^\varepsilon(t)).$$

The joint continuity of the first two terms on the right is immediate, leaving only the integrals. Writing

$$\Psi(s,a;t)\equiv\int_{t\wedge T}^{s\wedge T} I_{\{B_u-\psi_u^\varepsilon(t)-a>0\}}\,\mathrm{d}B_u,$$

we have that for $p>1$, with C_p denoting a constant depending on p (and changing from line to line),

$$(19)\qquad \begin{aligned} &E\,|\Psi(s,a;t)-\Psi(w,b;v)|^p\\ &\leqq C_p\{E\,|\Psi(s,a;t)-\Psi(w,a;t)|^p+E\,|\Psi(w,a;t)-\Psi(w,a;v)|^p\\ &\quad +E\,|\Psi(w,a;v)-\Psi(w,b;v)|^p\}. \end{aligned}$$

Estimating the terms one by one, the first is trivially at most $C_p|s-w|^{p/2}$. For the second, assuming $t\geqq v$ without loss of generality, we have an upper bound of

$$(20)\qquad \begin{aligned} &C_p\left\{|t-v|^{p/2}+E\left|\int_{t\wedge T}^{w\wedge T} I_{\{\psi_u^\varepsilon(v)<B_u-a\leqq\psi_u^\varepsilon(t)\}}\,\mathrm{d}u\right|^{p/2}\right\}\\ &\leqq C_p\left\{|t-v|^{p/2}+E\left|\int_{t\wedge T}^{w\wedge T} I_{\{0<B_u-a-\psi_u^\varepsilon(v)\leqq(t-v)/\varepsilon\}}\,\mathrm{d}u\right|^{p/2}\right\} \end{aligned}$$

since $\psi_u^\varepsilon(\cdot)$ is globally Lipschitz with constant ε^{-1}. Now the estimation on pp. 100–101 of [RW] shows that if X is a continuous semimartingale whose martingale part M is bounded by K, and such that the variation of $A\equiv X-M$ is also bounded by K, then the inequality

$$E\left|\int_0^\infty I_{\{a<X_s\leqq b\}}\,\mathrm{d}[X]_s\right|^{p/2}\leqq\delta^{p/2}\,C_p\,K^p$$

holds, where $\delta\equiv b-a>0$. This inequality is applicable to (20) because B^T is bounded by N, and because *$u\mapsto\psi_u^\varepsilon(t)$ is Lipschitz continuous with constant* $2\varepsilon^{-1}$. To see this, suppose that $\xi\equiv\psi_s^\varepsilon(t)$, $\eta\equiv\psi_{s+h}^\varepsilon(t)$, where $h>0$. By definition, $A^\varepsilon(s,\xi)=t=A^\varepsilon(s+h,\eta)$, so

$$\begin{aligned} 0&=A^\varepsilon(s+h,0)-A^\varepsilon(s,0)+\int_0^\eta(L(s+h,y)\vee\varepsilon)\,\mathrm{d}y-\int_0^\xi(L(s,y)\vee\varepsilon)\,\mathrm{d}y\\ &=A^\varepsilon(s+h,0)-A^\varepsilon(s,0)+\int_0^\eta\{L(s+h,y)\vee\varepsilon-L(s,y)\vee\varepsilon\}\,\mathrm{d}y\\ &\quad-\int_\eta^\xi(L(s,y)\vee\varepsilon)\,\mathrm{d}y, \end{aligned}$$

whence

$$\int_\eta^\xi(L(s,y)\vee\varepsilon)\,\mathrm{d}y=A^\varepsilon(s+h,0)-A^\varepsilon(s,0)+\int_0^\eta\{L(s+h,y)\vee\varepsilon-L(s,y)\vee\varepsilon\}\,\mathrm{d}y.$$

Now the integrand on the right-hand side is non-negative, and is at most $L(s+h,y)-L(s,y)$, so the modulus of the integral is at most h, and we conclude that $|\xi-\eta|\leqq 2h/\varepsilon$. Returning to (20) we have an upper bound of the form

$$C(p,N,\varepsilon)\,|t-v|^{p/2}.$$

The estimation of the final term on the right-hand side of (19) follows exactly similar lines, yielding

$$E|\Psi(s,a;t)-\Psi(w,b;v)|^p \leqq C(p,N,\varepsilon)\{|s-w|^{p/2}+|t-v|^{p/2}+|b-a|^{p/2}\}.$$

By taking $p>6$, we get the kind of estimate needed to feed into Kolmogorov's Lemma, and give a jointly continuous version of Ψ. It remains now only to establish the existence of a jointly continuous version of the finite variation integral in (18):

(21) $$\Phi(s,a;t) \equiv \int_{t\wedge T}^{s\wedge T} I_{\{B_u-\psi_u^\varepsilon(t)-a>0\}}\,\mathrm{d}\psi_u^\varepsilon(t).$$

Because $\psi_\cdot^\varepsilon(t)$ is Lipschitz with constant $2\varepsilon^{-1}$, we have

$$|\Phi(s,a;t)-\Phi(w,a;t)| \leqq 3\varepsilon^{-1}|s-w|,$$

leaving only the continuity in a, t for some fixed s. This turns out to be quite a bit more difficult than might at first sight appear. We need a subsidiary result.

Lemma. *Suppose that f_n: $[0,N]\to[-1,1]$ are measurable, and that $f_n\to f$ in measure. Let μ_n be signed measures on $[0,N]$ with densities ρ_n with respect to Lebesgue measure, $|\rho_n|\leqq 1$, and assume that $\mu_n \Rightarrow \mu$. Then*

$$\int_0^N f_n\,\mathrm{d}\mu_n \to \int_0^N f\,\mathrm{d}\mu.$$

Proof. Firstly, note that $\mathscr{A}\equiv\{$bounded Borel g: $[0,N]\to\mathbb{R}$ s.t. $\int g\,\mathrm{d}\mu_n\to\int g\,\mathrm{d}\mu\}$ is a vector space containing constants, closed under uniform limits, and also closed under bounded monotone limits; since $\mathscr{A}$ contains the algebra of continuous functions, by the monotone-class theorem (see, for example, [DM], I.21) $\mathscr{A}$ contains all bounded Borel g. To finish the proof, define $f_n^\varepsilon\equiv(f_n\wedge(f+\varepsilon))\vee(f-\varepsilon)$ and notice that $|f-f_n^\varepsilon|\leqq\varepsilon$, and that for all large enough n, $\mathrm{Leb}(\{f_n\neq f_n^\varepsilon\})\leqq\varepsilon$. Thus

$$|\int f_n\,\mathrm{d}\mu_n-\int f\,\mathrm{d}\mu| \leqq |\int(f_n-f_n^\varepsilon)\,\mathrm{d}\mu_n|+|\int(f_n^\varepsilon-f)\,\mathrm{d}\mu_n|+|\int f\,\mathrm{d}\mu_n-\int f\,\mathrm{d}\mu|$$

which tends to 0 as $n\to\infty$. □

Now the measures with distribution functions $u\mapsto\psi_u^\varepsilon(t)$ vary continuously (in the weak topology) with the parameter t, and have bounded densities, so the existence of a jointly continuous version of Ψ will be achieved once we can prove that the functions

$$u\mapsto I_{\{B_u-\psi_u^\varepsilon(t)-a>0\}}$$

vary continuously in measure with the parameters t, a. To this end, define

$$\tilde{\Phi}(t,t',s,a,a') \equiv \int_{(t\vee t')\wedge T}^{s\wedge T} |I_{\{B_u-\psi_u^\varepsilon(t)-a>0\}}-I_{\{B_u-\psi_u^\varepsilon(t')-a'>0\}}|\,\mathrm{d}u.$$

We shall prove that this has a jointly continuous version in all five variables, and this will be enough. Without loss of generality, assume that $0<t\leqq t'<s\leqq T$, to ease notation.

$$E|\tilde{\Phi}(t,t',s,a,a')-\tilde{\Phi}(v,v',w,b,b')|^p \leqq C\{|s-w|^p+E|\tilde{\Phi}(t,t',s,a,a')-\tilde{\Phi}(v,v',s,b,b')|^p\},$$

the second term of which is bounded above by

$$|t'-v'|^p+E\left|\int_{v'\vee t'}^{s\wedge T}\{|I_{\{B_u-\psi_u^\varepsilon(t)-a>0\}}-I_{\{B_u-\psi_u^\varepsilon(t')-a'>0\}}| - |I_{\{B_u-\psi_u^\varepsilon(v)-b>0\}}-I_{\{B_u-\psi_u^\varepsilon(v')-b'>0\}}|\,\mathrm{d}u\}\right|^p$$

$$\leqq |t'-v'|^p+2^{p-1}E\left|\int_{v'\vee t'}^{s\wedge T}|I_{\{B_u-\psi_u^\varepsilon(t)-a>0\}}-I_{\{B_u-\psi_u^\varepsilon(v)-b>0\}}|\,\mathrm{d}u\right|^p + 2^{p-1}E\left|\int_{v'\vee t'}^{s\wedge T}|I_{\{B_u-\psi_u^\varepsilon(t')-a'>0\}}-I_{\{B_u-\psi_u^\varepsilon(v')-b'>0\}}|\,\mathrm{d}u\right|^p,$$

and both of these terms can be estimated as they were for the integral with respect to B. Once again, an application of Kolmogorov's Lemma completes the proof. □

4. Obtaining the compensator of $\tilde{L}(t,\cdot)$ by excursion theory

Since the compensator of $\tilde{L}(t,\cdot)$ is quite singular, it seems worthwhile to obtain instead the compensator of $\int_0^\infty e^{-\lambda t}\tilde{L}(t,\cdot)\,\mathrm{d}t$, and then unravel the Laplace transform. This is the method of McGill [M], by which he obtains (2). Our aim in this section is to establish the following result, which is at the heart of McGill's analysis.

Theorem 5. *Fix $\theta>0$, $a<b$, and let $\lambda\equiv\frac{1}{2}\theta^2$. Then*

$$E\left[\int_0^\infty \lambda e^{-\lambda t}\tilde{L}(t,b)\,\mathrm{d}t\,|\mathscr{E}_a\right] \tag{22}$$

$$=E\left[\int_0^\infty e^{-\lambda A(s,b)}L(\mathrm{d}s,b)\,|\mathscr{E}_a\right]$$

$$=\alpha\,\mathrm{sech}\,\theta\delta\int_0^\infty \exp\{-\lambda t-\tfrac{1}{2}\theta\tanh\theta\delta\,\tilde{L}(t,a)\}\,\tilde{L}(\mathrm{d}t,a), \tag{23}$$

where $\delta \equiv b-a$ *and*

$$\alpha = 1 \quad \text{if } a \geqq 0$$
$$= \cosh \theta b^+ \operatorname{sech} \theta \delta \quad \text{if } a < 0.$$

Remarks. From (23), with $a \geqq 0$,

$$\delta^{-1} E\left[\int_0^\infty \lambda \mathrm{e}^{-\lambda t}(\tilde{L}(t,b)-\tilde{L}(t,a))\,\mathrm{d}t \,|\, \mathscr{E}_a\right]$$
$$= \int_0^\infty \mathrm{e}^{-\lambda t}\delta^{-1}\{\operatorname{sech}\theta\delta \exp(-\tfrac{1}{2}\theta \tanh\theta\delta\,\tilde{L}(t,a))-1\}\,\tilde{L}(\mathrm{d}t,a)$$
$$\to -\int_0^\infty \mathrm{e}^{-\lambda t}\lambda \tilde{L}(t,a)\,\tilde{L}(\mathrm{d}t,a) \quad \text{as } \delta \downarrow 0;$$
$$= -\int_0^\infty \lambda^2 \mathrm{e}^{-\lambda s}\tfrac{1}{2}\tilde{L}(s,a)^2\,\mathrm{d}s,$$

which makes it look very plausible that the compensator of

$$(\zeta_x)_{x\geqq 0} \equiv \left(\int_0^\infty \lambda \mathrm{e}^{-\lambda t}\tilde{L}(t,x)\,\mathrm{d}t\right)_{x \geqq 0}$$

should be

$$\int_0^\infty \left(\int_0^\infty \lambda^2 \mathrm{e}^{-\lambda s}\tfrac{1}{2}\tilde{L}(s,y)^2\,\mathrm{d}s\right)\mathrm{d}y. \tag{24}$$

A little care is needed over this; see Sect. 4 of McGill [M]. Given this, the statement that (2) is an expression for the compensator of $\tilde{L}(t,\cdot)$ looks eminently credible – and, as we have proved, is true.

Proof. The equality (22) is obtained by a trivial integration by parts. The cases $a \geqq 0$ and $a<0$ of (23) need slightly different treatment, according as $\{B_t: 0 \leqq t \leqq H_a\}$ is $\mathscr{E}_a$-measurable or not ($H_a = \inf\{t: B_t = a\}$). We shall leave to the reader the trouble of bolting on the modifications required for the initial part of the path, and shall treat only the case $a=0$, to which all others can be reduced.

Let U denote the space of Brownian excursions:

$$U = \{f \in C(\mathbb{R}^+, \mathbb{R}): f^{-1}(\mathbb{R}\setminus\{0\}) = (0,\zeta) \text{ for some } \zeta > 0\};$$

where for $f \in U$, $\zeta \equiv \inf\{t>0: f(t)=0\}$ is called the *lifetime of the excursion f.* The space U decomposes naturally into $U_+ \cup U_-$, where

$$U_+ = \{f \in U: f(t) \geqq 0 \text{ for all } t\}.$$

The Brownian path can be decomposed into its excursions, yielding a point process in $(0,\infty)\times U$, which turns out to be a Poisson process with expectation measure $\mathrm{d}t \times \mathrm{d}n$, where the σ-finite measure n is the so-called *excursion law* of Brownian motion. For more information on Brownian excursion theory, consult Part 8 of Chapter VI in Rogers and Williams [RW]; in particular, the characterisation of n given in Theorem VI.55.11 will be used without further comment.

Let N denote the Poisson random measure of excursions and let $N_\pm$ denote the restriction of N to $U_\pm$. Then $\mathscr{E}_0$ is the σ-field generated by N_- and every zero-mean $L^2(\mathscr{E}_0)$ random variable Y has a stochastic integral representation

$$Y = \iint_{(0,\infty)\times U_-} \phi(s,f)\,\tilde{N}_-(\mathrm{d}s,\mathrm{d}f), \tag{25}$$

where

$$E\left[\int_0^\infty \mathrm{d}s \int_{U_-} n(\mathrm{d}f)\,\phi(s,f)^2\right] < \infty,$$

and $\tilde{N}_-$ is the compensated Poisson random measure

$$\tilde{N}_-(\mathrm{d}s,\mathrm{d}f) \equiv N_-(\mathrm{d}s,\mathrm{d}f) - \mathrm{d}s\, n(\mathrm{d}f).$$

Define

$$\gamma_t \equiv \inf\{u: L(u,0) > t\}.$$

Then it is easy to see that

$$\int_0^\infty e^{-\lambda A(s,b)} L(\mathrm{d}s,b) = \iint_{[0,\infty)\times U} V_{t-}\,\phi(f)\,N(\mathrm{d}t,\mathrm{d}f) \tag{26}$$

$$\equiv X, \tag{27}$$

say, where

$$V_t \equiv \exp\{-\lambda A(\gamma_t, b)\},$$

and

$$\phi(f) = \int_0^\zeta \exp\left(-\lambda \int_0^u I_{(-\infty,b]}(f_s)\,\mathrm{d}s\right) l_f(\mathrm{d}u, b),$$

where $l_f(\cdot,\cdot)$ is the local time process of the excursion f. Notice that ϕ is supported in U_+, so that the N-stochastic integral representation of X is actually a stochastic integral with respect to N_+. Notice also that since

$$X = \int_0^\infty \lambda e^{-\lambda t} \tilde{L}(t,b)\,\mathrm{d}t$$

$$\leq \int_0^\infty \lambda e^{-\lambda t} \tilde{L}(t+H_b, b)\,\mathrm{d}t,$$

a random variable with an exponential law, X possesses all moments.

Thus

$$M_t \equiv \iint_{(0,t]\times U_+} V_{s-}\,\phi(f)\,\tilde{N}_+(\mathrm{d}s,\mathrm{d}f)$$

is an L^2-bounded martingale which is purely discontinuous and orthogonal to any stochastic integral with respect to $\tilde{N}_-$, because the jumps of N_+ and

N_- never coincide in time. Because of the stochastic integral representation (25), then,

$$E[M_\infty | \mathscr{E}_0] = 0,$$

and hence

(28) $$E[X | \mathscr{E}_0] \equiv E[\iint V_{t-}\, \phi(f)\, N_+(\mathrm{d}t, \mathrm{d}f) | \mathscr{E}_0] = \int_U \phi(f)\, n(\mathrm{d}f) . E\left[\int_0^\infty V_t\, \mathrm{d}t | \mathscr{E}_0\right],$$

expressing the thing we are interested in as the product of two factors. The first we shall evaluate presently using Theorem VI.55.11 of Rogers and Williams [RW], but before that, we deal with the second.
We prove that

(29) $$E[V_t | \mathscr{E}_0] = \exp(-\lambda A(\gamma_t, 0) - c t),$$

where $c = \frac{1}{2}\theta \tanh \theta \delta$. The argument was supplied by a referee, and replaces our longer original. The function

$$\Phi(x) = \cosh \theta (x-b)^-$$

is a bounded positive solution to

$$\tfrac{1}{2}\Phi'' = \lambda I_{(0,b]} \Phi$$

and hence by Itô's formula, with $c = \Phi'(0)/\Phi(0) = \frac{1}{2}\theta \tanh \theta \delta$,

$$Y_t = \Phi(B_t) \exp\left\{-\lambda \int_0^t I_{(0,b]}(B_s)\, \mathrm{d}s + c L(t, 0)\right\}$$

is a continuous martingale,

$$Y_t = \Phi(0) + \int_0^t \Phi'(B_s) \exp\left\{-\lambda \int_0^s I_{(0,b]}(B_u)\, \mathrm{d}u + c L(s, 0)\right\} I_{(B_s > 0)}\, \mathrm{d}B_s .$$

But then $E(Y(\gamma_t) | \mathscr{E}_0) = \Phi(0)$ by Lemma 1.2 of McGill [M], as at the end of Sect. 2. Hence

$$E[\exp\{-\lambda(A(\gamma_t, b) - A(\gamma_t, 0))\} | \mathscr{E}_0] = \mathrm{e}^{-ct},$$

from which (29) is immediate. Now a little calculus takes us from (29) to

(30) $$E\left[\int_0^\infty V_t\, \mathrm{d}t | \mathscr{E}_0\right] = \int_0^\infty \exp(-\lambda t - c \tilde{L}(t, 0))\, \tilde{L}(\mathrm{d}t, 0).$$

So, in summary, the result (22) which we want follows from (28) and (30) once we can prove that

(31) $$\int_U \phi(f)\, n(\mathrm{d}f) = \operatorname{sech} \theta \delta.$$

Note that ϕ is zero unless the maximum of the excursion f exceeds b; by Williams' characterisation of the Brownian excursion law, the n-measure of excursions which exceed b is $1/2b$, and, given that the maximum is greater than b, the excursion behaves as a BES(3) process until it reaches b, and thereafter it behaves as a Brownian motion.

Thus

$$\int_U \phi(f)\, n(\mathrm{d}f) = \frac{1}{2b} \cdot \frac{\theta b}{\sinh \theta b} E^b\left[\int_0^{H_0} L(\mathrm{d}s, b)\, \mathrm{e}^{-\lambda A(s,b)}\right], \tag{32}$$

since $\lambda \mapsto \sqrt{2\lambda}\, b \operatorname{cosech} \sqrt{2\lambda}\, b$ is the Laplace transform of the BES(3) first passage law. By Lévy's characterisation of reflecting Brownian motion, we have that

$$E^b\left[\int_0^{H_0} L(\mathrm{d}s, b)\, \mathrm{e}^{-\lambda A(s,b)}\right] = E^0\left[\int_0^T 2L(\mathrm{d}s, 0)\, \mathrm{e}^{-\lambda s}\right],$$

where $T = \inf\{u \colon |B_u| = b\}$, and this is equal to

$$2E^0\left[\int_0^\infty \mathrm{e}^{-\lambda s} L(\mathrm{d}s, 0)\right](1 - E^0\, \mathrm{e}^{-\lambda S}), \tag{33}$$

using the strong Markov property at the stopping time $S = \inf\{t > T \colon B_t = 0\}$. Hence (33) is equal to

$$\frac{2}{\theta}(1 - \mathrm{e}^{-\theta\delta} \operatorname{sech} \theta\delta) = 2\theta^{-1} \tanh \theta\delta. \tag{34}$$

Assembling (32) and (34) yields (31). □

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