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Probability Theory and Related Fields

The heat kernel formula in a geodesic chart and some applications to the eigenvalue problem of the 3-sphere

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Summary. This paper deals with a heat kernel formula in a geodesic chart with some applications to the standard n-sphere. Our emphasis will be on the special case of the 3-sphere which exhibits some identities linking spherical harmonics and certain homogeneous polynomials harmonic on $\mathbb{R}^4$. In particular, we will deduce an expression for $P_x(\zeta > t)$ where ζ is the first (random) time that the bridge process in S^3 hits the south pole. Another easy consequence will be a special case of the H.P. McKean and I.M. Singer expansion of the heat kernel.

1. Introduction

We will first give the notation for a general (complete connected) Riemannian manifold $M.c|f$ [6]. Let $L = \frac{1}{2}\Delta + b + V$ be a differential operator on M where Δ is the Laplace-Beltrami operator on M, b a smooth vector field on M and V a continuous potential term supposed bounded above.

Let θ_y be the Jacobian determinant of the exponential map $\exp_y\colon T_y M \to M$ at $y \in M$: $\theta_y(x) = |\det_U T_v \exp_y|$ with $x = \exp_y(v) \in U$ where $U \subset M - \mathrm{Cut}(y)$ is star-shaped from y and $\mathrm{Cut}(y)$ is the cut-locus of M at y. Star-shaped here means that for any $x \in U$ there exists a unique geodesic joining x and y and lying entirely in U. Let

$$(1.1) \qquad B_y(x) = \exp\left\{\int_0^1 \langle \dot\gamma(s), b(\gamma(s))\rangle \, ds\right\},$$

where γ is the unique minimal geodesic from x to y parameterized to take unit time.

Then let,

$$(1.2) \qquad C_y(x) = B_y(x)\, \theta_y(x)^{-\frac{1}{2}}, \qquad x \in M - \mathrm{Cut}(y)$$

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and let $q_t(x, y) = (2\pi t)^{-\frac{n}{2}} C_y(x) \exp\left\{-\frac{d(x, y)}{2t}\right\}^2$, where n is the dimension of M and d is the distance compatible with the Riemannian metric on M. Let $P_t^M(-,-)$ be the heat kernel of M relative to L and $P_t^U(-,-)$ the Dirichlet heat kernel of U relative to L. Let $S^n = (S^n(1), g_0)$ be the standard n-sphere. Fix a point (the north pole) $y \in S^n$ and let $\bar{y}$ (the south pole) be the point anti-podal to y.

The notion of the semi-classical Brownian Riemannian bridge is better understood via the construction of the canonical Brownian motion (with time-dependent drift) as carried out in [7], Chap. VII, §1 and §12 or in [9].

To start with, consider the underlying probability space $(\Omega, \mathbf{F}, P)$. We can construct $(\Omega, \mathbf{F}, P)$ as follows: Let $t > 0$, define $\Omega = C_0([0, t]; R^n)$ = space of continuous paths from $[0, t]$ to R^n starting from $0 \in R^n$. $\mathbf{F}$ is the σ-algebra generated by the Borel cylinder sets. A Borel cylinder set $B \subset \Omega$ is a set of the form.

(1.3) $$B = \{\omega \in \Omega: (\omega(t_1), \ldots, \omega(t_m)) \in E\}$$

for $0 < t_1 < t_2 < \ldots < t_m$ and $E \in B(R^n)$.

P is the Wiener measure on $(\Omega, \mathbf{F})$. Let $\Pi: O(M) \to M$ be the orthonormal frame bundle and let $T\Pi: TO(M) \to TM$ be the derivative map. For $u \in O(M)$, we have the subspace $\ker T_u \Pi$ of $T_u O(M)$. Then set

(1.4) $$VTO(M) = \bigcup_{u \in O(M)} \ker T_u \Pi$$

to define a subbundle $VTO(M)$ of $TO(M)$. The Levi-Civita connection on M determines a unique complementary horizontal subbundle which we denote by $HTO(M)$.

Hence we have:

(1.5) $$TO(M) = VTO(M) \oplus HTO(M).$$

In terms of fibres we have:

(1.6) $$T_u O(M) = VT_u O(M) \oplus HT_u O(M), \qquad u \in O(M).$$

This determines a map:

(1.7) $$X:\ O(M) \times R^n \to HTO(M) \subset TO(M)$$

which is a trivialization (i.e. a diffeomorphism) of the horizontal (sub)bundle $HTO(M)$: X is defined as follows:

(1.8) $$X(u, e) = (T\Pi \mid HTO(M))^{-1}(u(e)),$$

where $T\Pi \mid HTO(M)$ is the restriction of $T\Pi$ to $HTO(M)$.

Equivalently, X is the inverse of the map:

(1.9) $$\tilde{\theta}\colon HTO(M) \to O(M) \times R^n$$

defined by:

(1.10) $$\tilde{\theta}(v) = (u, \theta(v)); \quad v \in HT_u O(M),$$

where $\theta\colon TO(M) \to R^n$ is the canonical 1-form restricted to $HTO(M)$.

Given a C^1 time-dependent vector field on M

$$A_s\colon M \to TM, \quad 0 \leqq s < t$$

let $\tilde{A}_s\colon O(M) \to TO(M)$ be its horizontal lift:

(1.11) $$\tilde{A}_s(u) \in HT_u O(M) \quad \text{and} \quad T\Pi(\tilde{A}_s(u)) = A_s(\Pi(u)); \quad 0 \leqq s < t.$$

Now fix $x \in M$ and take $u_0 \in \Pi^{-1}(x)$. Let $(u_s)\, 0 \leqq s < t \wedge \zeta$ be the solution of the Stratonovich stochastic differential equation for $u\colon [0, t \wedge \zeta) \times \Omega \to O(M)$

(1.12) $$\mathrm{d}u_s = X(u_s, 0\,\mathrm{d}B_s) + \tilde{A}_s(u_s)\,\mathrm{d}s$$

$$u(0, \omega) = u_0, \quad \omega \in \Omega,$$

where $(B_s)\, 0 \leqq s \leqq t$ is the n-dimensional Euclidean Brownian motion defined on the basic space $(\Omega, \mathbf{F}, P)$ and ζ is the explosion time of the solution process $(u_s)\, 0 \leqq s < t \wedge \zeta$.

Finally $x_s^t = \Pi(u_s)$, $0 \leqq s < t \wedge \zeta$ is a diffusion process starting from $x = \Pi(u_0)$ with differential generator $\frac{1}{2}\Delta + A_s$. We distinguish four cases:

(i) When $A_s = 0$, then the process $(x_s)\, 0 \leqq s \leqq t \wedge \zeta$ (which is independent of t) is just Brownian motion on M.
(ii) When $A_s = b$ (b independent of time s), then $(x_s)\, 0 \leqq s \leqq t \wedge \zeta$ (independent of t) is Brownian motion with drift b on M.
(iii) When $A_s = b + \nabla \log P_{t-s}^M(-, y)$, then $(x_s^t)\, 0 \leqq s \leqq t \wedge \zeta$ is the classical Brownian Riemannian bridge with drift b starting from $x \in M$ and ending at $y \in M$ in time t.
(iv) When $A = b + \nabla \log q_{t-s}(-, y)$, then $(x_s^t)\, 0 \leqq s \leqq t \wedge \zeta$ is the semi-classical Brownian Riemannian bridge with drift b starting from $x \in M$ and ending at $y \in M$ in time t.

Since the semi-classical Brownian Riemannian bridge with drift b $(x_s^t)\, 0 \leqq s \leqq t \wedge \zeta$ defined on the basic space $(\Omega, \mathbf{F}, P)$ starts from the point $x \in M$, we will write $(\Omega, \mathbf{F}, P_x)$ instead of $(\Omega, \mathbf{F}, P)$ to emphasize the fact that $P\,(\omega \in \Omega\colon x_0^t(\omega) = x) = 1$. The corresponding mathematical expectation will be denoted by E_x.

2. The heat kernel formula in a geodesic chart

Here we will prove the heat kernel formula for $U \subset M - \mathrm{Cut}(y)$ star-shaped from y. This was proved in [16] and given in [15] for $b = 0$. We will first assume that U has compact closure $\bar{U}$ in $M - \mathrm{Cut}(y)$.

Let f_t^λ be the solution of the diffusion equation in U with Dirichlet boundary conditions:

$$\text{(2.1)} \qquad \frac{\partial f_t^\lambda}{\partial t} = L f_t^\lambda$$

$$f_0^\lambda = (2\pi\lambda)^{-\frac{n}{2}} T_0 \exp\left\{-\frac{d(-, y)^2}{2\lambda}\right\}$$

$$f_t^\lambda(x) = 0 \ \forall x \in \partial U; \qquad t > 0,$$

where T_0 is of compact support in U with $T_0(y) = 1$ and ∂U is the boundary of U.

2.1. Theorem

$$P_t^U(x, y) = \lim_{\lambda \downarrow 0} f_t^\lambda(x)$$

Proof.

$$\text{(2.2)} \qquad f_t^\lambda(x) = \int_U f_0^\lambda(z)\, P_t^U(x, z)\, dz,$$

where dz denotes the volume element measure on M. Replacing f_0^λ by its value given in (2.1) above, we have:

$$\text{(2.3)} \qquad f_t^\lambda(x) = (2\pi\lambda)^{-\frac{n}{2}} \int_U T_0(z) \exp\left\{-\frac{d(z, y)^2}{2\lambda}\right\} P_t^U(x, z)\, dz$$

$$\text{(2.4)} \qquad = (2\pi\lambda)^{-\frac{n}{2}} \int_{T_y M} T_0(\exp_y v) \exp\left\{-\frac{\|v\|^2}{2\lambda}\right\} P_t^U(x, \exp_y v)\, \theta_y(v)\, dv$$

by Lemma (2A) of [10] and the fact that T_0 has compact support in U.

Setting $v = \omega\sqrt{\lambda}$, (2.4) becomes:

$$\text{(2.5)} \qquad f_t^\lambda(x) = (2\pi)^{-\frac{n}{2}} \int_{T_y M} T_0(\exp_y \omega\sqrt{\lambda}) \exp\left\{-\frac{\|\omega\|^2}{2}\right\} \cdot P_t^U(x, \exp_y \omega\sqrt{\lambda})\, \theta_y(\omega\sqrt{\lambda})\, d\omega.$$

Thus,

$$\text{(2.6)} \qquad \lim_{\lambda \downarrow 0} f_t^\lambda(x) = (2\pi)^{-\frac{n}{2}} \int_{T_y M} T_0(y) \exp\left\{-\frac{\|\omega\|^2}{2}\right\} P_t^U(x, y)\, \theta_y(0)\, d\omega = P_t^U(x, y)$$

since

$$\int_{T_y M} \exp\left\{-\frac{\|\omega\|^2}{2}\right\} d\omega = (2\pi)^{\frac{n}{2}}$$

and $T_0(y) = 1 = \theta_y(0)$.

A probabilistic representation of the solution f_t^λ of the diffusion equation in (2.1) is given by the Feynman-Kac formula:

$$f_t^\lambda(x) = E_x\left(\chi_{\tau>t} f_0^\lambda(y_t) \exp\left\{\int_0^t V(y_s)\, ds\right\}\right), \tag{2.7}$$

where $\tau = \tau(x)$ is the first exit time from U of the diffusion process $(y_s)\, 0 \leqq s < \infty$ with generator $L_0 = \frac{1}{2}\Delta + b$. By the Girsanov-Cameron-Martin formula:

$$f_t^\lambda(x) = E_x\left(\chi_{\tau^\lambda>t} f_0^\lambda(x_t^\lambda)\, M_t^\lambda \exp\left\{\int_0^t V(x_s^\lambda)\, ds\right\}\right), \tag{2.8}$$

where

(i) For each $t > 0$, $(x_s^\lambda)\, 0 \leqq s < t \wedge \zeta^\lambda$ is now a diffusion process starting from $x \in U$ with generator $L_0 + A_s^\lambda$ where A_s^λ is defined in geodesic normal coordinates by:

$$A_s^\lambda(x) = -\frac{x}{\lambda + t - s} + \nabla \log C_y(x) \quad \text{i.e.} \quad A_s^\lambda(x) = \nabla \log q_{\lambda+t-s}(x, y).$$

Thus $(x_s^\lambda)\, 0 \leqq s \leqq t \wedge \zeta^\lambda$ is the semi-classical Brownian Riemannian bridge with drift b in $M - \mathrm{Cut}(y)$ starting from $x \in U$ and ending at y in time $\lambda + t$.
(ii) ζ^λ is the first hitting time of the cut-locus by the bridge process x_s^λ started at $x \in U$ and τ^λ is the first exit time of x_s^λ from U.
(iii) $M_t^\lambda\, 0 \leqq s < \infty$ is the exponential local martingale given by the stochastic differential equation

$$d M_s^\lambda = -M_s^\lambda \langle A_s^\lambda(x_s^\lambda), u_s^\lambda\, dB_s\rangle_{x_s^\lambda}$$

$$M_0^\lambda = 1$$

i.e.

$$M_t^\lambda = \exp\left\{-\int_0^t \langle A_s^\lambda(x_s^\lambda), u_s^\lambda\, dB_s\rangle_{x_s^\lambda} - \frac{1}{2}\int_0^t \|A_s^\lambda(x_s^\lambda)\|^2\, ds\right\}, \tag{2.9}$$

where $(u_s^\lambda)\, 0 \leqq s \leqq t \wedge \zeta^\lambda$ is the horizontal lift of $(x_s^\lambda)\, 0 \leqq s \leqq t \wedge \zeta^\lambda$ on the orthonormal frame bundle $O(M)$ of M: $x_s^\lambda = \Pi(u_s^\lambda)$.

Much of the proof of the heat kernel formula in a geodesic chart is based on the computations of M_t^λ:

2.2. Lemma

$$M_t^\lambda = \left(\frac{\lambda}{\lambda+t}\right)^{-\frac{n}{2}} \exp\left\{-\frac{d(x,y)^2}{2(\lambda+t)} + \frac{d(x_t^\lambda, y)^2}{2\lambda}\right\}$$

$$\cdot C_y(x)\, C_y(x_t^\lambda)^{-1} \exp\left\{\int_0^t \frac{L_0 C_y}{C_y}(x_s^\lambda)\, ds\right\}$$

where $L_0 = \frac{1}{2}\Delta + b$.

Proof. Define $Y_s^\lambda\colon M \to R$ by

$$Y_s^\lambda(x) = -\frac{d(x, y)^2}{2(\lambda+t-s)} + \log C_y(x).$$

Then $\nabla Y_s^\lambda = A_s^\lambda$ and so by Itô's formula,

$$(2.10)\quad Y_t^\lambda(x_t^\lambda) = Y_0^\lambda(x) + \int_0^t \frac{\partial Y^\lambda}{\partial s}(x_s^\lambda)\, ds + \int_0^t \langle A_s^\lambda(x_s^\lambda), u_s^\lambda\, dB_s\rangle_{x_s^\lambda}$$

$$+ \int_0^t \langle \nabla Y_s^\lambda(x_s^\lambda), b(x_s^\lambda) + A_s^\lambda(x_s^\lambda)\rangle_{x_s^\lambda}\, ds + \int_0^t \tfrac{1}{2}\Delta Y_s^\lambda(x_s^\lambda)\, ds \quad \text{a.s.}$$

Now, using (2.10) above we substitute for

$$-\int_0^t \langle A_s^\lambda(x_s^\lambda), u_s^\lambda\, dB_s\rangle_{x_s^\lambda}$$

in (2.9), to have:

$$(2.11)\quad M_t^\lambda = \exp\left\{Y_0^\lambda(x) - Y_t^\lambda(x_t^\lambda) + \int_0^t \frac{\partial Y_s^\lambda}{\partial s}(x_s^\lambda)\, ds + \tfrac{1}{2}\int_0^t \|A_s^\lambda(x_s^\lambda)\|^2\, ds\right.$$

$$\left. + \int_0^\lambda \langle A_s^\lambda(x_s^\lambda), b(x_s^\lambda)\rangle\, ds + \tfrac{1}{2}\int_0^t \Delta Y_s^\lambda(x_s^\lambda)\, ds\right\} \quad \text{a.s.}$$

Setting $r(x) = d(x, y)$, we have:

$$A_s^\lambda(x) = \nabla Y_s^\lambda(x) = -\frac{r(x)\,\nabla r(x)}{\lambda+t-s} + \nabla \log C_y(x).$$

Hence we have:

$$(2.12)\quad \tfrac{1}{2}\|A_s^\lambda(x)\|^2 = \frac{r^2(x)}{2(\lambda+t-s)^2} - \frac{r(x)}{\lambda+t-s}\langle \nabla r(x), \nabla \log C_y(x)\rangle + \tfrac{1}{2}\|\nabla \log C_y(x)\|^2$$

$$(2.13)\quad = \frac{r^2(x)}{2(\lambda+t-s)^2} - \frac{r(x)}{\lambda+t-s}\frac{\partial}{\partial r}\log C_y(x) + \tfrac{1}{2}\|\nabla \log C_y(x)\|^2.$$

Recall that if a C^2-function $f\colon M \to \mathbb{R}$ depends only on $r(x)$, then:

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \left(\frac{n-1}{r} + \frac{\partial}{\partial r}\log\theta_y\right)\frac{\partial f}{\partial r}$$

and hence,

$$(2.14)\quad \tfrac{1}{2}\Delta Y_s^\lambda = -\frac{1}{4(\lambda+t-s)}\left[2 + \left(\frac{n-1}{r} + \frac{\partial}{\partial r}\log\theta_y\right)2r\right] + \tfrac{1}{2}\Delta \log C_y$$

$$-\frac{1}{2(\lambda+t-s)}\left[n + r\frac{\partial}{\partial r}\log\theta_y\right] + \tfrac{1}{2}\Delta \log C_y.$$

It is easily shown (see for example [18]) that

$$\langle \nabla r(x), b(x)+\nabla \log B_y(x)\rangle = 0 \tag{2.15}$$

and so,

$$\begin{aligned}\langle A_s^\lambda(x), b(x)\rangle &= -\frac{r(x)}{\lambda+t-s}\langle \nabla r(x), b(x)\rangle + \langle \nabla \log C_y(x), b(x)\rangle \\ &= -\frac{r(x)}{\lambda+t-s}\langle \nabla r(x), b(x)+\nabla \log B_y(x)\rangle \\ &\quad + \frac{r(x)}{\lambda+t-s}\langle \nabla r(x), \nabla \log B_y(x)\rangle + \langle \nabla \log C_y(x), b(x)\rangle\end{aligned} \tag{2.16}$$

$$\begin{aligned}&= \frac{r(x)}{\lambda+t-s}\langle \nabla r(x), \nabla \log B_y(x)\rangle + \langle \nabla \log C_y(x), b(x)\rangle \\ &= \frac{r(x)}{\lambda+t-s}\frac{\partial}{\partial r}\log B_y(x) + \langle \nabla \log C_y(x), b(x)\rangle.\end{aligned} \tag{2.17}$$

Thus by (2.14) and (2.17), we have:

$$\begin{aligned}\langle A_s^\lambda(x), b(x)\rangle + \tfrac{1}{2}\Delta Y_s^\lambda(x) = -\frac{n}{2(\lambda+t-s)} + \frac{r(x)}{\lambda+t-s}\frac{\partial}{\partial r}\log C_y(x) \\ + (\tfrac{1}{2}\Delta + b)\log C_y(x).\end{aligned} \tag{2.18}$$

Finally, we have:

$$\frac{\partial Y_s^\lambda}{\partial s}(x) = -\frac{r^2(x)}{2(\lambda+t-s)^2} \tag{2.19}$$

and so by (2.13), (2.18) and (2.19), M_t^λ in (2.11) becomes:

$$\begin{aligned}M_t^\lambda = \exp\Bigg\{ -\frac{r^2(x)}{2(\lambda+t)} + \frac{r^2(x_t^\lambda)}{2\lambda} + \log C_y(x) - \log C_y(x_t^\lambda) \\ + \int_0^t \left(\tfrac{1}{2}\|\nabla \log C_y(x_s^\lambda)\|^2 + (\tfrac{1}{2}\Delta + b)\log C_y(x_s^\lambda) - \frac{n}{2(\lambda+t-s)}\right) \mathrm{d}s\Bigg\}.\end{aligned} \tag{2.20}$$

A direct computation shows that:

$$\tfrac{1}{2}\|\nabla \log C_y(x)\|^2 + \tfrac{1}{2}\Delta \log C_y(x) = \frac{1}{2}\frac{\Delta C_y(x)}{C_y(x)} \tag{2.21}$$

and since $\langle b(x), \nabla \log C_y(x)\rangle = \left\langle b(x), \frac{\nabla C_y(x)}{C_y(x)}\right\rangle$. We have:

$$\tfrac{1}{2}\|\nabla \log C_y(x)\|^2 + (\tfrac{1}{2}\Delta + b)\log C_y(x) = \frac{L_0 C_y(x)}{C_y(x)}. \tag{2.22}$$

Therefore (2.20) becomes:

$$(2.23)\quad M_t^\lambda = \left(\frac{\lambda}{\lambda+t}\right)^{\frac{n}{2}} \exp\left\{-\frac{r^2(x)}{2(\lambda+t)} + \frac{r(x_t^\lambda)}{2\lambda}\right\} \frac{C_y(x)}{C_y(x_t^\lambda)} \exp\left\{\int_0^t \frac{L_0 C_y}{C_y}(x_t^\lambda)\, ds\right\}$$

and so the Lemma is proved.

The following Lemma is proved in [10].

2.3. Lemma. *$(x_s^\lambda)\, 0 \leqq s \leqq t$ converges as $\lambda \downarrow 0$ uniformly on $[0, t]$ in probability to a process $(x_s)\, 0 \leqq s \leqq t$ where*

$$x_s = x_s^0 \qquad \forall s \in [0, t)$$

$$x_t = y \quad \text{a.s.}$$

i.e. $(x_s)\, 0 \leqq s \leqq t \wedge \zeta$ is the semi-classical Brownian Riemannian bridge with drift b from x to y in time t.

2.4. Lemma. *The indicator function $\chi_{\tau^\lambda > t}$ of the first exit time τ^λ of the bridge process $(x_s^\lambda)\, 0 \leqq s \leqq t \wedge \zeta^\lambda$ from U has the property:*

$$\chi_{\tau^\lambda > t} = \lim_p \downarrow \exp\left\{\int_0^t W_p(x_s)^\lambda\, ds\right\} \quad \text{a.s.}$$

where $(W_p)\, p \geqq 1$ is a certain decreasing sequence of bounded continuous functions on M.

Proof (by Construction). Define the sequence of open subsets of M:

$$U_p = \left\{x \in M : d(x, \bar{U}) < \frac{1}{p}\right\}; \qquad p \geqq 1$$

and define the sequence $(W_p)\, p \geqq 1$ on M as follows:

$$(2.24)\qquad W_p = \begin{cases} 0 & \text{on } \bar{U} \\ -\min[p^2\, d(-, \bar{U}), p] & \text{on } U_p - U \\ -p & \text{on } U_p^c. \end{cases}$$

Then clearly $(W_p)\, p \geqq 1$ is a decreasing sequence of continuous function on M each of which is bounded above (in fact it is uniformly bounded above by 0). Moreover from the definition of W_p, we have clearly:

$$(2.25)\qquad \chi_{\bar{\tau}^\lambda \geqq t} = \lim_p \downarrow \exp\left\{\int_0^t W_p(x_s^\lambda)\, ds\right\}$$

where $\bar{\tau}^\lambda$ is the first exit time of the bridge process $(x_s^\lambda)\, 0 \leqq s \leqq t \wedge \zeta^\lambda$ from $\bar{U}$. Since U has smooth boundary,

$$(2.26)\qquad \bar{\tau}^\lambda = \tau^\lambda \quad \text{a.s.}$$

and hence,

$$\chi_{\tau^\lambda \geqq t} = \lim_p \downarrow \exp\left\{\int_0^t W_p(x_s^\lambda)\,\mathrm{d}s\right\} \quad \text{a.s.} \tag{2.27}$$

Now, by [11], Theorem (5.2), p. 47, the Cauchy problem in U for the parabolic equation:

$$\frac{\partial g_s^\lambda}{\partial s} = \frac{1}{2}\Delta g_s^\lambda + b(g_s^\lambda) + A_s^\lambda(g_s^\lambda) \tag{2.28}$$

$$g_0^\lambda = 1$$

$g_s^\lambda = 0$ on the boundary ∂U of U has a $C^{1,2}(U)$-solution given by

$$g_s^\lambda(x) = P_x(\tau^\lambda > s)$$

and hence the measure $P_x(\tau^\lambda \in \cdot)$ has a density with respect to Lebesgue measure on $[0, t]$. Consequently, we have:

$$\chi_{\tau^\lambda \geqq t} = \chi_{\tau > t} \quad \text{a.s.} \tag{2.29}$$

The Lemma follows by (2.27).

2.5(a). Theorem. *If U has compact closure and smooth boundary such that $\bar{U} \subset M - \mathrm{Cut}(y)$ is star-shaped from y, then we have the inequality:*

$$P_t^U(x, y) \leqq (2\pi t)^{-\frac{n}{2}} \exp\left\{-\frac{\mathrm{d}(x, y)^2}{2t}\right\} C_y(x)\, E_x\left(\chi_{\tau > t} \exp\left\{\int_0^t \frac{LC_y}{C_y}(x_s)\,\mathrm{d}s\right\}\right)$$

where $(x_s)\, 0 \leqq s \leqq t \wedge \zeta$ is the semi-classical Brownian Riemannian bridge with drift b (whose differential generator is $\frac{1}{2}\Delta + b + A_s$) where

$$A_s(x) = A_s^0(x) = \nabla \log q_{t-s}(x, y).$$

$\zeta = \zeta(t, x, y)$, is the first hitting time of the cut-locus by the above bridge process started at $x \in U$ and reaching y at time t. $\tau = \tau(t, x, y)$ is its first exit time from U.

Proof. By (2.8) and Theorem (2.1), we have:

$$P_t^U(x, y) = \lim_{\lambda \downarrow 0} E_x\left(\chi_{\tau^\lambda > t} f_0^\lambda(x_t^\lambda)\, M_t^\lambda \exp\left\{\int_0^t V(x_s^\lambda)\,\mathrm{d}s\right\}\right) \tag{2.30}$$

$$= (2\pi t)^{-\frac{n}{2}} \exp\left\{-\frac{\mathrm{d}(x, y)^2}{2t}\right\} C_y(x) \tag{2.31}$$

$$\cdot \lim_{\lambda \downarrow 0} E_x\left(\chi_{\tau^\lambda > t}\, T_0(x_t^\lambda)\, C_y(x_t^\lambda)^{-1} \exp\left\{\int_0^t \frac{LCy}{Cy}(x_s^\lambda)\,\mathrm{d}s\right\}\right)$$

by Lemma (2.2).

The function $C_y\left(\text{resp. } \frac{LC\,y}{C\,y}\right)$ is defined and is continuous in $\bar{U}$ and hence can be continuously extended to all of M such that the extension be equal to $C_y\left(\text{resp. } \frac{LC\,y}{C\,y}\right)$ in $\bar{U}$ and 0 outside a bounded neighbourhood of $\bar{U}$. We will denote the extension simply by $C_y\left(\text{resp. } \frac{LC\,y}{C\,y}\right)$. Hence the expression:

$$T_0(x_t^\lambda)\,C_y(x_t^\lambda)^{-1}\exp\left\{\int_0^t\left(\frac{LC\,y}{C\,y}(x_s^\lambda)+W_p(x_s^\lambda)\right)\mathrm{d}\,s\right\}$$

is bounded by a constant $C(t)$. By Lemma (2.3), $(x_s^\lambda)\,0\leqq s\leqq t\wedge\zeta^\lambda$ converges (uniformly on $[0, t]$) in probability to $(x_s)\,0\leqq s\leqq t\wedge\zeta$. Now, set:

$$I^\lambda=E_x\left(\chi_{\tau^\lambda>t}\,T_0(x_t^\lambda)\,C_y(x_t^\lambda)^{-1}\exp\left\{\int_0^t\frac{LC\,y}{C\,y}(x_s^\lambda)\,\mathrm{d}\,s\right\}\right).$$

By Lemma (2.4), we have:

$$\operatorname*{Lim}_{\lambda\downarrow 0} I^\lambda\leqq E_x\left(T_0(x_t)\,C_y(x_t)^{-1}\exp\left\{\int_0^t\left(\frac{LC\,y}{C\,y}(x_s)+W_p(x_s)\right)\mathrm{d}\,s\right\}\right)$$

$x_t=y$ a.s. by Lemma (2.3) and $T_0(y)=1=C_y(y)$.

The above inequality thus becomes:

$$\operatorname*{Lim}_{\lambda\downarrow 0} I^\lambda\leqq E_x\left(\exp\left\{\int_0^t\left(\frac{LC\,y}{C\,y}(x_s)+W_p(x_s)\right)\mathrm{d}\,s\right\}\right).$$

By taking limits as $p\uparrow\infty$, we have by Lemma (2.4) again:

$$\operatorname*{Lim}_{\lambda\downarrow 0} I^\lambda\leqq E_x\left(\chi_{\tau>t}\exp\left\{\int_0^t\frac{LC\,y}{C\,y}(x_s)\,\mathrm{d}\,s\right\}\right) \tag{2.32}$$

and so the inequality of the Theorem is obtained

2.5(b). Theorem. *The reverse inequality of Theorem* (2.5(a)) *is also valid under the same hypotheses:*

$$P_t^U(x, y)\geqq(2\pi t)^{\frac{n}{2}}\exp\left\{-\frac{\mathrm{d}(x, y)^2}{2t}\right\}C_y(x)\,E\left(\chi_{\tau>t}\exp\left\{\int_0^t\frac{LC\,y}{C\,y}(x_s)\,\mathrm{d}\,s\right\}\right).$$

Proof. The proof follows that of Theorem 2B in [10]: We know by (2.8) and Theorem (2.1) that:

$$P_t^U(x,y)=\lim_{\lambda\downarrow 0} E_x\left(\chi_{\tau^\lambda>t} f_0^\lambda(x_t^\lambda) M_t^\lambda \exp\left\{\int_0^t V(x_s^\lambda)\,\mathrm{d}s\right\}\right) \tag{2.33}$$

$$=(2\pi t)^{-\frac{n}{2}}\exp\left\{-\frac{\mathrm{d}(x,y)^2}{2t}\right\} C_y(x) \tag{2.34}$$

$$\cdot\lim_{\lambda\downarrow 0} E_x\left(\chi_{\tau^\lambda>t}\, T_0(x_t^\lambda)\, C_y(x_t^\lambda)^{-1} \exp\left\{\int_0^t \frac{LCy}{Cy}(x_s^\lambda)\,\mathrm{d}s\right\}\right)$$

$$=(2\pi t)^{-\frac{n}{2}}\exp\left\{-\frac{\mathrm{d}(x,y)^2}{2t}\right\} C_y(x)\lim_{\lambda\downarrow 0} I^\lambda. \tag{2.35}$$

Now, let Γ be the space of continuous paths σ: $[0,t]\to M^*$ where M^* is the one-point compactification of M (when M is non-compact) with $\sigma(0)=x\in U$.

Let τ: $\Gamma\to\mathbb{R}_+ U\{+\infty\}$ be the map defined by: $\tau(\sigma)$ is the first exit time of the path σ from U.

Then we have:

$$\lim_{\lambda\downarrow 0} I^\lambda=\lim_{\lambda\downarrow 0}\int_\Gamma \chi_{\tau>t}(\sigma)\, T_0(\sigma(t)\, C_y(\sigma(t))^{-1}\exp\left\{\int_0^t \frac{LC}{C_y}(\sigma(s))\,\mathrm{d}s\right\}\mathrm{d}P_x^\lambda(\sigma), \tag{2.36}$$

where $P_x^\lambda = x_\cdot^\lambda(P_x)$ is the image measure of $\mathbb{P}_x$ by $x_\cdot^\lambda$. We know that convergence in probability implies convergence in law. Hence since $x_\cdot^\lambda$ converges in probability to $x.$ by Lemma (2.3), we conclude that P_x^λ converges weakly (or narrowly) to $x.(P_x)$. Consequently, by ([17], Appendix, Proposition 1), we have the inequality of the Theorem.

2.6. Theorem. *For $\bar{U}\subset M\,\mathrm{Cut}(y)$ star-shaped from y and compact, with smooth boundary, we have:*

$$P_t^U(x,y)=q_t(x,y)\,E_x\left(\chi_{\tau>t}\exp\left\{\int_0^t \frac{LCy}{Cy}(x_s)\,\mathrm{d}s\right\}\right).$$

Proof. The result is immediately by the inequality of Theorem (2.5(a)) and the reverse inequality of Theorem (2.5(b)).

2.7. Corollary

(i) *For any $U\subset M-\mathrm{Cut}(y)$ star-shaped from y, we have:*

$$P_t(x,y)=q_t(x,y)\,E_x\left(\chi_{\tau>t}\exp\left\{\int_0^t \frac{LCy}{Cy}(x(s))\,\mathrm{d}s\right\}\right).$$

(ii) *When M has a pole at y, then we have:*

$$P_t^M(x, y) = q_t(x, y)\, E_x\left(\exp\left\{\int_0^t \frac{LCy}{Cy}(x(s))\, ds\right\}\right).$$

Proof. (i) Let $(U_m)_{m \geqq 0}$ be an increasing sequence of open subsets of U exhausting U such that each U_m is star-shaped from y, has compact closure with smooth boundary and $\bar{U}_m \subset U_{m+1}$ for each $m \geqq 0$. Then each $P_t U_m(x, y)$ satisfies the equality of the theorem:

$$P_t U_m(x, y) = q_t(x, y)\, E_x\left(\chi_{\tau_m > t} \exp\left\{\int_0^t \frac{LCy}{Cy}(x(s))\, ds\right\}\right), \tag{2.37}$$

where τ_m is the first exit time of the bridge process

$$(x(s))\, 0 \leqq s \leqq t \wedge \zeta \quad \text{from } U_m.$$

Since $\tau_m \uparrow \tau$ as $m \uparrow \infty$ where τ is the first exit time from U of the above bridge process, we have, by taking limits as $m \uparrow \infty$:

$$\operatorname*{Lim}_m \uparrow P_t^U m(x, y) = q_t(x, y)\, E_x\left(\chi_{\tau > t} \exp\left\{\int_0^t \frac{LCy}{Cy}(x(s))\, ds\right\}\right). \tag{2.38}$$

Let us show that:

$$\lim_m \uparrow P_t^U m(x, y) = P_t^U(x, y). \tag{2.39}$$

If $b=0$, then (2.39) is just Theorem (4), Chap. 8 of [4]. If $b \neq 0$ then (2.39) is proved as follows: Let $(y_s)\, 0 \leqq s < +\infty$ be as in (2.7) and let $\beta(M)$ be the σ-algebra of Borel subsets of M. Then for any positive $\beta(M)$-measurable function f, we have ($c|f$ [1], proof of Theorem (1.6)):

$$\int_U f(z)\, P_t^U(x, z)\, dz = E_x\left(\chi_{\tau > t} f(y_t) \exp\left\{\int_0^t V(y_s)\, ds\right\}\right) \tag{2.40}$$

$$= \lim_m \uparrow E_x\left(\chi_{\tau_m > t} f(y_t) \exp\left\{\int_0^t V(y_s)\, ds\right\}\right) \tag{2.41}$$

$$= \int_U f(z)\, \bar{P}_t^U(x, z)\, dz. \tag{2.42}$$

Where τ (resp. τ_m) is now the first exit time of $(y_t)\, 0 \leqq t < +\infty$ from U (resp. U_m) and where we set:

$$\lim_m \uparrow P_t^U m(x, z) = \bar{P}_t^U(x, z) \ldots \tag{2.43}$$

In particular, taking $f = X_B$, $B \in \beta(M)$, we have:

$$\int_{U \cap B} P_t^U(x, z)\, dz = \int_{U \cap B} \bar{P}_t^U(x, z)\, dz \quad \text{for all } B \in \beta(M). \tag{2.44}$$

Consequently,

(2.45) $$P_t^U(x, z) = \bar{P}_t^U(x, z) \quad \text{for } \pi\text{-almost all } z \in U,$$

where π is the volume element measure dz on M.

Since $\bar{U}_0 \subset U = \bigcup_{m \geqq 0} U_m$, we have:

(2.46) $$P_t^U(x, z) = \bar{P}_t^U(x, z) \quad \text{for } \pi\text{-almost all } z \in \bar{U}_0.$$

We want to remove the "π-almost" condition above in (2.46): Set $f_m(z) = P_t^U m(x, z)$.

Then $(f_m) m \geqq 0$ is an increasing sequence of continuous functions on the compact set $\bar{U}_0$ whose limit is $\bar{P}_t^U(x, -)$ for each $x \in U$. Hence by Dini's theorem, the sequence converges uniformly on $\bar{U}_0$ to $\bar{P}_t^U(x, -)$. Thus, the limit $\bar{P}_t^U(x, -)$ is continuous on $\bar{U}_0$ for each $x \in U$. Since both sides of (2.46) are now continuous in z on $\bar{U}_0$, we finally conclude that:

(2.47) $$P_t^U(x, z) = \bar{P}_t^U(x, z) \quad \text{for all } z \in \bar{U}_0.$$

In particular, since $y \in U_0$,

(2.48) $$P_t^U(x, y) = \bar{P}_t^U(x, y)$$

and so (i) of the corollary is proved.

(ii) We take $U = M$ in (i) since $\text{cut}(y) = \emptyset$ in this case. The result then follows since the semi-classical Brownian Riemannian bridge (with drift b) on M is non-explosive (when M has a pole) $c|f$ [18] p. 66.

2.7. Remark. (ii) of the above corollary is Theorem (7.14) in [18]. Note that we have removed the boundedness assumption on $\frac{LCy}{Cy}$ contained in that theorem and that the expectation on the R.H.S. of (ii) is finite without it.

2.8. Remark. We can say much more about the equality in (2.47). In fact it is valid in all of U and not just in U_0. To prove this, set

$$f_m(z) = P_t\, U_m(x, z) \quad \text{as before.}$$

Choose and fix any integer $m_0 \geqq 0$. Then, by definition (e.g. see [4], p. 188).

(2.49) $$f_m(z) = 0 \quad \text{for } z \in U - U_m$$

and hence

(2.50) $$f_m(z) = 0 \quad \text{for } z \in \bar{U}_{m_0} - U_m \quad \text{where} \quad 0 \leqq m \leqq m_0 - 1.$$

Hence $(f_m) m \geqq 0$ is an increasing sequence of continuous functions on the compact set $\bar{U}_{m_0}$. It converges (simply) to $\bar{P}_t^U(x, -)$ on $\bar{U}_{m_0}$.

Hence the above convergence is uniform by Dini's theorem (see e.g. [6], p. 86).

Now, let K be a compact subset of U. Then there exists an integer $m_0 \geqq 0$ such that $K \subset U_{m_0}$ and so $(f_m) m \geqq 0$ converges uniformly to $\bar{P}_t^U(x, -)$ on K i.e. $(f_m) m \geqq 0$ converges uniformally on compact subsets of U to the limit $\bar{P}_t^U(x, -)$

and so $\bar{P}_t^U(x, z)$ is continuous in $z \in U$ by ([6], p. 84). Both sides of (2.45) are now continuous in z on all of U and we conclude that

$$P_t^U(x, z) = \bar{P}_t^U(x, z) \qquad \text{for all } z \in U. \tag{2.51}$$

2.9. Remark. The equality in (2.51) shows that the upper bound condition imposed on V at the beginning is not necessary if $P_t^M(-, -)$ exists.

3. Application to the standard 3-sphere

In the case of the standard n-sphere, we have, for $n \geqq 2$ and $x \neq \bar{y}$ where $\bar{y}$ is the point in S^n anti-podal to y:

$$P_t^{S^n - \{\bar{y}\}}(x, y) = q_t(x, y)\, E_x\left(\chi_{\zeta > t} \exp\left\{\int_0^t \frac{L C y}{C y}(x^t(s))\, \mathrm{d}s\right\}\right), \tag{3.1}$$

where $(x^t(s))\; 0 \leqq s \leqq t \wedge \zeta$ is the semi-classical Brownian Riemannian bridge with drift b, from x to y in time t and ζ is its hitting time of the cut-locus $\mathrm{Cut}(y) = \{\bar{y}\}$.

Let $b \equiv 0$ and $V \equiv 0$. Then

$$P_t^{S^n - \{\bar{y}\}}(x, y) = (2\pi t)^{-\frac{n}{2}}\, \theta_y(x)^{-\frac{1}{2}} \exp\left\{-\frac{\mathrm{d}(x, y)^2}{2t}\right\} \cdot E_x\left(\chi_{\zeta > t} \exp\left\{\int_0^t \tfrac{1}{2}\theta_y^{\frac{1}{2}}(x^t(s))\, \Delta\, \theta_y^{-\frac{1}{2}}(x^t(s))\, s\right\}\right). \tag{3.2}$$

$\mathrm{Cut}(y) = \{\bar{y}\}$ has codimension $\geqq 2$ and hence has capacity zero. Consequently, Brownian motion starting from $x \neq \bar{y}$ never hits $\bar{y}$ and hence:

$$P_t^{S^n}(x, y) = P_t^{S^n - \{\bar{y}\}}(x, y) \qquad \text{for } x \neq \bar{y}. \tag{3.3}$$

Consequently, for $x \neq \bar{y}$, we have:

$$P_t^{S^n}(x, y) = (2\pi t)^{-\frac{n}{2}}\, \theta_y^{-\frac{1}{2}}(x) \exp\left\{-\frac{\mathrm{d}(x, y)^2}{2t}\right\} \cdot E_x\left(\chi_{\zeta > t} \exp\left\{\int_0^t \tfrac{1}{2}\theta_y^{\frac{1}{2}}(x^t(s))\, \Delta\, \theta_y^{-\frac{1}{2}}(x^t(s))\, \mathrm{d}s\right\}\right). \tag{3.4}$$

Set $r = \mathrm{d}(x, y)$, then,

$$\theta_y(x) = \left(\frac{\sin r}{r}\right)^{n-1}; \qquad 0 < r < \pi \tag{3.5}$$

and

$$\tfrac{1}{2}\theta_y^{\frac{1}{2}}(x)\, \Delta\, \theta_y^{-\frac{1}{2}}(x) = \frac{(n-1)^2}{8} + \frac{(n-1)(n-3)}{8}\left(\frac{1}{r^2} - \frac{1}{\sin^2 r}\right) \tag{3.6}$$

for $n \geqq 2$.

In particular, for $n=3$,

$$P_t^{S^3}(x, y)=(2\pi t)^{-\frac{3}{2}}\frac{r}{\sin r}\,e^{-\frac{r^2}{2t}}\,e^{\frac{t}{2}}\,p_x(\zeta>t). \tag{3.7}$$

The radial component $r_s^t=d(x^t(s), y)$ of the bridge process $(x^t(s))\,0\leqq s\leqq t\wedge\zeta$ from x to y in time t has the same distribution as the radial component of the corresponding Euclidean Brownian bridge from $x_0=\exp_y^{-1}(x)$ to $0=\exp_y^{-1}(y)$ in time t ([7], Chap. IX, proof of Theorem 12 C(i)).

Hence we have:

$$P_x(\zeta>t)=P_{x_0}(\zeta_0>t). \tag{3.8}$$

Where $\zeta_0=\zeta_0(x_0, t)$ is now the first exit time from the Euclidean ball $D=D(0, \pi)$ of the n-dimensional Euclidean Brownian bridge from x_0 to 0 in time t.

Thus by (3.7), we have:

$$P_t^{S^3}(x, y)=(2\pi t)^{-\frac{3}{2}}\frac{r}{\sin r}\,e^{-\frac{r^2}{2t}}\,e^{\frac{t}{2}}\,P_{x_0}(\zeta_0>t). \tag{3.9}$$

By (2.6) and (3.6), we have for $M=R^3$, $U=D$,

$$P_t^D(x_0, 0)=(2\pi t)^{-\frac{3}{2}}\,e^{-\frac{r^2}{2t}}\,P_{x_0}(\zeta_0>t). \tag{3.10}$$

Now, consider the eigenvalue problem in D:

$$\Delta\phi+\lambda\phi=0 \tag{3.11}$$

$$\phi\,|\,\partial D\equiv 0.$$

Then by ([5]; Chap. V, § 8),

$$P_t^D(x_0, 0)=\sum_{n=0}^{\infty}\sum_{p=1}^{\infty} e^{-\lambda_{n,p}t}\,\phi_{n,p}(x_0)\,\phi_{n,p}(0) \tag{3.12}$$

where

1. $\phi_{n,p}(x)=Y_n(\theta, \varphi)\,S_n(r\sqrt{\lambda_{n,p}})$ is the eigenfunction corresponding to the eigenvalue $\lambda_{n,p}$ and $x\to(\theta, \varphi, r)$ is the change from Cartesian to spherical coordinates.
2. $S_n(r\sqrt{\lambda})=\dfrac{J_{n+\frac{1}{2}}}{r}$

(which is regular at $r=0$), J_n being the Bessel function of order n and the eigenvalues $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,p}, \ldots$ are solutions of

$$J_{n+\frac{1}{2}}(\sqrt{\lambda})=0.$$

By (3.9), (3.10) and (3.12),

$$P_t^{S^3}(x, y) = \frac{r}{\sin r} e^{\frac{t}{2}} P_t^D(x_0, 0) \tag{3.13}$$

$$= \frac{r}{\sin r} e^{\frac{t}{2}} \sum_{n=0}^{\infty} \sum_{p=1} e^{-\lambda_{n,p} t} \phi_{n,p}(x_0) \phi_{n,p}(0). \tag{3.14}$$

On the other hand, we have the eigenvalue problem in S^3:

$$\Delta \Phi + \mu \Phi = 0$$

and hence,

$$P_t^{S^3}(x, y) = \sum_{k=0}^{\infty} e^{-\mu_k} \Phi_k(x) \Phi_k(y) \tag{3.15}$$

where by ([3], Chap. III, Proposition C.I.1).

1. $\mu_k = k(k+2)$ is the eigenvalue corresponding to the eigenfunction Φ_k.
2. The eigenfunction Φ_k is a homogeneous polynomial of degree k harmonic on R^4.

Lastly we have a third formula recently proved by K.D. Elworthy in [8] by using (3.4) above and the method of images (as a special case of the formula for compact Lie groups):

$$P_t^{S^3}(x, y) = (2\pi t)^{-\frac{3}{2}} e^{\frac{t}{2}} \sum_{\gamma} \frac{l(\gamma)}{\sin(l(\gamma))} \exp\left\{-\frac{l(\gamma)^2}{2t}\right\} \tag{3.16}$$

where the sum is taken over all geodesics γ from x to y and $l(\gamma)$ is the length of γ. These lengths are given by:

$$\gamma_k = 2\pi k + r; \qquad k = 0, 1, 2, \ldots$$

$$\gamma_k = 2\pi k - r; \qquad k = 1, 2, 3, \ldots$$

Thus (3.16) becomes:

$$P_t^{S^3}(x, y) = (2\pi t)^{-\frac{3}{2}} e^{\frac{t}{2}} \left[\sum_{k=0}^{\infty} \frac{2\pi k + r}{\sin(2\pi k + r)} \exp\left\{-\frac{(2\pi k + r)^2}{2t}\right\} + \sum_{k=1}^{\infty} \frac{2\pi k - r}{\sin(2\pi k - r)} \exp\left\{-\frac{(2\pi k - r)^2}{2t}\right\} \right] \tag{3.17}$$

$$= (2\pi t)^{-\frac{3}{2}} e^{\frac{t}{2}} \frac{r}{\sin r} \exp\left\{-\frac{r^2}{2t}\right\} \cdot \left[1 - 2\sum_{k=1}^{\infty} \exp\left\{-\frac{2\pi^2 k^2}{t}\right\} \left(\frac{2\pi k}{r} \sinh\left(\frac{2\pi k r}{t}\right) - \cosh\left(\frac{2\pi k r}{t}\right)\right)\right]. \tag{3.18}$$

Consequently, by (3.14), (3.15) and (3.18), we have the identities:

(3.19)
$$e^{\frac{t}{2}} \frac{r}{\sin r} \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} e^{-\lambda_{n,p} t} \Phi_{n,p}(x_0)\, \Phi_{n,p}(0)$$
$$= \sum_{k=0}^{\infty} e^{-k(k+2)t} \Phi_k(x)\, \Phi_k(y)$$
$$= (2\pi t)^{-\frac{3}{2}} e^{\frac{t}{2}} \frac{r}{\sin r} \exp\left\{-\frac{r^2}{2t}\right\} \left[1 - 2 \sum_{k=1}^{\infty} \exp\left\{-\frac{2\pi^2 k^2}{t}\right\}\right.$$
$$\left.\cdot \left(\frac{2\pi k}{r} \sinh\left(\frac{2\pi k r}{t}\right) - \cosh\left(\frac{2\pi k r}{t}\right)\right)\right].$$

4. Some consequences

(i) A direct consequence of (3.7) and (3.18) is the formula:

(4.1)
$$P_x(\zeta > t) = 1 - 2 \sum_{k=1}^{\infty} \exp\left\{-\frac{2\pi^2 k^2}{t}\right\} \left(\frac{2\pi k}{r} \sinh\left(\frac{2\pi k r}{t}\right) - \cosh\left(\frac{2\pi k r}{t}\right)\right).$$

Let $(\beta_s^t)\, 0 \leqq s \leqq t \wedge \zeta_0$ be the n-dimensional Euclidean Brownian bridge from x_0 to 0 in time t. Since

$$P_{x_0}\left(\sup_{0 \leqq s \leqq t} |\beta_s^t| < \pi\right) = P_x(\zeta > t),$$

we conclude by (3.8) that $P_{x_0}\left(\sup_{0 \leqq s \leqq t} |\beta_s^t| < \pi\right)$ is equal to the R.H.S. of (4.1) above. Thus by computing on the standard 3-sphere $S^3\left(\frac{\varepsilon}{\pi}\right)$ instead of $S^3(1)$, we have:

(4.2)
$$P\left(\sup_{0 \leqq s \leqq t} |\beta_s^t| < \varepsilon\right) = 1 - 2 \sum_{k=1}^{\infty} \exp\left\{-\frac{2\varepsilon^2 k^2}{t}\right\}$$
$$\cdot \left(\frac{2\varepsilon k}{t} \sinh\left(\frac{2\varepsilon k r}{t}\right) - \cosh\left(\frac{2 k r}{t}\right)\right)$$

for $0 < r < \varepsilon$.

Let $t = 1$ and take limits as $r \downarrow 0$ in (4.2); then

(4.3)
$$P_0\left(\sup_{0 \leqq s \leqq 1} |\beta_s^1| < \varepsilon\right) = 1 - 2 \sum_{k=1}^{\infty} \exp\{-2\varepsilon^2 k^2\}(4\varepsilon^2 k^2 - 1).$$

Let $(w(t))\, 0 \leqq t \leqq \infty$ be the 1-dimensional Brownian motion on R. Then set:

$$\tau_1 = \sup\{t < 1 : w(t) = 0\}; \qquad \tau_2 = \inf\{t > 1 : w(t) = 0\}.$$

Define the process $(w_1(s))\, 0 \leqq s \leqq 1$ by

$$(4.4) \qquad w_1(s) = \frac{|w(\tau_2 s - \tau_1(1-s))|}{(\tau_2 - \tau_1)^{\frac{1}{2}}}.$$

Then the process $(w_1(s))\, 0 \leqq s \leqq 1$ is called the unsigned scaled Brownian excursion process.

By (1.1) of Theorem 1 in [13], we have:

$$(4.5) \qquad P_0\left(\sup_{0 \leqq s \leqq 1} w_1(s) < \varepsilon\right) = 1 - 2 \sum_{k=1}^{\infty} \exp\{-2\varepsilon^2 k^2\}(4\varepsilon^2 k^2 - 1)$$

where $(w_1(s))\, 0 \leqq s \leqq 1$ is the unsigned scaled Brownian excursion defined above. The above results in (4.3) and (4.5) confirm D. William's observation in [19] that $(w_1(s))\, 0 \leqq s \leqq 1$ has the same distribution as the 3-dimensional Bessel bridge $(|\beta_s^1|)\, 0 \leqq s \leqq 1$. Thus (4.2) is a generalisation of (1.1) of Theorem 1 in [13] which is given here in (4.5).

(ii) By (3.18), we have:

$$(4.6) \qquad P_t^{S^3}(y, y) = (2\pi t)^{-\frac{3}{2}} e^{\frac{t}{2}} \left[1 - 2 \sum_{k=1}^{\infty} \exp\left\{-\frac{2\pi^2 k^2}{t}\right\}(4\pi^2 k^2 - 1)\right].$$

Since $-2 \sum_{k=1}^{\infty} \exp\left\{-\frac{2\pi^2 k^2}{t}\right\}(4\pi^2 k^2 - 1) = o(t^n)$ for all $n \geqq 1$,

$$(4.7) \qquad (2\pi^{\frac{3}{2}} P_t^{S^3}(y, y) = e^{\frac{t}{2}} + o(t) \qquad \text{for all } n \geqq 1$$

$$(4.8) \qquad = 1 + \frac{1}{2} t + \frac{1}{2!}\left(\frac{1}{2}\right)^2 t^2 + \ldots + \frac{1}{n!}\left(\frac{1}{2}\right)^n t^n + o(t^n) \qquad \text{for all } n \geqq 1,$$

and so we obtain all the terms of the expansion to any order of H.P. McKean and I.M. Singer for S^3 given in [14].

4.1. Remark. We notice that the coefficient of t above in (3.8) is $\frac{1}{2} = \frac{S(y)}{12}$ where $S(y) = 6$ is the scalar curvature of S^3 as is well known.

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