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Correction to: SK_1 for Finite Group Rings: I

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Ulf Rehmann has pointed out an error in the proof of Lemma 9 in [2]. A certain field extension was assumed there to be Galois; and although this was the case in later applications, it did not follow from the hypotheses. In addition to correcting the proof, we take the opportunity (as a result of comments by various people) to fill in some of the details which were omitted in [2].

We use N and nr to denote the norm and reduced norm maps, respectively. A lemma is first needed.

Lemma. *Let $E \supseteq F$ be finite extensions of the p-adic rationals $\hat{Q}_p$, such that E/F is a cyclic Galois extension. Then $K_2(F)$ is generated by symbols of the form $\{x, N_{E/F}(y)\}$ for $x \in F^*$ and $y \in E^*$.*

Proof. Let $n = [E:F]$; then ([1], p. 140, Theorem 2):

$$F^*/N_{E/F}(E^*) \cong \mathrm{Gal}(E/F) \cong \mathbb{Z}/n.$$

We must show that the surjection (induced by the symbol map)

$$F^*/N(E^*) \otimes F^*/N(E^*) \cong \mathbb{Z}/n \twoheadrightarrow K_2(F)/\{F^*, N(E^*)\}$$

is trivial. If for some $a \in F^*$, a and $-a$ both generate $F^*/N(E^*)$, we are done since $\{a, -a\} = 1$.

The only case this does not occur is when $-1 \notin N(E^*)$ and $n \equiv 2 \pmod 4$. In this case let $\bar{E} \subseteq E$ be the degree two extension of F. Since for any $a \in F^*$ generating $F^*/N(E^*)$, $-a$ is the square of a generator, $-a$ generates the subgroup

$$N(\bar{E}^*)/N(E^*) \subseteq F^*/N(E^*)$$

of index two. Hence it remains only to find $a, b \in F^* - N(\bar{E}^*)$ such that $\{a, b\} = 1$.

So assume that $\{a, b\} \neq 1$ for any $a, b \notin N(\bar{E}^*)$. Since $-1 \notin N(\bar{E}^*)$,

$$2 \in N(\bar{E}^*) \quad (\text{since } \{-1, 2\} = 1), \quad \text{so} \quad -2 \notin N(\bar{E}^*);$$

$$3 \in N(\bar{E}^*) \quad (\text{since } \{-2, 3\} = 1), \quad \text{so} \quad -3 \notin N(\bar{E}^*);$$

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etc. But this contradicts the fact that $N_{E/F}(\bar{E}^*)$ is closed ([1], p. 143, Theorem 3). □

Proposition (Lemma 9 in [2]). *Assume A is a finite dimensional simple $\hat{Q}_p$-algebra with center F, and let $E \subseteq A$ be any maximal subfield containing F such that $[E\colon F] = [A\colon F]^{\frac{1}{2}}$. Then the image of the induction map*

$$i_*\colon K_2(E) \to K_2(A)$$

contains the image of the symbol map

$$c^A\colon F^* \otimes A^* \to K_2(A).$$

Proof. It follows from ([1], p. 143, Proposition 4) that either $N_{E/F}$ is onto, or there is a cyclic extension $F_1 \supsetneqq F$ in E. Repeating this gives a sequence of fields

$$F = F_0 \subseteq F_1 \subseteq \ldots \subseteq F_k = E;$$

such that F_i/F_{i-1} is a cyclic extension for all $1 \leqq i \leqq k-1$, and such that $N_{F_k/F_{k-1}}$ is onto. Let A_i be the centralizer in A of F_i. Then by Theorems 7.11 and 7.13 in [4], F_i is the centralizer (in particular, the center) of A_i for all i; and $[F_i\colon F] = [A\colon A_i]$. Note in particular that $A_k = F_k = E$.

For $1 \leqq i \leqq k$, consider the following diagram:

$$\begin{array}{ccccccc}
F_{i-1}^* \otimes A_{i-1}^* & \xrightarrow[\cong]{1 \otimes nr} & F_{i-1}^* \otimes F_{i-1}^* & \xrightarrow{c^{F_{i-1}}} & K_2(F_{i-1}) & \xrightarrow{\psi} & K_2(A_{i-1}). \\
\uparrow \scriptstyle{\text{incl}} & & \uparrow \scriptstyle{1 \otimes N_{F_i/F_{i-1}}} & & & & \\
F_{i-1}^* \otimes A_i^* & \xrightarrow[\cong]{1 \otimes nr} & F_{i-1}^* \otimes F_i^* & & & &
\end{array}$$

Here, ψ is the map defined in [3] so that the composite of the top row is the symbol map $c^{A_{i-1}}$. Although ψ was defined in [3] only for division algebras, the construction clearly extends to matrix algebras.

The square commutes by ([5], p. 28). If $i < k$, then the composite

$$c^{F_{i-1}} \circ (1 \otimes N_{F_i/F_{i-1}})$$

is surjective by the lemma; and if $i = k$ it is surjective by the assumption on $N_{F_i/F_{i-1}}$. It follows that

$$\{F_{i-1}^*, A_{i-1}^*\} = \{F_{i-1}^*, A_i^*\} \subseteq \{F_i^*, A_i^*\} \quad \text{in } K_2(A)$$

for all $1 \leqq i \leqq k$; and hence

$$\operatorname{Im}(c^A) = \{A_0^*, F_0^*\} \subseteq \{A_k^*, F_k^*\} = i_*(K_2(E)). \quad □$$

References

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