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Inventiones mathematicae

The fundamental group of the complement of a union of complex hyperplanes: correction

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In this note we correct the main result (Theorem 1) of [1]. The error involves the local description in the neighborhood of a singularity. In order to correct this we must assume that the arrangement is real and specify an ordering of its hyperplanes.

Thus we consider a collection of linear forms $\phi_1, \ldots, \phi_n, \phi_{n+1}$ in the complex variables (z_1, z_2, z_3), with zero loci $V_1, \ldots, V_n, V_{n+1}$ respectively. Without loss of generality we may assume that $\phi_{n+1}(z_1, z_2, z_3) = z_3$, so that if we identify $\mathbb{C}^2$ (with coordinates $Z_1 = z_1/z_3$, $Z_2 = z_2/z_3$) with $\mathbb{CP}^2 - V_{n+1}$, we have $\mathbb{CP}^2 - (V_1 \cup \ldots \cup V_n \cup V_{n+1}) \cong \mathbb{C}^2 - (V_1 \cup \ldots \cup V_n)$. We henceforth work with the latter space, which we call N. Notice that if $\phi_1, \ldots, \phi_n, \phi_{n+1}$ are real forms (their coefficients are real), we may make the appropriate change of coordinates as above so that $V_1, \ldots, V_n$ are defined by real forms.

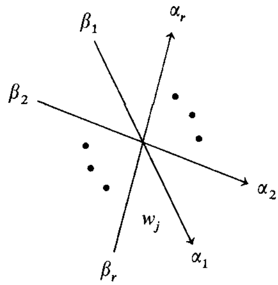
We now specialize to the real case. First write $Z_1 = u_1 + iv_1$, $Z_2 = u_2 + iv_2$. Then we have the canonical $\mathbb{R}^2 \subset \mathbb{C}^2$ given by $v_1 = v_2 = 0$, and letting $L_i = V_i \cap \mathbb{R}^2$, $i = 1, \ldots, n$ we have the representation of the arrangement by n lines in $\mathbb{R}^2$.

To specify the algorithm, we first order and orient these lines. Then we specify generators and relations for $\pi_1(N)$, based upon this choice of order and orientation:

By a change of coordinates we may assume without loss of generality that no line L_i is vertical or horizontal in $\mathbb{R}^2$. Thus each line L_i is defined in $\mathbb{R}^2$ by a linear equation $u_2 = m_i u_1 + d_i$, where $m_i \in \mathbb{R} - \{0\}$, $d_i \in \mathbb{R}$. We order the lines by the lexicographical order for (m_i, d_i). Thus $L_i < L_j$ if and only if $i < j$ if and only if $m_i < m_j$ or $m_i = m_j$ and $d_i < d_j$. We orient the line L_i by taking the positive direction to be that of increasing u_1.

Next we specify generators for $\pi_1(N)$. Note that $\mathbb{R}^2 \cap (V_1 \cup \ldots \cup V_n) = \Gamma \subset \mathbb{R}^2$ is a planar "graph" (allowing rays). Let $W = \{w_1, \ldots, w_k\}$ denote the set of vertices of Γ. Then $\Gamma - W$ has several components. For each component we introduce a generator of $\pi_1(N)$ as follows. For the components of L_i we will have generators $a_i, b_i, c_i, \ldots$ where a_i corresponds to the component of $L_i - W$ which is farthest to the right along L_i, b_i corresponds to the next farthest to the right, etc. Let G denote the set of such generators.

Finally, we specify relations for $\pi_1(N)$. They are of three types, all of which arise from vertices of Γ. Suppose we consider a vertex w_j of Γ:



Then we have relations:

$$
\begin{aligned}
R_{1j}&:\ \alpha_1^{-1}\alpha_2^{-1}\dots\alpha_r^{-1}\beta_1\beta_2\dots\beta_r=1,\\
R_{2j}&:\ \beta_1=\alpha_1\\
&\quad\ \beta_2=\alpha_1^{-1}\alpha_2\alpha_1\\
&\quad\ \vdots\\
&\quad\ \beta_r=\alpha_1^{-1}\alpha_2^{-1}\dots\alpha_{r-1}^{-1}\alpha_r\alpha_{r-1}\dots\alpha_2\alpha_1,\\
R_{3j}&:\ \alpha_1\alpha_2\dots\alpha_r=\alpha_2\alpha_3\dots\alpha_r\alpha_1=\alpha_3\dots\alpha_r\alpha_1\alpha_2=\dots=\alpha_r\alpha_1\dots\alpha_{r-1}.
\end{aligned}
$$

Relations R_{1j} arise from a Wirtinger-type presentation of $\pi_1(\mathbb{R}^3-\Gamma)$ where $\mathbb{R}^3=\{v_2=0\}\subset\mathbb{C}^2$. Since β_1 is a conjugate of α_1, relations R_{2j} specify precisely how this occurs. And the relations R_{3j} are the relations for the group of the Hopf link as in [1].

The correct statement is then:

Theorem 1. *Suppose N is the complement of a real arrangement of lines in $\mathbb{CP}^2$ as above. Then*

$$\pi_1(N)\cong\langle G\,|\,R_{1j},R_{2j},R_{3j},j=1,\dots,k\rangle.$$

Comment. The relation R_{1j} is a consequence of the set of R_{2j}, and R_{2j}, $j=1,2,\dots,k$ may be used to obtain a presentation of $\pi_1(N)$ with generators $a_1,\dots,a_n$ and relations R'_{3j}, where R'_{3j} is simply R_{3j} written in terms of $a_1,\dots,a_n$.

In [1] the relations R_{2j} were incorrectly given (in other notation) as $\beta_1=\alpha_1$, $\beta_2=\alpha_2,\dots,\beta_r=\alpha_r$.

Reference

1. Randell, R.: The fundamental group of the complement of a union of complex hyperplanes. Inventiones math. **69**, 103–108 (1982)

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