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On the existence of positive solutions for semilinear elliptic equations with Neumann boundary conditions*

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Summary. We give sufficient conditions for the existence of positive solutions to some semilinear elliptic equations in unbounded Lipschitz domains $D \subset \mathbb{R}^d$ $(d \geq 3)$, having compact boundary, with nonlinear Neumann boundary conditions on the boundary of D. For this we use an implicit probabilistic representation, Schauder's fixed point theorem, and a recently proved Sobolev inequality for $W^{1,2}(D)$. Special cases include equations arising from the study of pattern formation in various models in mathematical biology and from problems in geometry concerning the conformal deformation of metrics.

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1 Introduction

Let D be an unbounded Lipschitz domain in $\mathbb{R}^d$ $(d \geq 3)$ with compact boundary ∂D, σ be the surface measure on ∂D, and n be the unit inward normal vector field defined σ-a.e. on ∂D. We shall let $\overline{D}$ denote the Euclidean closure of D, $C(\overline{D})$ the space of real-valued continuous functions defined on $\overline{D}$, $C_b(\overline{D})$ the space of bounded functions in $C(\overline{D})$, and $C_c^2(\mathbb{R}^d)$ the space of twice continuously differentiable real-valued functions defined on $\mathbb{R}^d$ which have compact support. The Laplacian in $\mathbb{R}^d$ will be denoted by Δ. In the following a positive solution on $\overline{D}$ means a solution that is strictly positive on $\overline{D}$.

In this paper, we study the existence of positive continuous solutions on $\overline{D}$ to the following semilinear elliptic equation with nonlinear Neumann boundary condition:

$$(1.1) \qquad \frac{1}{2}\Delta u + F_1(\cdot\,, u) + g \;=\; 0 \qquad \text{in } D,$$

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$$(1.2)\qquad \frac{\partial u}{\partial n} + F_2(\cdot, u) + \phi = 0 \qquad \text{on } \partial D,$$

$$(1.3)\qquad \lim_{\substack{|x|\to\infty \\ x\in D}} u(x) = \alpha,$$

where $\frac{\partial u}{\partial n}$ is the inward normal derivative of u, F_1 is a real-valued Borel measurable function defined on $D \times (0, \beta)$ for some constant $\beta \in (0, \infty]$ such that $F_1(x, \cdot)$ is continuous on $(0, \beta)$ for each $x \in D$ and

$$(1.4)\qquad -U_1(x)u \le F_1(x, u) \le V_1(x)f_1(u) \quad \text{for all } (x, u) \in D \times (0, \beta),$$

where U_1 and V_1 are non-negative Green-tight functions on D (see Definition 1.1 below), and f_1 is a non-negative Borel measurable function defined on $(0, \beta)$. The function F_2 is a real-valued Borel measurable function defined on $\partial D \times (0, \beta)$ such that $F_2(x, \cdot)$ is continuous on $(0, \beta)$ for each $x \in \partial D$ and

$$(1.5)\qquad -U_2(x)u \le F_2(x, u) \le V_2(x)f_2(u) \quad \text{for all } (x, u) \in \partial D \times (0, \beta),$$

where U_2 and V_2 are non-negative functions in the class Γ to be specified below in Definition 1.2 and f_2 is a non-negative Borel measurable function defined on $(0, \beta)$. The function g is a non-negative Green-tight function on D, ϕ is a non-negative function in Γ and $\alpha \ge 0$. The precise definitions of Green-tightness on D and of the class Γ are made below. However, to give the reader some concrete conditions to keep in mind, we note that if U_1, V_1, g are in $L^p(D)$ with $p > d/2$ and have bounded support then they are Green-tight on D, and if U_2, V_2, ψ are in $L^p(\partial D, \sigma)$ with $p > d - 1$ then they are in Γ. Here $L^p(D)$ (respectively, $L^p(\partial D, \sigma)$) is the space of real-valued Borel measurable functions defined on D (respectively, ∂D) whose absolute pth power is integrable with respect to Lebesgue measure on D (respectively, surface measure σ on ∂D).

Solutions of (1.1)–(1.2) are to be interpreted in the weak sense (i.e., in the sense of distributions). The factor of $\frac{1}{2}$ appears in Eq. (1.1) only for the technical reason that our method of proof uses (reflecting) Brownian motion which has $\frac{1}{2}\Delta$ as its infinitesimal generator. In order for our theorem on the existence of positive solutions for (1.1)–(1.3) to apply, we shall need further restrictions on f_1, f_2, g, ϕ and α, which are specified precisely in Theorem 1.2 below. But, for example, if $f_i(u) = u^{\gamma_i}$ with $\gamma_i > 1$ for $i = 1, 2$, and g, ϕ, α are sufficiently small (in a certain potential norm) and at least one of them is positive, then these conditions are satisfied.

Definition 1.1. *A function w is Green-tight on D if and only if w is a real-valued Borel measurable function defined on D such that the family of functions $\{w(\cdot)/|x - \cdot|^{d-2}, x \in D\}$ defined on D is uniformly integrable in the sense that w satisfies*

$$(1.6)\qquad \lim_{\substack{m(A)\to 0 \\ A\subset D}} \left\{ \sup_{x\in D} \int_A \frac{|w(y)|}{|x-y|^{d-2}}\, dy \right\} = 0,$$

and

$$\lim_{M\to\infty}\left\{\sup_{x\in D}\int_{\substack{|y|>M\\ y\in D}}\frac{|w(y)|}{|x-y|^{d-2}}\,dy\right\}=0, \tag{1.7}$$

where m denotes Lebesgue measure on $\mathbb{R}^d$. Note that the limit in (1.6) is uniform in sets $A\subset D$ such that $m(A)\to 0$.

It follows easily from (1.6)–(1.7) that if w is Green-tight on D then

$$\|w\|_D\equiv\sup_{x\in D}\int_D\frac{|w(y)|}{|x-y|^{d-2}}\,dy<\infty. \tag{1.8}$$

It is known [32] (see also [4]) that a Borel measurable function w is Green-tight on D if and only if $1_D w\in K_d^\infty$, where

$$K_d^\infty=\left\{v\in K_d:\lim_{M\to\infty}\left[\sup_{x\in\mathbb{R}^d}\int_{|y|>M}\frac{|v(y)|}{|x-y|^{d-2}}\,dy\right]=0\right\}. \tag{1.9}$$

Here K_d denotes the Kato class for $\mathbb{R}^d$ which consists of all those real-valued Borel measurable functions v defined on $\mathbb{R}^d$ such that

$$\lim_{r\downarrow 0}\left[\sup_{x\in\mathbb{R}^d}\int_{|x-y|\le r}\frac{|v(y)|}{|x-y|^{d-2}}\,dy\right]=0. \tag{1.10}$$

A real-valued Borel measurable function v defined on $\mathbb{R}^d$ is in K_d^{loc} if and only if $1_B v\in K_d$ for each bounded ball B in $\mathbb{R}^d$.

A sufficient condition (see [32]) for a real-valued Borel measurable function w to be Green-tight on D is that $w\in L^p(D)$ with $p>d/2$ and that there is $M>0$ such that

$$|(1_D w)(x)|\le\frac{\psi(|x|)}{|x|^2}\quad\text{for all } |x|\ge M, \tag{1.11}$$

where ψ is a positive function defined on the interval $[M,\infty)$ with $\int_M^\infty s^{-1}\psi(s)\,ds<\infty$.

To study the Neumann boundary value problem (1.1)–(1.3), we need to introduce the class Γ which is an analogue of the Green-tight class but for the boundary ∂D in place of D.

Definition 1.2. *A function w is in the class $\Gamma=\Gamma(\partial D)$ if and only if w is a real-valued Borel measurable function defined on ∂D such that the family of functions $\{w(\cdot)/|x-\cdot|^{d-2},\ x\in\partial D\}$ is uniformly integrable with respect to the surface measure σ on ∂D, i.e.,*

$$\lim_{\substack{\sigma(A)\to 0\\ A\subset\partial D}}\left\{\sup_{x\in\partial D}\int_A\frac{|w(y)|}{|x-y|^{d-2}}\,\sigma(dy)\right\}=0. \tag{1.12}$$

In (1.12) the limit is uniform in sets $A\subset\partial D$ such that $\sigma(A)\to 0$. It is not difficult to show (see Proposition 2.1 below) that "$x\in\partial D$" in (1.12) can be replaced by "$x\in\overline{D}$" and therefore if $w\in\Gamma$,

$$\|w\|_{\partial D} \equiv \sup_{x\in\overline{D}} \int_{\partial D} \frac{|w(y)|}{|x-y|^{d-2}}\,\sigma(dy) < \infty. \tag{1.13}$$

Since we assume ∂D is compact, $\Gamma \subset L^1(\partial D, \sigma)$. It is shown in Proposition 2.2 that Γ contains all functions in $L^p(\partial D, \sigma)$ for $p > d-1$.

The notion of a weak solution of (1.1)–(1.3) can be made precise as follows. Let

$$W^{1,2}(D) = \left\{ f \in L^2(D) : \ \frac{\partial f}{\partial x_i} \in L^2(D) \ \text{ for } i = 1, 2, \ldots, d \right\},$$

where $\frac{\partial f}{\partial x_i}$ denotes the distributional derivative of f with respect to x_i. In this paper, a continuous function u on $\overline{D}$ is said to be a positive solution of (1.1)–(1.2) if $u > 0$ on $\overline{D}$, $1_B\, u \in W^{1,2}(D \cap B)$ for any bounded ball B in $\mathbb{R}^d$, and for any $\psi \in C_c^2(\mathbb{R}^d)$ we have

$$\begin{aligned} &\frac{1}{2}\int_D \nabla u(x)\cdot\nabla\psi(x)\,dx \\ &= \int_D \Big(F_1(x,u) + g(x)\Big)\psi(x)\,dx + \frac{1}{2}\int_{\partial D}\Big(F_2(x,u) + \phi(x)\Big)\psi(x)\,\sigma(dx). \end{aligned} \tag{1.14}$$

It is known [18] that there exists a unique symmetric strong Markov process $(X, \{P_x, x \in \overline{D}\})$ with continuous paths in $\overline{D}$ and associated Dirichlet form $(W^{1,2}(D), \mathscr{E})$ defined by

$$\mathscr{E}(f,g) = \frac{1}{2}\int_D \nabla f\cdot\nabla g\,dx \qquad \text{for } f, g \in W^{1,2}(D). \tag{1.15}$$

This process is called normally reflecting Brownian motion on $\overline{D}$. To state our main theorem, we need the following result which is proved in [5].

Proposition 1.1. *The transition density function $(t,x,y) \to p(t,x,y)$ of X exists as a continuous function on $(0,\infty)\times\overline{D}\times\overline{D}$. Furthermore, there exist constants $c_1 = c_1(D) > 0$ and $c_2 = c_2(D) > 0$ such that*

$$p(t,x,y) \le \frac{c_1}{t^{d/2}} \exp\left(-\frac{|x-y|^2}{c_2 t}\right) \quad \textit{for all } t > 0,\ x, y \in \overline{D}. \tag{1.16}$$

Let $G(x,y) = \int_0^\infty p(t,x,y)\,dt$, the Green function for X. Then $G(x,y)$ is finite and continuous on $\overline{D}\times\overline{D}$, except on the diagonal. Furthermore, there exists a constant $c = c(D) > 0$ such that

$$G(x,y) \le \frac{c}{|x-y|^{d-2}}, \quad \textit{for all } x, y \in \overline{D}. \tag{1.17}$$

For a non-negative Borel measurable function w defined on D and a non-negative Borel measurable function ψ defined on ∂D, write

$$(Gw)(x) = \int_D G(x,y)w(y)dy \quad \text{for all } x \in \overline{D},$$

and

$$(\tilde{G}\psi)(x) = \int_{\partial D} G(x,y)\psi(y)\sigma(dy) \quad \text{for all } x \in \overline{D}.$$

The notation $\|\cdot\|_\infty$ will denote the supremum norm of a real-valued function over its domain of definition. Note that by Proposition 1.1, for a Green-tight function w on D and a function ψ in Γ, $\|Gw\|_\infty \leq c\,\|w\|_D < \infty$ and $\|\tilde{G}\psi\|_\infty \leq c\,\|\psi\|_{\partial D} < \infty$.

We now formulate our existence theorem. For $\lambda \in (0, 1)$, let

$$(1.18) \qquad C_\lambda = \sup\left\{\varepsilon \in (0,\beta) : \left(\sup_{0<y\leq\varepsilon} \frac{f_1(y)}{y}\right)\|GV_1\|_\infty + \frac{1}{2}\left(\sup_{0<y\leq\varepsilon} \frac{f_2(y)}{y}\right)\|\tilde{G}V_2\|_\infty \leq \lambda\right\}.$$

Here $\sup\emptyset \equiv 0$ and $1/0 \equiv +\infty$. The factor $1/2$ appears in front of $\|\tilde{G}V_2\|_\infty$ in (1.18) only for a technical reason which becomes clear in (4.9) below. We shall only consider situations in which $C_\lambda > 0$ for some $\lambda \in (0, 1)$. Note that in the special case where $f_i(u) = u^{\gamma_i}$ with $\gamma_i > 1$ for $i = 1, 2$, or $f_i(u) = u$ and $\|GV_1\|_\infty + \frac{1}{2}\|\tilde{G}V_2\|_\infty < 1$, then $C_\lambda > 0$ for some $\lambda \in (0, 1)$.

Theorem 1.2. *Suppose that either $g > 0$ on a subset of D of positive Lebesgue measure or $\phi > 0$ on a subset of ∂D of positive σ-measure or $\alpha > 0$, and $\|Gg\|_\infty + \frac{1}{2}\|\tilde{G}\phi\|_\infty + \alpha < (1-\lambda)C_\lambda$ for some $\lambda \in (0, 1)$ such that $C_\lambda > 0$. Then the boundary value problem (1.1)–(1.3) has a positive continuous weak solution.*

In general, uniqueness may not hold for the solutions found in Theorem 1.2 (see the remark following Corollary 1.3 below). However, if $F_1(x,u)$ and $F_2(x,u)$ are monotone decreasing as functions of $u \in (0,\beta)$ for each $x \in \overline{D}$ and $x \in \partial D$, respectively, we show in Sect. 4 that there is uniqueness of continuous bounded solutions to (1.1)–(1.3). (It is implicit here that a solution u takes values in $(0,\beta)$, so that $F_1(\cdot,u)$, $F_2(\cdot,u)$ are well defined.)

In order to keep the exposition as concrete and transparent as possible, we have not attempted to optimize the bound on $\|Gg\|_\infty + \frac{1}{2}\|\tilde{G}\phi\|_\infty + \alpha$, the space of Green-tight functions on D, or the class Γ. In fact, in the Definition 1.1 for Green-tight functions and in the Definition 1.2 for the class Γ, one may replace the kernel $|x-y|^{2-d}$ by $G(x,y)$, the Green function for the reflecting Brownian motion on $\overline{D}$. Theorem 1.2 remains valid with these modifications.

Special cases of Eqs. (1.1)–(1.2) arise in pattern formation in various models in mathematical biology (see the introduction to [1] and the references therein) and in Riemannian geometry in connection with the problem of conformal deformation of metrics for manifolds with boundary (cf. [12], [13]). More precisely, for the latter, let M be a Riemannian manifold with smooth boundary, having metric $\mathscr{G}_0$ and dimension $d \geq 3$. Let k denote the scalar curvature on M and h the mean curvature on the boundary ∂M. (If $(M, \mathscr{G}_0)$ is a Euclidean domain, then $k \equiv 0$.) The Yamabe problem in Riemannian geometry asks whether, given smooth functions K and H defined on M and ∂M, respectively, one can find a Riemannian metric $\mathcal{G}$ on M which is pointwise conformal to $\mathcal{G}_0$ such that $(M, \mathcal{G})$ has K as its scalar curvature and H as its mean curvature on the boundary. If one writes $\mathcal{G} = u^{4/(d-2)}\mathcal{G}_0$, then (cf. [13]) the above deformation problem is equivalent to the existence of a positive solution for the Eqs. (1.1)–(1.2) with

$$
\begin{aligned}
(1.19) \qquad F_1(x,u) &= -\frac{d-2}{4(d-1)}k(x)\,u + \frac{d-2}{4(d-1)}K(x)\,u^{(d+2)/(d-2)}, \\
(1.20) \qquad F_2(x,u) &= -\frac{d-2}{2}h(x)\,u + \frac{d-2}{2}H(x)\,u^{d/(d-2)},
\end{aligned}
$$

$\phi = 0$ and $\Delta = \Delta_{\mathcal{G}_0}$, the Laplace-Beltrami operator on $(M, \mathcal{G}_0)$. Escobar [12], [13] has studied the problem of conformal deformation of metrics for *compact* Riemannian manifolds with boundary. In contrast, our Theorem 1.2 is for an *unbounded* Lipschitz domain D with compact boundary. Since $g \equiv 0$ and $\phi \equiv 0$, for the conformal deformation problem, we need to specify $\alpha > 0$ in order for our Theorem 1.2 to give sufficient conditions for conformal deformability of the Euclidean metric on $\overline{D}$. As a simple example of applying Theorem 1.2 in this case, we have the following corollary for deforming the mean curvature on the boundary of the exterior of a ball of radius $a > 0$. In this case, $h \equiv -1/a$, $k \equiv 0$ and for simplicity we take $K \equiv 0$.

Corollary 1.3. *Let $D = \{x \in \mathbb{R}^d : |x| > a\}$ with $a > 0$ and $d \geq 3$. For any $H \in L^p(\partial D, \sigma)$ with $p > d-1$, there exists a metric $\mathcal{G}$ on $\overline{D}$ which is pointwise conformal to the Euclidean metric and such that it has zero scalar curvature in D and mean curvature H on ∂D.*

Remark. The metric $\mathcal{G}$ found in the proof of Corollary 1.3 has a limiting value at infinity of α times the Euclidean metric, where $\alpha > 0$. Even if one fixes the value of α, there may be more than one metric $\mathcal{G}$ with this limiting value and the other properties described in Corollary 1.3. Indeed, for $a = 1$, $d = 3$ and $H = 1$, it is shown in Sect. 4 that for α sufficiently small there are two different rotationally invariant metrics $\mathcal{G}$ satisfying the aforementioned conditions.

There is a wealth of literature on solutions of semilinear elliptic equations. However, we have not been able to find results that entirely subsume ours. Most of the existing literature employs analytic methods for solving semilinear equations, such as variational methods or methods of sub- and super- solutions (see e.g., [1], [2], [23], [24] [26], [30], and the references therein). On the other hand, it is well known that one can solve certain linear elliptic equations with boundary conditions by running suitable diffusion processes (see e.g., [8], [10], [21], [25] and [27]) and there are a few works on the use of probabilistic methods for solving semilinear elliptic equations (see e.g., [4], [11], [16], [17], [20], [25] and [32]). The methods used in these works basically fall into the following four categories.

1. Measure-valued branching processes have been used to solve equations of the form (1.1) with Dirichlet boundary conditions and $F_1(x,u) = -w(x)u^p$, where w is non-negative and bounded, and $p \in (1,2]$ (see e.g., Dynkin [11]).
2. A probabilistic potential theoretic refinement of the analytic method of sub- and super-solutions has been used where $F_i(x,u) = F_i(u)$ and $u\,F_i(u) \leq 0$ for $i = 1,2$ (see Glover and McKenna [20] for the case $D = \mathbb{R}^d$, and Ma and Song [25] for bounded domains with boundary conditions).
3. An implicit probabilistic representation coupled with Picard iteration or a contraction mapping has been used to solve parabolic and (degenerate) elliptic semilinear equations where the coefficients are assumed to be at least Lipschitz continuous (see e.g., Freidlin [16], [17]).
4. An implicit probabilistic representation together with Schauder's fixed point theorem has been used to find positive solutions to equations of the form (1.1) with Dirichlet boundary conditions (see Zhao [32], Chen, Williams and Zhao [4]).

Both methods 3 and 4 use an implicit probabilistic representation, but method 4 uses the potentially broader mechanism of Schauder's fixed point theorem rather than a contraction mapping argument. In this paper, we adapt method 4 to the case of Neumann boundary conditions. Our approach also heavily uses the theory of Dirichlet spaces. Of course, Schauder's fixed point theorem has been used before in solving partial differential equations. It is the combination of an implicit probabilistic representation and Schauder's fixed point theorem which is new and we find appealing. In particular, it allows us to deal with a Lipschitz boundary and to allow semilinear terms that may be locally unbounded in x.

Before proving Theorem 1.2, we develop some preliminaries concerning the class Γ and probabilistic representations of solutions of *linear* Schrödinger equations with possibly singular potential terms and Neumann boundary conditions in *unbounded* Lipschitz domains. This we do in Sects. 2–3 below. The idea of our method of proof of Theorem 1.2 is as follows. Suppose that u is a positive continuous solution of (1.1)–(1.3). Then u solves the linear Schrödinger boundary value problem

$$\frac{1}{2}\Delta u + q\,u + g = 0 \quad \text{in } D, \tag{1.21}$$

$$\frac{\partial u}{\partial n} + \kappa\,u + \phi = 0 \quad \text{on } \partial D, \tag{1.22}$$

$$\lim_{\substack{|x|\to\infty \\ x\in D}} u(x) = \alpha, \tag{1.23}$$

where

$$q = \frac{F_1(\,\cdot\,,u)}{u}, \qquad \kappa = \frac{F_2(\,\cdot\,,u)}{u}. \tag{1.24}$$

Let q^+ and κ^+ denote the positive parts of q and κ respectively. Then, provided that $\|Gq^+\|_\infty + \frac{1}{2}\|\tilde{G}\kappa^+\|_\infty < 1$, the solution of (1.21)–(1.23) has an (implicit) probabilistic representation in terms of reflecting Brownian motion on $\overline{D}$ (see Lemma 3.3). If we denote this representation by Tu, the idea of our method is to define a suitable space Λ of positive bounded continuous functions on $\overline{D}$ such that T maps Λ into Λ and has a fixed point there. It then follows that such a fixed point solves the original Eqs. (1.1)–(1.3). The full proofs of Theorem 1.2 and Corollary 1.3 are given in Sect. 4, as well as a uniqueness theorem for solutions to (1.1)–(1.3) under the condition that $F_1(x,u)$, $F_2(x,u)$ are monotone decreasing in u. Some extensions of the results in this paper are listed in Sect. 5.

For convenience, throughout this paper, all functions are extended to be zero off their domains of definition and are still denoted by the same symbols, unless otherwise specified. In the sequel, we use v^+ and v^- to denote the positive and negative part of a real-valued Borel measurable function v, respectively; that is, $v^+ = \max\{v, 0\}$ and $v^- = \max\{-v, 0\}$.

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2 The class Γ

In this section, we present several equivalent definitions for the class Γ and then we give a sufficient condition for functions to be in this class. Let $B(x,r)$ denote the open ball in $\mathbb{R}^d$ centered at x with radius r. In part (c) of the following proposition, A is Borel measurable. We omit the proof of this proposition, since it is a straight forward exercise in real analysis.

Proposition 2.1. *Suppose w is a real-valued Borel measurable function defined on ∂D. Then the following four statements are equivalent:*
(a) w is in the class Γ;
(b) w satisfies:

$$\lim_{r\downarrow 0}\left\{\sup_{x\in\partial D}\int_{\partial D\cap B(x,r)}\frac{|w(y)|}{|x-y|^{d-2}}\,\sigma(dy)\right\}=0;$$

(c) w satisfies:

$$\lim_{\substack{\sigma(A)\to 0\\ A\subset\partial D}}\left\{\sup_{x\in\overline{D}}\int_{A}\frac{|w(y)|}{|x-y|^{d-2}}\,\sigma(dy)\right\}=0;$$

(d) w satisfies:

$$\lim_{r\downarrow 0}\left\{\sup_{x\in\overline{D}}\int_{\partial D\cap B(x,r)}\frac{|w(y)|}{|x-y|^{d-2}}\,\sigma(dy)\right\}=0.$$

Proposition 2.2. *$L^p(\partial D,\sigma)\subset\Gamma$ for $p>d-1$.*

Proof. Suppose $w\in L^p(\partial D,\sigma)$ with $p>d-1$. Since ∂D is compact and D is Lipschitz, there exists $r_0>0$ such that for each fixed $x\in\partial D$, there is a coordinate system $(y',t)\in\mathbb{R}^{d-1}\times\mathbb{R}$ under which x has coordinate $(0,0)$ and there is a Lipschitz continuous function $\psi:\mathbb{R}^{d-1}\times\mathbb{R}$ with Lipschitz constant M (independent of $x\in\partial D$) such that

$$D \cap B(x, r_0) = \{y = (y', t) : t > \psi(y')\} \cap B(x, r_0).$$

For $0 < r < r_0$, let $U_r = \{y' \in \mathbb{R}^{d-1} : (y', \psi(y')) \in \partial D \cap B(x, r)\}$. Clearly, $U_r \subset \{y' \in \mathbb{R}^{d-1} : |y'| < r\}$. Then for $q = p/(p-1)$,

$$\begin{aligned}
&\int_{\partial D \cap B(x,r)} \frac{|w(y)|}{|x-y|^{d-2}}\,\sigma(dy) \\
\leq\ & \left(\int_{\partial D \cap B(x,r)} |w(y)|^p\,\sigma(dy)\right)^{1/p} \left(\int_{\partial D \cap B(x,r)} \frac{1}{|x-y|^{(d-2)q}}\,\sigma(dy)\right)^{1/q} \\
\leq\ & \|w\|_{L^p(\partial D)} \left(\int_{U_r} \frac{1}{|(y', \psi(y'))|^{(d-2)q}}\,(1 + |\nabla\psi(y')|^2)^{1/2}\,dy'\right)^{1/q} \\
\leq\ & (1+M^2)^{1/2q}\,\|w\|_{L^p(\partial D)} \left(\int_{|y'|<r} \frac{1}{|y'|^{(d-2)q}}\,dy'\right)^{1/q} \\
\leq\ & \left(\frac{p-1}{p-d+1}\right)^{1/q} \omega_{d-2}^{1/q}\,(1+M^2)^{1/2q}\,\|w\|_{L^p(\partial D)}\, r^{(1-\frac{d-2}{p-1})/q},
\end{aligned}$$

where ω_{d-2} is the surface area of the unit ball in $\mathbb{R}^{d-1}$. Thus w is in Γ by Proposition 2.1(b) since $1 - \frac{d-2}{p-1} > 0$. $\square$

Let

$$\hat{C}(\overline{D}) = \left\{ u \in C_b(\overline{D}) : \lim_{\substack{|x|\to\infty \\ x\in\overline{D}}} u(x) = 0 \right\},$$

endowed with the topology of uniform convergence on $\overline{D}$. For any Green-tight function w on D and ρ in class Γ, let

$$\begin{aligned}
\text{(2.1)}\qquad \Lambda_w &= \{v : D \to \mathbb{R},\ v \text{ is Borel measurable} \\
&\qquad \text{and } |v(x)| \leq |w(x)| \text{ for all } x \in D\}, \\
\text{(2.2)}\qquad \tilde{\Lambda}_\rho &= \{\psi : \partial D \to \mathbb{R},\ \psi \text{ is Borel measurable} \\
&\qquad \text{and } |\psi(x)| \leq |\rho(x)| \text{ for all } x \in \partial D\}.
\end{aligned}$$

Proposition 2.3. *For any Green-tight function w on D and $\rho \in \Gamma$, the families of functions $G\Lambda_w \equiv \{Gv : v \in \Lambda_w\}$ and $\tilde{G}\tilde{\Lambda}_\rho \equiv \{\tilde{G}\psi : \psi \in \tilde{\Lambda}_\rho\}$ are relatively compact in $\hat{C}(\overline{D})$.*

Proof. It follows from (1.17) that for each $y \in \overline{D}$,

$$\text{(2.3)}\qquad \lim_{\substack{|x|\to\infty \\ x\in\overline{D}}} G(x, y) = 0.$$

For any $v \in \Lambda_w$,

$$\text{(2.4)}\qquad \lim_{\substack{|x|\to\infty \\ x\in\overline{D}}} |Gv(x)| \leq \lim_{\substack{|x|\to\infty \\ x\in\overline{D}}} \int_D G(x,y)|w(y)|dy = 0,$$

and so the limit in (2.4) is uniform for $v \in \Lambda_w$. It is clear that $G\Lambda_w$ is uniformly bounded by $c\|w\|_D$. For $x, z \in \overline{D}$, $v \in \Lambda_w$,

$$(2.5) \qquad |Gv(x) - Gv(z)| \le \int_D |G(x,y) - G(z,y)||w(y)|dz.$$

By the continuity of $G(\cdot,\cdot)$ off the diagonal of $\overline{D} \times \overline{D}$, (2.4) and the uniform integrability of $\{G(x,\cdot)w(\cdot),\, x \in \overline{D}\}$, the right member of (2.5) can be made as small as we like by choosing $|x-z|$ sufficiently small (but independent of x and z). Hence $G\Lambda_w$ is equicontinuous on $\overline{D}$. Thus $G\Lambda_w$ is a family of functions in $\hat{C}(\overline{D})$ that are uniformly bounded and equicontinuous on $\overline{D}$, and have a uniform limit of zero at infinity. It follows that $G\Lambda_w$ is relatively compact in $\hat{C}(\overline{D})$. Similarly, one can show that $\tilde{G}\tilde{\Lambda}_\rho$ is relatively compact in $\hat{C}(\overline{D})$. □

3 Reflecting Brownian motion and linear elliptic equations

Recall that the reflecting Brownian motion $(X, \{P_x, x \in \overline{D}\})$ on $\overline{D}$ is a symmetric continuous strong Markov process on $\overline{D}$ that is associated with the Dirichlet form $(W^{1,2}(D), \mathscr{E})$, (cf. (1.15)). This process X behaves like a free Brownian motion in the interior of D and is instantaneously reflected at the boundary of D in the inward normal direction n. Indeed, under each P_x, $x \in \overline{D}$, we have the following Skorokhod decomposition for X (cf. [3]):

$$(3.1) \qquad X_t = X_0 + W_t + \int_0^t n(X_s)dL_s \quad \text{for all} \quad t \ge 0,$$

where W is a d-dimensional Brownian motion starting from the origin and L is a continuous increasing additive functional of X which increases only when X is on ∂D. The process L is called the boundary local time for X and has Revuz measure $\frac{1}{2}\sigma$. For any non-negative Borel measurable function ψ defined on ∂D, it is known from Theorem 3.2.3 and Lemma 5.1.4 in [18] that

$$(3.2) \qquad E^x\left[\int_0^\infty \psi(X_s)dL_s\right] = \frac{1}{2}\int_{\partial D} G(x,y)\psi(y)\sigma(dy) = \frac{1}{2}\tilde{G}\psi(x),$$

where E^x denotes expectation under P_x. It is well known, especially for bounded domains, that one can solve certain linear elliptic equations with boundary conditions by running suitable diffusion processes (see e.g., [8], [10], [21], [25] and [27]). In connection with this, the following four lemmas may be known to experts. However, we could not find proofs for them in the literature and so for completeness we provide proofs here. Denote $B(0,r)$ by B_r and define $D_r = D \cap B_r$. Let $\tau_r = \inf\{t > 0 : X_t \notin B_r\}$, the first exit time of X from B_r.

Lemma 3.1. *Suppose $r > 0$ such that $\partial D \subset B_r$ and ψ is a bounded Borel measurable function defined on ∂B_r. Then $h_0(x) = E^x\left[\psi(X_{\tau_r})\right]$ is a continuous bounded function of $x \in \overline{D} \cap B_r$ which is harmonic in D_r, $h_0 \in W^{1,2}(D_a)$ for any $0 < a < r$ such that $\partial D \subset B_a$, and h_0 satisfies $\frac{\partial h_0}{\partial n} = 0$ on ∂D in the distributional sense. If ψ is continuous on ∂B_r, then h_0 is continuous on $\overline{D}_r$ with $h = \psi$ on ∂B_r.*

Proof. It is clear that h_0 is a bounded function on $\overline{D}_r$ with $\|h_0\|_\infty \leq \|\psi\|_\infty$. We extend h_0 to be zero off $\overline{D}_r$ and still denote it by h_0. Then it can be readly shown that $(P_t h_0)(x) = E^x\,[h_0(X_t)]$ converges uniformly for $x \in \overline{D}_a$ to $h_0(x)$ as $t \to 0$, for any $0 < a < r$ such that $B_a \supset \partial D$. By Proposition 1.1, $P_t h_0$ is continuous on $\overline{D}$ and hence h_0 is continuous on $\overline{D} \cap B_r$. When ψ is continuous on ∂B_r, that h_0 is continuous on $\overline{D}_r$ and equals ψ on ∂B_r follows from the same argument as that for the Brownian motion case since ∂B_r is regular (cf. [10], p. 247).

Since X behaves like a Brownian motion inside D, the strong Markov property of X reveals that h_0 is harmonic in D_r (see, e.g., [7], Chapter 4). In particular, h_0 is C^∞ in D_r. Let $0 < a < r$ be such that $\partial D \subset B_a$. To see that $h_0 \in W^{1,2}(D_a)$, we proceed as follows. Let $W_a^{1,2}(D_a)$ be the closure in $(W^{1,2}(D_a), \mathscr{E}_1^a)$ of the set of restrictions to D_a of all $C^\infty(\mathbb{R}^d)$ functions having compact support in B_a, where $\mathscr{E}_1^a(f,g) = \frac{1}{2}\int_{D_a} \nabla f \cdot \nabla g\,dx + \int_{D_a} fg\,dx$. By Lemma 5 of [5], there is a constant $c_3 > 0$ such that

$$\|f\|_2 \leq c_3\,\|\nabla f\|_2, \quad \text{for } f \in W_a^{1,2}(D_a). \tag{3.3}$$

Thus the inner product $\mathscr{E}^a$ is equivalent to $\mathscr{E}_1^a$ on $W_a^{1,2}(D_a)$ and therefore $W_a^{1,2}(D_a)$ is a Hilbert space with respect to $\mathscr{E}^a$. Now let $\tilde{h}_0$ be a smooth function with compact support in D_r such that $\tilde{h}_0 = h_0$ on ∂B_a. By the Riesz representation theorem there is a unique $g_1 \in W_a^{1,2}(D_a)$ such that

$$\mathscr{E}^a(g_1,\phi) = \frac{1}{2}\int_{D_a} \nabla g_1 \cdot \nabla\phi\,dx = \frac{1}{2}\int_{D_a} \phi\,\Delta\tilde{h}_0\,dx, \quad \forall\phi \in W_a^{1,2}(D_a). \tag{3.4}$$

Hence g_1 is a function in $W_a^{1,2}(D_a)$ that weakly satisfies $\Delta g_1 = -\Delta\tilde{h}_0$ in D_a and $\frac{\partial g_1}{\partial n} = 0$ on ∂D. Therefore $g_2 \equiv g_1 + \tilde{h}_0$ is a function in $W^{1,2}(D_a)$ that weakly satisfies $\Delta g_2 = 0$ in D_a and $\frac{\partial g_2}{\partial n} = 0$ on ∂D. Notice that g_1 admits a quasi-continuous version on $(\overline{D} \cap B_a) \cup \{\delta\}$ by Theorem 3.1.3 of [18], which we still denote by g_1. Then by Theorem 4.3.2 of [18], we have $\lim_{t\to\infty} g_1(X_t^a) = 0$, P_x-a.s. for quasi-every (q.e. in abbreviation) $x \in \overline{D} \cap B_a$. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers in $(0,a)$ that increases to a, and let $S_n = \tau_{a_n} \wedge n$, where $\tau_{a_n} = \inf\{t > 0 : X(t) \notin B_{a_n}\}$. Then $\{S_n\}_{n\geq 1}$ is an increasing sequence of stopping times relative to $\{\mathscr{F}_t\}_{t\geq 0}$ that announces τ_a, where $\{\mathscr{F}_t\}_{t\geq 0}$ is the filtration generated by X. Then by (3.4), Theorems 5.2.2 and 5.3.2 in [18], and Doob's stopping theorem,

$$\left\{ g_1(X_{S_n}) - g_1(X_0) + \frac{1}{2}\int_0^{S_n} \Delta\tilde{h}_0(X_s)ds,\ \mathscr{F}_{S_n},\ n \geq 1 \right\} \tag{3.5}$$

is a martingale under P_x for q.e. $x \in \overline{D} \cap B_a$. Note that by (3.4) and the analogue of Lemma 4.4.2 of [18] (the $\alpha = 0$ version) for the transient Dirichlet space $(W_a^{1,2}(D_a), \mathscr{E}^a)$, we have that

$$g_1(x) = \frac{1}{2}E^x\left[\int_0^{\tau_a} \Delta\tilde{h}_0(X_s)ds\right]$$

for q.e. $x \in \overline{D} \cap B_a$. Since $\tilde{h}_0$ is a smooth function with compact support in D_r, $\Delta\tilde{h}_0$ is Green-tight on D. Thus by (1.17), for q.e. $x \in \overline{D} \cap B_a$,

$$|g_1(x)| \leq \frac{1}{2}E^x\left[\int_0^\infty |\Delta\tilde{h}_0|(X_s)ds\right] \leq \frac{c}{2}\|\Delta\tilde{h}_0\|_D < \infty.$$

Hence g_1 is bounded on $\overline{D} \cap B_a$ and so is $g_2 = g_1 + \tilde{h}_0$. On the other hand, by Ito's formula, (3.1) and Doob's stopping theorem,

$$\left\{\tilde{h}_0(X_{S_n}) - \tilde{h}_0(X_0) - \frac{1}{2}\int_0^{S_n} \Delta\tilde{h}_0(X_s)ds,\ \mathscr{F}_{S_n},\ n \geq 1\right\} \tag{3.6}$$

is a martingale under P_x for each $x \in \overline{D}$. Thus by (3.5)–(3.6),

$$\left\{g_2(X_{S_n}) = g_1(X_{S_n}) + \tilde{h}_0(X_{S_n}),\ \mathscr{F}_{S_n},\ n \geq 1\right\}$$

is a bounded martingale under P_x for q.e. $x \in \overline{D} \cap B_a$. Therefore for q.e. $x \in \overline{D} \cap B_a$,

$$\begin{aligned} g_2(x) &= \lim_{n\to\infty} E^x\left[g_1(X_{S_n}) + \tilde{h}_0(X_{S_n})\right] \\ &= E^x\left[\tilde{h}_0(X_{T_a})\right] = E^x\left[h_0(X_{T_a})\right] = h_0(x). \qquad \square \end{aligned}$$

Lemma 3.2. *Suppose that w is a Green-tight function on D and $\rho \in \Gamma$. Then the equation*

$$\frac{1}{2}\Delta h_1 + w = 0 \quad in\ D, \tag{3.7}$$

$$\frac{\partial h_1}{\partial n} + \rho = 0 \quad on\ \partial D, \tag{3.8}$$

$$\lim_{\substack{|x|\to\infty \\ x\in D}} h_1(x) = \gamma, \tag{3.9}$$

has a unique bounded continuous weak solution on $\overline{D}$ which is given by the formula

$$\begin{aligned} h_1(x) &= E^x\left[\int_0^\infty w(X_s)ds\right] + E^x\left[\int_0^\infty \rho(X_t)dL_t\right] + \gamma \\ &= \int_D G(x,y)w(y)dy + \frac{1}{2}\int_{\partial D} G(x,y)\rho(y)\sigma(dy) + \gamma, \end{aligned} \tag{3.10}$$

for all $x \in \overline{D}$.

Proof. We first prove the uniqueness. Suppose that u is a bounded continuous weak solution on $\overline{D}$ for (3.7)–(3.9) with $w = \rho = \gamma = 0$. Then by (4.1) and Theorem A2 in [19], for q.e. $x \in \overline{D}$, under P_x, $u(X)$ is a bounded local martingale. By the martingale convergence theorem and the transience of reflecting Brownian motion X on $\overline{D}$, we have for q.e. $x \in \overline{D}$,

$$u(x) = \lim_{t\to\infty} E^x\left[u(X_t)\right] = E^x\left[\lim_{t\to\infty} u(X_t)\right] = 0.$$

Hence $u = 0$ m-a.e. on $\overline{D}$ and since u is continuous, $u \equiv 0$ on $\overline{D}$. This proves the uniqueness.

For $r > 0$ with $\partial D \subset B_r$, set $X_t^r = X_t$ if $t < \tau_r$ and $X_t^r = \delta$ if $t \geq \tau_r$, where δ is a cemetery point which is added to $\overline{D} \cap B_r$ as a one-point compactification. Any function ψ defined on $\overline{D} \cap B_r$ is extended to $(\overline{D} \cap B_r) \cup \{\delta\}$ by setting $\psi(\delta) = 0$. Let $W_r^{1,2}(D_r)$ and $\mathscr{E}^r$ be defined in the same way as $W_a^{1,2}(D_a)$ and $\mathscr{E}^a$ in the proof of Lemma 3.1 but with r in place of a. By Theorem 4.4.2 of [18], $(W_r^{1,2}(D_r), \mathscr{E}^r)$ is the Dirichlet space associated with the process $(X^r, \{P_x, x \in \overline{D} \cap B_r\})$. Note that $L_{\cdot \wedge \tau_r}$ can be viewed as a positive continuous additive functional of X^r whose corresponding Revuz measure is $\frac{1}{2}\sigma$ (see [15], Theorem 2.22). Define

$$(\tilde{G}^r \rho)(x) = 2E^x \left[\int_0^{\tau_r} \rho(X_t) dL_t \right], \quad x \in \overline{D}.$$

For $x \in \overline{D} \cap B_r$, by (3.2),

$$|\tilde{G}^r \rho(x)| \leq 2E^x \left[\int_0^{\infty} |\rho(X_s)| dL_s \right] \leq \tilde{G}|\rho|(x) \leq c \|\rho\|_{\partial D} < \infty.$$

Thus $\tilde{G}^r \rho$ is bounded and therefore L^2-integrable on D_r. By the strong Markov property of X,

$$\tilde{G}\rho(x) = \tilde{G}^r \rho(x) + E^x \left[\tilde{G}\rho(X_{\tau_r}) \right], \quad x \in \overline{D}_r. \tag{3.11}$$

By Propositions 1.1 and 2.1, $\tilde{G}\rho$ is a bounded continuous function on $\overline{D}$. Hence by Lemma 3.1, $g_1(x) = E^x \left[\tilde{G}\rho(X_{\tau_r}) \right]$ is a bounded continuous function on $\overline{D}_r$ that is harmonic in D_r, $\frac{\partial g_1}{\partial n} = 0$ on ∂D and $g_1 = \tilde{G}\rho$ on ∂B_r. In particular, this implies through (3.11) that $\tilde{G}^r \rho$ is a bounded continuous function on $\overline{D}_r$.

By using Lemma 1.3.4 of [18], it can be shown that $\tilde{G}^r \rho \in W_r^{1,2}(D_r)$ and that for $\psi \in C^2(\mathbb{R}^d)$ with compact support in B_r,

$$\begin{aligned} \frac{1}{2} \int_{D_r} \nabla(\tilde{G}^r \rho)(x) \cdot \nabla \psi(x) dx &= \lim_{t \to 0} \frac{1}{t} \int_{D_r} (\tilde{G}^r \rho(x) - \tilde{P}_t \tilde{G}^r \rho(x)) \psi(x) dx \\ &= \int_{\partial D} \rho(x) \psi(x) \sigma(dx). \end{aligned} \tag{3.12}$$

Since (3.11)–(3.12) hold for any $r > 0$ such that $\partial D \subset B_r$, $\tilde{G}\rho$ weakly solves the following equation

$$\Delta(\tilde{G}\rho) = 0 \quad \text{in } D, \tag{3.13}$$

$$\frac{\partial(\tilde{G}\rho)}{\partial n} + 2\rho = 0 \quad \text{on } \partial D, \tag{3.14}$$

$$\lim_{\substack{|x| \to \infty \\ x \in D}} (\tilde{G}\rho)(x) = 0. \tag{3.15}$$

The last property follows from the definition of the class Γ and the estimate (1.17).

Similarly, it can be shown that Gw is a continuous bounded function on $\overline{D}$ that weakly satisfies

$$\frac{1}{2}\Delta(Gw) = -w \quad \text{in } D \tag{3.16}$$

$$\frac{\partial(Gw)}{\partial n} = 0 \quad \text{on } \partial D, \tag{3.17}$$

$$\lim_{\substack{|x|\to\infty \\ x\in D}} Gw(x) = 0. \tag{3.18}$$

Thus combining (3.13)–(3.15) with (3.16)–(3.18), we see that the function h_1 given by (3.10) solves Eqs. (3.7)–(3.9). □

Lemma 3.3. *Suppose that q and w are Green-tight functions on D, κ and ρ are in Γ, and γ is a real number. When $\|Gq^+\|_\infty + \frac{1}{2}\|\tilde{G}\kappa^+\|_\infty \leq \lambda < 1$,*

(3.19)
$$h_2(x) \equiv E^x\left[\int_0^\infty e(t)w(X_t)dt\right] + E^x\left[\int_0^\infty e(t)\rho(X_t)dL_t\right] + \gamma E^x\left[e(\infty)\right], \; x \in \overline{D},$$

where

(3.20)
$$e(t) = \exp\left(\int_0^t q(X_s)ds + \int_0^t \kappa(X_s)dL_s\right), \quad t \in [0,\infty],$$

is the unique bounded continuous solution of the following (reduced) Schrödinger equation with Neumann boundary condition:

$$\frac{1}{2}\Delta h_2 + q\,h_2 + w = 0 \quad \text{in } D, \tag{3.21}$$

$$\frac{\partial h_2}{\partial n} + \kappa\,h_2 + \rho = 0 \quad \text{on } \partial D, \tag{3.22}$$

$$\lim_{\substack{|x|\to\infty \\ x\in D}} h_2(x) = \gamma. \tag{3.23}$$

Furthermore, for $x \in \overline{D}$, h_2 satisfies

$$\begin{aligned} h_2(x) &= h_1(x) + \int_D G(x,y)q(y)h_2(y)\,dy \\ &\quad + \frac{1}{2}\int_{\partial D} G(x,y)\kappa(y)h_2(y)\,\sigma(dy), \end{aligned} \tag{3.24}$$

and when w, ρ and γ are non-negative,

$$0 \leq h_2(x) \leq \frac{1}{1-\lambda}\left(\|Gw\|_\infty + \frac{1}{2}\|\tilde{G}\rho\|_\infty + \gamma\right). \tag{3.25}$$

Remark 3.1. It will be shown below under the given conditions on q and κ that $e(\infty) = \lim_{t\to\infty} e(t)$ exists P_x-a.s. for each $x \in \overline{D}$.

Proof. We first show the uniqueness. Since the equations are linear, for this it suffices to show that a bounded continuous weak solution $\hat{u}$ of (3.21)–(3.23) with $w = \rho = \gamma = 0$ is identically zero. Such a solution $\hat{u}$ is in $W^{1,2}(D \cap B)$ for each ball B in $\mathbb{R}^d$, and for each $\psi \in C_c^2(\mathbb{R}^d)$,

$$\begin{aligned}(3.26) \qquad & \frac{1}{2}\int_D \nabla\hat{u}(x)\cdot\nabla\psi(x)dx + \int_D \hat{u}(x)\psi(x)q^-(x)dx \\ & +\frac{1}{2}\int_{\partial D} \hat{u}(x)\psi(x)\kappa^-(x)\sigma(dx) \\ = & \int_D \hat{u}(x)\psi(x)q^+(x)dx + \frac{1}{2}\int_{\partial D} \hat{u}(x)\psi(x)\kappa^+(x)\sigma(dx).\end{aligned}$$

Let $(Y, \{Q_x, x \in \overline{D}\})$ be the process on $\overline{D}$ obtained by killing X according to the Revuz measure $q^- dx + \frac{1}{2}\kappa^- \sigma(dx)$; that is, for any non-negative Borel measurable function f on $\overline{D}$,

$$E^{Q_x}\left[f(Y_t)\right] = E^x\left[\exp\left(-\int_0^t q^-(X_s)ds - \int_0^t \kappa^-(X_s)\,dL_s\right)f(X_t)\right].$$

Let ζ denote the life-time of Y and δ be the one-point compactification of $\overline{D}$. Then $Y_t = \delta$ for $t \geq \zeta$ and we use the convention that any function f defined on $\overline{D}$ is extended to $\overline{D} \cup \{\delta\}$ by taking $f(\delta) = 0$. The left hand side of (3.26) is the Dirichlet inner product for the pair $(\hat{u}, \psi)$ with respect to the process $(Y, \{Q_x, x \in \overline{D}\})$ (cf. Proposition 3 of [14] and the Appendix of [19]). Thus by Theorem A2 of [19],

$$\left\{\hat{u}(Y_t) + \int_0^t q^+(Y_s)\hat{u}(Y_s)ds + \int_0^t \kappa^+(Y_s)\hat{u}(Y_s)d\hat{L}_s, \quad t \geq 0\right\}$$

is a Q_x-local martingale for m-a.e. $x \in \overline{D}$, where $\hat{L}$ is the positive continuous additive functional of Y with associated Revuz measure $\frac{1}{2}\sigma$. Denote by $\hat{G}(x,y)$ the Green function of Y. Clearly, $\hat{G} \leq G$ on $\overline{D} \times \overline{D}$. Thus

$$\begin{aligned}(3.27) \qquad E^{Q_x}\left[\int_0^\infty |q^+(Y_s)\hat{u}(Y_s)|ds\right] & \leq \|\hat{u}\|_\infty \int_D \hat{G}(x,y)q^+(y)dy \\ & \leq \|\hat{u}\|_\infty \,\|Gq^+\|_\infty,\end{aligned}$$

and

$$\begin{aligned}(3.28)\ E^{Q_x}\left[\int_0^\infty |\kappa^+(Y_s)\hat{u}(Y_s)|d\hat{L}_s\right] & \leq \frac{1}{2}\|\hat{u}\|_\infty \int_{\partial D} \hat{G}(x,y)\kappa^+(y)\sigma(dy) \\ & \leq \frac{1}{2}\|\hat{u}\|_\infty \,\|\tilde{G}\kappa^+\|_\infty.\end{aligned}$$

Combining the above with the fact that $\hat{u}$ is bounded on $\overline{D}$, we conclude that

$$\left\{\hat{u}(Y_t) + \int_0^t q^+(Y_s)\hat{u}(Y_s)ds + \int_0^t \kappa^+(Y_s)\hat{u}(Y_s)d\hat{L}_s, \quad t \geq 0\right\}$$

is a uniformly integrable martingale under Q_x for m-a.e. $x \in \overline{D}$. Thus for m-a.e. $x \in \overline{D}$,

$$\begin{aligned} \hat{u}(x) &= \lim_{t\to\infty} E^{Q_x}\left[\hat{u}(Y_t) + \int_0^t q^+(Y_s)\hat{u}(Y_s)ds + \int_0^t \kappa^+(Y_s)\hat{u}(Y_s)d\hat{L}_s\right] \\ &= E^{Q_x}\left[\lim_{t\to\infty} \hat{u}(Y_t) + \int_0^\infty q^+(Y_s)\hat{u}(Y_s)ds + \int_0^\infty \kappa^+(Y_s)\hat{u}(Y_s)d\hat{L}_s\right] \\ &= E^{Q_x}\left[\int_0^\infty q^+(Y_s)\hat{u}(Y_s)ds + \int_0^\infty \kappa^+(Y_s)\hat{u}(Y_s)d\hat{L}_s\right]. \end{aligned}$$

Combining this with (3.27)–(3.28) and the hypothesis of the Lemma, we have for m-a.e. $x \in \overline{D}$,

$$|\hat{u}(x)| \leq \|\hat{u}\|_\infty \left(\|Gq^+\|_\infty + \frac{1}{2}\|\tilde{G}\kappa^+\|_\infty\right) \leq \lambda\|\hat{u}\|_\infty.$$

This implies that $\|\hat{u}\|_\infty = 0$ and therefore $\hat{u} = 0$. The uniqueness is thus proved. For the existence, by the linearity of (3.21)–(3.23) and by considering h_2 for (w^+, ρ^+, γ^+) and (w^-, ρ^-, γ^-), we may assume that $w \geq 0$, $\rho \geq 0$, and $\gamma \geq 0$. Since

$$\begin{aligned} \text{(3.29)}\quad \sup_{x\in\overline{D}} E^x\left[\int_0^\infty q^+(X_s)ds + \int_0^\infty \kappa^+(X_s)dL_s\right] &\leq \|Gq^+\|_\infty + \frac{1}{2}\|\tilde{G}\kappa^+\|_\infty \\ &\leq \lambda < 1, \end{aligned}$$

$e(\infty)$ is well defined and is finite P_x-a.s. for each $x \in \overline{D}$. In the course of the following proof of (3.25), it will become apparent that h_2 is finite. Let h_1 be the function given by (3.10) and

$$e_+(t) = \exp\left(\int_0^t q^+(X_s)\,ds + \int_0^t \kappa^+(X_s)\,dL_s\right), \quad \text{for } t \geq 0.$$

It follows from (3.29) and Khasminskii's lemma (see [9] or [29]) that

$$\sup_{x\in\overline{D}} E^x[e_+(\infty)] \leq \frac{1}{1-\lambda}.$$

Using ordinary calculus, Fubini's theorem and the Markov property of X, we have for $x \in \overline{D}$,

$$\begin{aligned} & h_2(x) - h_1(x) \\ \leq\ & E^x\left[\int_0^\infty (e_+(t) - 1)w(X_t)dt\right] + E^x\left[\int_0^\infty (e_+(t) - 1)\,\rho(X_t)\,dL_t\right] \\ & +\gamma E^x[e_+(\infty) - 1] \\ \leq\ & E^x\left[\int_0^\infty \left(\int_0^t q^+(X_s)e_+(s)ds + \int_0^t \kappa^+(X_s)e_+(s)dL_s\right) w(X_t)dt\right] \\ & +E^x\left[\int_0^\infty \left(\int_0^t q^+(X_s)e_+(s)ds + \int_0^t \kappa^+(X_s)e_+(s)dL_s\right) \rho(X_t)dL_t\right] \end{aligned}$$

$$
\begin{aligned}
&+\gamma E^x[e_+(\infty)-1] \\
= \quad & E^x\left[\int_0^\infty e_+(s)\,q^+(X_s)\left(\int_s^\infty w(X_t)dt+\int_s^\infty \rho(X_t)\,dL_t+\gamma\right)ds\right] \\
& +E^x\left[\int_0^\infty e_+(s)\,\kappa^+(X_s)\left(\int_s^\infty w(X_t)dt+\int_s^\infty \rho(X_t)\,dL_t+\gamma\right)dL_s\right] \\
= \quad & E^x\left[\int_0^\infty e_+(s)\,q^+(X_s)\,h_1(X_s)\,ds\right]+E^x\left[\int_0^\infty e_+(s)\,\kappa^+(X_s)\,h_1(X_s)\,dL_s\right] \\
\le \quad & \|h_1\|_\infty\,E^x\,[e_+(\infty)-1] \\
\le \quad & \frac{\lambda}{1-\lambda}\|h_1\|_\infty .
\end{aligned}
$$

The equality in the third last line (especially for the second expectation) follows from an optional projection theorem (see [28], Theorem 28.7) and the Markov property of X. Thus by noting that $h_2 \ge 0$,

$$
\|h_2\|_\infty \le \frac{1}{1-\lambda}\,\|h_1\|_\infty \le \frac{1}{(1-\lambda)}\left(\|Gw\|_\infty+\frac{1}{2}\|\tilde{G}\rho\|_\infty+\gamma\right). \tag{3.30}
$$

Now we are going to prove that the function h_2 defined by (3.19) satisfies the implicit Eq. (3.24). Using ordinary calculus, Fubini's theorem, and the Markov property of X, we have

$$
\begin{aligned}
& h_2(x)-h_1(x) \\
& = E^x\left[\int_0^\infty e(s)w(X_s)ds+\int_0^\infty e(s)\rho(X_s)dL_s+\gamma\,e(\infty)\right]-h_1(x) \\
& = E^x\left[\int_0^\infty (e(s)-1)w(X_s)\,ds+\int_0^\infty (e(s)-1)\rho(X_s)\,dL_s+\gamma\,(e(\infty)-1)\right] \\
& = E^x\left[\int_0^\infty e(s)w(X_s)\left(\int_0^s e(t)^{-1}\,(q(X_t)\,dt+\kappa(X_t)\,dL_t)\right)ds\right] \qquad (3.31)\\
& \quad +E^x\left[\int_0^\infty e(s)\rho(X_s)\left(\int_0^s e(t)^{-1}\,(q(X_t)\,dt+\kappa(X_t)\,dL_t)\right)dL_s\right] \\
& \quad +\gamma E^x\left[\left(\int_0^\infty e(t)^{-1}\,(q(X_t)\,dt+\kappa(X_t)\,dL_t)\right)e(\infty)\right] \\
& = E^x\Bigg[\int_0^\infty e(t)^{-1}\left(\int_t^\infty e(s)w(X_s)ds+\int_t^\infty e(s)\rho(X_s)dL_s+\gamma\,e(\infty)\right) \qquad (3.32)\\
& \qquad\qquad \cdot\,(q(X_t)dt+\kappa(X_t)dL_t)\Bigg] \\
& = E^x\left[\int_0^\infty h_2(X_t)\,(q(X_t)dt+\kappa(X_t)dL_t)\right] \qquad (3.33)\\
& = \int_D G(x,y)q(y)h_2(y)\,dy+\frac{1}{2}\int_{\partial D} G(x,y)\kappa(y)h_2(y)\,\sigma(dy),
\end{aligned}
$$

where for the equality in (3.31) we used that fact that $e(t)^{-1}\,(q(X_t)dt+\kappa(X_t)dL_t)$ is the exact differential of $-e(t)^{-1}$, and for the passage from (3.32) to (3.33) we used an optional projection theorem. Since qh_2 is a Green-tight function on D and $\kappa h_2 \in \Gamma$, h_2 is a continuous function on $\overline{D}$ by the above identity and Lemma 3.2. Furthermore by Lemma 3.2,

$$u(x) = \int_D G(x,y)q(y)h_2(y)\,dy + \frac{1}{2}\int_{\partial D} G(x,y)\kappa(y)h_2(y)\sigma(dy)$$

satisfies

$$\begin{aligned}
&(3.34) && \frac{1}{2}\Delta u + q\,h_2 &&= 0 && \text{in } D,\\
&(3.35) && \frac{\partial u}{\partial n} + \kappa\,h_2 &&= 0 && \text{on } \partial D,\\
&(3.36) && \lim_{\substack{|x|\to\infty\\ x\in D}} u(x) &&= 0\,.
\end{aligned}$$

Note that h_1 is the solution of (3.7)–(3.9). Combining this with (3.24) and (3.34)–(3.36), we see that h_2 is a weak solution of (3.21)–(3.23). □

The following lemma, which is used in [6], is a generalization of Lemma 3.3 under a weaker assumption on q.

Lemma 3.4. *Let q be a real-valued Borel measurable function on D such that q^+ is Green-tight on D and $1_D q^- \in K_d^{loc}$. Assume the remaining conditions in Lemma 3.3 hold and $\gamma = 0$, then the conclusion of Lemma 3.3 still holds.*

Proof. Without loss of generality, we may assume that $w \geq 0$ and $\rho \geq 0$. The same argument as in the proof of Lemma 3.3 up to (3.30) shows the uniqueness and that the function h_2 defined by (3.19) with $\gamma = 0$ satisfies (3.25). The identity (3.24) can be proved in the same way as that in Lemma 3.3, except that in the passage from (3.32) to (3.33), we first apply Fubini's Theorem and the Markov property of X to the integrand with respect to the non-negative integrators $q^+(X_t)dt + \kappa^+(X_t)dL_t$ and $q^-(X_t)dt + \kappa^-(X_t)dL_t$ respectively, and then take their difference which is permitted under the assumption that $\|Gq^+\|_\infty + \frac{1}{2}\|\tilde{G}\kappa^+\|_\infty < 1$ and the fact that $h_2 \geq 0$ is bounded.

Since $h_2 \geq 0$, it follows from (3.24) that for $x \in \overline{D}$,

$$(3.37)\quad G(q^-h_2)(x) + \frac{1}{2}\tilde{G}(\kappa^-h_2)(x) \leq h_1(x) + G(q^+h_2)(x) + \frac{1}{2}\tilde{G}(\kappa^+h_2)(x).$$

Because h_2 is bounded on $\overline{D}$, q^+h_2 is Green-tight on D and $\kappa^+h_2 \in \Gamma$. Thus by Lemma 3.2, $v_1 \equiv G(q^+h_2) + \frac{1}{2}\tilde{G}(\kappa^+h_2)$ satisfies

$$\begin{aligned}
&(3.38) && \frac{1}{2}\Delta v_1 + q^+h_2 &&= 0 && \text{in } D\\
&(3.39) && \frac{\partial v_1}{\partial n} + \kappa^+h_2 &&= 0 && \text{on } \partial D,\\
&(3.40) && \lim_{\substack{|x|\to\infty\\ x\in D}} v_1(x) &&= 0,
\end{aligned}$$

Put $v_2 \equiv G(q^- h_2) + \frac{1}{2}\tilde{G}(\kappa^- h_2)$. By (3.37) and Lemma 3.2, v_2 is a bounded non-negative function on $\overline{D}$ with

$$\lim_{\substack{|x|\to\infty \\ x\in D}} v_2(x) \le \lim_{\substack{|x|\to\infty \\ x\in D}} h_1(x) + \lim_{\substack{|x|\to\infty \\ x\in D}} v_1(x) = 0. \tag{3.41}$$

By the strong Markov property of X, we have for any $k > 0$ and $x \in \overline{D}$,

$$\begin{aligned} v_2(x) &= G(1_{B_k} q^- h_2)(x) + \frac{1}{2}\tilde{G}(1_{B_k}\kappa^- h_2)(x) \\ &\quad + E^x \left[G(1_{B_k^c} q^- h_2)(X_{\tau_k}) + \frac{1}{2}\tilde{G}(1_{B_k^c}\kappa^- h_2)(X_{\tau_k}) \right], \end{aligned} \tag{3.42}$$

where $\tau_k = \inf\{t > 0 : X_t \notin B_k\}$ and $B_k^c = \mathbb{R}^d \setminus B_k$. Since $|X_{\tau_k}| \ge k$ on $\{\tau_k < \infty\}$ and $G(1_{B_k^c} q^- h_2) + \frac{1}{2}\tilde{G}(1_{B_k^c}\kappa^- h_2) \le v_2$, it follows from (3.41)–(3.42) that $g_k \equiv G(1_{B_k} q^- h_2)+\frac{1}{2}\tilde{G}(1_{B_k}\kappa^- h_2)$ converges uniformly on $\overline{D}$ to v_2 as $k \to \infty$. Note that $1_{B_k}\kappa^- \in \Gamma$ and $1_{B_k} q^-$ is a Green-tight function on D for all $k > 0$ since $q^- \in K_d^{loc}$. Therefore by Lemma 3.2, for large k such that $\partial D \subset B_k$, g_k is a bounded continuous function on $\overline{D}$ with $\frac{1}{2}\Delta g_k + 1_{B_k} q^- h_2 = 0$ in D, $\frac{\partial g_k}{\partial n} + \kappa^- h_2 = 0$ on ∂D. In particular, v_2, the uniform limit of g_k as $k \to \infty$, is continuous on $\overline{D}$. On the other hand by Lemma 3.1, $\eta_k(x) = E^x \left[G(1_{B_k^c} q^- h_2)(X_{\tau_k}) + \frac{1}{2}\tilde{G}(1_{B_k^c}\kappa^- h_2)(X_{\tau_k}) \right]$ is a continuous function on $\overline{D} \cap B_k$ that is harmonic in $D \cap B_k$ and $\frac{\partial \eta_k}{\partial n} = 0$ on ∂D in the distributional sense. Therefore $v_2 = g_k + \eta_k$ (for each $k \ge 1$) weakly satisfies the following equations:

$$\frac{1}{2}\Delta v_2 + q^- h_2 = 0 \qquad \text{in } D, \tag{3.43}$$

$$\frac{\partial v_2}{\partial n} + \kappa^- h_2 = 0 \qquad \text{on } \partial D. \tag{3.44}$$

Thus combining (3.38)–(3.41), (3.43)–(3.44) and Lemma 3.2, we see that $h_2 = h_1 + v_1 - v_2$ is a bounded continuous function on $\overline{D}$ which is the weak solution of (3.21)–(3.23) with $\gamma = 0$. □

4 Semilinear elliptic equations

Proof of Theorem 1.2. Let $\lambda \in (0, 1)$ be as in Theorem 1.2. Let

$$\begin{aligned} v_1(x) &= E^x \left[\int_0^\infty e_1(t) g(X_t)\, dt \right] \\ &\quad + E^x \left[\int_0^\infty e_1(t)\phi(X_t)\, dL_t \right] + \alpha E^x \left[e_1(\infty) \right], \end{aligned} \tag{4.1}$$

where

$$e_1(t) = \exp\left(-\int_0^t U_1(X_s)\, ds - \int_0^t U_2(X_s)\, dL_s \right). \tag{4.2}$$

Since the Green function $G(x,y)$ is strictly positive on $\overline{D} \times \overline{D}$, it follows from the conditions on D, g, ϕ and α that the function

$$\begin{aligned} E^x\left[\int_0^\infty g(X_t)dt\right] + E^x\left[\int_0^\infty \phi(X_t)dL_t\right] + \alpha \\ = \int_D G(x,y)g(y)dy + \frac{1}{2}\int_{\partial D} G(x,y)\phi(y)\sigma(dy) + \alpha \end{aligned}$$

is strictly positive for all $x \in \overline{D}$. Combining this with the fact that $e_1(t) > 0$ for all $t \geq 0$ (since $\|GU_1\|_\infty < \infty$ and $\|\tilde{G}U_2\|_\infty < \infty$), we see that v_1 is well defined and strictly positive on $\overline{D}$. Let

$$\text{(4.3)} \quad \Lambda = \left\{ u \in C_b(\overline{D}) : v_1 \leq u \leq \frac{1}{1-\lambda}\left(\|Gg\|_\infty + \frac{1}{2}\|\tilde{G}\phi\|_\infty + \alpha\right)\right\}.$$

For $u \in \Lambda$, let

$$\text{(4.4)} \qquad q_u = \frac{F_1(\cdot, u)}{u} \quad \text{on } D, \qquad \kappa_u = \frac{F_2(\cdot, u)}{u} \quad \text{on } \partial D.$$

Then by (1.4)–(1.5), (1.18) and the assumptions of Theorem 1.2,

$$\text{(4.5)} \qquad -U_1 \leq q_u \leq \left(\sup_{0<y\leq C_\lambda} \frac{f_1(y)}{y}\right) V_1 \equiv \tilde{V}_1 \quad \text{on } D$$

and

$$\text{(4.6)} \qquad -U_2 \leq \kappa_u \leq \left(\sup_{0<y\leq C_\lambda} \frac{f_2(y)}{y}\right) V_2 \equiv \tilde{V}_2 \quad \text{on } \partial D.$$

For $u \in \Lambda$, define

$$\text{(4.7)} \quad e_u(t) = \exp\left(\int_0^t q_u(X_s)ds + \int_0^t \kappa_u(X_s)dL_s\right) \quad \text{for all } t \in [0,\infty],$$

and for $x \in \overline{D}$,

$$\text{(4.8)} \quad \begin{aligned} (Tu)(x) = E^x\left[\int_0^\infty e_u(t)g(X_t)dt\right] + E^x\left[\int_0^\infty e_u(t)\phi(X_t)dL_t\right] \\ +\alpha E^x\left[e_u(\infty)\right]. \end{aligned}$$

Clearly, $Tu \geq v_1$. Since

$$\text{(4.9)} \qquad \|Gq_u^+\|_\infty + \frac{1}{2}\|\tilde{G}\kappa_u^+\|_\infty \leq \|G\tilde{V}_1\|_\infty + \frac{1}{2}\|\tilde{G}\tilde{V}_2\|_\infty \leq \lambda,$$

it follows from (3.25) of Lemma 3.3 that for all $x \in \overline{D}$,

$$\text{(4.10)} \qquad Tu(x) \leq \frac{1}{1-\lambda}\left(\|Gg\|_\infty + \frac{1}{2}\|G\phi\|_\infty + \alpha\right).$$

Thus $T\Lambda \subset \Lambda$. By (3.24),

$$\text{(4.11)} \qquad Tu(x) = h_1(x) + \int_D G(x,y) q_u(y) Tu(y)\, dy + \frac{1}{2}\int_{\partial D} G(x,y)\kappa_u(y) Tu(y)\, \sigma(dy).$$

By (4.5)-(4.6) and (4.10), we have

$$\text{(4.12)} \quad |q_u(x)Tu(x)| \le \frac{1}{1-\lambda}\left(\|Gg\|_\infty + \frac{1}{2}\|\tilde{G}\phi\|_\infty + \alpha\right)\left(U_1 + \tilde{V}_1\right)(x), \qquad x \in D,$$

$$\text{(4.13)} \quad |\kappa_u(x)Tu(x)| \le \frac{1}{1-\lambda}\left(\|Gg\|_\infty + \frac{1}{2}\|\tilde{G}\phi\|_\infty + \alpha\right)\left(U_2 + \tilde{V}_2\right)(x), \qquad x \in \partial D,$$

and so $q_u Tu$ is Green-tight on D and $\kappa_u Tu$ is in Γ. It then follows from (4.11)–(4.13) and Proposition 2.3 that $T\Lambda$ is a family of functions in Λ which is uniformly bounded and equicontinuous, and all functions there have limit α at infinity. Hence $T\Lambda$ is relatively compact in $C_b(\overline{D})$ with respect to the uniform norm $\|\cdot\|_\infty$. Suppose that u and $\{u_k\}_{k=1}^\infty$ are in Λ such that $\|u_k - u\|_\infty \to 0$ as $k \to \infty$. Since $F_1(x,u)$ and $F_2(x,u)$ are continuous in u for each fixed x, $q_{u_k} \to q_u$ in D and $\kappa_{u_k} \to \kappa_u$ on ∂D pointwise as $k \to \infty$. Since

$$e_u(t) \le \exp\left(\int_0^t \tilde{V}_1(X_s)\, ds + \int_0^t \tilde{V}_2(X_s)\, dL_s\right) \quad \text{for all } t \in [0,\infty],$$

by (4.8), Lemma 3.3 and the Lebesgue dominated convergence theorem, as $k \to \infty$, Tu_k converges to Tu pointwise on $\overline{D}$ and therefore uniformly, by the equicontinuity. Thus T is a compact and continuous mapping from the nonempty, convex, closed bounded set $\Lambda \subset C_b(\overline{D})$ into itself. By Schauder's fixed point theorem (cf. [31]), there is a function $u_0 \in \Lambda$ such that $Tu_0 = u_0$. It follows from (4.8) and Lemma 3.3 that u_0 solves (1.1)–(1.3). □

Proof of Corollary 1.3. We seek a positive continuous solution u of (1.1)–(1.3) with $F_1 \equiv g \equiv \phi \equiv 0$ and

$$F_2(x,u) = \frac{d-2}{2a}u + \frac{d-2}{2}H(x)u^{d/d-2}.$$

If we further require that u be such that $0 < u \le \beta \le 1$, then (1.4)–(1.5) hold with $U_1 \equiv 0$, $V_1 \equiv 0$, $f_1 = 0$, $f_2(u) = u$, $U_2(x) = \frac{d-2}{2}\beta^{2/d-2}H^-(x)$ and $V_2(x) = \frac{d-2}{2a} + \frac{d-2}{2}\beta^{2/d-2}H^+(x)$, for $x \in \partial D$. Let

$$h_1(x) = \tilde{G}\left(\frac{1}{a}\right)(x) = \int_{\partial D} G(x,y)a^{-1}\sigma(dy), \quad \text{for } x \in \overline{D}.$$

Then h_1 is clearly a radial function, i.e. $h_1(x) = \psi(|x|)$, where by using Eqs. (3.7)–(3.9) we see that ψ satisfies

$$\psi''(r) + \frac{d-1}{r}\psi'(r) = 0 \quad \text{for } r > a, \tag{4.14}$$

$$\psi'(a) + \frac{2}{a} = 0, \tag{4.15}$$

$$\lim_{r \to \infty} \psi(r) = 0. \tag{4.16}$$

The solution of these equations is given by

$$\psi(r) = \frac{2}{d-2}\left(\frac{a}{r}\right)^{d-2}, \qquad r \geq a.$$

This implies that

$$\|\tilde{G}V_2\|_\infty \leq \frac{d-2}{2}\|h_1\|_\infty + \frac{d-2}{2}\beta^{2/d-2}\|\tilde{G}H^+\|_\infty = 1 + \frac{d-2}{2}\beta^{2/d-2}\|\tilde{G}H^+\|_\infty.$$

Since $H \in \Gamma(\partial D)$ (see Proposition 2.2), by choosing β sufficiently small, we can ensure that $1 + \frac{d-2}{2}\beta^{2/d-2}\|\tilde{G}H^+\|_\infty < 2$. Then there is $\lambda \in (0,\ 1)$ such that $1 + \frac{d-2}{2}\beta^{2/d-2}\|\tilde{G}H^+\|_\infty = 2\lambda$ and in this case $C_\lambda = \beta > 0$. Then by Theorem 1.2, for any positive number $\alpha < (1-\lambda)\beta$, Eqs. (1.1)–(1.3) with F_1, F_2, g and ψ as above, have a positive continuous solution u on $\overline{D} = \{x \in \mathbb{R}^d : |x| \geq a\}$ with values in $(0, \beta)$. $\square$

In general, uniqueness may not hold for the solutions found in Theorem 1.2. Indeed, as we illustrate below, there may be more than one metric $\mathcal{G}$ as described in Corollary 1.3 with a prescribed limiting value at infinity of $\alpha > 0$ times the Euclidean metric. Consider the semilinear equations for this problem with $d = 3$, $a = 1$ and $H = 1$. Then we are seeking a positive continuous solution u of (1.1)–(1.3) with $F_1 \equiv g \equiv \phi \equiv 0$ and $F_2(x,u) = \frac{1}{2}u + \frac{1}{2}u^3$. We shall focus on radial solutions, i.e., $u(x) = \psi(|x|)$, and so $\psi > 0$ should satisfy:

$$\psi''(r) + \frac{2}{r}\psi'(r) = 0 \quad \text{for } r > 1, \tag{4.17}$$

$$\psi'(1) + \frac{1}{2}\psi(1) + \frac{1}{2}\psi(1)^3 = 0, \tag{4.18}$$

$$\lim_{r \to \infty} \psi(r) = \alpha. \tag{4.19}$$

The solution of Eqs. (4.17) and (4.19) is $\psi(r) = cr^{-1} + \alpha$. For (4.18) to hold, c should satisfy $(c+\alpha)^3 - c + \alpha = 0$. A straightforward calculation shows that for each fixed $\alpha \in (0, \frac{1}{3\sqrt{3}})$, this equation has two real positive solutions c. Hence there is more than one positive solution ψ for (4.17)–(4.19).

Despite the lack of uniqueness in general, we do have the following theorem.

Theorem 4.1. *Suppose, in addition to the conditions in Theorem 1.2, that $F_1(x,u)$, $F_2(x,u)$ are monotonically decreasing as functions of $u \in (0,\beta)$ for each fixed $x \in D$ and $x \in \partial D$, respectively. Then any bounded continuous weak solution of equations (1.1)–(1.3) is unique.*

Proof. Suppose that u_1 and u_2 are two different continuous bounded positive weak solutions on $\overline{D}$ of (1.1)–(1.3). Without loss of generality, we may assume that $\hat{D} = \{x \in \overline{D} : u_1(x) > u_2(x)\}$ is nonempty. Then $\hat{D}$ is a relatively open subset of $\overline{D}$ and $\hat{u} = u_1 - u_2$ satisfies:

$$\begin{aligned} &(4.20) \quad & \frac{1}{2}\Delta\hat{u}(x) &= F_1(x,u_2) - F_1(x,u_1) \geq 0 && \text{in } D \cap \hat{D}, \\ &(4.21) \quad & \frac{\partial \hat{u}}{\partial n} &= F_2(x,u_2) - F_2(x,u_1) \geq 0 && \text{on } \partial D \cap \hat{D}, \\ &(4.22) \quad & \lim_{\substack{|x|\to\infty \\ x\in\hat{D}}} \hat{u}(x) &= 0 && \text{if } \hat{D} \text{ is unbounded.} \end{aligned}$$

Thus for any $\psi \in C_c^2(\mathbb{R}^d)$ such that $\psi = 0$ in a neighborhood of $\overline{D} \setminus \hat{D}$,

$$\begin{aligned} \frac{1}{2}\int_{\hat{D}} \nabla\hat{u}(x)\cdot\nabla\psi(x)dx &= -\int_{\hat{D}} \left(F_1(x,u_2(x)) - F_1(x,u_1(x))\right)\psi(x)dx \\ (4.23) \qquad & \quad -\frac{1}{2}\int_{\partial D\cap\hat{D}} \left(F_2(x,u_2(x)) - F_2(x,u_1(x))\right)\psi(x)\sigma(dx). \end{aligned}$$

Let $\left(X, \{P_x, x \in \overline{D}\}\right)$ be (normally) reflecting Brownian motion on $\overline{D}$ and let $\tau = \inf\{t \geq 0 : X_t \notin \hat{D}\}$. Clearly, $\hat{u}(X_\tau) = 0$ with the convention that $X_\infty = \delta$ and $\hat{u}(\delta) = 0$. It follows from (4.20)–(4.21), (4.23) and Theorem A2 of [19] that $\{\hat{u}(X_{t\wedge\tau}), t \geq 0\}$ is a bounded P_x-submartingale for m-a.e. $x \in \hat{D}$. Thus for m-a.e. $x \in \hat{D}$,

$$\hat{u}(x) \leq \lim_{t\to\infty} E^x\left[\hat{u}(X_{t\wedge\tau})\right] = E^x\left[\hat{u}(X_\tau)\right] = 0.$$

Thus by continuity, $\hat{u} \leq 0$ on $\hat{D} = \{x \in \overline{D} : \hat{u} > 0\}$, which is a contradiction. □

5 Extensions

It can be seen from the proof of Theorem 1.2 that if g and ϕ are allowed to take negative values, by prescribing $\alpha > 0$ to be sufficiently large, we may still obtain positive solutions for the Eqs. (1.1)–(1.3). The functions g and ϕ can even be be replaced by measures (cf. [20], [25]).

In this paper, we assumed that D is an unbounded Lipschitz domain with compact boundary. This assumption ensures that the Green function for $\frac{1}{2}\Delta$ on D with zero Neumann boundary condition exists and is controlled by the Newtonian potential $|x-y|^{2-d}$. Our method would work for D, a bounded Lipschitz domain, whenever (1.1) can be written as

$$\left(\frac{1}{2}\Delta + H\right) u + F_1(\cdot, u) + g = 0 \quad \text{in } D$$

and the spectrum of $\frac{1}{2}\Delta + H$ with zero Neumann boundary condition lies strictly in the negative half-line, where H is a function in K_d^{loc}. In this case, the Green function of $\frac{1}{2}\Delta + H$ exists on D and is bounded above and below by constant multiples of the Newtonian potential $|x-y|^{2-d}$ (cf. [22]).

Suppose that U is an unbounded Lipschitz domain with compact boundary and A is a closed subset of $\overline{U}$ such that $D = U \setminus A$ is a regular domain. Set $I_1 = \partial D \setminus A$ and $I_2 = \partial D \cap A$. Note that if $x \in I_1$, then $x \in \partial U$. The method used in proving Theorem 1.2 can be easily modified to find positive solutions for the following semilinear elliptic equation with mixed boundary conditions:

$$\begin{aligned}
&(5.1) & \frac{1}{2}\Delta u + F_1(\cdot, u) + g &= 0 && \text{in } D,\\
&(5.2) & \frac{\partial u}{\partial n} + F_2(\cdot, u) + \phi &= 0 && \text{on } I_1,\\
&(5.3) & u &= \psi && \text{on } I_2,\\
&(5.4) & \lim_{\substack{|x|\to\infty \\ x\in D}} u(x) &= \alpha,
\end{aligned}$$

where $g, \phi, \alpha, F_1, F_2$ are the same as before, ψ is a non-negative continuous function on I_2. The only difference in solving the above mixed boundary value problem is that we run a Brownian motion in D that is reflected on I_1 and is absorbed on I_2. For this, let $(X, \{P_x, x \in \overline{U}\})$ be the (normally) reflecting Brownian motion on $\overline{U}$ with local time process L, and denote by G the Green function of X. Let $\tau = \inf\{t > 0 : X_t \in I_2\}$, the first hitting time of I_2, and let $X_t^\tau = X_t$ for $t < \tau$ and $X_t^\tau = \delta$ for $t \geq \tau$, where δ is a cemetery point. Denote by G^* the Green function for the process $(X^\tau, \{P_x, x \in \overline{D} \setminus I_2\})$. Clearly $G^* \leq G$. Suppose q and g are Green-tight functions on D, κ and ϕ are in the class Γ for I_1, ψ is a bounded continuous function on I_2, and α is a constant. Then, whenever $\|G^* q^+\|_\infty + \frac{1}{2}\|\tilde{G}^* \kappa^+\|_\infty < 1$, the solution of the counterpart linear elliptic equation with boundary conditions:

$$\begin{aligned}
&(5.5) & \frac{1}{2}\Delta u + qu + g &= 0 && \text{in } D,\\
&(5.6) & \frac{\partial u}{\partial n} + \kappa u + \phi &= 0 && \text{on } I_1,\\
&(5.7) & u &= \psi && \text{on } I_2,\\
&(5.8) & \lim_{\substack{|x|\to\infty \\ x\in D}} u(x) &= \alpha,
\end{aligned}$$

exists and is given by:

$$(5.9) \qquad u(x) = E^x\left[\int_0^\tau e(t) g(X_t)dt\right] + E^x\left[\int_0^\tau e(t)\phi(X_t)dL_t\right] + E^x\left[\psi(X_\tau)\right] + \alpha E^x\left[e(\tau)\right]$$

for $x \in \overline{D}$, where

$$e(t) = \exp\left(\int_0^{t\wedge\tau} q(X_s)ds + \int_0^{t\wedge\tau} \kappa(X_s)\, dL_s\right), \quad t \geq 0.$$

Using the right hand side of (5.9) to define the mapping T instead of using (4.8), the proof of Theorem 1.2 carries over to give a positive solution of Eqs. (5.1)–(5.4). We leave the details to the interested reader. We only remark here that in solving semilinear elliptic mixed boundary value problems, D need not necessarily be unbounded with compact boundary. In fact, D can be a bounded regular domain provided that it can be expressed as $U \setminus A$ where U is a bounded Lipschitz domain and A is a closed subset of $\overline{U}$ such that $I_2 = \partial D \cap A$ has positive capacity. In this case, the Green function of the Brownian motion in D that is reflected on I_1 and killed upon hitting I_2 is bounded above by a multiple of $|x-y|^{2-d}$. The latter estimate can be derived by proving a Sobolev inequality in a similar manner to that in [5] (see especially Lemma 5 and Theorem 1 there).

The results of this paper can be extended to second order uniformly elliptic operators of divergence form with conormal derivative boundary conditions in place of the Laplacian with normal derivative boundary condition, by using the corresponding reflecting diffusion process instead of reflecting Brownian motion.

In this paper we only allowed nonlinearity in the zero order term in (1.1). By using a change of measure transformation (Girsanov theorem), it is possible to study nonlinear elliptic equations where nonlinearity also occurs in the coefficients of first order derivative terms. We shall address this problem in a separate article.

When α is zero, it is possible to relax the condition on the lower bound for F_1 in (1.4). The condition that U_1 in (1.4) is Green-tight on D can be replaced by the condition that $1_D U_1 \in K_d^{loc}$. The proof of the existence result is then a bit more involved. For this we refer the reader to Theorem 5.1 in [6].

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