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On Financial Guarantee Insurance under Stochastic Interest Rates

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Abstract

We extend the financial guarantee insurance literature by modeling, under stochastic interest rates, private financial guarantees when the guarantor potentially defaults. By performing numerical simulations under plausible parameters values, we characterize the differential impact of the incorporation of stochasticity of interest rates on the valuation of both public and private guarantees.

Key words: Financial guarantees, Default risk, Interest-rate risk, Credit enhancement, Private and public guarantees, Deposit insurance, Contingent claims analysis

Financial guarantee insurance is a commitment by a third party to make payment in the event of a default in a financial contract. Typically, a parent company, a bank, or an insurance company and often different levels of government stand as the third party. Financial guarantees have become increasingly widespread with the development of securitization of various types of loans and the growth of off-balance-sheet guarantees by commercial banks and insurance companies. For example, default insurance on corporate and municipal bonds, credit enhancement features found in mortgage-backed securities, letters of credit, interest rate and currency swaps, and auto loan receivables constitute financial guarantee insurance (for more details see Insurance Information Institute, [1986]).

One traditional reason for the popularity of financial guarantees is that they constitute off-balance-sheet items. For instance, while loan guarantees by the government represent taxpayers' contingent liabilities, they are still not included in the budget (Baldwin, Lessard, and Mason [1983]; Selby, Franks, and Karki [1988]). Likewise, bank stand-by letters of credit are recorded off balance sheet. Nevertheless, bank regulators recently started to monitor off-balance-sheet liabilities and require banks to maintain sufficient capital to support them. The burgeoning demand of municipal demand of municipal bond insurance (see Quigley and Rubinfeld [1991]) and other financial guarantee insurance (surety bonds, commercial paper insurance, and so on) from insurance companies have also forced regulators to devise safeguards to ensure that losses resulting from financial guarantees do not affect the insurer's other insurance businesses.

According to Merton and Bodie [1992], implicit or explicit guarantees are also ubiquitous in the world of corporate finance—for example, parent company guarantees of the debt or other contractual obligations of a subsidiary, involvement in swap and other derivative-security contracts, pension obligations under defined-benefit pension plans.

In a financial engineering perspective, Merton [1990] and Merton and Bodie [1992] show that the purchase of any loan is equivalent in both a functional sense and a valuation sense to the purchase of a pure default-free loan and the simultaneous issue of a default-free guarantee of that loan. They conclude that the analysis of guarantees has relevance to the evaluation of virtually all financial contracts, whether or not guarantees are explicit. Clearly, the valuation of loan guarantees is of interest to *all* economic agents involved in financial contracting and not only to the largest provider of financial guarantees, the U.S. government.

Since the seminal works of Black and Scholes [1973] and Merton [1973], contingent claims analysis (CCA) has been used to value insurance contracts. For instance, Doherty and Garven [1986] and Cummins [1988] use CCA to respectively price property-liability insurance and derive risk-based premiums for insurance guarantee funds. In a lecture to the Geneva Association, Merton [1989] demonstrates how CCA contributes to the enrichment of the theory of financial intermediation and insurance. Merton's [1992] book on continuous-time finance provides a comprehensive discussion of most CCA applications.

In line with the above works, we apply continuous-time CCA to the valuation of financial guarantee insurance. Specifically, we extend the existing literature by allowing potential default of the guarantor and incorporating a stochastic interest-rate regime. While assuming a nonstochastic interest rate, financial economists have focused their studies on the valuation of loan guarantees by the federal government or its affiliated agencies, which may be considered riskless or default-free guarantors (see Merton [1977], Jones and Mason [1980], Sosin [1980], Chen, Chen, and Sears [1986], Selby, Franks, and Karki [1988]). Recently, after reviewing option pricing and the valuation of loan guarantees, Lai [1992] uses a discrete-time framework to analyze guarantees by a risky guarantor, but still in a nonstochastic interest-rate environment.

Loan guarantees are subject not only to credit risk but also, as financial claims, to interest-rate risk, which, to our knowledge, has not been taken into account in existing models. The ensuing question is whether the explicit incorporation of stochastic interest rates gives rise to economically meaningful effects on the valuation of loan guarantees. The answer to this question is by no means obvious. In the related risk-adjusted deposit insurance literature, Ronn and Verma [1986] show that the incorporation of stochastic processes for the riskless rate of interest does not materially affect the valuation of such insurance. On the other hand, McCulloch [1985] and Pennacchi [1987] find that the volatility of interest rates does affect the value of deposit insurance. Following the work of Merton [1973], Jones and Mason [1980] conjecture that since stochastic interest rates could be treated as an increase in total risk, guarantee values computed under nonstochastic interest rates are low estimates of the "exact" values.

To investigate the effect of the stochasticity of interest rate in the valuation of loan guarantees, we develop a general model that explicitly accounts for both credit risk and interest-rate risk using Merton's [1973] interest-rate process. Our numerical simulations under plausible parameters values demonstrate that (1) the incorporation of a stochastic interest-rate regime does affect significantly the value of loan guarantees and (2) the elasticity of the value of guarantees with respect to the volatility of interest rate is larger for public guarantees than for private guarantees. We are able to verify Jones and Mason's [1980] conjecture about the underestimation of loan guarantees when they are computed with deterministic interest rates. The rest of the paper is organized as follows.

In Section 1, we derive a model of vulnerable loan guarantees under Merton's [1973] interest-rate process. We position our model in relation to the loan guarantees literature in Section 2; in particular, we show that existing models with deterministic interest rates are special cases of our extended model. We present and discuss our simulation results in Section 3. Section 4 concludes the paper.

1. A simplified model of vulnerable loan guarantees

1.1. Assumptions

Under the standard framework of continuous-time finance, we assume there is no violation of the absolute priority rule and ignore all potential agency problems inherent to financial contracting (for a discussion of agency problems, see Campbell [1988], Smith [1980]).[1] We assume that the capital structure of the guaranteed firm consists solely of equity and the single issue of the debt being valued. More complex bond characteristics such as call and sinking funds features are not considered.

1.1.1. Bond price dynamics. As in Merton [1973], Schwartz [1982], Carr [1987], and Chance [1990], among others, let $Q[\tau)$ be the price of a default-free unit discount bond with the same time to maturity, τ, as the debt to be valued. Assume that $Q(\tau)$ satisfies

$$dQ/Q = \alpha_Q(\tau)dt + \sigma_Q(\tau)dz_Q(t;\ \tau), \tag{1}$$

where α_Q is the instantaneous expected return on the bond, σ_Q is the instantaneous standard deviation, deterministic function of time, t, and $dz_Q(t;\ \tau)$ is a Gauss-Wiener process for maturity τ. We also denote r, as the instantaneous riskless rate of interest.[2]

This interest-rate process leads us to a tractable and pedagogical approach to financial guarantees problems involving three stochastic state variables. It allows us not only to reduce the problem by one dimension but also to produce integral forms (say, quasi-closed form integrals) and propositions that are easily interpreted in relation to the existing and familiar literature such as the classical Merton's [1973, 1974, 1977] approach to corporate debt and deposit insurance pricing.

We recognize that other more appropriate stochastic interest-rate dynamics could have been used with Monte Carlo simulation technique from the outset. This would have been, however, at the cost of a less elegant and much more computing time intensive approach than our three state variables CCA while not changing materially the results discussed later in Section 3.[3]

1.1.2. Dynamics of the guarantor and guarantee's firm value. Let W be the value of the guarantor firm assets and V be the value of the assets of the firm issuing the debt to be guaranteed. The continous paths these asset values follow are described by the stochastic differential equations

$$dW/W = \alpha_W dt + \sigma_W dz_W \tag{2}$$

and

$$dV/V = \alpha_V dt + \sigma_V dz_V, \tag{3}$$

where α_W and α_V are the instantaneous returns on the assets, and σ_W and σ_V are the deterministic instantaneous standard deviations of respectively the guarantor and the insured firm asset returns. The Gauss-Wiener processes dz_Q, dz_W, and dz_V are such that their correlation, ρ, are given by $dz_W \cdot dz_Q = \rho_{WQ}\, dt$; $dz_V \cdot dz_Q = \rho_{VQ}\, dt$; $dz_V \cdot dz_W = \rho_{VW}\, dt$, and $\sigma_{WQ} = \rho_{WQ}\sigma_W\sigma_Q$; $\sigma_{VQ} = \rho_{VQ}\sigma_V\sigma_Q$; $\delta_{VW} = \rho_{VW}\sigma_V\sigma_W$.

1.1.3. No dividends or coupons. For simplicity we assume that there are no payouts from either the firm or its guarantor to shareholders and debtholders before the loan maturity, although the model can be easily extended to account for instantaneous payouts proportional to assets.

To calculate the value of the guarantee, G, we first compute the value of the guaranteed debt, Bg, from which we subtract the value of the debt without guarantee, B.

1.2. Value of the guaranteed debt

Consider a pure discount (zero coupon) debt, Bg, with promised principal F. Under the assumptions A.1 to A.3 and perfect capital markets, the value of a guaranteed debt, Bg, can be obtained using a generalized "risk-neutralized" approach (see Cox, Ingersoll, and Ross [1985a] and Ingersoll [1987] for more detailed descriptions of the equivalent martingale methodology), where the value $\Omega(x_1, x_2, \ldots x_n)$ of an n-asset contingent claim that pays off $\Omega_T(\cdot)$ at time T is given by

$$\Omega(\cdot) = \hat{E}\left[\exp\left(-\int_t^T r(s)ds \mid r(t)\right)\Omega_T(\cdot)\right] = \int\cdot\int\cdot\int \Omega_T(\cdot)\, L(\cdot)dx_1dx_2 \ldots dx_n,$$

where $\hat{E}$ is the expectation operator in a risk neutralized world, $r(s)$ the instantaneous risk-free rate at time s, and $L(\cdot)$ the n-joint probability density function. Under our assumption Ω_T is a function of V, W, and Q and $L(\cdot) = L(V, W, Q)$ joint lognormal.

As shown in Table 1, the value of the insured debt Bg can be decomposed in three parts according to the states of nature at maturity. The three rows of Table 1 correspond respectively to the situations when the firm is solvent, the firm is insolvent but the guarantor is not, and both the debt issuing firm and the guarantor are bankrupt.

Table 1. Value of debts and guarantee at maturity.

Case	Debt with Guarantee (B_g)	Debt with Guarantee (B)	Guarantee (G)
$0 < W < \infty$; $V \geq F$	F	F	0
$0 < W < F$; $V + W \geq F$	F	V	$F-V$
$0 < V < F$; $V + W < F$	$V + W$	V	W

1.2.1. The firm is solvent. If the firm is solvent ($V \geq F$), the bondholder gets F, regardless of the wealth of the guarantor:

$$Bg_i = \int_0^\infty \int_0^\infty \int_0^\infty FL(\cdot)dVdWdQ.$$

1.2.2. The firm is bankrupt and the guarantor has sufficient funds to honor its commitment. When the firm is bankrupt ($0 \leq V \leq F$) and the value of the guarantor exceeds the shortfall, $W \geq F - V$, the bondholder gets F:

$$Bg_{ii} = \int_0^\infty \int_{F-V}^\infty \int_0^F FL(\cdot)dVdWdQ.$$

For integration purposes, it is convenient to rewrite the value of the insured debt in these two situations when the bondholder gets F, as the difference of two integrals:

$$Bg_i + Bg_{ii} = Bg_1 - Bg_2,$$

where

$$Bg_1 = \int_0^\infty \int_0^\infty \int_0^\infty FL(\cdot)dVdWdQ = FQ, \tag{4a}$$

$$Bg_2 = \int_0^\infty \int_0^{F-V} \int_0^F FL(\cdot)dVdWdQ. \tag{4b}$$

The first integral Bg_1 represents the loan face value regardless of the states of the firm and its guarantor. The second integral Bg_2 is the correction made when both firms go bankrupt.

1.2.3. The firm is bankrupt but the guarantor does not have sufficient funds to honor its obligation. In this case the value of the guarantor is lower than the shortfall ($0 < W < F - V$) and the bondholder receives the salvage values of the two companies V and W, instead of F:

$$Bg_3 = \int_0^\infty \int_0^{F-V} \int_0^F VL(\cdot)dVdWdQ. \tag{4c}$$

$$Bg_4 = \int_0^\infty \int_0^{F-V} \int_0^F WL(\cdot)dVdWdQ. \tag{4d}$$

The value of the guaranteed loan is then

$$Bg = Bg_1 - Bg_2 + Bg_3 + Bg_4. \tag{5}$$

This leads to the main proposition of this paper where three of the above four integrals are written in analytical forms amenable to numerical solutions.

Proposition 1: *The value of a loan with private guarantee under Merton's [1973] stochastic unit bond price dynamics is given by*

$$Bg = Bg_1 - Bg_2 + Bg_3 + Bg_4, \tag{5}$$

where

$$Bg_1 = FQ, \tag{5.1}$$

$$Bg_2 = FQ \int_{-\infty}^{\ln(FQ/V)} \int_{-\infty}^{\ln[(FQ/W)-(V/W)\exp(x)]} n\left[\begin{matrix} -.5\sigma_{V/Q}^2 \\ -.5\sigma_{W/Q}^2 \end{matrix}; \Sigma\right] dydx, \tag{5.2}$$

with

$$\sigma_{V/Q}^2 = \int_0^T (\sigma_V^2 + \sigma_Q^2 - 2\sigma_{VQ})ds,$$

$$\sigma_{W/Q}^2 = \int_0^T (\sigma_W^2 + \sigma_Q^2 - 2\sigma_{WQ})ds,$$

$$\Sigma = \begin{bmatrix} \sigma_{V/Q}^2 & \sigma_{V/Q\cdot W/Q} \\ \sigma_{V/Q\cdot W/Q} & \sigma_{W/Q}^2 \end{bmatrix},$$

$$\sigma_{V/Q\cdot W/Q} = \int_0^T (\sigma_{VW} - \sigma_{VQ} - \sigma_{WQ} + \sigma_Q^2)ds, \text{ and}$$

$n(\mu; \Sigma)$ = Bivariate normal density function with vector mean μ and covariance matrix Σ.

$$Bg_3 = V \int_{-\infty}^{\ln(FQ/V)} \int_{-\infty}^{\ln[(FQ/W)-(V/W)\exp(x)]} n\left[\begin{matrix} +.5\sigma_{V/Q}^2 \\ -.5\sigma_{WQV}^2 \end{matrix}; \Sigma\right] dydx, \tag{5.3}$$

where

$$\sigma_{WQV}^2 = \int_0^T (\sigma_W^2 - \sigma_Q^2 - 2\sigma_{VW} + 2\sigma_{VQ})ds.$$

and

$$Bg_4 = W \int_{-\infty}^{\ln(FQ/V)} \int_{-\infty}^{\ln[(FQ/W)-(V/W)\exp(x)]} n\left[\begin{matrix} +.5\sigma_{W/Q}^2 \\ -.5\sigma_{VQW}^2 \end{matrix}; \Sigma\right] dydx, \tag{5.4}$$

where

$$\sigma_{VQW}^2 = \int_0^T (\sigma_V^2 - \sigma_Q^2 - 2\sigma_{VW} + 2\sigma_{WQ})ds.$$

For ease of notation, parameters involving Q have not beren indexed for time. The variables x and y are dummy variables resulting from a change of variables, V and W normalized by Q in logarithm form to reduce the dimension of the problem by one.

Proof: Available upon request. ■

Bg_1, the value of government guaranteed debt, represents the present value of the face amount F discounted at a stochastic (Gaussian) interest rate. Note that each of the other three integrals has the same integration limits and involves cumulative bivariate normal distributions with the same covariance matrix but different means or location parameters corresponding to risk adjustment in a risk-neutralized world. Closed form solutions for the integrals cannot be obtained because the dependent dummy variable x appears in the upper limit of integration. Specifically, the inner upper integration limit is variable in iterated integrals, unlike the Black and Scholes [1973] type of integrals which has fixed limits of integration. We use Drezner's [1978] and Simpson algorithms to compute Bg_2, Bg_3 and Bg_4. This is easier and faster than trying to solve partial differential equations (PDE).[4]

1.3. *Value of the debt without guarantee*

Following Merton [1973], the value of a loan without a guarantee, $B(V, Q, \tau)$, is

$$B = V[1 - (erfc(-h_1))/2] + [FQ\ erfc(-h_2)]/2, \tag{6}$$

where

$$h_1 = [\ln(V/FQ) + .5I]/\sqrt{2I},$$

$$h_2 = [\ln(V/FQ) - .5I]/\sqrt{2I},$$

$$I = \int_0^{\tau} \sigma_{V/Q}^2 ds = \int_0^{\tau} (\sigma_V^2 + \sigma_Q^2 - 2\sigma_{VQ})ds = \int_0^{\tau} (\sigma_V^2 + \sigma_Q^2 - 2\rho_{VQ}\sigma_V\sigma_Q)ds,$$

$$erfc(z) = 1 - erf(z) \text{ with } erf(z) = (2/\sqrt{\pi}) \int_0^{z} \exp(-u^2)du,$$

and *erfc* and *erf* stand for error function complementary and error function, respectively.

Note that equation (6) follows directly from Proposition 1 by setting W and σ_W equal to zero—that is, the value of the debt without guarantee is the value of the guaranteed debt when there is no guarantor. To check our numerical algorithm, we use both Proposition 1 (equation (5)) and equation (6) above to derive the value of B.

1.4. *Value of the guarantee*

The value of the guarantee, G, is the difference between Bg and B as given by equations (5) and (6). The value of the guarantee is the risk-adjusted present value of the terminal payoff discounted at Merton's [1973] unit bond prices.

The next section situates our model with stochastic interest rates in the loan guarantees literature.

2. Contribution to the loan guarantees literature

The existing financial guarantees literature that uses CCA can be categorized by the nature of the guarantor (risky/private/vulnerable versus default-free/public/government) and by the interest-rate regime (stochastic versus deterministic). Table 2 presents such a classification of this literature, as categorized above, labelled by four propositions. Proofs of the imbedded Propositions 2, 3, and 4 follow from successive applications of Proposition 1, under appropriate conditions, to obtain the value of the debt with guarantee Bg and the debt without guarantee B. The value of the guarantee, G, is simply the difference between these two values.

2.1. Government guarantee and constant interest rate

Following the seminal work of Merton [1977], most of the loan guarantees literature falls into this category. As a matter of fact, there now exists a vast literature related to the valuation of deposit insurance. The government is commonly viewed as a riskless institution ($\sigma_W = 0$) with unlimited assets ($W = \infty$). This condition combined with a constant interest rate ($\sigma_Q = 0$) leads to the following known proposition first derived by Merton [1977].

Proposition 2: *The value of a government guarantee under constant interest rate regime, $G_{f,r}$, is given by $G_{f,r} = Fe^{-rT}(1 - N(h_2^*)) - V(1 - N(h_1^*))$, where $N(\cdot)$ is the cumulative normal function, and*

$$h_1^* = (\ln(V/F) + rT)/\sigma_V\sqrt{T} + 1/2\sigma_V\sqrt{T},$$

$$h_2^* = h_1^* - \sigma_V\sqrt{T}. \qquad \blacksquare$$

Table 2. Loan guarantees literature.

	Government Guarantee	Private Guarantee
Deterministic interest rate	*Proposition 2* Merton [1977], Sosin [1980], Jones and Mason [1980], Chen et al. [1986], Selby et al. [1988] Deposit insurance: Merton [1977] and others	*Proposition 4* Johnson and Stulz [1987], Lai [1992]
Stochastic interest rate	*Proposition 3* Deposit insurance: McCulloch [1985], Ronn and Verma [1986], Pennacchi [1987]	*Proposition 1* This paper

The first term is the present value in a deterministic interest-rate environment of the face value times the probability that the debt issuing firm is insolvent. The second term is interpreted as the expected value of the borrowing firm conditional upon it being bankrupt.

2.2. *Government guarantee and stochastic interest rate*

The introduction of a stochastic interest rate ($\sigma_Q = 0$) modifies the formulas as follows.

Proposition 3: *The value of a government guaranteed debt under Merton's [1973] unit bond price process is given by the following expression where h_1 and h_2 are as defined in Section 1.3, equation (6)*

$$G_{f,s} = FQ[1 - erfc(-h_2)/2] - V[1 - erfc(-h_1)/2]. \qquad \blacksquare$$

The terms in the expression for $G_{f,s}$ are the "stochastic" counterparts of the two terms in Proposition 2.

2.3. *Private guarantee and constant interest rate*

When the guarantor is risky, the state variable W comes into play.[5]

Proposition 4: *The value of a private guarantee under constant interest rate, $G_{p,r}$, is given by*

$$G_{p,r} = G_{f,r} + (-Bg_2 + Bg_3 + Bg_4),$$

where

$$Bg_2 = Fe^{-rT} \int_{-\infty}^{\ln(Fe^{-rT}/V)} \int_{-\infty}^{\ln[(Fe^{-rT}/W)-(V/W)e^x)]} n\left[\begin{matrix} -1/2\sigma_V^2 T \\ -1/2\sigma_W^2 T \end{matrix}; \Sigma\right] dydx,$$

$$Bg_3 = V \int_{-\infty}^{\ln(Fe^{-rT}/V)} \int_{-\infty}^{\ln[(Fe^{-rT}/W)-(V/W)e^x)]} n\left[\begin{matrix} +(1/2)\sigma_V^2 T \\ -(1/2)(\sigma_W^2 - 2\sigma_{VW})T \end{matrix}; \Sigma\right] dydx,$$

$$Bg_4 = W \int_{-\infty}^{\ln(Fe^{-rT}/V)} \int_{-\infty}^{\ln[(Fe^{-rT}/W)-(V/W)e^x)]} n\left[\begin{matrix} -(1/2)(\sigma_V^2 - 2\sigma_{VW})T \\ +(1/2)\sigma_W^2 T \end{matrix}; \Sigma\right] dydx,$$

and

$$\Sigma = \begin{bmatrix} \sigma_V^2 & \sigma_{VW} \\ \sigma_{VW} & \sigma_W^2 \end{bmatrix} T. \qquad \blacksquare$$

Note that the difference between the value of the public and private guarantees, Propositions 2 and 4, respectively, represents the credit risk premium attached to an insurer with a full faith and credit rating. It is given by the expression $+Bg_2 - Bg_3 - Bg_4$.

Corollary: *The value of a government guarantee is equal to or greater than a private guarantee.*

Proof: Since the expected proceeds to the bondholder when both the debt insuring firm and its guarantor go bankrupt ($Bg_3 + Bg_4$) is less than the expected value of the face value of the discount debt (Bg_2) in the state of joint bankruptcy of the two firms $F \geq (V + W)$, we have $Bg_2 \geq Bg_3 + Bg_4$. Therefore, from Proposition 4, $G_{f,r} \geq G_{p,r}$. ■

We turn next to simulations to verify whether the incorporation of the stochasticity of interest rates affects significantly the valuation of loan guarantees by both private and public guarantors.

3. Simulation results

We first derive numerical comparative statics of the value of the guarantee within our framework. We then check the robustness of our results by using Monte Carlo simulations under the Cox, Ingersoll, and Ross [1985b] (CIR) interest-rate specification.

3.1. Results

We conduct numerical simulations to derive the comparative statics for the value of the loan with private guarantee (Bg), the one with riskless guarantee (Bgf), the value of the uninsured loan (B), and the two corresponding guarantees (G) and (Gf).[6] To simulate the cases of default-free government guarantees, we set both W and σ_W arbitrarily large and small, respectively. Recall the presence of a stochastic interest-rate regime that departs from the extant literature on government loan guarantees (for example, Merton [1977], Sosin [1980], Jones and Mason [1980]) under deterministic interest rates.

We use Drezner's [1978] algorithm for the computation of the bivariate normal integral for the portion where the integration limits (bounds) are constant.[7] For the variable bound part, we use a composite Simpson procedure (see Burden and Faires [1989]).

For Merton's [1973] interest-rate dynamics, following Chan, Karolyi, Longstaff, and Sanders [1992], we use $\alpha_r = 0.0055$, $\sigma_r = 0.02$ and $r = 0.067$ per annum.[8] The present value of the borrowing firm (V) is set at \$1,100 with a volatility (σ_V) at 0.3. The private guarantor value (W) is \$1,500 with a risk (σ_V) of 0.3. Their cross correlations are put equal to 0.3. The principal of the loan (F) is \$1,000 with a basic maturity (T) of three years.

We focus our discussion of the comparative statics results on the effect of the stochasticity of interest rates on the value of financial guarantee insurance. However, note that the comparative statics related to other variables, under our continuous-time and stochastic interest-rate framework, are qualitatively the same as those of Lai [1992], who employs a discrete-time methodology and a constant interest rate. For instance, the more the guarantor resembles the insured firm ($\rho_{VW} > 0$), the less valuable the private guarantee is in the credit enhancement. One notable difference is that, for the range of economically plausible firm's asset risk, the borrowing firm risk σ_V has an increasing effect on the value of both the private and government guarantee.[9]

Figure 1 first shows the impact of the loan maturity T on the debt values and guarantees. Both the insured and uninsured debts, Bg and B, decrease with the maturity T. However, their difference, which is the guarantee, increases and decreases with longer maturity.[10]

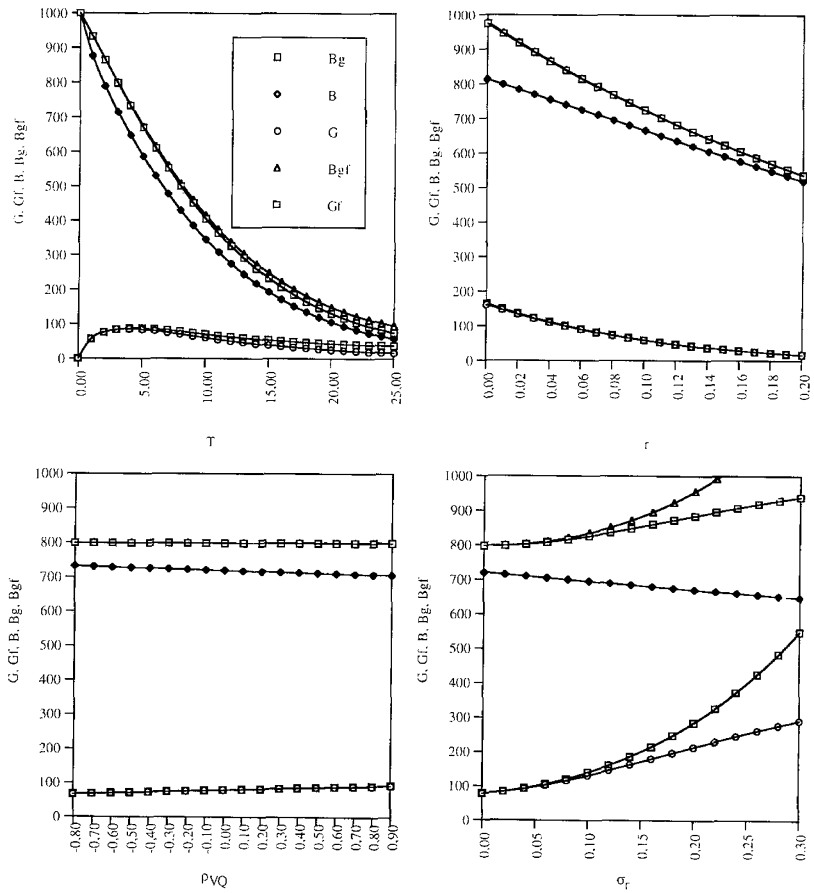

Figure 1. Impact of the time to maturity (T), the short-term interest rate (r), the coefficient of correlation (ρ_{VQ}) between the firm (V) and the value of a default-free unit discount bond (Q), the volatility of interest rates (σ_r) on the value of the loan with guarantee (Bg), the one without guarantee (B), and the guarantee (G). The subscript (f) denotes the cases corresponding to the default-free guarantee.

Note: We conduct the experiments using numerical integration under Merton's [1973] interest-rate process with $V = 1100$, $\sigma_V = 0.3$, $W = 1500$, $\sigma_W = 0.3$, $r = 0.067$, $\alpha_r = 0.0055$, $\sigma_r = 0.02$, $\rho_{WV} = 0.3$, $\rho_{VQ} = 0.3$, $\rho_{WQ} = 0.3$, $T = 3$, and $F = 1000$, while varying the studied parameters in the comparative statics ceteris paribus.

The other two plots on the right depict the effect of the interest-rate level and its volatility on the loan guarantee variables. An increase in the short-term rate r, for a given volatility $\sigma_r = .02$, reduces the present value of the price of the default-free unit discount bond Q and hence reduces the value of both the private and public guarantee. For Merton's [1973] interest-rate dynamics, our simulation results indicate that both the value of the public and private guarantees increase monotonically with the volatility of interest rate. For the loan without guarantee, an increase in σ_r augments the rotal risk ($\sigma^2_{V/Q}$), which increases the value of the equity (the call on the borrowing firm's assets), hence reduces the value of the debt. On the other hand, a larger total risk increases the put option value on the sum of the value of the firm and its guarantor, this augmentation is less than in the present value of the debt face value. The difference between Bg and B, (G), increases with σ_r.

For our baseline parameters, the underestimation of the value of the loan guarantee associated with the use of a constant interest rate instead of a stochastic one with a volatility level of $\sigma_r = 0.02$ (the average volatility documented by Chan, Karolyi, Longstaff, and Sanders [1992]), is roughly 20 percent. At level of σ_r of 12 percent the government guarantee Gf is twice the guarantee computed with a deterministic rate ($\sigma_r = 0$). For the private guarantee an error of 100 percent occurs with an interest-rate volatility of 14 percent. In economic parlance, for high interest-rates volatilities, the elasticity of the government guarantee to changes in interest rate volatilities is greater than the one of the private guarantee. Our result concurs qualitatively with empirical findings of McCulloch [1985] and Pennacchi [1987] but not with those of Ronn and Verma [1986], who find under their models, that the incorporation of stochastic interest rates does not affect significantly the valuation of deposit insurance.[11]

As an illustration of the effect of the covariance between asset values and the interest rate, Figure 1 also shows the effect of the coefficients of correlation between the firm and the unit interest price Q on the loan guarantee. Given that Q varies inversely with the interest rate, our results show that the more the interest rate and firm value vary in the same direction (positive correlation), the higher the guarantee.

3.2. Results under the CIR interest-rate process

To check the robustness of our results, we repeated the above exercise with the well-known single-factor CIR interest-rate process. As indicated earlier, this process does not yield a simple quasi-closed form solution amenable to direct numerical integration, therefore requiring Monte Carlo simulations (see, for example, Boyle [1977], Hull [1993], Abken [1993]). We also ran Monte Carlo simulations with Merton's interest-rate process to isolate the effect of a change in interest-rate specification from the effect of a change in the numerical procedure.[12]

Following again Chan, Karolyi, Longstaff, and Sanders [1992], we calibrated the CIR process ($dr = (\alpha_r + \beta r)dt + \sigma_r\sqrt{r}\, ds$) with $\alpha_r = 0.0189$, $\beta = -0.2339$, and $\sigma_r = 0.08544$, assuming the price of risk $\lambda = 0$ (the local expectation hypothesis).[13]

Figure 2 shows a comparison of the values of the vulnerable guarantee computed under both CIR and Merton's interest-rate specification by Monte Carlo simulations (MC) and Merton by numerical integration (NI). Three points can be highlighted. First, comparative

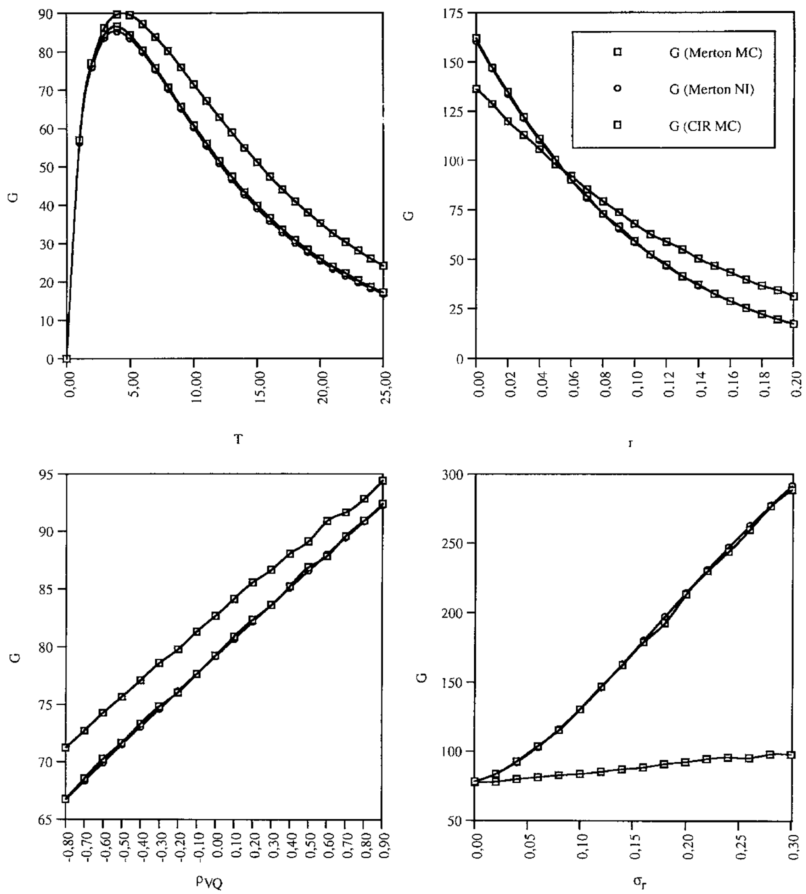

Figure 2. Impact of the time to maturity (T), the short-term interest rate (r), the coefficient of correlation (ρ_{VQ}) between the firm (V) and the value of a default-free unit discount bond (Q), the volatility of interest rates (σ_r) on the value of the loan with guarantee (G), under Merton's [1973] and Cox, Ingersoll, and Ross [1985a, 1985b] (CIR) interest-rate dynamics.

Note: We conduct the experiments using both Monte Carlo simulations (MC) and numerical integration (NI) with $V = 1100$, $\sigma_V = 0.3$, $W = 1500$, $\sigma_W = 0.3$, $r = 0.067$, $\lambda = 0$, Merton ($\alpha_r = 0.0055$, $\sigma_r = 0.02$), CIR ($\alpha_r = 0.0189$, $\beta = -0.2339$, $\sigma_r = 0.08544$), $\rho_{WV} = 0.3$, $\rho_{VQ} = 0.3$, $\rho_{WQ} = 0.3$, $T = 3$, and $F = 1000$, while varying the studied parameters in the comparative statics ceteris paribus.

statics are the same, regardless of the numerical procedure or the interest-rate process. Second, there are no material differences between the three cases when the calculations are made with the baseline parameters values. Differences show up with larger deviations from baseline values. Substantial differences in the guarantee estimates occur with long maturities (T) or excessive interest-rate volatilities (σ_r). Third, for sufficient iterations, there are no differences due to numerical procedures, as estimates obtained under Merton by NI are very close to those obtained by MC.

Table 3. Numerical comparative static results.

	V	W	σ_V	σ_W	r	σ_r	ρ_{VQ}	ρ_{WQ}	ρ_{VW}	F	T
$Bgf = Bgf$	0	0	0	0	−	+	0	0	0	+	−
$Bg = Bg$	+	+/0	0/−	−	−	+	0	−	−	+	−
$B = B$	+	0	−	0	−	−	−	0	0	+	−
$Gf = Gf$	−	0	+	0	−	+	+	0	0	+	±
$G = G$	−	+/0	+	−	−	+	+	−	−	+	±

Although Figure 1 and Figure 2 focus on new results related to the effect of the stochasticity of interest rates on loan guarantees, for completeness we summarize in Table 3 all numerical comparative statics results by the signs of the partial derivatives for *Bgf*, *Bg*, *B*, *Gf*, and *G*.

4. Conclusions

Using the continuous-time option-pricing methodology, we develop a "simplified" model of private financial guarantees that takes account of both stochastic interest rates and potential default by the guarantor. The extant literature generally assumes a nonstochastic interest rate and a riskless default-free guarantor as buttressed by the works of Merton [1977], Jones and Mason [1980], Chen, Chen, and Sears [1986], Selby, Franks, and Karki [1988] in the context of government loan guarantees. With the exception of Johnson and Stulz's [1987] work on pricing options with default risk and Lai's [1992] analysis of private loan guarantees, both under nonstochastic interest rates, evaluating loan guarantees by vulnerable guarantors under stochastic interest rates has not been attempted in the literature. To demonstrate the effect of stochastic interest rates on the valuation of loan guarantees, we derive a model under Merton's [1973] lognormal bond price process that is used to compute both the values of insured and uninsured debts. We obtain a model that necessitates numerical integration of the bivariate normal density function to circumvent the need to solve partial differential equations with three state variables or to use Monte Carlo simulations.

Using numerical simulations, we obtain comparative statics for the value of the debts, uninsured and insured, and the value of the guarantees. The effect of incorporating the term structure of interest rates is gauged. We find that consistent with the conjecture of Jones and Mason [1980], guarantee values computed under nonstochastic interest rates are low estimates of the "correct" values. In other words, their conjecture is substantiated, and loan guarantees are increasing monotonically with the interest-rate volatilities. The value of both the private and public guarantees are found to be an ambiguous function of term of the loan. Our other results are consistent with those both in Lai [1992] and in the literature on government loan guarantees.

To provide support for our results with Merton's interest-rate process, we compare them with those obtained from Monte Carlo simulations under the well-known CIR interest-rate process, which precludes negative interest rates. Regardless of the numerical procedure or the interest-rate dynamics employed, we find that comparative statics of the guarantees remain unchanged. Also under plausible baseline parameters values estimates of the guarantees are close.

Among other possibilities, our framework can be used for instance to study the case of a company that directly issues a commercial paper that is guaranteed by a stand-by letter of credit from a private bank. It could also be used to study a private deposit insurance system as advocated by Merton and Bodie [1992], Kane [1986], and Kane, Hickman and Burger [1993], among others, and to value the liabilities of government-state enterprise liabilities such as pension benefit guarantee corporation.

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Notes

1. Following Black and Scholes [1973], we adopt the continuous-time approach, which leads to a valuation relationship independent of investors' preferences. Even if some underlying assets and insurance contracts (such as stand-by letters of credit and surety bonds) are not traded continuously and publicly, it is only needed that capital markets are sufficiently complete to ensure the existence of assets for replication of the underlying assets.
2. Following the work of Merton [1973], Vasicek [1977], and Dothan [1978], a substantial body of literature has emerged for the characterization of the term structure of interest rates in arbitrage or equilibrium settings (see, for instance, Cox, Ingersoll, and Ross [1985b] (CIR), Brennan and Schwartz [1979, 1982], Courtadon [1982], Ball and Torous [1983], Schaefer and Schwartz [1984, 1987], Longstaff and Schwartz [1992], Ho and Lee [1986], Heath, Jarrow, and Morton [1992]). See Chan, Karolyi, Longstaff, and Sanders [1992] for an empirical comparison of a variety of competing continuous-time models of the short-term riskless rate. In spite of its drawback, the lognormality assumption for bond prices, which can imply negative interest rates, enables us to obtain tractable model of vulnerable financial guarantees under stochastic interest rates. In her analysis of the feasibility of arbitrage-based option pricing when stochastic bond prices are involved, Cheng [1991] recognizes that *the simple log-normal bond price process is likely to satisfy the bond price specification*, even though it implies negative interest rates with positive probability.

   As in Merton [1973, n. 43], we assume the short-rate r follows a Gauss-Wiener process according to $dr = \alpha_r dt + \sigma_r dz$. This stochastic process is simply a Brownian motion with drift α_r and volatility σ_r (see Chan, Karolyi, Longstaff, and Sanders [1992] for an empirical specification). Many authors have assumed Merton's unit bond price process to account for stochasticity of interest rates in pricing derivative securities (Schwartz [1982, eq. 31, p. 534] and Carr [1987, eq. (1), p. 1071] have used this assumption to price commodity-linked bonds; Chance [1990, p. 268] explicitly assumed lognormal unit discount bond price to analyze the duration of zero coupon bonds subject to default risk; Grabbe [1983] and Jarrow [1987] used the lognormal unit bond assumption to price options on foreign exchange and commodity options, respectively). Although Vasicek's [1977] model of interest rate could be used, the derivation becomes cluttered, gaining little insight in demonstrating our basic point regarding the effect of the stochasticity of interest rates on values of guarantees. Furthermore, for our baseline parameters values, problems related to pinning the bond price to a fixed value at maturity would occur only with excessive interest-rate volatility. Anyhow, such potential limitation would not change qualitatively the results of our compariative statics as evidenced by the Monte Carlo simulations under the CIR interest-rate specification discussed in Section 3.

3. A numerical integration solution, as obtained in this paper, could not be derived in a known compact form under more realistic interest-rate processes, such as CIR, because of the complex resulting from the combination of joint distributions other than normal. Although complex European options under different stochastic interest-rate regimes may be priced using Monte Carlo simulations, this raw approach does not lead to an elegant synthesis and review of theoretical well-known results, which insights furthermore get buried with computer runs. Notwithstanding the issue of realism of interest-rate dynamics, Monte Carlo simulations are more burdensome and computing time consuming than direct numerical integration methods (see Chen and Scott [1992, p. 620]). Including results from Monte-Carlo simulations would entail comparing results from different numerical approaches as well as different interest-rate processes. Since the spirit of the paper is not of a computational nature, this exercise is undertaken in Section 3 only for the purpose of gauging the robustness of our comparative statics results.
4. In their survey of option pricing applied to mortgages pricing, Hendershott and Van Order [1987] indicate that solving PDE involving three state variables has not been attempted so far, due to the lack of computational technology (tractability and affordability). For an example of resolution of PDE containing two state variables, see, for instance, Titman and Torous [1989] among others. Note also that there exist a number of models for valuing option-like claims in the presence of stochastic interest rates such as those of Rabinovitch [1989], Jamshidian [1989], and Hull and White [1990, 1993]; however, these models are limited to two state variables representing one underlying asset and the interest rate.
5. This expression for the value of the guarantee can be reconciled with the expression obtained by Johnson and Stulz ([1987], eq. 27, p. 279) with an appropriate change of the variables to standardized normal bivariates.
6. As a cross check, we compare the value B obtained from equation (5) (Proposition 1) and the one computed directly from equation (6) employing approximation of the error function (see Abramowitz and Stegun [1972]). The results are identical to the order of 10^{-9}.
7. Hull ([1993], app. 10B) and Stoll and Whaley ([1993], app. 13.1) provide summaries of Drezner's [1978] algorithm. There exist alternative algorithms for approximating the bivariate normal integral (see, for instance, Divgi [1979]).
8. As specified by Merton [1973], the bond price volatility written as a deterministic function of time, is $\sigma_Q(T) = -\sigma_r T$. Hence the volatility of bond price at maturity is zero.
9. This result contrasts with Lai [1992] in a discrete time framework where assets volatilities compound the time dimension (see also Stulz and Johnson [1985] in a somewhat different context of a case of debt with a security provision).
10. This result differs from the one of Sosin [1980], who finds that the guarantee is everywhere an increasing function of the term of the discount loan. Note, however, that Sosin allows for dividend payments.
11. Note that there are numerical differences, albeit imperceptible in our figures, between the government and private guarantees.
12. Details of the mechanics of the Monte Carlo simulations and computer setup using time increments of one month with 100,000 independent sets of realizations of the state variables are available upon request. For an example, see, for instance, Abken [1993]. We found that the computing time required for Monte Carlo simulation is about 200-fold the time used in the numerical integration.
13. We also experimented with $\lambda = -0.06$ as estimated by Chen and Scott [1992]. As expected, in this case the values of the guarantee decrease, but the results do not change qualitatively.

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