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Anticipative Girsanov transformations

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Summary. The transformations of measures induced by $\left(W+\int_0^{\bullet} K_s \,\mathrm{d}s\right)$ with (K_s) possibly anticipating the Wiener process (W_s) is discussed and a Girsanovtype theorem under rather weak assumptions on (K_s) is derived.

1 Introduction

Let $(\Omega, \mathscr{F}, P)$, $\Omega = C_0([0, 1])$, be the standard Wiener space and $W_t(\omega)$ the canonical process. We study transformations $T: \Omega \to \Omega$ of the form

$$T\omega = \omega + \int_0^{\bullet} K_s(\omega)\,\mathrm{d}s,$$

where the process (K_s) may anticipate the process (W_s). In the framework of an abstract Wiener space this problem has been considered by Ramer [9], and under more general conditions, but still with the assumption of the invertibility of T and with ω-wise assumptions on $K_s(\omega)$, by Kusuoka [4], following earlier work of Cameron and Martin, Gross, Shepp and Kuo. Using the generalized anticipating stochastic calculus Nualart and Zakai [8] have studied transformations T which are the limit of a sequence of invertible transformations.

Instead of such an assumption, we impose on (K_s) some Novikov-type condition and suppose that its Fréchet derivative is bounded by a constant which is less than one. Under these conditions we show that T induces a measure which is absolutely continuous w.r.t. P, prove the existence of another transformation $A: \Omega \to \Omega$ with $T(A\omega) = A(T\omega) = \omega$, $P(d\omega)$-a.e., and compute the density of $P \circ [A]^{-1}$ w.r.t. P. This will be done on the basis of the extended stochastic calculus developed by Nualart, Pardoux and Zakai, [6], [7] and [8].

The paper is organized as follows. In Sect. 2, we give a short review on the notions of derivation on the Wiener space and the Skorohod integral, and we present some basic statements for transformations inducing absolutely continuous measures on $(\Omega, \mathscr{F}, P)$. In Sect. 3, we apply Kusuoka's Theorem to our problem in the case of a smooth step process (K_s), and in Sect. 4, we extend the results of Sect. 3 by approximation of the process (K_s) by smooth step processes.

2 Stochastic calculus for transformations

Let $\Omega = C_0([0,1])$ be equipped with the supremum norm, $\mathscr{F}$ denote the Borel σ-field, P be the standard Wiener measure and $W_t(\omega) = \omega(t)$ the coordinate process. For all $1 \leqq p \leqq +\infty$ denote by $L_p(\Omega)$ the space of the p-integrable random variables on Ω and by $\|\cdot\|_p$ its norm.

Let $\mathscr{S}$ be the dense subset of $L_2(\Omega)$ consisting of those random variables F of the form

$$F = f(W(\Delta_1), \ldots, W(\Delta_n)), \tag{2.1}$$

where $n \in \mathbb{N}$, $f \in C_b^\infty(R^n)$, and $\Pi = \{\Delta_1, \ldots, \Delta_n\}$ is a partition of $[0,1]$ into subintervals. Here $W(\Delta_j)$ denotes the increment of the coordinate process on Δ_j.

If F has the form (2.1), we define its derivative

$$D_t F = \sum_{j=1}^{n} \left(\frac{\partial}{\partial x_j} f\right)(W(\Delta_1), \ldots, W(\Delta_n)) I_{\Delta_j}(t), \qquad 0 \leqq t \leqq 1. \tag{2.2}$$

Then $DF = (D_t F)$ is an element of $L_2([0,1] \times \Omega)$.

Proposition 2.1 *D is an unbounded closable linear operator from $L_2(\Omega)$ into $L_2([0,1] \times \Omega)$. We identify D with its closed extension and denote by $\mathbb{D}_{1,2}$ its domain which we endow with the norm*

$$\|F\|_{1,2} = \|F\|_2 + \left\| \left(\int_0^1 |D_t F|^2 \, dt \right)^{1/2} \right\|_2 . \quad \square$$

Note that the derivative D obeys the chain rule:

Proposition 2.2 *Suppose the $F = (F^1, \ldots, F^n)$ is a random vector whose components belong to $\mathbb{D}_{1,2}$ and assume that $f \in C_b^1(R^n)$. Then, $f(F) \in \mathbb{D}_{1,2}$, and $D_t[f(F)]$ $= \sum_{j=1}^{n} \left(\frac{\partial}{d x_j} f\right)(F) D_t F^j$, a.e.* $\square$

We now associate to $\mathbb{D}_{1,2}$ the space $L_{1,2} = L_2([0,1], dt; \mathbb{D}_{1,2})$ of all processes $(K_s) \subset \mathbb{D}_{1,2}$ which are such that

$$\left\| \left(\int_0^1 K_s^2 \, ds \right)^{1/2} \right\|_2 + \left\| \left(\int_0^1 \int_0^1 |D_t K_s|^2 \, dt \, ds \right)^{1/2} \right\|_2$$

is finite.

Proposition 2.3 *For $t\in[0,1]$ we can define a linear continuous mapping from $L_{1,2}$ into $L_2(\Omega)$ with operator norm 1 which to $(K_s)\in L_{1,2}$ associates the Skorohod integral*

$$\int_0^t u_s \,\mathrm{d} W_s.$$

This linear mapping is characterized by the following property:

$$E\left[\int_0^t K_s \,\mathrm{d} W_s \cdot F\right]=E\left[\int_0^t K_s D_s F \,\mathrm{d} s\right], \qquad F\in\mathscr{S}. \quad \square$$

We now introduce the notion of an absolutely continuous and invertible transformation:

Definition 2.4 We say that the mapping T: $\Omega\to\Omega$ is a transformation, if it has the form

$$T\omega=\omega+\int_0^{\bullet} K_s(\omega)\,\mathrm{d} s, \qquad (K_s)\in L_2([0,1]\times\Omega).$$

Moreover, we call

(i) the transformation T absolutely continuous, if the measure $P\circ[T]^{-1}$ with

$$P\circ[T]^{-1}(B)=P\{\omega\colon T\omega\in B\}, \qquad B\in\mathscr{F},$$

is absolutely continuous relative to P, and,

(ii) the absolutely continuous transformation T invertible, if there exists an absolutely continuous transformation A with

$$T(A\omega)=A(T\omega)=\omega, \qquad P(d\omega)-\text{a.e.} \quad \square$$

Note that for any absolutely continuous and invertible transformation T the measures P, $P\circ[T]^{-1}$ and $P\circ[A]^{-1}$ are equivalent.

Before describing properties of absolutely continuous transformations we present a statement how to reduce the studies of transformations of general type to those transformations whose shift (K_s) is a smooth step process.

Proposition 2.5 *Let $F\in\mathbb{D}_{1,2}$. Then, for any $\varepsilon>0$, there exists a sequence of functionals $(F^n)\subseteq\mathscr{S}$ with $\|F-F^n\|_{1,2}\to 0 (n\to\infty)$ such that, for all n,*

$$\|F^n\|_\infty\leqq\|F\|_\infty \quad \textit{and} \quad \left\|\left(\int_0^1 |D_s F^n|^2 \,\mathrm{d} s\right)^{1/2}\right\|_\infty\leqq\varepsilon+\left\|\left(\int_0^1 |D_s F|^2 \,\mathrm{d} s\right)^{1/2}\right\|_\infty. \quad \square$$

For the proof we refer to the Appendix.

The statement of Proposition 2.5 is extended in Proposition 2.6 to processes from $L_{1,2}$, where the role of the elements of $\mathscr{S}$ in Proposition 2.5 will be replaced by smooth step processes.

A process (K_s) is called smooth step process, if there exists a partition Π of [0, 1] into subintervals Δ and random variables $F_\Delta \in \mathcal{S}$, $\Delta \in \Pi$, such that

$$K_s(\omega) = \sum_{\Delta \in \Pi} F_\Delta(\omega)\, I_\Delta(s), \quad \text{a.e.} \tag{2.3}$$

Proposition 2.6 *Let $(K_s) \in L_{1,2}$. Then, for any $\varepsilon > 0$, there exists a sequence (K_s^n) of smooth step processes with*

$$\int_0^1 \|K_s - K_s^n\|_{1,2}^2 \, \mathrm{d}s \to 0 \qquad (n \to \infty)$$

such that

(i) $\left\| \left(\int_0^1 |K_s^n|^2 \, \mathrm{d}s \right)^{1/2} \right\|_\infty \leqq \left\| \left(\int_0^1 |K_s|^2 \, \mathrm{d}s \right)^{1/2} \right\|_\infty$, *and*

(ii) $\left\| \left(\int_0^1 \int_0^1 |D_t K_s^n|^2 \, \mathrm{d}s \, \mathrm{d}t \right)^{1/2} \right\|_\infty \leqq \varepsilon + \left\| \left(\int_0^1 \int_0^1 |D_t K_s|^2 \, \mathrm{d}s \, \mathrm{d}t \right)^{1/2} \right\|_\infty$, $\quad n = 1, 2, 3, \ldots$ □

We now give some direct consequences of the above Propositions. The first one is a Lipschitz-type statement:

Proposition 2.7 *Let T^1, T^2 be two transformations with shift process (K_s^1) and (K_s^2), respectively. Assume that either*

(i) *$F \in \mathcal{S}$, or*

(ii) *$F \in \mathbb{D}_{1,2}$, and T^1, T^2 are absolutely continuous.*

Then,

$$\begin{aligned} &|F(T^1\omega) - F(T^2\omega)| \\ &\leqq \left\| \left(\int_0^1 |D_s F|^2 \, \mathrm{d}s \right)^{1/2} \right\|_\infty \left(\int_0^1 |K_s^1(\omega) - K_s^2(\omega)|^2 \, \mathrm{d}s \right)^{1/2}, \textit{ a.e.} \end{aligned} \tag{2.4}$$ □

Proof. Under condition (i) the inequality (2.4) can be derived by a straightforward calculus starting from the special form (2.1) of $F \in \mathcal{S}$.

Let now (ii) be satisfied and choose an $\varepsilon > 0$. Due to Proposition 2.5 there is a sequence $(F^n) \subseteq \mathcal{S}$ with $\|F - F^n\|_{1,2} \to 0$ $(n \to \infty)$ such that

$$\left\| \left(\int_0^1 |D_s F^n|^2 \, \mathrm{d}s \right)^{1/2} \right\|_\infty \leqq \varepsilon + \left\| \left(\int_0^1 |D_s F|^2 \, \mathrm{d}s \right)^{1/2} \right\|_\infty, \qquad n = 1, 2, 3, \ldots$$

Clearly,

$$\begin{aligned} &|F^n(T^1\omega) - F^n(T^2\omega)| \\ &\leqq \left\| \left(\int_0^1 |D_s F^n|^2 \, \mathrm{d}s \right)^{1/2} \right\|_\infty \left(\int_0^1 |K_s^1(\omega) - K_s^2(\omega)|^2 \, \mathrm{d}s \right)^{1/2} \\ &\leqq \left(\varepsilon + \left\| \left(\int_0^1 |D_s F|^2 \, \mathrm{d}s \right)^{1/2} \right\|_\infty \right) \left(\int_0^1 |K_s^1(\omega) - K_s^2(\omega)|^2 \, \mathrm{d}s \right)^{1/2}, \text{ a.e.} \end{aligned} \tag{2.5}$$

On the other hand, the transformations T^1, T^2 are supposed to be absolutely continuous. Then, for any $\delta>0$, and with the notation $L^i=\frac{\mathrm{d}P\circ(T^i)^{-1}}{\mathrm{d}P}$, we have,

$$P\{|F^n(T^i)-F(T^i)|>\delta\}=E[I_{\{|F^n-F|>\delta\}}L^i]\to 0\,(n\to\infty),\qquad i=1,2.$$

Hence, (2.5) provides, for $n\to\infty$,

$$\begin{aligned}&|F(T^1\omega)-F(T^2\omega)|\\ &\leqq\left(\varepsilon+\left\|\left(\int_0^1|D_sF|^2\,\mathrm{d}s\right)^{1/2}\right\|_\infty\right)\left(\int_0^1|K_s^1(\omega)-K_s^2(\omega)|^2\,\mathrm{d}s\right)^{1/2},\text{ a.e.}\end{aligned}$$

Since $\varepsilon>0$ was arbitrary choosen we can pass to the limit $\varepsilon\downarrow 0$ and get (2.4). □

Proposition 2.8 *Let $F\in\mathbb{D}_{1,2}$ and T be a transformation with shift process (K_s) such that*

$$\left\|\left(\int_0^1\int_0^1|D_tK_s|^2\,\mathrm{d}s\,\mathrm{d}t\right)^{1/2}\right\|_\infty<+\infty.$$

Assume that either

(i) *$F\in\mathcal{S}$, or*

(ii) *T is absolutely continuous, $F(T\omega)\in L_2(\Omega)$ and $\left\|\left(\int_0^1|D_sF|^2\,\mathrm{d}s\right)^{1/2}\right\|_\infty<+\infty$.*

Then, $F(T\omega)\in\mathbb{D}_{1,2}$, and

$$D_t[F(T\omega)]=(D_tF)(T\omega)+\int_0^1(D_sF)(T\omega)\,D_tK_s(\omega)\,\mathrm{d}s,\text{ a.e.}\tag{2.6}$$ □

Proof. If condition (i) is satisfied, we can suppose that F has form (2.1), and thus,

$$F(T\omega)=f\left(\omega(\Delta_1)+\int_{\Delta_1}K_s(\omega)\,\mathrm{d}s,\ldots,\omega(\Delta_n)+\int_{\Delta_n}K_s(\omega)\,\mathrm{d}s\right).$$

Since

$$\omega(\Delta_j)+\int_{\Delta_j}K_s(\omega)\,\mathrm{d}s\in\mathbb{D}_{1,2},\qquad j=1,2,\ldots,n,$$

we can apply Proposition 2.2 and obtain that $F(T\omega)$ belongs to $\mathbb{D}_{1,2}$ and satisfies (2.6). Assume now condition (ii) and, additionally, suppose that F is essentially bounded. With regard to Proposition 2.5 we find a sequence $(F^n)\subseteq\mathcal{S}$ with $\|F-F^n\|_{1,2}\to 0(n\to\infty)$ such that

$$\|F^n\|_\infty\leqq\|F\|_\infty\quad\text{and}$$
$$\left\|\left(\int_0^1|D_sF^n|^2\,\mathrm{d}s\right)^{1/2}\right\|_\infty\leqq 1+\left\|\left(\int_0^1|D_sF|^2\,\mathrm{d}s\right)^{1/2}\right\|_\infty,\qquad n=1,2,3\ldots$$

If L denotes the density of T, we now get from the dominating convergence theorem

$$(2.7)\qquad \|F(T)-F^n(T)\|_2^2+\left\|\left(\int_0^1 |(D_s F)(T)-(D_s F^n)(T)|^2\,\mathrm{d}s\right)^{1/2}\right\|_2^2$$
$$=\left\|\left\{|F-F^n|^2+\int_0^1 |D_s F-D_s F^n|^2\,\mathrm{d}s\right\}\cdot L\right\|_1 \to 0\ (n\to\infty).$$

Thus, making use of (2.6) which is already proved for all $F^n\in\mathcal{S}$, we can deduce that $F(T\omega)$ is the limit of $(F^n(T))$ in $\mathbb{D}_{1,2}$. Substituting F^n in (2.6) and passing to the limit we see that (2.6) holds also for $F\in\mathbb{D}_{1,2}\cap L_\infty(\Omega)$ such that

$$\left\|\left(\int_0^1 |D_s F|^2\,\mathrm{d}s\right)^{1/2}\right\|_\infty < +\infty.$$

Finally, we consider possibly unbounded $F\in\mathbb{D}_{1,2}$ under condition (ii). We choose for each natural n a function $\varphi_n\in C_b^1(R^1)$ which coincides on $[-n,n]$ with the identity function and has a derivative $\varphi_n'(t)\in[0,1]$, $t\in R^1$. Clearly, $F^n=\varphi_n(F)\in\mathbb{D}_{1,2}$ such that

$$|F^n|\leqq|F|,$$

$$\left(\int_0^1 |D_s F^n|^2\,\mathrm{d}s\right)^{1/2}\leqq\left(\int_0^1 |D_s F|^2\,\mathrm{d}s\right)^{1/2},\ \text{a.e.},\qquad n=1,2,3,\ldots$$

Moreover,

$$F(T)=L_2(\Omega)-\lim_{n\to\infty} F^n(T),$$

and

$$(DF)(T)=L_2([0,1]\times\Omega)-\lim_{n\to\infty}(DF^n)(T).$$

Since (2.6) is satisfied for all F^n, we can pass to the limit there and see that (2.6) holds also for F. □

The following statement is due to [2], Theorems 2 and 3, and is concerned with the convergence of sequences of transformations:

Proposition 2.9 *Let* $\left(T^n\omega=\omega+\int_0^\cdot K_s^n(\omega)\,\mathrm{d}s\right)$ *be a sequence of absolutely continuous transformations such that*

(i) *the sequence of processes* (K_s^n) *is convergent in* $L_2([0,1]\times\Omega)$ *to some process* (K_s), *and*

(ii) *the sequence of densities* $\left(L^n=\dfrac{\mathrm{d}P\circ[T^n]^{-1}}{\mathrm{d}P}\right)$ *is uniformly integrable.*

Then, the transformation

$$T\omega=\omega+\int_0^\cdot K_s(\omega)\,\mathrm{d}s$$

is absolutely continuous, and the density L of T is the limit of (L^n) *in the weak topology* $\sigma(L_1, L_\infty)$. □

Thanks to Proposition 2.9 we can state:

Proposition 2.10 *Let* $\left(T^n\omega=\omega+\int_0^{\bullet} K_s^n(\omega)\,\mathrm{d}s\right)$ *be a sequence of absolutely continuous transformations such that*

(i) *the sequence* (K_s^n) *is convergent in* $L_2([0,1]\times\Omega)$ *to some* (K_s), *and*

(ii) $\left(L^n=\dfrac{\mathrm{d}P\circ[T^n]^{-1}}{\mathrm{d}P}\right)$ *is uniformly integrable.*

Then, with the notation

$$T\omega=\omega+\int_0^{\bullet} K_s(\omega)\,\mathrm{d}s,$$

the convergence of any sequence (F^n) *in probability to some F implies*

$$F(T\omega)=\lim_{n\to\infty} F^n(T^n\omega), \quad \textit{in probability.} \quad \square$$

Proof. Since the densities L^n form a uniformly integrable set, there is for any $\varepsilon>0$ a natural M_ε such that

$$\sup_n E[L^n I\{L^n>M_\varepsilon\}]\leqq\varepsilon/2.$$

Then, for any $\delta>0$,

$$P\{|F(T^n)-F^n(T^n)|\geqq\delta\}=E[I\{|F-F^n|\geqq\delta\}\,L^n]\leqq\varepsilon/2+M_\varepsilon P\{|F-F^n|\geqq\delta\},$$

i.e.,

$$P\{|F(T^n)-F^n(T^n)|\geqq\delta\}\leqq\varepsilon,$$

if n is sufficiently large.

Thus, it remains to prove

$$F(T)=\lim_{n\to\infty} F(T^n), \quad \text{in propability.}$$

Let $(G^n)\subset\mathcal{S}$ be a sequence which converges in $L_2(\Omega)$ to F and L the density of transformation T. Then, for any δ, $\varepsilon>0$, there exists a natural k such that

$$\begin{aligned}&P\{|F(T^n)-G^k(T^n)|\geqq\delta\}+P\{|F(T)-G^k(T)|\geqq\delta\}\\&=E[I\{|F-G^k|\geqq\delta\}(L^n+L)]\\&\leqq\varepsilon+2M_\varepsilon P\{|F-G^k|\geqq\delta\}\leqq2\varepsilon, \qquad n=1,2,3,\dots\end{aligned}$$

On the other hand, from the Lipschitz condition we can deduce that, for sufficiently large n,

$$\begin{aligned} &P\{|G^k(T)-G^k(T^n)|\geqq\delta\}\\ &\leqq\frac{1}{\delta}\left\|\left(\int_0^1|D_s G^k|^2\,\mathrm{d}s\right)^{1/2}\right\|_\infty\cdot\left\|\left(\int_0^1|K_s-K_s^n|^2\,\mathrm{d}s\right)^{1/2}\right\|_2\\ &\leqq\varepsilon. \end{aligned}$$

Therefore,

$$P\{|F(T)-F(T^n)|\geqq 3\delta\}\leqq 3\varepsilon,\qquad \text{if } n \text{ is large enough.}$$

This completes the proof. □

3 Computation of the density of transformations with smooth step shift process

In this section we study transformations $T\colon \Omega\to\Omega$ of the form

(3.1) $$T\omega=\omega+\int_0^{\bullet} K_s(\omega)\,\mathrm{d}s,$$

where we assume that (K_s) is a smooth step process with

(3.2) $$c_K=\left\|\left(\int_0^1\int_0^1|D_t K_s|^2\,\mathrm{d}s\,\mathrm{d}t\right)^{1/2}\right\|_\infty<1.$$

We will show that this condition is sufficient for the absolute continuity and the invertibility of the transformation, and we will compute its density.

For this we shall exploit the extended Girsanov theorem of Kusuoka (Theorem 6.4 of [4]).

Proposition 3.1 *Let* $T\colon \Omega\to\Omega$ *be a transformation of the form* (3.1) *with* $(K_s)\in L_2([0,1]\times\Omega)$, *and suppose that the following conditions are satisfied:*

(i) *T is bijective.*
(ii) *There exists a version of $(D_t K_s)$ such that, for each $\omega\in\Omega$, $(D_t K_s(\omega))$ is a Hilbert-Schmidt operator from $L_2([0,1])$ into itself with*

(1) $\int_0^1\left|K_s\left(\omega+\int_0^{\bullet} h_r\,\mathrm{d}r\right)-K_s(\omega)-\int_0^1 D_t K_s(\omega)\,h_t\,\mathrm{d}t\right|^2\mathrm{d}s=o\left(\int_0^1 h_r^2\,\mathrm{d}r\right)$, *as h tends in $L_2([0,1])$ to zero,*

(2) $h\mapsto\left(D_t K_s\left(\omega+\int_0^{\bullet} h_r\,\mathrm{d}r\right)\right)$ *is continuous from $L_2([0,1])$ into $L_2([0,1])$, and*

(3) $I+(D_t K_s(\omega))$ *is invertible, where I denotes the unit operator from $L_2([0,1])$ into itself.*

Then, transformation T and its inverse transformation A are absolutely continuous,

$$\frac{dP\circ[A]^{-1}}{dP}=|d_c(-DK)|\exp\left\{-\int_0^1 K_s\,\mathrm{d}W_s-\frac{1}{2}\int_0^1 K_s^2\,\mathrm{d}s\right\},$$

where $d_c(-DK)$ is the Carleman-Fredholm determinant of the Hilbert-Schmidt operator $(-D_t K_s)$. □

The Carleman-Fredholm determinant of a Hilbert-Schmidt operator B from $L_2([0,1])$ into itself is defined by the product expansion

$$d_c(B)=\prod_j(1-\lambda_j)\exp(\lambda_j).$$

Here the λ_j's are the nonzero eigenvalues of B counted with their multiplicities. In particular, if the operator B is nuclear,

$$d_c(B)=\det(I-B)\exp(\operatorname{trace} B). \tag{3.3}$$

We now want to apply Proposition 3.1 to transformation T with a smooth step process (K_s) with (3.2) as shift process. Note that for such a (K_s) there are $n\in\mathbb{N}$, $f_1,\ldots,f_n\in C_b^\infty(R^n)$ and a partition $0=t_0<t_1\ldots<t_n=1$ of $[0,1]$ into subintervals $\Delta_j=(t_{j-1},t_j]$ of the length $|\Delta_j|=t_j-t_{j-1}$ such that

$$K_s(\omega)=\sum_{j=1}^n f_j(\omega(\Delta_1),\ldots,\omega(\Delta_n))\,I_{\Delta_j}(s), \tag{3.4}$$

and

$$D_t K_s(\omega)=\sum_{i,j=1}^n \frac{\partial}{\partial x_i} f_j(\omega(\Delta_1),\ldots,\omega(\Delta_n))\,I_{\Delta_i}(t)\,I_{\Delta_j}(s),\qquad \omega\in\Omega.$$

Then transformation T and condition (3.2) take the form

$$T\omega=\omega+\sum_{j=1}^n f_j(\omega(\Delta_1),\ldots,\omega(\Delta_n))\int_0^{\bullet} I_{\Delta_j}(s)\,\mathrm{d}s \tag{3.5}$$

with

$$c_K^2=\sup_{x\in R^n}\sum_{i,j=1}^n |\Delta_i|\,|\Delta_j|\left(\frac{\partial}{\partial x_i}f_j(x)\right)^2<1.$$

Lemma 3.2 *Transformation T defined in* (3.5) *satisfies the conditions of Proposition* 3.1. □

Proof. The correctness of this assertion for Proposition 3.1 (ii) follows immediately from (3.4) and (3.5). For the proof of Proposition 3.1 (i) put

$$|x|_2=\left(\sum_{j=1}^n |\Delta_j|^{-1}x_j^2\right)^{1/2},\qquad x=(x_1,\ldots,x_n)\in R^n.$$

Then, for $f=(|\Delta_1|\cdot f_1, \ldots, |\Delta_n|\cdot f_n)\in C_b(R^n\to R^n)$, we can estimate

$$|f(x)-f(y)|_2 \leqq c_K|x-y|_2, \qquad \text{for all } x, y\in R^n.$$

Since $c_K<1$, the mapping f is contractive, i.e., there exists a function $\psi\in C^\infty(R^n\to R^n)$ which is inverse to $\varphi=(\varphi(x)=x+f(x))$. Let now $A:\Omega\to\Omega$ be the transformation

$$A\omega=\omega-\sum_{j=1}^{n} f_j\circ\psi(\omega(\Delta_1), \ldots, \omega(\Delta_n))\int_0^{\cdot} I_{\Delta_j}(s)\,\mathrm{d}s, \qquad \omega\in\Omega. \tag{3.6}$$

In order to complete the proof it suffices to show

$$T(A\omega)=A(T\omega)=\omega, \qquad \omega\in\Omega.$$

For this denote by I the unit matrix in R^n. From (3.5) we see that

$$\begin{aligned}(A\omega(\Delta_1), \ldots, A\omega(\Delta_n))&=(I-f\circ\psi)(\omega(\Delta_1), \ldots, \omega(\Delta_n))\\ &=\psi(\omega(\Delta_1), \ldots, \omega(\Delta_n)).\end{aligned}$$

Hence,

$$\begin{aligned}T(A\omega)&=A\omega+\sum_{j=1}^{n} f_j(A\omega(\Delta_1), \ldots, A\omega(\Delta_n))\int_0^{\cdot} I_{\Delta_j}(s)\,\mathrm{d}s\\ &=\omega, \qquad \omega\in\Omega.\end{aligned}$$

On the other hand,

$$(T\omega(\Delta_1), \ldots, T\omega(\Delta_n))=\varphi(\omega(\Delta_1), \ldots, \omega(\Delta_n))$$

provides

$$\begin{aligned}A(T\omega)&=T\omega-\sum_{j=1}^{n} (f_j\circ\psi)(T\omega(\Delta_1), \ldots, T\omega(\Delta_n))\int_0^{\cdot} I_{\Delta_j}(s)\,\mathrm{d}s\\ &=\omega, \qquad \omega\in\Omega. \quad \square\end{aligned}$$

From Proposition 3.1 we now know that a transformation $T:\Omega\to\Omega$ of the form (3.1) is absolutely continuous and invertible, whenever the shift process (K_s) is a smooth step process which satisfies (3.2). For the presentation of the density of the inverse transformation A we still have to compute the Carleman-Fredholm determinant of $(-D_t K_s(\omega))$.

Computation of $d_c(-DK)$: Remark that $(-D_t K_s)$ given in (3.4) is a nuclear operator acting on the subspace of $L_2([0, 1])$ generated by the orthogonal system $\{|\Delta_j|^{-1/2} I_{\Delta_j}, j=1, 2, \ldots, n\}$, so that (3.3) gives

$$\begin{aligned}d_c(-DK(\omega))&=\det\left(\delta_{i,j}+|\Delta_i|^{1/2}|\Delta_j|^{1/2}\frac{\partial}{\partial x_i} f_j(\omega(\Delta_1), \ldots, \omega(\Delta_n))\right)\\ &\quad\times\exp\left\{-\sum_{j=1}^{n} |\Delta_j|\frac{\partial}{\partial x_j} f_j(\omega(\Delta_1), \ldots, \omega(\Delta_n))\right\}, \qquad \omega\in\Omega.\end{aligned} \tag{3.7}$$

The hard problem consists in the computation of the determinant of the matrix $\left(\delta_{i,j}+|\Delta_i|^{1/2}|\Delta_j|^{1/2}\frac{\partial}{\partial x_i}f_j(x)\right)$.

For

$$\begin{aligned}\varphi(x)&=(\varphi_1(x),\ \dots,\ \varphi_n(x))\\&=(x_1+|\Delta_1|\,f_1(x),\ \dots,\ x_n+|\Delta_n|\,f_n(x))\end{aligned}$$

denote by $\left[\frac{\partial\varphi}{\partial x}\right](x)$ the Jacobian matrix

$$\left(\frac{|\Delta_j|^{-1/2}\,\partial\varphi_j}{\partial(|\Delta_i|^{-1/2}x_i)}(x)\right)$$

of the transformation

$$(|\Delta_1|^{-1/2}x_1,\ \dots, |\Delta_n|^{-1/2}x_n)\to(|\Delta_1|^{-1/2}\varphi_1(x),\ \dots, |\Delta_n|^{-1/2}\varphi_n(x)).$$

Then, obviously,

$$\det\left[\frac{\partial\varphi}{\partial x}\right](x)=\det\left(\delta_{i,j}+|\Delta_i|^{1/2}|\Delta_j|^{1/2}\frac{\partial}{\partial x_i}f_j(x)\right).$$

For the computation of $\det\left[\frac{\partial\varphi}{\partial x}\right](x)$ we need for each $t\in[0,1]$ the mapping

$$\varphi_t(x)=(x_1+|\Delta_1\cap[0,t]|\,f_1(x),\ \dots,\ x_n+|\Delta_n\cap[0,t]|\,f_n(x)),\qquad x\in R^n.$$

The mappings $\varphi_t\colon R^n\to R^n$ are invertible; the proof is analogous to that of the invertibility of $\varphi(=\varphi_1)$. Let $\psi_t\colon R^n\to R^n$ denote the inverse to φ_t.

Lemma 3.3 *With the above notations we have*

$$\det\left[\frac{\partial\varphi}{\partial x}\right](x)=\exp\left\{\sum_{j=1}^{n}\int_{\Delta_j}\frac{\partial}{\partial x_j}(f_j\circ\psi_t)(\varphi_t(x))\,dt\right\},\qquad x\in R^n.\quad\square$$

Proof. Choose for each $j=1,2,\dots,n$ any subpartition $t_{j-1}=t_0^j<t_1^j,\ \dots<t_{k_j}^j=t_j$ of Δ_j, and denote by Δ_k^j the interval $(t_{k-1}^j, t_k^j]$. We put

$$\theta_k^j(x)=(x_1,\ \dots,\ x_{j-1},\ x_j+|\Delta_k^j|\,f_j(\psi_{t_{k-1}^j}(x)),\ x_{j+1},\ \dots,\ x_n),$$
$$k=1,2,\ \dots,\ k_j,\qquad j=1,2,\ \dots,\ n.$$

Then, we have

$$\varphi_{t_k^j}(x)=\theta_k^j(\theta_{k-1}^j(\dots\theta_1^j(\theta_{k_{j-1}}^{j-1}(\theta_{k_{j-1}-1}^{j-1}(\dots(\theta_2^1(\theta_1^1(x))\dots)))\dots))),$$

so that the form of the determinant

$$\det\left[\frac{\partial\theta_k^j}{\partial x}\right](x)=1+|\Delta_k^j|\frac{\partial}{\partial x_j}(f_j\circ\psi_{t_{k-1}^j}(x))$$

of the Jacobian matrix of the transformation

$$(|\Delta_1|^{-1/2}x_1, \dots, |\Delta_n|^{-1/2}x_n) \to (|\Delta_1|^{-1/2}x_1, \dots, |\Delta_{j-1}|^{-1/2}x_{j-1}, \\ |\Delta_j|^{-1/2}(x_j + |\Delta_k^j| f_j(\psi_{t_{k-1}^j}(x))), \\ |\Delta_{j+1}|^{-1/2}x_{j+1}, \dots, |\Delta_n|^{-1/2}x_n)$$

implies the following relation:

$$\text{(3.8)} \qquad \det\left[\frac{\partial \varphi}{\partial x}\right](x) = \prod_{j=1}^{n} \prod_{k=1}^{k_j} \left(1 + |\Delta_k^j| \frac{\partial}{\partial x_j}(f_j \circ \psi_{t_{k-1}^j})(\varphi_{t_{k-1}^j}(x))\right), \quad x \in R^n.$$

Now fix any $j = 1, 2, \dots, n$. Since $\frac{\partial}{\partial x_j}(f \circ \psi_t)(\varphi_t(x))$ is continuous and bounded in $[0, 1] \times R^n$, we get

$$\lim_{\max_k |\Delta_k^j| \to 0} \prod_{k=1}^{k_j} \left(1 + |\Delta_k^j| \frac{\partial}{\partial x_j}(f_j \circ \psi_{t_{k-1}^j})(\varphi_{t_{k-1}^j}(x))\right)$$

$$= \lim_{\max_k |\Delta_k^j| \to 0} \exp\left\{\sum_{k=1}^{k_j} |\Delta_k^j| \frac{\partial}{\partial x_j}(f_j \circ \psi_{t_{k-1}^j})(\varphi_{t_{k-1}^j}(x))\right\}$$

$$= \exp\left\{\int_{\Delta_j} \frac{\partial}{\partial x_j}(f_j \circ \psi_t)(\varphi_t(x))\, dt\right\}.$$

From (3.8) it now becomes clear that the statement is correct. □

If we substitute Lemma 3.3 in (3.7), then we obtain

$$\text{(3.9)} \qquad d_c(-DK(\omega)) = \exp\left\{\sum_{j=1}^{n} \int_{\Delta_j} \frac{\partial}{\partial x_j}(f_j \circ \psi_t(\varphi_t(\omega(\Delta_1), \dots, \omega(\Delta_n)))\, dt \right. \\ \left. - \sum_{j=1}^{n} \frac{\partial}{\partial x_j} f_j(\omega(\Delta_1), \dots, \omega(\Delta_n))|\Delta_j|\right\}.$$

In order to rewrite (3.9) in terms of (K_s) we introduce for each $t \in [0, 1]$ the transformation

$$T_t\,\omega = \omega + \int_0^{\bullet} I_{[0,t]}(s)\, K_s(\omega)\, ds$$

$$\left(= \omega + \sum_{j=1}^{n} f_j(\omega(\Delta_1), \dots, \omega(\Delta_n)) \int_0^{\bullet} I_{\Delta_j \cap [0,t]}(s)\, ds\right).$$

Since $(I_{[0,t]}(s)\, K_s)$ satisfies (3.2), T_t is absolutely continuous and invertible. From (3.6) we see that $A_t = T_t^{-1}$ has the form

$$\text{(3.10)} \qquad A_t\,\omega = \omega - \sum_{j=1}^{n} f_j \circ \psi_t(\omega(\Delta_1), \dots, \omega(\Delta_n)) \int_0^{\bullet} I_{\Delta_j \cap [0,1]}(s)\, ds, \qquad \omega \in \Omega.$$

Then,

$$K_t(A_t\,\omega)=\sum_{j=1}^{n} f_j\circ\psi_t(\omega(\Delta_1),\,\dots,\,\omega(\Delta_n))\,I_{\Delta_j}(t).$$

Since $f_j\circ\psi_t\in C_b^\infty(R^n)$, $j=1,2,\,\dots,\,n$, the random variable $K_t(A_t\,\omega)$ belongs to $\mathscr{S}$. Due to (2.2) we can compute $D_t[K_t(A_t\,\omega)]$,

$$D_t[K_t(A_t\,\omega)]=\sum_{j=1}^{n}\frac{\partial}{\partial x_j}(f_j\circ\psi_t)(\omega(\Delta_1),\,\dots,\,\omega(\Delta_n))\,I_{\Delta_j}(t).$$

Finally, substituting

$$(T_t\,\omega(\Delta_1),\,\dots,\,T_t\,\omega(\Delta_n))=\varphi_t(\omega(\Delta_1),\,\dots,\,\omega(\Delta_n)),$$

we see that (3.9) takes the form

$$d_c(-DK(\omega))=\exp\left\{\int_0^1 D_t[K_t(A_t)](T_t\,\omega)\,\mathrm{d}t-\int_0^1 D_t\,K_t\,(\omega)\,\mathrm{d}t\right\}.$$

Summarizing the above results we can state:

Proposition 3.4 *Let (K_s) be a smooth step process which is such that*

$$\left\|\left(\int_0^1\int_0^1 |D_t\,K_s|^2\,\mathrm{d}s\,\mathrm{d}t\right)^{1/2}\right\|_\infty<1.$$

Then the transformation

$$T\omega=\omega+\int_0^{\bullet} K_s(\omega)\,\mathrm{d}s$$

is absolutely continuous and invertible. Its inverse transformation A has the density

$$\text{(3.11)}\quad \frac{\mathrm{d}P\circ[A]^{-1}}{\mathrm{d}P}$$
$$=\exp\left\{-\int_0^1 K_s\,\mathrm{d}W_s-\frac{1}{2}\int_0^1 K_s^2\,\mathrm{d}s+\int_0^1 [D_s[K_s(A_s)](T_s)-D_s\,K_s]\,\mathrm{d}s\right\},$$

where

$$T_t\,\omega=\omega+\int_0^{\bullet} K_s(\omega)\,I_{[0,t]}(s)\,\mathrm{d}s,\qquad A_t=T_t^{-1},\qquad t\in[0,1].\quad\square$$

Remark 3.5 Making use of Proposition 2.8 we get

$$D_s[K_s(A_s\,\omega)]=(D_s\,K_s)(A_s\,\omega)$$
$$-\int_0^s (D_t\,K_s)(A_s\,\omega)\,D_s\,[K_t(A_s\,\omega)]\,\mathrm{d}t,\qquad s\in[0,1].$$

Therefore, we can rewrite (3.11) as follows:

$$(3.12)\quad \frac{\mathrm{d}P\circ[A]^{-1}}{\mathrm{d}P} = \exp\left\{-\int_0^1 K_s\,\mathrm{d}W_s - \frac{1}{2}\int_0^1 K_s^2\,\mathrm{d}s - \int_0^1\int_0^s D_t K_s D_s[K_t(A_s)](T_s)\,\mathrm{d}t\,\mathrm{d}s\right\}.$$

The purpose of this equation is that here the right-hand side can make sense also in the case, where (K_s) is not a smooth step process. □

4 Absolute continuity of transformations. General case

We consider now the transformation

$$(4.1)\quad T\omega=\omega+\int_0^{\bullet} K_s(\omega)\,\mathrm{d}s$$

for processes $(K_s)\in L_{1,2}$ such that

$$(4.2)\quad \left\|\left(\int_0^1\int_0^1 |D_t K_s|^2\,\mathrm{d}s\,\mathrm{d}t\right)^{1/2}\right\|_\infty<1.$$

We will see that condition (4.2) is sufficient for the absolute continuity of T. In order to deduce the invertibility of T and to compute its density we will need an additional Novikov-type condition on (K_s).

Proposition 4.1 *Let T: $\Omega\to\Omega$ be a transformation of the form* (4.1) *and assume* (4.2) *to be satisfied. Then T is absolutely continuous.* □

Proof. Due to Proposition 2.6 there exists a sequence (K_s^n) of smooth step processes which converge in $L_{1,2}$ to (K_s) and are such that

$$c_K^*=\sup_n\left\|\left(\int_0^1\int_0^1 |D_t K_s^n|^2\,\mathrm{d}s\,\mathrm{d}t\right)^{1/2}\right\|_\infty<1.$$

From Proposition 3.4 we know that the transformations

$$T^n\omega=\omega+\int_0^{\bullet} K_s^n(\omega)\,\mathrm{d}s,\qquad n=1,2,3,\dots$$

are absolutely continuous and invertible, the inverse transformation A^n of T^n has the density

$$(4.3)\quad L^n=\exp\left\{-\int_0^1 K_s^n\,\mathrm{d}W_s-\frac{1}{2}\int_0^1 (K_s^n)^2\,\mathrm{d}s-\int_0^1\int_0^t D_s K_t^n D_t[K_s^n(A_t^n)](T_t^n)\,\mathrm{d}s\,\mathrm{d}t\right\},$$

where

$$T_t^n\,\omega=\omega+\int_0^{\bullet} I_{[0,t]}(s)\,K_s^n(\omega)\,\mathrm{d}s \quad \text{and} \quad A_t^n=(T_t^n)^{-1}.$$

In order to apply now Proposition 2.9 and to conclude to the absolute continuity of T, it is sufficient to show that the sequence $\left(\mathscr{L}^n=\frac{\mathrm{d}P\circ[T^n]^{-1}}{\mathrm{d}P}\right)$ of densities is uniformly integrable or to check the stronger condition

$$\sup_n E[\mathscr{L}^n|\ln \mathscr{L}^n|]<+\infty \qquad \text{(cf. [5])}.$$

Since $\mathscr{L}^n(T^n)^{-1}=L^n$ and $E[\mathscr{L}^n|\ln \mathscr{L}^n|]=E[|\ln \mathscr{L}^n(T^n)|]=E[|\ln L^n|]$, this condition is equivalent to

$$\sup_n E[|\ln L^n|]<+\infty.$$

Obviously, $\left(E\left[\left|\int_0^1 K_s^n\,\mathrm{d}W_s+\frac{1}{2}\int_0^1 (K_s^n)^2\,\mathrm{d}s\right|\right]\right)$ is bounded. Thus, it remains to estimate

$$\int_0^1\int_0^t D_s K_t^n\, D_t[K_s^n(A_t^n)](T_t^n)\,\mathrm{d}s\,\mathrm{d}t.$$

For this recall from (3.10) that $K_s^n(A_t^n\,\omega)\in\mathscr{S}$, so that Proposition 2.8 provides

$$\begin{aligned}(4.4)\qquad D_r K_s^n(\omega)&=D_r[(K_s^n\circ A_t^n)(T_t^n\,\omega)]\\ &=D_r(K_s^n\circ A_t^n)(T_t^n\,\omega)+\int_0^t D_u(K_s^n\circ A_t^n)(T_t^n\,\omega)\,D_r K_u^n(\omega)\,\mathrm{d}u.\end{aligned}$$

Consequently,

$$\begin{aligned}&\left(\int_0^1\int_0^1 |D_r(K_s^n\circ A_t^n)(T_t^n\,\omega)|^2\,\mathrm{d}s\,\mathrm{d}r\right)^{1/2}\\ &\leqq\frac{\left(\int_0^1\int_0^1 |D_r K_s^n(\omega)|^2\,\mathrm{d}s\,\mathrm{d}r\right)^{1/2}}{1-\left(\int_0^1\int_0^1 |D_r K_s^n(\omega)|^2\,\mathrm{d}s\,\mathrm{d}r\right)^{1/2}}\leqq\frac{1}{1-c_K^*}<+\infty.\end{aligned}$$

Setting $r=t$ in (4.4) and using the last result we obtain

$$\begin{aligned}(4.5)\qquad &\left(\int_0^1\int_0^1 |D_t(K_s^n\circ A_t^n)(T_t^n\,\omega)|^2\,\mathrm{d}s\,\mathrm{d}t\right)^{1/2}\\ &\leqq\left(1+\frac{1}{1-c_K^*}\right)\left(\int_0^1\int_0^1 |D_r K_s^n(\omega)|^2\,\mathrm{d}s\,\mathrm{d}r\right)^{1/2}\leqq 1+\frac{1}{1-c_K^*}.\end{aligned}$$

From here it is easy to see that

$$\left(E\left[\left|\int_0^1 \int_0^t D_s K_t^n D_t[K_s^n(A_t^n)](T_t^n)\,\mathrm{d}s\,\mathrm{d}t\right|\right]\right)$$

is bounded. □

For our further studies let us fix any $(K_s)\in L_{1,2}$ which satisfies (4.2) and assume that there exists a $q>1$ such that

$$E\left[\exp\left\{\frac{q}{2}\int_0^1 K_s^2\,\mathrm{d}s\right\}\right]<+\infty. \tag{4.6}$$

We want to prove the invertibility of the corresponding transformation T with shift process (K_s). For this we need an approximation of (K_s) in $L_{1,2}$ by a sequence of smooth step processes satisfying (4.2) and (4.6) uniformly, in order to pass in Proposition 3.4 to the limit. This will be established in Lemmata 4.2–4.7 and the main result will be presented in Theorem 4.9.

Lemma 4.2 *For the process $(K_s)\in L_{1,2}$ with* (4.2) *and* (4.6) *there exists a sequence of smooth step processes (K_s^n) approximating (K_s) in $L_{1,2}$ such that*

$$c_K^* = \sup_n \left\|\left(\int_0^1 \int_0^1 |D_t K_s^n|^2\,\mathrm{d}s\,\mathrm{d}t\right)^{1/2}\right\|_\infty <1 \tag{4.7}$$

and for some $q>1$,

$$\sup_n E\left[\exp\left\{\frac{q}{2}\int_0^1 (K_s^n)^2\,\mathrm{d}s\right\}\right]<+\infty. \quad \square \tag{4.8}$$

Proof. For each natural m we set

$$\varphi_m(u)=\max\{\min\{u,m\},-m\}, \qquad u\in R^1.$$

Then, clearly, $(\varphi_m(K_s))$ converges in $L_{1,2}$ to (K_s),

$$|\varphi_m(K_s)|\leqq |K_s|, \quad \text{and} \quad |D_t[\varphi_m(K_s)]|\leqq |D_t K_s|, \qquad \text{a.e.}$$

From Proposition 2.6 we now can deduce that the processes $(\varphi_m(K_s))$ can be approximated in $L_{1,2}$ by sequences of smooth step processes $(K_s^{m,n})$ such that

$$\sup_{n,m}\left\|\left(\int_0^1 \int_0^1 |D_t K_s^{m,n}|^2\,\mathrm{d}s\,\mathrm{d}t\right)^{1/2}\right\|_\infty <1$$

and

$$\sup_n \left\|\left(\int_0^1 |K_s^{m,n}|^2\,\mathrm{d}s\right)^{1/2}\right\|_\infty \leqq \left\|\left(\int_0^1 \varphi_m(K_s)^2\,\mathrm{d}s\right)^{1/2}\right\|_\infty, \qquad m=1,2,3,\ldots$$

Hence, we can choose a diagonal sequence $(K_s^{m,n(m)})$ which converges in $L_{1,2}$ to (K_s) and satisfies the inequality

$$E\left[\exp\left\{\frac{q}{2}\int_0^1 (K_s^{m,n(m)})^2\,\mathrm{d}s\right\}\right] \leqq 1 + E\left[\exp\left\{\frac{q}{2}\int_0^1 \varphi_m(K_s)^2\,\mathrm{d}s\right\}\right],$$
$$m=1,2,3,\ldots$$

Clearly, this sequence of smooth step processes has the required properties. □

Together with the process $(K_s)\in L_{1,2}$ let us fix now a sequence of smooth step processes (K_s^n) which approximates (K_s) in the sense of Lemma 4.2. To these processes (K_s^n) we associate the absolutely continuous and invertible transformations

$$T_t^n\,\omega = \omega + \int_0^{\bullet} K_s^n(\omega)\, I_{[0,t]}(s)\,\mathrm{d}s,$$

and we denote their inverse transformations by A_t^n. Then, for each $t\in[0,1]$, A_t^n is absolutely continuous and has the density

$$\text{(4.9)}\quad L_t^n = \exp\left\{-\int_0^t K_s^n\,\mathrm{d}W_s - \frac{1}{2}\int_0^t (K_s^n)^2\,\mathrm{d}s - \int_0^t\int_0^s D_r K_s^n\, D_s[K_r^n(A_s^n)](T_s^n)\,\mathrm{d}r\,\mathrm{d}s\right\}.$$

In order to show the invertibility of T, we want to apply Proposition 2.9 to the sequence of transformations (A_t^n). For this we need some auxiliary results:

Lemma 4.3 *Under the above assumptions there exists a $p>1$ with*

$$\sup_{n,t} \|L_t^n\|_p < +\infty.$$

Then, in particular, the family L_t^n, $t\in[0,1]$, $n=1,2,3,\ldots$, is uniformly integrable. □

Proof. Let $\varepsilon>0$ be such that $c_K^* < 1-\varepsilon$ and $(1-\varepsilon)\,q>1$. Set

$$p' = 1 + \frac{\varepsilon}{(1-\varepsilon)^2 q} \qquad \text{and fix any } p\in\left(1, \frac{(1-\varepsilon)\,q}{(1-\varepsilon)^2 q + \varepsilon}\right).$$

Then, $p\,p'\,c_K^* < 1$ and

$$\frac{p\,p'(p\,p'-1)}{p'-1} < q.$$

Obviously, we have

$$\text{(4.10)}\qquad \|L_t^n\|_p^p \leqq e^{pC_1} E\left[\exp\left\{-p\,p'\int_0^t K_s^n\,\mathrm{d}W_s - \tfrac{1}{2}(p\,p')^2\int_0^t (K_s^n)^2\,\mathrm{d}s\right\}\right]^{1/p'}$$
$$\times E\left[\exp\left\{\frac{p\,p'(p\,p'-1)}{2(p'-1)}\int_0^1 (K_s^n)^2\,\mathrm{d}s\right\}\right]^{\frac{p'-1}{p'}}$$

where

$$c_1 = \sup_n \left\| \int_0^1 \int_0^t |D_s K_t^n D_t[K_s^n(A_t^n)](T_t^n)| \, \mathrm{d}s \, \mathrm{d}t \right\|_\infty \leqq 1 + \frac{1}{1-c_K^*} < +\infty. \tag{4.11}$$

Here we have used (4.5) in the last line. From the above assumption for pp' we see that the transformation

$$\bar{T}_t^n \omega = \omega + pp' \int_0^{\bullet} K_s^n(\omega) I_{[0,t]}(s) \, \mathrm{d}s$$

is absolutely continuous and invertible. Due to (4.9) its inverse transformation $\bar{A}_t^n$ has the density

$$\bar{L}_t^n = \exp\left\{ -pp' \int_0^t K_s^n \, \mathrm{d}W_s - \tfrac{1}{2}(pp')^2 \int_0^t (K_s^n)^2 \, \mathrm{d}s - (pp')^2 \int_0^t \int_0^s D_r K_s^n D_s[K_r^n(\bar{A}_s^n)](\bar{T}_s^n) \, \mathrm{d}r \, \mathrm{d}s \right\}. \tag{4.12}$$

Moreover, (4.5) gives

$$\left\| (pp')^2 \int_0^1 \int_0^t |D_s K_t^n D_t[K_s^n(\bar{A}_t^n)](\bar{T}_t^n)| \, \mathrm{d}s \, \mathrm{d}t \right\|_\infty \leqq 1 + \frac{1}{1 - pp' c_K^*} < +\infty. \tag{4.13}$$

Finally, substituting (4.12) and (4.13) in (4.10) we get the existence of a real constant C such that

$$\| L_t^n \|_p^p \leqq C E\left[\exp\left\{ \frac{q}{2} \int_0^1 (K_s^n)^2 \, \mathrm{d}s \right\} \right]^{\frac{p'-1}{p'}}.$$

Now the correctness of the statement follows immediately from (4.6). □

Lemma 4.4 *For each* $t \in [0,1]$, *the sequence* $(K_s^n(A_t^n))$ *is convergent in* $L_2([0,1] \times \Omega)$. □

Proof. Fix any $t \in [0,1]$. Since A_t^n is the inverse to T_t^n, it has the form

$$A_t^n \omega = \omega - \int_0^{\bullet} K_s^n(A_t^n \omega) I_{[0,t]}(s) \, \mathrm{d}s.$$

Consequently, Proposition 2.7 permits to estimate

$$E\left[\int_0^1 |K_s^n(A_t^n)-K_s^m(A_t^m)|^2\,\mathrm{d}s\right]^{1/2}$$
$$\leqq \left\|\left(\int_0^1\int_0^1 |D_t K_s^n|^2\,\mathrm{d}s\,\mathrm{d}t\right)^{1/2}\right\|_\infty E\left[\int_0^t |K_s^n(A_t^n)-K_s^m(A_t^m)|^2\,\mathrm{d}s\right]^{1/2}$$
$$+E\left[\int_0^1 |K_s^n(A_t^m)-K_s^m(A_t^m)|^2\,\mathrm{d}s\right]^{1/2}, \qquad n,m=1,2,3,\ldots,$$

and from (4.7) we obtain

$$E\left[\int_0^1 |K_s^n(A_t^n)-K_s^m(A_t^m)|^2\,\mathrm{d}s\right]^{1/2} \leqq \frac{1}{1-c_K^*} E\left[\int_0^1 |K_s^n-K_s^m|^2\,\mathrm{d}s\, L_t^m\right]^{1/2},$$
$$n,m=1,2,3,\ldots$$

We now can derive from Lemma 4.3 and (4.8) that the right-hand side of the above inequality tends to zero, if $n,m\to\infty$. Hence, the sequence $(K_s^n(A_t^n))$ is convergent in $L_2([0,1]\times\Omega)$. □

Denote by T_t the transformation

$$T_t\,\omega=\omega+\int_0^{\bullet} I_{[0,t]}(s)\,K_s(\omega)\,\mathrm{d}s, \qquad t\in[0,1],$$

which is absolutely continuous due to Proposition 4.1. We now state:

Proposition 4.5 *Let* $t\in[0,1]$ *and denote by* $(\bar{K}_s)$ *the limit of* $(K_s^n(A_t^n))$ *in* $L_2([0,1]\times\Omega)$. *Then the transformation*

$$A_t\,\omega=\omega-\int_0^{\bullet} I_{[0,t]}(s)\,\bar{K}_s(\omega)\,\mathrm{d}s$$

is absolutely continuous and inverse to T_t. □

Proof. Since the absolute continuity of A_t follows immediately from Proposition 2.9, Lemmata 4.3 and 4.4, it remains to prove that A_t is inverse to T_t. For this note that by making use of Proposition 2.7 we can estimate

$$\int_0^1 P\{|K_s(A_t)-K_s^n(A_t^n)|>\varepsilon\}\,\mathrm{d}s$$
$$\leqq \frac{2c_K^*}{\varepsilon} E\left[\int_0^1 |\bar{K}_s-K_s^n(A_t^n)|^2\,\mathrm{d}s\right]^{1/2}+E\left[\int_0^1 I_{\{|K_s-K_s^n|>\varepsilon/2\}}\,\mathrm{d}s\cdot L_t\right],$$

where L_t denotes the density of A_t. As we have established in Lemma 4.4, the right-hand side of this estimation tends to zero, for any $\varepsilon>0$, i.e.,

$$K_s(A_t)=\lim_{n\to\infty} K_s^n(A_t^n)=\bar{K}_s.$$

But, this relation implies immediately

$$T_t(A_t\,\omega)=A_t\,\omega+\int_0^{\cdot} K_s(A_t\,\omega)\,I_{[0,t]}(s)\,\mathrm{d}s=\omega, \qquad \text{a.e.}$$

On the other hand, if $\varphi\in C_b^1(R^1)$, then the proof of Proposition 4.1 and statement Lemma 4.4 allow to apply Proposition 2.10 to the sequence $(F^n=\varphi(K_s^n(A_t^n)))$ of random variables and to the sequence $\left(T_t^n\,\omega=\omega+\int_0^{\cdot} I_{[0,t]}(s)\,K_s^n(\omega)\,\mathrm{d}s\right)$ of transformations. This yields

$$\begin{aligned}\varphi(\bar{K}_s)(T_t)&=L_2(\Omega)-\lim_{n\to\infty}\varphi(K_s^n\circ A_t^n)(T_t^n)\\&=L_2(\Omega)-\lim_{n\to\infty}\varphi(K_s^n)=\varphi(K_s).\end{aligned}$$

Hence,

$$K_s=\bar{K}_s(T_t),$$

and

$$A_t(T_t\,\omega)=T_t\,\omega-\int_0^{\cdot} I_{[0,t]}(s)\,\bar{K}_s(T_t\,\omega)\,\mathrm{d}s=\omega, \qquad \text{a.e.} \quad \square$$

It remains only to compute the density of the transformation A_t. For this we still need the following auxiliary statements:

Lemma 4.6 *For each* $t\in[0,1]$, *the process* $(K_s^n(A_t^n))$ *converges in* $L_{1,2}$ *to* $(K_s(A_t))$.

Proof. Since $F=K_s^n$ belongs to $\mathcal{S}$ and the shift process $(K_r^n(A_t^n)\,I_{[0,t]}(r))$ of A_t^n is a smooth step process, Proposition 2.8 can be applied to compute $D_r[K_s^n(A_t^n)]$. This yields

$$(4.14)\quad \begin{aligned}D_r[K_s^n(A_t^n\,\omega)]&=(D_r\,K_s^n)(A_t^n\,\omega)\\&\quad-\int_0^t (D_u\,K_s^n)(A_t^n\,\omega)\,D_r\,[K_u^n(A_t^n\,\omega)]\,\mathrm{d}u, \qquad r,s\in[0,1].\end{aligned}$$

Hence,

$$(4.15)\qquad \left(\int_0^1\int_0^1 |D_r[K_s^n(A_t^n)]|^2\,\mathrm{d}r\,\mathrm{d}s\right)^{1/2}\leqq\frac{1}{1-c_K^*}.$$

A renewed estimation in (4.14) with substitution of (4.15) gives

$$(4.16)\qquad \begin{aligned}&E\left[\int_0^1\int_0^1 |D_r[K_s^n(A_t^n)]-D_r[K_s^m(A_t^m)]|^2\,\mathrm{d}r\,\mathrm{d}s\right]^{1/2}\\&\leqq\frac{2-c_K^*}{(1-c_K^*)^2}\cdot E\left[\int_0^1\int_0^1 |(D_r\,K_s^n)(A_t^n)-(D_r\,K_s^m)(A_t^m)|^2\,\mathrm{d}r\,\mathrm{d}s\right]^{1/2}.\end{aligned}$$

Thus, by virtue of Proposition 2.10 it suffices to show the uniform square integrability of $(DK^n)(A_t^n)$ in $[0,1]^2\times\Omega$ in order to conclude that $D[K^n(A_t^n)]$ converges in $L_2([0,1]^2\times\Omega)$. For this note that due to Lemma 4.3, for some $q>1$ and some real C_q it holds

$$\sup_n E\left[\int_0^1\int_0^1 |(D_r K_s^n)(A_t^n)|^2 I\{|(D_r K_s^n)(A_t^n)|\geqq M\}\,\mathrm{d}r\,\mathrm{d}s\right]$$
$$=\sup_n E\left[\left(\int_0^1\int_0^1 |D_r K_s^n|^2 I\{|D_r K_s^n|\geqq M\}\,\mathrm{d}r\,\mathrm{d}s\right)L_t^n\right]$$
$$\leqq C_q \sup_n E\left[\left(\int_0^1\int_0^1 |D_r K_s^n|^2 I\{|D_r K_s^n|\geqq M\}\,\mathrm{d}r\,\mathrm{d}s\right)^q\right]^{1/q}, \qquad M>0,$$

and that due to Lemma 4.2 the right-hand side of this estimation tends to zero, as $M\to\infty$. On the other hand, in Lemma 4.4 we have already established that $(K_s^n(A_t^n))$ converges in $L_2([0,1]\times\Omega)$ to $(K_s(A_t))$. Consequently, $(K_s(A_t))\in L_{1,2}$ and $(K_s(A_t))=L_{1,2}-\lim_{n\to\infty}(K_s^n(A_t^n))$. □

Lemma 4.7 *The process $(D_r[K_s(A_t)])$ has a version for which the function $(t\mapsto D_r[K_s(A_t)]\in L_2(\Omega))$ is continuous, for $r,s\in[0,1]$.* □

Proof. If we pass in (4.14) to the limit, then we get

$$D_r[K_s(A_t)]=(D_r K_s)(A_t)-\int_0^t (D_u K_s)(A_t)\,D_r[K_u(A_t)]\,\mathrm{d}u, \qquad \text{a.e.} \tag{4.17}$$

From this equation and condition (4.2) on (K_s) it becomes clear that it suffices to show the mean-square continuity of $t\mapsto(D_r K_s)(A_t)$ in order to conclude to the correctness of the statement. This mean-square continuity we want to verify by Proposition 2.10. For this we only have to check that

(i) the family $(K_s(A_t))$ of the shift processes of the transformations A_t is continuous in $L_2([0,1]\times\Omega)$ and

(ii) the set of the densities $\left(L_t=\dfrac{\mathrm{d}P\circ[A_t]^{-1}}{\mathrm{d}P}\right)$ is uniformly integrable.

From Proposition 2.9, Lemmata 4.3 and 4.4 we see that

$$L_t=\sigma(L_1,L_\infty)-\lim_{n\to\infty}L_t^n, \qquad t\in[0,1]. \tag{4.18}$$

Hence, the uniform integrability of the set $\{L_t^n, t\in[0,1], n=1,2,3,\ldots\}$ (cf. Lemma 4.3) implies (ii). For the proof of (i) we apply Proposition 2.7 to estimate

$$E\left[\int_0^1 |K_r(A_s)-K_r(A_t)|^2\,\mathrm{d}r\right]^{1/2}, \qquad s<t.$$

Since

$$A_t\,\omega=\omega-\int_0^{\bullet} I_{[0,t]}(r)\,K_r(A_t\,\omega)\,\mathrm{d}r, \qquad \text{a.e.,}$$

we obtain

$$E\left[\int_0^1 |K_r(A_s)-K_r(A_t)|^2\,dr\right]^{1/2}$$
$$\leqq c_K\left\{E\left[\int_0^1 |K_r(A_t)|^2\,dr\right]^{1/2}+E\left[\int_0^s |K_r(A_s)-K_r(A_t)|^2\,dr\right]^{1/2}\right\}.$$

Thus

$$E\left[\int_0^1 |K_r(A_s)-K_r(A_t)|^2\,dr\right]^{1/2}\leqq\frac{1}{1-c_K}E\left[\int_s^t |K_r|^2\,dr\cdot L_t\right]^{1/2},$$

and if we consider (4.18) and Lemma 4.2, then we have for some $p, p'>1$ with $\frac{1}{p}+\frac{1}{p'}=1$ the following estimation:

$$\text{(4.19)}\qquad E\left[\int_0^1 |K_r(A_s)-K_r(A_t)|^2\,dr\right]^{1/2}$$
$$\leqq\frac{1}{1-c_K}\left(\sup_{n,t}\|L_t^n\|_p\right)^{1/2}\cdot E\left[\left(\int_s^t |K_r|^2\,dr\right)^{p'}\right]^{1/2p'}.$$

But from condition (4.6) we now can deduce that the right-hand side tends to zero, if $|t-s|\to 0$. This completes the proof. □

Remark 4.8 Due to Lemma 4.7 we can define the process $(D_r[K_s(A_r)])$ by

$$\text{(4.20)}\qquad D_r[K_s(A_r)]=L_2(\Omega)-\lim_{t\to r} D_r[K_s(A_t)].$$

By passing in (4.17) to the limit $t\mapsto r$ we see that $(D_r[K_s(A_r)])$ and $(D_r[K_s(A_r)](T_r))$ belong to $L_2([0,1]^2\times\Omega)$. □

Remark 4.8 gives sense to the right-hand side of (3.12) also in that case, where $(K_s)\in L_{1,2}$ is not a smooth step process. Thus, we now can formulate the main result:

Theorem 4.9 *Let $(K_s)\in L_{1,2}$ such that*

(i) $\left\|\left(\int_0^1\int_0^1 |D_t K_s|^2\,ds\,dt\right)^{1/2}\right\|_\infty<1,$

and

(ii) *there is a $q>1$ with* $E\left[\exp\left\{\frac{q}{2}\int_0^1 K_s^2\,ds\right\}\right]<+\infty.$

Then, the transformation

$$T\omega=\omega+\int_0^{\cdot} K_s(\omega)\,ds$$

is absolutely continuous and invertible. Its inverse transformation A has the density

$$(4.21)\quad L=\exp\left\{-\int_0^1 K_s\,\mathrm{d}W_s-\frac{1}{2}\int_0^1 K_s^2\,\mathrm{d}s-\int_0^1\int_0^t D_s K_t\,D_t[K_s(A_t)](T_t)\,\mathrm{d}s\,\mathrm{d}t\right\},$$

where

$$T_t\,\omega=\omega+\int_0^{\bullet} I_{[0,t]}(s)\,K_s(\omega)\,\mathrm{d}s \quad \textit{and} \quad A_t=T_t^{-1}. \quad \square$$

Proof. In Proposition 4.5 we habe already shown that the absolutely continuous transformations T_t, $t\in[0,1]$, are invertible. Hence, we only have still to compute the density L of A.

Let (K_s^n) be the sequence of smooth step processes we have associated to (K_s) by Lemma 4.2, and keep the notations introduced above. Since due to (4.18) L is the limit of the uniformly integrable sequence (L^n) in the weak topology $\sigma(L_1, L_\infty)$, it suffices to prove

$$\begin{aligned}\int_0^1 K_s\,\mathrm{d}W_s+\frac{1}{2}\int_0^1 K_s^2\,\mathrm{d}s+\int_0^1\int_0^t D_s K_t\,D_t[K_s(A_t)](T_t)\,\mathrm{d}s\,\mathrm{d}t \\ =L_1(\Omega)-\lim_{n\to\infty}\left\{\int_0^1 K_s^n\,\mathrm{d}W_s+\frac{1}{2}\int_0^1 (K_s^n)^2\,\mathrm{d}s\right. \\ \left.+\int_0^1\int_0^t D_s K_t^n\,D_t[K_s^n(A_t^n)](T_t^n)\,\mathrm{d}s\,\mathrm{d}t\right\}.\end{aligned}$$

Since (K_s^n) tends to (K_s) in $L_{1,2}$, the problem reduces to the proof of

$$(4.22)\qquad E\left[\int_0^1\int_0^1 |D_t[K_s(A_t)](T_t)-D_t[K_s^n(A_t^n)](T_t^n)|^2\,\mathrm{d}s\,\mathrm{d}t\right]^{1/2}\to 0,$$

as $n\to\infty$.

In order to verify (4.22) we deduce from (4.17) for $t\to r$ by substitution of $T_r\,\omega$ that

$$(4.23)\qquad D_r\,[K_s(A_r)](T_r)=D_r\,K_s-\int_0^r D_u\,K_s\,D_r[K_u(A_r)](T_r)\,\mathrm{d}u, \qquad \text{a.e.}$$

Analogously we obtain

$$(4.24)\quad D_r[K_s^n(A_r^n)](T_r^n)=D_r\,K_s^n-\int_0^r D_u\,K_s^n\,D_r[K_u^n(A_r^n)](T_r^n)\,\mathrm{d}u, \qquad n=1,2,3,\dots$$

From here we see

$$\left\|\left(\int_0^1\int_0^1 |D_r[K_s^n(A_r^n)](T_r^n)|^2\,\mathrm{d}r\,\mathrm{d}s\right)^{1/2}\right\|_\infty \leqq \frac{1}{1-c_K^*}, \qquad n=1,2,3,\dots,$$

and so we can derive from (4.23) and (4.24)

$$E\left[\int_0^1\int_0^1 |D_r[K_s(A_r)](T_r)-D_r[K_s^n(A_r^n)](T_r^n)|^2\,\mathrm{d}r\,\mathrm{d}s\right]^{1/2}$$

$$\leqq\frac{2-c_K^*}{(1-c_K^*)^2}\cdot E\left[\int_0^1\int_0^1 |D_r K_s-D_r K_s^n|^2\,\mathrm{d}r\,\mathrm{d}s\right]^{1/2}.$$

Clearly, the right-hand side of this estimation tends to zero. Therefore, (4.22) is true. This completes the proof. □

Appendix

The *proof of Proposition 2.5* will be carried out in two steps. For the first step fix any partition

$$\Pi=\{0=t_0<t_1<\ldots<t_n=1\}$$

and put

$$\Delta_j=(t_{j-1},t_j],\quad |\Delta_j|=t_j-t_{j-1},\quad j=1,2,\ldots,n.$$

Let $\mathscr{F}^\pi$ be the σ-field on Ω generated by $W(\Delta_1),\ldots,W(\Delta_n)$, and $\mathscr{G}^\pi$ the σ-field on $[0,1]\times\Omega$ generated by all step processes $I_{\Delta_i}W(\Delta_j)$, $i,j=1,2\ldots,n$. Denote by μ the product measure of the Lebesgue measure and the Wiener measure P on $[0,1]\times\Omega$, and by E_μ its expectation.

Moreover, we need the notion of the Sobolev space $W_{1,2}(R^n,\mathscr{N}(0,B))$ of measurable functions $\varphi\colon R^n\to R$ such that $|\varphi(x)|^2+\sum_{j=1}^n\left|\frac{\partial}{\partial x_j}\varphi(x)\right|^2$ is integrable w.r.t. the mean zero Gaussian distribution with covariance matrix B.

We now state:

Lemma A.1 *Let $F\in\mathbb{D}_{1,2}$ and put $F^\pi=E[F|\mathscr{F}^\pi]$. Then, $F^\pi\in\mathbb{D}_{1,2}$, and*

(A.1) $$DF^\pi=E_\mu[DF|\mathscr{G}^\pi].$$

Moreover, there exists a $\varphi\in W_{1,2}(R^n,\mathscr{N}(0,B))$, $B=(|\Delta_j|\delta_{ij})$, such that

(A.2) $$F^\pi=\varphi(W(\Delta_1),\ldots,W(\Delta_n)).\quad\square$$

Proof. We first assume that $F\in\mathscr{S}$, i.e., there exist a partition of $[0,1]$ into subintervals $\Delta_1',\ldots,\Delta_m'$, a function $f\in C_b^\infty(R^m)$ and a natural m such that

(A.3) $$F=f(W(\Delta_1'),\ldots,W(\Delta_m')).$$

For each $k=1,2,\ldots,m$, denote by $\alpha_1^k,\ldots,\alpha_n^k$ those reals for which the function

(A.4) $$g_k=I_{\Delta_k'}-\sum_{j=1}^n\alpha_j^k I_{\Delta_j}$$

is orthogonal to $I_{\Delta_1}, \ldots, I_{\Delta_n}$ in $L_2([0,1])$, and set

$$\varphi(x_1, \ldots, x_m) = E[f(x_1 + \int g_1(t)\,\mathrm{d}W_t, \ldots, x_m + \int g_m(t)\,\mathrm{d}W_t)]. \tag{A.5}$$

Then obviously, $\varphi \in C_b^\infty(R^m)$ and

$$F^\pi = \varphi\left(\sum_{j=1}^n \alpha_j^1 W(\Delta_j), \ldots, \sum_{j=1}^n \alpha_j^m W(\Delta_j)\right), \tag{A.6}$$

i.e., (A.2) holds for $F \in \mathscr{S}$. Substituting

$$\int g_k(t)\,\mathrm{d}W_t = W(\Delta_k') - \sum_{j=1}^n \alpha_j^k W(\Delta_j)$$

in (A.3) we can deduce

$$DF = \sum_{k=1}^m \left(\frac{\partial}{\partial x_k} f\right)\left(\sum_{j=1}^n \alpha_j^1 W(\Delta_j) + \int g_1(t)\,\mathrm{d}W_t, \ldots, \sum_{j=1}^n \alpha_j^m W(\Delta_j) + \int g_m(t)\,\mathrm{d}W_t\right) \cdot \left\{\sum_{j=1}^n \alpha_j^k I_{\Delta_j} + g_k\right\},$$

so that (A.4) and (A.5) now yield

$$\begin{aligned} &E_\mu[DF \mid \mathscr{G}^\pi] \\ &= \sum_{k=1}^m \left(\frac{\partial}{\partial x_k}\varphi\right)\left(\sum_{j=1}^n \alpha_j^1 W(\Delta_j), \ldots, \sum_{j=1}^n \alpha_j^m W(\Delta_j)\right) \\ &\quad \cdot \left\{\sum_{j=1}^n \alpha_j^k I_{\Delta_j} + \int_0^1 g_k(t)\,\mathrm{d}t\right\}. \end{aligned} \tag{A.7}$$

Using again (A.4) in the last line,

$$\int_0^1 g_k(t)\,\mathrm{d}t = \sum_{j=1}^n \int_0^1 g_k(t) I_{\Delta_j}(t)\,\mathrm{d}t = 0,$$

we get from (A.6) that the right-hand side of (A.7) coincides with DF.

Now, in the second step, let $F \in \mathbb{D}_{1,2}$. Then there exists a sequence $(F_j) \subseteq \mathscr{S}$ which tends to F in $\mathbb{D}_{1,2}$. Put $F_j^\pi = E[F_j \mid \mathscr{F}^\pi]$. Then, due to the first step, $DF_j^\pi = E_\mu[DF_j \mid \mathscr{G}^\pi]$, and passing to the limit $j \to \infty$ shows that $F^\pi \in \mathbb{D}_{1,2}$ and (A.1) holds. Moreover, there are φ_j's of $C_b^\infty(R^n)$ such that

$$F_j^\pi = \varphi_j(W(\Delta_1), \ldots, W(\Delta_n)), \qquad j = 1, 2, 3, \ldots$$

Hence the convergence of (F_j^π) in $\mathbb{D}_{1,2}$ is equivalent to the convergence of (φ_j) in $W_{1,2}(R^n, \mathcal{N}(0, B))$. This completes the proof. □

We can prove now Proposition 2.5: Let $F \in \mathbb{D}_{1,2}$ and fix any $\varepsilon > 0$. Assume that $(\Pi_n = \{\Delta_1^n, \ldots, \Delta_n^n\})$ is an increasing sequence of partitions such that

$$\max_{j=1,2,\ldots,n} |\Delta_j^n| \to 0, \qquad \text{as } n \to \infty,$$

and put

$$F_n = E[F \mid \mathscr{F}^{\Pi_n}], \qquad n = 1, 2, 3, \ldots$$

From Lemma A.1 we see that

$$\|F_n\|_\infty \leqq \|F\|_\infty$$

and

$$\left\| \left(\int_0^1 |D_s F_n|^2 \, ds \right)^{1/2} \right\|_\infty \leqq \left\| \left(\int_0^1 |D_s F|^2 \, ds \right)^{1/2} \right\|_\infty, \qquad n = 1, 2, 3, \ldots$$

Hence, it suffices to prove Proposition 2.5 for any $F \in \mathbb{D}_{1,2}$ of the form

$$F = \varphi(W(\Delta_1), \ldots, W(\Delta_n)), \tag{A.8}$$

where $\Pi = \{\Delta_1, \ldots, \Delta_n\}$ is a partition of $[0, 1]$, $\varphi \in W_{1,2}(R^n, \mathscr{N}(0, B))$ and $B = (|\Delta_j| \cdot \delta_{ij})$.

We first assume additionally that φ has a compact support. Then we choose any nonnegative function $\psi \in C_0^\infty(R^n)$ with compact support such that $\int \psi(x)\, dx = 1$ and define the $C_0^\infty(R^n)$-functions

$$\varphi_h(x) = \frac{1}{h^n} \int \psi\left(\frac{1}{h} y\right) \varphi(x - y)\, dy, \qquad h > 0,$$

which converge to φ in $W_{1,2}(R^n, \mathscr{N}(0, B))$. If we now put

$$F_h = \varphi_h(W(\Delta_1), \ldots, W(\Delta_n)), \qquad h > 0,$$

it is not hard to see that the so defined sequence $(F_h) \subseteq \mathscr{S}$ tends to F in $\mathbb{D}_{1,2}$ as $h \to 0$, and

$$\|F_h\|_\infty \leqq \|F\|_\infty,$$

and

$$\left\| \left(\int_0^1 |D_s F_h|^2 \, ds \right)^{1/2} \right\|_\infty \leqq \left\| \left(\int_0^1 |D_s F|^2 \, ds \right)^{1/2} \right\|_\infty, \qquad h > 0.$$

Thus, it remains to consider $F \in \mathbb{D}_{1,2}$ given in the form (A.8), where φ does not have a compact support. Let $\psi \in C_0^\infty(R^n)$ be a nonnegative function with a compact support and assume

$$\psi(x) = 1, \quad \text{for } |x| \leqq 1, \quad \text{and} \quad 0 \leqq \psi(x) \leqq 1, \quad \text{for all } x \in R^n.$$

Then we put

$$\varphi_k(x) = \varphi(x)\, \psi\left(\frac{1}{k} x\right)$$

and

$$F_k = \varphi_k(W(\Delta_1), \ldots, W(\Delta_n)), \qquad k = 1, 2, 3, \ldots$$

Obviously, $|F_k|$ tends to F in $\mathbb{D}_{1,2}$, F_k has the form (A.8), where φ_k has a compact support

$$\|F_k\|_\infty \leqq \|F\|_\infty,$$

and

$$\left\|\left(\int_0^1 |D_s F_k|^2 \mathrm{d}s\right)^{1/2}\right\|_\infty$$
$$\leqq \left\|\left(\int_0^1 |D_s F|^2 \mathrm{d}s\right)^{1/2}\right\|_\infty + \frac{1}{k}\|F\|_\infty \cdot \sup_x \left(\sum_{j=1}^n \left|\frac{\partial}{\partial x_j}\psi(x)\right|^2 |\Delta_j|\right)^{1/2}$$
$$\leqq \varepsilon + \left\|\left(\int_0^1 |D_s F|^2 \mathrm{d}s\right)^{1/2}\right\|_\infty,$$

for all k's which are large enough.

But, for those F_k's we already know that they can be approximated in the required manner. □

The *proof of Proposition 2.6* can be reduced to Proposition 2.5, if we approximate $(K_s) \in L_{1,2}$ by the step process

$$\left(K_s^\pi = \sum_{j=1}^n \frac{1}{|\Delta_j|} \int_{\Delta_j} K_r \,\mathrm{d}r \cdot I_{\Delta_j}(s)\right)$$

for a partition $\Pi = \{\Delta_1, \ldots, \Delta_n\}$ of $[0, 1]$, since

(i) (K_s^Π) tends to (K_s) in $L_{1,2}$ if (Π) is monotonically increasing such that $|\Pi| = \max |\Delta_j| \to 0$, and

(ii) $\left\|\left(\int_0^1 |K_s^\Pi|^2 \mathrm{d}s\right)^{1/2}\right\|_\infty \leqq \left\|\left(\int_0^1 K_s^2 \,\mathrm{d}s\right)^{1/2}\right\|_\infty$

and

$$\left\|\left(\int_0^1 \int_0^1 |D_t K_s^\Pi|^2 \mathrm{d}s\,\mathrm{d}t\right)^{1/2}\right\|_\infty \leqq \left\|\left(\int_0^1 \int_0^1 |D_t K_s|^2 \mathrm{d}s\,\mathrm{d}t\right)^{1/2}\right\|_\infty.$$

In order to complete now the proof of Proposition 2.6 we only have to approximate $\frac{1}{|\Delta_j|} \int_{\Delta_j} K_r \,\mathrm{d}r$ in the sense of Proposition 2.5. □

References

1. Buckdahn, R.: Transformations on the Wiener space and Skorohodtype stochastic differential equations. Seminarbericht 105, Sekt. Math., Humboldt-Universität Berlin, 1989
2. Gihman, I.I., Skorohod, A.W.: Densities of probability measures in functional spaces. Usp. Mat. Nauk **6**, 83–156 (1966)

3. Ikeda, N., Watanabe, S.: Stochastic differential equations and diffusion processes. Amsterdam: North-Holland 1981
4. Kusuoka, S.: The non-linear transformation of Gaussian measure on Banach space and its absolute continuity (1). J. Fac. Sci., Univ. Tokyo, Sect. IA, **29**, 567–597 (1982)
5. Meyer, P.A.: Probability and potentials. Waltham: Blaisdell 1966
6. Nualart, D.: Noncausal stochastic integrals and calculus. In: Korezlioglu, H., Ustunel, A.S. (eds.) Stochastic analysis and related fields. Proceedings, Silivri 1986 (Lect. Notes Math., vol. 1316, pp. 80–129) Berlin Heidelberg New York: Springer 1988
7. Nualart, D., Pardoux, E.: Stochastic calculus with anticipating integrands. Probab. Th. Rel. Fields **78**, 535–581 (1988)
8. Nualart, D., Zakai, M.: Generalized stochastic integrals and the Malliavin calculus. Probab. Th. Rel. Fields **73**, 255–280 (1986)
9. Ramer, R.: On non-linear transformations of Gaussian measures. J. Funkt. Anal. **15**, 166–187 (1974)