Probab. Th. Rel. Fields 89, 201–210 (1991)

Probability Theory and Related Fields


# On the domain of attraction of an operator between supremum and sum

**Priscilla E. Greenwood**[1] **and Gerard Hooghiemstra**[2]

[1] University of British Columbia, Mathematical Institute, 121-1984 Mathematics Road, Vancouver, B.C. Canada V6T 1Y4
[2] Delft University of Technology, Faculty of Mathematics and Informatics, Department of Statistics, Probability and Operations Research, Mekelweg 4, 2628 CD Delft, The Netherlands

Received September 19, 1990; in revised form March 26, 1991

**Summary.** Between the operations which produce partial maxima and partial sums of a sequence $Y_1, Y_2, \ldots,$ lies the inductive operation: $X_n = X_{n-1} \vee (\alpha X_{n-1} + Y_n)$, $n \geqq 1$, for $0 < \alpha < 1$. If the $Y_n$ are independent random variables with common distribution $F$, we show that the limiting behavior of normed sequences formed from $\{X_n, n \geqq 1\}$, is, for $0 < \alpha < 1$, parallel to the extreme value case $\alpha = 0$. For $F \in D(\Phi_\gamma)$ we give a full proof of the convergence, whereas for $F \in D(\Psi_\gamma) \cup D(\Lambda)$, we only succeeded in proving tightness of the involved sequence. The process $X_n$ is interesting for some applied probability models.

## 1 Introduction

Let $Y_1, Y_2, \ldots,$ be a sequence of independent random variables with distribution $F$. We define

$$X_0 := x_0 \in \mathbb{R}$$
$$X_n := \max\{X_{n-1}, \alpha X_{n-1} + Y_n\}, \qquad n \geqq 1, \tag{1.1}$$

where $\alpha$ is a parameter with range $[0, 1)$. In this paper we study the existence of norming constants $a_n(\alpha)$ and $b_n(\alpha)$ such that

$$\{X_n - b_n(\alpha)\}/a_n(\alpha)$$

converges in distribution to a non-degenerate limit.

For $\alpha = 0$,

$$X_n = \max\{x_0, Y_1, \ldots, Y_n\}.$$

In this case the limit laws are well known and the domain of attraction problem has been solved by Gnedenko [3] and by de Haan [6]. For $\alpha=1$, a value *not* included in this study,

$$X_n = x_0 + \sum_{j=1}^{n} Y_j 1_{[0,\infty)}(Y_j),$$

in which case the limits are stable laws. A complete description of the domain of attraction problem for $\alpha=1$ can be found in Gnedenko and Kolmogorov [4].

Instead of studying the sequence $X_n$ in (1.1) directly it is more useful to study a stochastic process $X_n(\cdot)$ which we introduce below. We denote by $D(G)$ the domain of attraction of the distribution $G$, where $G$ is one of the extreme value distributions. For $F\in D(G)$ and $a_n>0$, $b_n\in\mathbb{R}$ such that $F^n(a_n x+b_n)\to G(x)$, for all $x$, we define for $n\geqq 1$,

$$(1.2)\qquad Y_{n,j} := (Y_j - b_n)/a_n, \qquad j=1,2,\ldots.$$

Then $X_n(\cdot)\in D[0,\infty)$ is defined by

$$(1.3)\; X_n(t) := a_n^{-1}(x_0 - b_n/(1-\alpha)), \quad 0\leqq t<n^{-1},$$

$$X_n(t) := \max\left\{X_n\left(\frac{j-1}{n}\right), \alpha X_n\left(\frac{j-1}{n}\right)+Y_{nj}\right\}, \quad \frac{j}{n}\leqq t<\frac{j+1}{n}, \quad j=1,2,\ldots,$$

where $x_0\in\mathbb{R}$. The process $X_n(\cdot)$ is called the $\alpha$-sun process with input sequence $Y_{nj}$, $j=1,2,\ldots$, and initial value $x_0$. The algebraic relation between $X_n$ and $X_n(\cdot)$ is obviously,

$$(1.4)\qquad X_n(j/n) = a_n^{-1}(X_j - b_n/(1-\alpha)).$$

In Sect. 2 we show that for $F\in D(\Phi_\gamma)$, where $\Phi_\gamma(x)=\exp(-x^{-\gamma})1_{[0,\infty)}(x)$, $\gamma>0$, and for all initial values $x_0$ the $\alpha$-sun processes $X_n(\cdot)$ converge weakly on $D[0,\infty)$. In Sect. 3 we formulate the results we obtained for $F\in D(\Psi_\gamma)$, where $\Psi_\gamma(x) = \exp(-(-x)^\gamma)1_{(-\infty,0]}(x)+1_{(0,\infty)}(x)$, $\gamma>0$, and $F\in D(\Lambda)$, where $\Lambda(x)=\exp(e^{-x})$. Basically the only thing we are able to show in these cases is tightness. Because the proofs are long and technical, and the results unsatisfactory, they are omitted. Details of these proofs can be obtained on request through the second author.

The interest in sequences of the type (1.1) originates from a storage problem for solar energy described by Haslett [7]. Additional studies are Daley and Haslett [1], Haslett [8], Hooghiemstra and Keane [9], Hooghiemstra and Scheffer [10] and Greenwood and Hooghiemstra [5]. In these studies the energy contents $X_n^{(\beta)}$ of a storage tank is analyzed, where $X_n^{(\beta)}$ satisfies the difference equation

$$(1.5)\qquad X_n^{(\beta)} = \max\{\beta X_{n-1}^{(\beta)}, \alpha\beta X_{n-1}^{(\beta)}+Y_n\}, \qquad \beta\in(0,1).$$

In [10] the limit behavior of $X_\infty^{(\beta)}$, the stationary solution of (1.5), for $\beta\to 1$ and $F(x)=1-x^{-\gamma}(x>1)$ was obtained. It turns out that the limits obtained in this paper for $F\in D(\Phi_\gamma)$ are of the same type as those in [10]. Intuitively this is because the order of the limit procedure has been interchanged, which apparently is allowed. The question whether this can be proved is not easy to answer and depends on the speed of convergence of $X_n^{(\beta)}$ to the stationary limit $X_\infty^{(\beta)}$. This matter will not be treated here.

The study of iterative sequences of the form (1.1) is also of some theoretical interest. The behavior of $X_n$ is somewhere between sups and sums. In fact they turn out to be close to suprema of i.i.d. random variables. The limit processes are new examples of self-similar Markov processes. In a forthcoming paper we study the situation where $\alpha$ depends on $n$ i.e., $\alpha_n = 1 - n^{-l}$, $l > 0$. For $0 < l < 1$ sequences $X_n$ satisfying (1.1) with $\alpha$ replaced by $\alpha_n$ behave similar to the sum of the $n^l$-largest order statistics.

## 2 Convergence for case I

We introduce an operator that enables us to write the $\alpha$-sun process $X_n(\cdot)$, defined in (1.3) as a function of point processes.

**Definition 1** For a countable collection $\xi = \{(x_j, y_j)\}_{j=1}^{\infty}$ of pairs of real numbers such that

(i) $0 < x_1 < x_2 < \ldots$,
(ii) card $\{x_j \colon x_j \leqq s\} < \infty$ for each positive $s$;

and for an arbitrary initial value $u \in \mathbb{R}$, we define the element $T_u(\xi)(\cdot) \in D[0, \infty)$ by

$$(2.1) \quad \begin{aligned} &T_u(\xi)(t) = u, \qquad x_0 = 0 \leqq t < x_1, \\ &T_u(\xi)(t) = \max\{T_u(\xi)(x_{j-1}), \alpha T_u(\xi)(x_{j-1}) + y_j\}, \qquad x_j \leqq t < x_{j+1}, \qquad j \geqq 1. \end{aligned}$$

The operator $T_u$ will be called the $\alpha$-sun operator. If $u = 0$ we write $T$ instead of $T_0$; occasionally we abbreviate $T_{u_n}$ by $T_n$. The element $X_n(\cdot)$ defined in (1.3) can be expressed as

$$X_n(\cdot) = T_{u_n}(\xi_n)(\cdot),$$

where $u_n = a_n^{-1}(x_0 - b_n/(1-\alpha))$ and $\xi_n = \{(j/n, Y_{nj})\}_{j=1}^{\infty}$ with $Y_{nj}$ defined in (1.2).

The following simple property of $\alpha$-sun operators is used repeatedly.

**Lemma 1** *Let $\xi^{(1)} = \{(x_j, y_j^{(1)})\}_{j=1}^{\infty}$ and $\xi^{(2)} = \{x_j, y_j^{(2)})\}_{j=1}^{\infty}$ be two collections satisfying* (i) *and* (ii) *of Definition 1 and having identical x-values. Suppose that for some $j_0$, $y_j^{(1)} = y_j^{(2)}$ for all $j \geqq j_0$. If $T_i$, $i = 1, 2$, denotes the $\alpha$-sun operator applied to $\xi^{(i)}$ and if $T_1(x_{j_0}) - T_2(x_{j_0}) = c > 0$, then for all $t \geqq x_{j_0}$,*

$$0 \leqq T_1(t) - T_2(t) \leqq c,$$

*i.e., the paths only become closer but do not cross.*

*Proof.* The result is obviously independent of the initial values of $T_i$. The proof goes by induction. Suppose that for some $j \geqq j_0$,

$$0 \leqq T_1(x_j) - T_2(x_j) \leqq c.$$

If $y_{j+1}^{(1)} > (1-\alpha) T_1(x_j)$ then $y_{j+1}^{(2)} = y_{j+1}^{(1)} > (1-\alpha) T_2(x_j)$, both $T_1$ and $T_2$ jump at $x_{j+1}$ and paths move closer by a factor $\alpha$. If

$$(1-\alpha) T_1(x_j) \geqq y_{j+1}^{(1)} = y_{j+1}^{(2)} > (1-\alpha) T_2(x_j),$$

then

$$\alpha T_2(x_j) + y_{j+1}^{(2)} < \alpha T_1(x_j) + y_{j+1}^{(1)} \leqq T_1(x_j),$$

so $T_2$ moves closer but does not cross. Finally if $y_{j+1}^{(1)} = y_{j+1}^{(2)} \leqq (1-\alpha) T_2(x_j)$, then neither $T_1$ nor $T_2$ has a jump at $x_{j+1}$. □

**Theorem 1** *For $F \in D(\Phi_\gamma)$ the α-sun process $X_n(\cdot)$ with arbitrary initial value $x_0 \in \mathbb{R}$, converges weakly in $D[0, \infty)$ to a proper limit which will be denoted by $Z(\cdot)$.*

*Proof.* Assume $x_0 = 0$, the case $x_0 \neq 0$ will be treated at the end. Put $b_n = 0$ and $a_n = \inf\{y\colon 1 - F(y) \leqq n^{-1}\}$, then $X_n(\cdot)$ is defined without ambiguity and $X_n(0) = 0$. By hypothesis

$$(2.2) \qquad n(1 - F(a_n x)) \to \mu(x, \infty), \qquad x > 0,$$

where $\mu(x, \infty) := x^{-\gamma}$. Fix $s > 0$ and denote by $\xi_n$ the collection $(j/n, Y_j/a_n)$, $j = 1, 2, \ldots, [ns]$, and by $\xi_n(\delta)$, $\delta > 0$, the collection of points $(j/n, Y_j/a_n) \in \xi_n$ with $Y_j \geqq a_n \delta$. Further we denote by $\xi(\delta)$ the points (or better the support) of a Poisson random measure, with intensity $dt \times d\mu$, restricted to the set $[0, s] \times [\delta, \infty)$. Since $\mu(\delta, \infty) = \delta^{-\gamma} < \infty$ the number of points in $\xi(\delta)$ is a.s. finite. The point process (random measure) which gives unit measure to each of the points $(j/n, Y_j(\omega)/a_n) \in \xi_n(\delta)$ will also be notated by $\xi_n(\delta)$. According to Proposition 3.1 of Resnick [12] the convergence (2.2) implies that $\xi_n(\delta)$ converges vaguely to $\xi(\delta)$ on the space of point measures defined on $[0, s] \times [\delta, \infty)$. Since the map $(x, y) \to x \vee (\alpha x + y)$ from $\mathbb{R}^2 \to \mathbb{R}$ is continuous we conclude that for fixed $\delta > 0$,

$$(2.3) \qquad T(\xi_n(\delta))(\cdot) \xrightarrow{d} T(\xi(\delta))(\cdot) \quad \text{on } D[0, s].$$

We will show that

(a) $T(\xi(\delta))(\cdot)$ has an almost sure limit $Z(\cdot)$ as $\delta \to 0$, which is proper;
(b) $\sup_{0 \leqq t \leqq s} |X_n(t) - T(\xi_n(\delta))(t)| \leqq 2\delta/(1-\alpha)$.

From (b), (a) and (2.3) it follows that $X_n(\cdot) \xrightarrow{d} Z(\cdot)$ on $D[0, s]$, for each positive $s$.

*Proof of* (b). We have $X_n(t) = T(\xi_n)(t) \geqq T(\xi_n(\delta))(t)$ for all $t \in [0, s]$, since $\xi_n$ contains more points than $\xi_n(\delta)$. The points $(j/n, Y_j/a_n) \in \xi_n \setminus \xi_n(\delta)$ are discarded as soon as $X_n(t) > \delta/(1-\alpha)$. Hence we may assume that $X_n(\cdot)$ and $T(\xi_n(\delta))(\cdot)$ see the same input for $t > \tau(\omega) := \min\{t\colon X_n(t) > \delta/(1-\alpha)\}$.

For $t < \tau$, $0 \leqq X_n(t) - T(\xi_n(\delta))(t) \leqq \delta/(1-\alpha)$. For $t = \tau$, $X_n(\tau) = \alpha X_n(\tau-) + Y_{nj}$, where $j/n = \tau(\omega)$. Either $Y_{nj} > \delta$ in which case $X_n(\tau) - T_n(\xi(\delta))(\tau) \leqq \alpha\delta/(1-\alpha)$, or $Y_{nj} \leqq \delta$ in which case $X_n(\tau) - T(\xi_n(\delta))(\tau) \leqq \delta/(1-\alpha) + \delta \leqq 2\delta/(1-\alpha)$. Now we apply Lemma 1 to obtain (b).

*Proof of (a)*. If $0<\delta'<\delta$, then $T(\xi(\delta'))(\cdot)$ sees more input than $T(\xi(\delta))(\cdot)$, namely points of $\xi$ with second coordinates in $[\delta', \delta)$; hence $T(\xi(\delta'))(t) \geqq T(\xi(\delta))(t)$ for all $t \in [0, s]$. Because of monotonicity

$$Z = \lim_{\delta \downarrow 0} T(\xi(\delta)),$$

exists almost sure. We know that $T(\xi(\delta))(t)$ is proper, because $T(\xi(\delta))(t)$ operates on only finitely many points. For $\delta'<\delta$ we have by the argument in the proof of (b)

$$0 \leqq T(\xi(\delta'))(t) - T(\xi(\delta))(t) \leqq 2\delta/(1-\alpha),$$

so that

$$\sup_{\delta'<\delta} T(\xi(\delta'))(t) \leqq T(\xi(\delta))(t) + 2\delta/(1-\alpha).$$

Finally we turn to the case where $x_0 \neq 0$. Then $X_n(\cdot) = T_{u_n}(\xi_n)(\cdot)$, and $u_n = x_0/a_n \to 0$. $X_n(\cdot)$ and $T(\xi_n)(\cdot)$ have the same input so that according to Lemma 1,

$$0 \leqq \sup_{t \leqq s} |T_{u_n}(\xi_n)(t) - T(\xi_n)(t)| \leqq u_n.$$

Hence $T_{u_n}(\xi_n)(\cdot)$ will also converge to $Z(\cdot)$ on $D[0, s]$. □

It is seen from the proof of Theorem 1 that the process $Z(s)$, $s \geqq t$, given $Z(t) = z > 0$, can be defined as an $\alpha$-sun process with input the Poisson process $\xi$ restricted to $[t, \infty) \times [(1-\alpha)z, \infty)$. Moreover, as $Z(t)$ is the weak limit of $X_{[nt]}/a_{[nt]}$ and $\lim_{n \to \infty} a_{[nt]}/a_n = t^{1/\gamma}$, it follows that $Z(\cdot)$ is self-similar with index $H = \gamma^{-1}$. Self-similar Markov processes have been extensively studied by Lamperti [11]. We focus on the distribution of $Z(1)$.

**Theorem 2** *For* $F \in D(\Phi_\gamma)$, $\gamma > 0$, *the distribution of the limit random variable* $Z(1)$ *has a density* $h_\alpha$ *concentrated on* $(0, \infty)$. *This density is the unique density solution of the equation*

$$(2.4) \qquad h_\alpha(x) = \gamma x^{-1} \int_0^x (x-\alpha u)^{-\gamma} h_\alpha(u)\, \mathrm{d}u, \qquad x>0.$$

In order to prove Theorem 2 we use the self-similarity together with the following relation between the law of $Z(\cdot)$ and the measure $\mu(x, \infty) = x^{-\gamma}$, $x > 0$.

**Lemma 2** *The law of* $Z(\cdot)$ *satisfies for* $t>0$ *and* $x>0$,

$$(2.5) \qquad P\{Z(t)>x\} = \int_0^t \int_0^x (x-\alpha u)^{-\gamma} P\{Z(s) \in \mathrm{d}u\}\, \mathrm{d}s.$$

*Proof.* From Definition (1.3) with $x_0 = 0$ we have for $x > 0$,

$$P\{X_n(j/n) \leqq x\} - P\left\{X_n\left(\frac{j-1}{n}\right) \leqq x\right\}$$
$$= -\int_0^x \{1 - F(a_n(x-\alpha u))\}\, P\left\{X_n\left(\frac{j-1}{n}\right) \in \mathrm{d}u\right\},$$

and iteration of this formula yields

$$P\{X_n(j/n)\leqq x\}-P\left\{X_n\left(\frac{j-k}{n}\right)\leqq x\right\}$$
$$=-\sum_{m=1}^{k}\int_0^x\{1-F(a_n(x-\alpha u))\}\,P\left\{X_n\left(\frac{j-m}{n}\right)\in \mathrm{d}u\right\}.$$

Fix $t>0$ and put $k=j=[nt]$, then for $x>0$,

$$(2.6)\qquad P\{X_n(t)>x\}=P\left\{X_n\left(\frac{j}{n}\right)>x\right\}=P\left\{X_n\left(\frac{j}{n}\right)>x\right\}-P\{X_n(0)>x\}$$
$$=\sum_{m=1}^{[nt]}\int_0^x\{1-F(a_n(x-\alpha u))\}\,P\left\{X_n\left(\frac{[nt]-m}{n}\right)\in \mathrm{d}u\right\}$$
$$=\int_{s=0}^{s=\frac{[nt]-1}{n}}\int_0^x n\{1-F(a_n(x-\alpha u))\}\,P\{X_n(s)\in \mathrm{d}u\}.$$

The final equality in (2.6) follows since $X_n(s)$ is a stepfunction.

Now fix $x\in\mathrm{Cont}(Z(t))$, the set of continuity points of the distribution of $Z(t)$. The left-hand side of (2.6) converges to $P\{Z(t)>x\}$, so we focus on the right-hand side. Note that for $u\in[0, x]$,

$$\lim_{n\to\infty} n\{1-F(a_n(x-\alpha u))\}=(x-\alpha u)^{-\gamma},$$

and by Karamata's theory this convergence is uniform for $u\in[0, x]$ (cf. [2], Theorem 1.3). Hence for each $\varepsilon>0$ and $n$ sufficiently large

$$(2.7)\qquad \int_0^{\frac{[nt]-1}{n}}\int_0^x |n\{1-F(a_n(x-\alpha u))\}-(x-\alpha u)^{-\gamma}|\,P\{X_n(s)\in \mathrm{d}u\}\,\mathrm{d}s$$
$$\leqq\varepsilon\int_0^{\frac{[nt]-1}{n}}\int_0^x P\{X_n(s)\in \mathrm{d}u\}\,\mathrm{d}s\leqq\varepsilon t.$$

For $s\in(0, t)$ we define

$$f_n(s):=\int_0^x (x-\alpha u)^{-\gamma}P\{X_n(s)\in \mathrm{d}u\},$$

$$f(s):=\int_0^x (x-\alpha u)^{-\gamma}P\{Z(s)\in \mathrm{d}u\}.$$

For $x \in \mathrm{Cont}(Z(s))$ we have $f_n(s) \to f(s)$ (this is standard weak convergence theory and follows since $(x-\alpha u)^{-\gamma} 1_{[0,x]}(u)$ is bounded and is only discontinuous at $u=x$). Furthermore

$$|f_n(s)| = \int_0^x (x-\alpha u)^{-\gamma} P\{X_n(s) \in \mathrm{d}u\} \leqq x^{-\gamma}(1-\alpha)^{-\gamma},$$

and so

$$\int_0^t |f_n(s)|\,\mathrm{d}s \leqq \int_0^t x^{-\gamma}(1-\alpha)^{-\gamma}\,\mathrm{d}s = t x^{-\gamma}(1-\alpha)^{-\gamma} < \infty.$$

By Lebesgues dominated convergence theorem $\int_0^t f_n(s)\,\mathrm{d}s \to \int_0^t f(s)\,\mathrm{d}s$ if we can prove that $f_n(s) \to f(s)$ Lebesgue – almost everywhere in $(0, t)$. The latter statement is true because by self-similarity, the set,

$$\{s \in (0, t)\colon \text{ the distribution function of } Z(s) \text{ is discontinuous at } x\}$$

has the same cardinality as the set of all discontinuities of the distribution function of $Z(1)$, which set evidently is countable. We conclude that

$$\lim_{n\to\infty} \int_0^t \int_0^x (x-\alpha u)^{-\gamma} P\{X_n(s) \in \mathrm{d}u\}\,\mathrm{d}s = \int_0^t \int_0^x (x-\alpha u)^{-\gamma} P\{Z(s) \in \mathrm{d}u\}\,\mathrm{d}s. \tag{2.8}$$

Finally

$$\left| \int_{\frac{[nt]-1}{n}}^{t} \int_0^x (x-\alpha u)^{-\gamma} P\{X_n(s) \in \mathrm{d}u\}\,\mathrm{d}s \right| \leqq 2x^{-\gamma}(1-\alpha)^{-\gamma}/n \to 0. \tag{2.9}$$

Combining (2.7), (2.8) and (2.9) it follows that the right-hand side of (2.6) converges to

$$\int_0^t f(s)\,\mathrm{d}s = \int_0^t \int_0^x (x-\alpha u)^{-\gamma} P\{Z(s) \in \mathrm{d}u\}\,\mathrm{d}s. \quad \square$$

*Proof of Theorem 2.* By self-similarity we have for each $t>0$,

$$P\{Z(t)>x\} = P\{Z(1)>x t^{-1/\gamma}\}. \tag{2.10}$$

Using (2.10) we can write the integral equation (2.5) entirely in terms of the distribution of $Z(1)$,

$$P\{Z(1)>x t^{-1/\gamma}\} = \int_0^t \int_0^x (x-\alpha u)^{-\gamma} P\{Z(1) \in s^{-1/\gamma}\,\mathrm{d}u\}\,\mathrm{d}s.$$

Next differentiate with respect to $t$ and set $t=1$ afterwards to obtain (2.4). Finally, Hooghiemstra and Scheffer [10] showed that the integral equation

$$g_\alpha(x)=x^{-1}\int_0^x (x-\alpha u)^{-\gamma} g_\alpha(u)\,\mathrm{d}u,$$

has a unique density solution. From this we obtain the remaining conclusion of Theorem 2, since if $X$ denotes an absolutely continuous random variable with density $g_\alpha$, then $Y:=\gamma^{1/\gamma}X$ is absolutely continuous with density $h_\alpha$, satisfying (2.4). □

## 3 Further results

As mentioned in the introduction the results for $F\in D(\Psi_\gamma)$ and $F\in D(\Lambda)$ are incomplete. In this section we formulate tightness results for the two indicated domains of attraction and convergence results for some special distributions $F$. We close the section with an elegant and short proof of thightness when $F$ is the uniform distribution on $[-1,0]$. We denote by $r$ the right end point of $F$, i.e., $r=\sup\{x\colon F(x)<1\}$.

**Theorem 3** *Let $F\in D(\Psi_\gamma)$, i.e., $r<\infty$ and $1-F(r-x^{-1})=x^{-\gamma}L(x)$ for some positive $\gamma$ and some slowly varying function L. Then there exist sequences $a_n>0$ and $b_n\in\mathbb{R}$ such that for all initial values $x_0<r/(1-\alpha)$ the sequence $a_n^{-1}(X_n-b_n/(1-\alpha))$ is tight on $(-\infty,0)$. A possible choice of $\{a_n\}$ and $\{b_n\}$ is*

$$a_n=r-\inf\{x\colon 1-F(x)\leqq n^{-1}\},\qquad b_n\equiv r.\quad □$$

**Theorem 4** *If $F\in D(\Lambda)$, then there exist sequences $\{a_n\}$ and $\{b_n\}$ such that for all initial values $x_0<r/(1-\alpha)$ the sequence $a_n^{-1}(X_n-b_n/(1-\alpha))$ is tight on $\mathbb{R}$. A possible choice of $\{a_n\}$ and $\{b_n\}$ is*

$$a_n=U^*(ne)-U^*(n),\qquad b_n=U^*(n),$$

*where for $x>1$,*

$$U^*(x)=\inf\{y\colon (1-F(y))^{-1}\geqq x\}.\quad □$$

**Theorem 5** *Let $Y_1, Y_2, \ldots,$ be independent with common distribution $F(x)=1-|x|^\gamma$, $-1\leqq x\leqq 0$, $F(x)=0$, $x<-1$ and $F(x)=1$, $x>0$. Let $a_n=n^{-1/\gamma}$, $b_n=0$. For each $x_0\in[-1,0)$ the $\alpha$-sun processes $X_n(\cdot)$, with initial value $x_0 n^{1/\gamma}$ converge weakly in $D(0,\infty)$ to a proper limit $Z(\cdot)$.* □

**Theorem 6** *Let $Y_1, Y_2, \ldots,$ be independent with common distribution F, given by $F(x)=1-\mathrm{e}^{-x}$ on $[0,\infty)$, and let $a_n=1$, $b_n=\log n$. For each $x_0\geqq 0$ the $\alpha$-sun processes $X_n(\cdot)$ with initial value $x_0-\log n/(1-\alpha)$ converge weakly in $D(0,\infty)$ to a proper limit $Z(\cdot)$.* □

**Theorem 7** *Suppose $Z(\cdot)$ is one of the limit processes of Theorem 5 or 6. (i) For $F(x)=1-|x|^\gamma$, on $[-1,0]$, the process $Z(\cdot)$ is a self-similar Markov process with index $H=-\gamma^{-1}$. The law of $Z(t)$ satisfies*

$$P\{Z(t)>x\}=\int_0^t \int_{x/\alpha}^x |x-\alpha u|^\gamma P\{Z(s)\in\mathrm{d}u\}\,\mathrm{d}s, \tag{3.1}$$

*for $t>0$ and $x<0$. In particular we obtain from* (3.1) *that the law of $Z$ is independent of $x_0 \in [-1, 0)$ and that the marginal distribution of $Z(1)$ admits a density on $(-\infty, 0)$, denoted by $h_\alpha$, where $h_\alpha$ is the unique density solution of*

$$h_\alpha(x)=\gamma|x|^{-1}\int_{x/\alpha}^{x}|x-\alpha u|^{\gamma}h_\alpha(u)\,du, \qquad x<0. \tag{3.2}$$

*(ii) For $F(x)=1-e^{-x}$, $x>0$, the process $\exp\{Z(t)\}$, $t>0$, is a self-similar Markov process with index $H=(1-\alpha)^{-1}$. The law of $Z(t)$ satisfies*

$$P\{Z(t)>x\}=\int_0^t\int_{-\infty}^{x} e^{-(x-\alpha u)}P\{Z(s)\in du\}\,ds, \tag{3.3}$$

*for $t>0$ and $x\in\mathbb{R}$. In particular we obtain from* (3.3) *that the law of $Z$ is independent of $x_0\in[0,\infty)$ and that the marginal distribution of $Z(1)$ has density,*

$$h_\alpha(x)=(1-\alpha)(\Gamma((1-\alpha)^{-1}))^{-1}\exp\{-x-e^{-x(1-\alpha)}\}, \qquad x\in\mathbb{R}, \tag{3.4}$$

*where $\Gamma(t):=\int_0^\infty x^{t-1}e^{-x}\,dx$, $t>0$.* □

The following lemma is the basis for the proof of Theorem 3 and 4. It shows tightness for the special case where $Y_1$ is uniformly distributed on $[-1, 0]$. The proof of the lemma originates from Tom Liggett (private communication).

**Lemma 3** *If $X_1=x_1\in[-1,0)$, $X_{n+1}=\max\{X_n, \alpha X_n+Y_{n+1}\}$, $n\geq 1$, where $Y_1, Y_2, \ldots$ is an* i.i.d. *sequence with $Y_1$ uniformly distributed on $[-1, 0]$, then*

$$\inf_{n\geq 1} EnX_n \geq -2(1-\alpha)^{-2}.$$

*Proof.* $E[X_n|X_{n-1}]=X_{n-1}+\frac{1}{2}(1-\alpha)^2X_{n-1}^2$, hence taking double expectations

$$EX_n=EX_{n-1}+\tfrac{1}{2}(1-\alpha)^2EX_{n-1}^2\geq EX_{n-1}+\tfrac{1}{2}(1-\alpha)^2(EX_{n-1})^2. \tag{3.5}$$

Put $g(u):=u+\frac{1}{2}(1-\alpha)^2u^2$, $u\in[-1,0]$. Then $g'(u)=1+(1-\alpha)^2u>0$ on $[-1,0]$ and hence $g$ is increasing on $[-1,0]$. Set $u_n=EX_n$ and $v_n=\dfrac{-2(1-\alpha)^{-2}}{n}$.

We shall prove by induction that $u_n\geq v_n$ for all $n\geq 1$. The statement is trivially true for $n=1$, so assume $u_{n-1}\geq v_{n-1}$. Then from (3.5) and the monotonicity of $g$,

$$u_n\geq g(u_{n-1})\geq g(v_{n-1}).$$

A simple calculation yields $g(v_{n-1})\geq v_n$, and this completes the proof. □

*Acknowledgements.* Discussions with Laurens de Haan, Frank den Hollander, Mike Keane and Wim Vervaat have been valuable to us. We thank Tom Liggett for the proof of Lemma 3.

## References


1. Daley, D.J., Haslett, J.: A thermal energy storage process with controlled input. Adv. Appl. Probab. **14**, 257–271 (1982)
2. Geluk, J.L., Haan, L. de: Regular variation, extensions and tauberian theorems. CWI Tract **40**, CWI, Amsterdam 1987
3. Gnedenko, B.: Sur la distribution limite du terme maximum d'une série aléatoire. Ann. Math. **44**, 423–453 (1943)
4. Gnedenko, B., Kolmogorov, A.N.: Limit distributions for sums of independent random variables. Reading, Mass: Addison Wesley 1968
5. Greenwood, P.E., Hooghiemstra, G.: An extreme-type limit law for a storage process. Math. Oper. Res. **13**, 232–242 (1988)
6. Haan, L. de: On regular variation and its application to the weak convergence of sample extremes. Mathematical Centre Tracts **32**, CWI, Amsterdam 1970
7. Haslett, J.: Problems in the storage of solar thermal energy. In: Jacobs, O.L.R., et al. (eds.) Analysis and optimization of stochastic systems. London: Academic Press 1980
8. Haslett, J.: New bounds for the thermal energy storage process with stationary input. J. Appl. Probab. **19**, 894–899 (1982)
9. Hooghiemstra, G., Keane, M.: Calculation of the equilibrium distribution for a solar energy storage model. J. Appl. Probab. **22**, 852–864 (1985)
10. Hooghiemstra, G., Scheffer, C.L.: Some limit theorems for an energy storage model. Stochastic Processes Appl. **22**, 121–128 (1986)
11. Lamperti, J.: Semi-stable Markov processes, I. Z. Wahrscheinlichkeitstheor. Verw. Geb. **22**, 205–225 (1972)
12. Resnick, S.I.: Point processes, regular variation and weak convergence. Adv. Appl. Probab. **18**, 66–138 (1986)