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Saddlepoint expansions for sums of Markov dependent variables on a continuous state space

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Summary. Based on the conjugate kernel studied in Iscoe et al. (1985) we derive saddlepoint expansions for either the density or distribution function of a sum $f(X_1)+\cdots+f(X_n)$, where the X_i's constitute a Markov chain. The chain is assumed to satisfy a strong recurrence condition which makes the results here very similar to the classical results for i.i.d. variables. In particular we establish also conditions under which the expansions hold uniformly over the range of the saddlepoint. Expansions are also derived for sums of the form $f(X_1, X_0)+f(X_2, X_1)+\cdots+f(X_n, X_{n-1})$ although the uniformity result just mentioned does not generalize.

1 Introduction

In this paper we will generalize some of the results in Höglund (1974) to continuous state Markov chains by combining structure results in Iscoe et al. (1985) with expansion results from Jensen (1987, 1988). We consider a Markov chain $X_0, X_1, X_2, \ldots$ on the state space $\mathbb{E}$ and the sum

$$S_n = f_0(X_0) + \sum_1^n f(X_i) \tag{1.1}$$

for functions $f_0, f\colon \mathbb{E} \to \mathbb{R}$. We then derive saddlepoint expansions for the density or tail probabilities of S_n at points of the form nz. The results may be illustrated by the following example.

Example 1.1. This example is an elaboration on an example from Cox and Miller (1965, Sect. 3.12). It has been chosen because it is of some practical interest and because the calculations can be done fairly easily.

We consider a Markov chain $X_n = (Y_n, W_n)$ on $\mathbb{E} = \{1, \ldots, d\} \times \mathbb{R}$ with transition density $p_{ij} f_{ij}(w)$ with respect to the product of counting measure and Lebesgue measure. Here p_{ij} are the transition probabilities for the discrete chain Y_n. We assume that $p_{ij} > 0$ for all i, j and that there exists a density g and constants $0<a<b<\infty$ such that $ag(w)\leqq f_{ij}(w)\leqq bg(w)$ for all i, j. Let $\varphi_{ij}(s) = \int \exp(sw) f_{ij}(w)\,dw$ be the Laplace transform of f_{ij}. Define $Q(s)$ as the $d\times d$ matrix with (i,j)'th element $p_{ij}\varphi_{ij}(s)$ and let $\lambda(s)$ be the maximal eigenvalue with corresponding right eigenvector $(r_1(s), \ldots, r_d(s))$ and left eigenvector $(c_1(s), \ldots, c_d(s))$. Finally define $l_j(w)=\lambda(s)^{-1}\exp(sw)\sum_i c_i(s)\,p_{ij} f_{ij}(w)$ and normalize such that $\max_i\{r_i(s)\}=1$ and $\sum_i r_i(s)\,c_i(s)=1$.

Considering the sum $S_n=\sum_1^n W_i$ we derive in this paper the following approximations

$$(1.2)\qquad \frac{dS_n(p)}{dm}(nA'(s))=\frac{\exp\{n(A(s)-A'(s))\}}{\{2\pi nA''(s)\}^{1/2}}\{\gamma_0(s)+O(n^{-1})\},$$

where m is Lebesgue measure on $\mathbb{R}$, and for $s>0$

$$(1.3)\qquad P(S_n>nA'(s))=\frac{\exp\{n(A(s)-A'(s))\}}{s\{nA''(s)\}^{1/2}}[\gamma_0(s)\,B_0\{s(nA''(s))^{1/2}\} + O(n^{-1})$$

with $B_0(u)=u\exp(\frac{1}{2}u^2)(1-\Phi(u))$, Φ being the standard normal distribution function. Here $A(s)=\log\lambda(s)$ and $\gamma_0(s)=(\sum_i c_i(s))(\sum_i r_i(s)\,P(Y_0=i))$. In general the approximations (1.2) and (1.3) hold uniformly for s in a compact subset of $\{s\in\mathbb{R}\,|\int\exp(sw)\; g(w)dw<\infty\}$ and under restriction on the tail behaviour of g the approximations hold uniformly in s.

We remark here that as compared to the classical formulas for the i.i.d. case, see e.g. Jensen (1988), the only difference is the appearance of the term $\gamma_0(s)$ in (1.2) and (1.3). This term is due both to the markov dependency and due to the influence of the initial distribution of X_0.

It is tempting to think that (1.3) should always follow from (1.2). However, it is important to note, as stated in Theorem 4.1 below, that (1.3) in general can be established under weaker conditions than (1.2). □

We have termed the expansions 'saddlepoint expansions' in analogy with Höglund (1974) and with reference to the classical saddlepoint expansions for sums of i.i.d. variables (Daniels 1954).

However, in the present author's view it would be better to use a name like conjugate expansions. This is because the basic ingredient in establishing the approximations is to shift the approximation problem by the use of a conjugate distribution in the i.i.d. case, or a conjugate transition kernel in the Markov chain case. The conjugate kernel is established from the maximal eigenvalue of a modified transition kernel and the associated eigenfunction, see formula (2.6) below. This conjugate kernel has been studied for a much more general setting than here in Iscoe et al. (1985) and by Ney and Nummelin (1987). Having shifted the problem by the use of the conjugate kernel we need to establish an Edgeworth type expansion under the conjugate measure. It turns out that the method in Jensen (1987) is particularly suited for this. When using the results in Jensen (1987) the main problem left is a study of a characteristic function for large values of the argument, i.e. we must establish either a Cramér condition or integrability of the characteristic function.

Let us also briefly mention the differences that appear when establishing local limit results like (1.2) and (1.3) as compared to the large deviation principles established in Iscoe et al. (1985). For the large deviation principle one obtains the limit of $\frac{1}{n}\log P(S_n/n\in A)$ through a rate function. For the set up here the rate function is found via the maximal eigenvalue $\lambda(s)$ mentioned above. Taking the logarithmic limit of the probability means that any factor of order one is discarded. However, to establish asymptotic relations like (1.2), (1.3) or the general version (4.3), one needs to take account of not only the maximal eigenvalue, but also the projection onto the corresponding eigenspace. This means that one has to study the properties of the eigenvector as well.

In Sect. 2 we give the results we need for the conjugate kernel. In Sect. 3 we then apply the results from Jensen (1987), and make the above mentioned study of the characteristic function. This leads us to the results in Sect. 4. As appears from Example 1.1 in certain cases the expansions hold uniformly in the parameter defining the conjugate kernel. Uniformity results are given in Theorem 4.2 and these are easily derived from the results in Jensen (1988, 1991).

Finally, we consider in Sect. 5 an extension to sums of the form $\sum_1^n f(X_i, X_{i-1})$. For these latter sums the results are not as complete as for the previous sums (1.1).

2 The conjugate idea

Let $X_0, X_1, X_2, \ldots$ be a homogeneous Markov chain on a state space $(\mathbb{E}, \mathscr{A})$ with initial distribution $P(X_0\in A)=\mu(A)$ and transition probabilities $P(X_1\in A\,|\,X_0=x)=P(A\,|\,x)$. Our fundamental assumption in this and the following two sections will be the existence of a probability measure ν on $(\mathbb{E}, \mathscr{A})$ and constants $0<a<b<\infty$ such that

$$(2.1)\qquad a\nu(A)\leqq P(A\,|\,x)\leqq b\nu(A),\qquad x\in\mathbb{E},\qquad A\in\mathscr{A}.$$

Defining $p(y\,|\,x)=\frac{dP(\cdot|x)}{d\nu}(y)$ we thus have $a\leqq p(y\,|\,x)\leqq b$. Let $f(\cdot)$ be a measurable function from $\mathbb{E}$ to $\mathbb{R}$ and define

$$(2.2)\qquad \mathscr{D}=\{s\in\mathbb{R}\,|\,\hat{\nu}(s)=\int_{\mathbb{E}}\exp\{sf(x)\}\,\nu(\mathrm{d}x)<\infty\}.$$

Precisely for $s\in\mathscr{D}$ we may define a finite measure $\hat{P}(A\,|\,x;s)$ by

$$(2.3)\qquad \hat{P}(A\,|\,x;s)=\int_A \exp\{sf(y)\}\,P(\mathrm{d}y\,|\,x).$$

From Iscoe et al. (1985) we collect the following results about the eigenvalue structure of $\hat{P}(\cdot\,|\,\cdot\,;s)$.

Lemma 2.1. *For each* $s \in \mathscr{D}$, $\hat{P}(\cdot|\cdot;s)$ *has a maximal simple real eigenvalue* $\lambda(s)$ *with associated eigenfunction* $r(\cdot;s)$ *and eigenmeasure* $L(\cdot;s)$, *i.e.* $\int r(y;s)\,\hat{P}(\mathrm{d}y|x;s)=\lambda(s)\,r(x;s)$ *and* $\int \hat{P}(A|x;s)\,L(\mathrm{d}x;s)=\lambda(s)\,L(A;s)$, *such that*

(i) $r(\cdot;s)$ *and* $l(\cdot;s)\exp\{-sf(\cdot)\}$ *are uniformly positive and bounded, where* $l(x;s)=\dfrac{dL(\cdot;s)}{dv}(x)$.

(ii) $\Lambda(s)=\log\lambda(s)$ *is analytic and strictly convex on* int $\mathscr{D}$.

(iii) $r(x;s)$ *is analytic on* int $\mathscr{D}$ *for each* $x\in\mathbb{E}$

(iv) $\Lambda'(\text{int }\mathscr{D})\subseteq\text{int }\mathscr{S}$, *where* $\mathscr{S}$ *is the convex hull of the support of* $f(v)$, *and equality holds if* $\mathscr{D}$ *is open. In the latter case we also have that* $\lambda(s)\to\infty$ *as* $s\to\partial\mathscr{D}$. □

We will always normalize $r(\cdot;s)$ and $L(\cdot;s)$ such that

$$\pi_s(A)=\int_A r(x;s)\,L(\mathrm{d}x;s) \tag{2.4}$$

becomes a probability measure, and

$$\sup_x r(x;s)=1. \tag{2.5}$$

Corollary 2.2. *We have the bounds*

$$a\hat{v}(s)\leqq\lambda(s)\leqq b\hat{v}(s),\qquad \frac{a}{b}\leqq r(x;s)\leqq 1$$

and

$$\frac{a}{b}\leqq l(x;s)\exp\{-sf(x)\}\,\hat{v}(s)\leqq\left(\frac{b}{a}\right)^2, \quad \square$$

Proof. We have

$$\begin{aligned}\lambda(s)&=\lambda(s)\sup_x r(x;s)\\&=\sup_x\int r(y;s)\hat{P}(\mathrm{d}y|x;s)\leqq b\int r(y;s)\exp\{sf(y)\}\,v(\mathrm{d}y)\leqq b\hat{v}(s)\end{aligned}$$

and

$$\begin{aligned}\lambda(s)\inf_x r(x;s)&=\inf_x\int r(y;s)\,\hat{P}(\mathrm{d}y|x;s)\\&\geqq a\int r(y;s)\exp\{sf(y)\}\,v(\mathrm{d}y)\geqq\begin{cases}a(\inf_x r(x;s))\,\hat{v}(s)\\ a\lambda(s)/b\end{cases}\end{aligned}$$

which give the first two statements. For the third statement we let $l_1 = \inf_x l(x;s)\exp\{-sf(x)\}$ and let l_2 be the supremum. Then

$$\lambda(s)\, l_2 = \sup_x \int p(y|x) l(x;s) v(\mathrm{d}x) \leqq b \int l(x;s) v(\mathrm{d}x)$$

$$\lambda(s)\, l_1 = \inf_x \int p(y|x) l(x;s) v(\mathrm{d}x) \geqq a \frac{\lambda(s) l_2}{b},$$

which together with (2.4) gives

$$\left(\frac{a}{b}\right)^2 l_2\, \hat{v}(s) \leqq \frac{a}{b} l_1\, \hat{v}(s) \leqq \int r(x;s) l(x;s) \exp\{-sf(x)\} \exp\{sf(x)\} v(dx) = 1 \leqq l_2\, \hat{v}(s).$$

Thus $l_2 \leqq \left(\frac{b}{a}\right)^2 \hat{v}(s)^{-1}$ and $l_1 \geqq \frac{a}{b} l_2 \geqq \frac{a}{b} \hat{v}(s)^{-1}$ which gives the statement. □

From the eigenfunction $r(\cdot, s)$ we define a new "conjugate" transition kernel,

$$P_s(A|x) = \int_A \exp\{-\Lambda(s) + sf(y)\} \frac{r(y;s)}{r(x;s)} P(\mathrm{d}y|x). \tag{2.6}$$

The invariant probability measure for this kernel is $\pi_s(\cdot)$ from (2.4), and the kernel shifts the mean of $f(X_i)$ in the following sense (see Iscoe et al. 1985)

$$\int f(x)\, \pi_s(\mathrm{d}x) = \Lambda'(s). \tag{2.7}$$

Thus from Lemma 2.1 we see, that if $\mathscr{D}$ is open we can for every $\xi \in \mathscr{S}$ find an $s \in \mathscr{D}$ such that the invariant mean of $f(X)$ under the conjugate kernel is ξ.

Let us define

$$v_s(A) = \int_A \hat{v}(s)^{-1} \exp\{sf(x)\} v(\mathrm{d}x)$$

and

$$p_s(y|x) = \frac{\mathrm{d}P_s(\cdot|x)}{d v_s}(y) = \frac{\hat{v}(s)}{\lambda(s)} \frac{r(y;s)}{r(x;s)} p(y|x). \tag{2.8}$$

Lemma 2.3. *We have*

$$\frac{a}{b^2} \leqq p_s(y|x) \leqq \frac{b}{a^2},$$

the conditional density w.r.t. v_s *of* X_1 *given* (X_0, X_2) *under the kernel* P_s *is bounded as follows*

$$\left(\frac{a}{b}\right)^6 \leqq \frac{p_s(x_2|x_1)\, p_s(x_1|x_0)}{\int p_s(x_2|x)\, p_s(x|x_0)\, v_s(\mathrm{d}x)} \leqq \left(\frac{b}{a}\right)^6,$$

and

$$\sup_{x,y \in \mathbb{E}, A \in \mathscr{A}} |P_s(A|x) - P_s(A|y)| \leqq \frac{b^3 - a^3}{b^3} = \rho < 1. \quad \square$$

Proof. The first statement follows from (2.1), Corollary 2.2 and (2.8). The second statement follows from the first. Finally the last statement follows from

$$|P_s(A|x)-P_s(A|y)|\leqq\begin{cases}(b/a^2-a/b^2)\,v_s(A) & \text{for } v_s(A)\leqq a^2/b\\ 1-(a/b^2)\,v_s(A) & \text{for } v_s(A)\geqq a^2/b\end{cases}. \quad \square$$

We want to establish asymptotic relations for the large deviation probabilities of

$$S_n=f_0(X_0)+\sum_1^n f(X_i) \tag{2.9}$$

where $f_0\colon \mathbb{E}\to\mathbb{R}$ is a measurable function. We want the large deviation properties to be determined by the kernel and not by the initial measure μ, and we therefore assume throughout that

$$\operatorname{int}\mathcal{D}\subseteq\{s\,|\,\hat{\mu}(s)=\int_{\mathbb{E}}\exp\{sf_0(x)\}\,\mu(\mathrm{d}x)<\infty\}. \tag{2.10}$$

In particular we have two cases in mind, namely $f_0\equiv 0$ and $f_0=f$ with $d\mu/dv$ bounded between a_1 and b_1, $0<a_1<b_1<\infty$, the latter condition being fulfilled for the invariant measure π_0 according to Corollary 2.2.
In accordance with the previous notation we introduce a conjugate initial measure μ_s by

$$\mu_s(A)=\hat{\mu}(s)^{-1}\int_A\exp\{sf_0(x)\}\,\mu(\mathrm{d}x). \tag{2.11}$$

We then define a probability measure $P_{n,s}$ on $\mathbb{E}^{n+1}$ by

$$P_{n,s}(A)=\int 1((x_0,\ldots,x_n)\in A)\,\varphi_n(s)^{-1}\frac{r(x_0;s)}{r(x_n;s)}\prod_1^n P_s(\mathrm{d}x_i|x_{i-1})\,\mu_s(\mathrm{d}x_0). \tag{2.12}$$

where $\frac{a}{b}\leqq\varphi_n(s)\leqq\frac{b}{a}$ is a norming constant. We have introduced $\varphi_n(s)$ here in order to express the formulae in terms of probability measures, but in the approximations below $\varphi_n(s)$ drops out. The next lemma gives the basic relations between probabilities in the original measure and in the conjugate measure.

Lemma 2.4. *Let B be a Borel set, then*

$$P(S_n\in B)=\exp\{nA(s)\}\,\hat{\mu}(s)\,\varphi_n(s)\int 1(S_n\in B)\exp\{-sS_n\}\,\mathrm{d}P_{n,s}.$$

In particular,

$$P(S_n>z)=\exp\{nA(s)\}\,\hat{\mu}(s)\,\varphi_n(s)\int_z^\infty\exp\{-s(u-z)\}\,S_n(P_{n,s})(\mathrm{d}u),$$

and

$$\frac{dS_n(P)}{dm}(z)=\exp\{nA(s)-z\}\,\hat{\mu}(s)\,\varphi_n(s)\,\frac{dS_n(P_{n,s})}{dm}(z),$$

where m is Lebesgue measure. □

Proof.

$$P(S_n\in B)=\int 1(S_n\in B)\prod_1^n P(\mathrm{d}x_i|x_{i-1})\mu(\mathrm{d}x_0)$$

$$=\exp\{nA(s)\}\,\hat{\mu}(s)\int 1(S_n\in B)\frac{r(x_0;s)}{r(x_n;s)}\exp\{-sS_n\}\prod_1^n\{\exp[-A(s)+sf(x_i)]$$

$$\cdot\frac{r(x_i;s)}{r(x_{i-1};s)}P(\mathrm{d}x_i|x_{i-1})\}\,\hat{\mu}(s)^{-1}\exp\{sf_0(x_0)\}\,\mu(\mathrm{d}x_0)$$

$$=\exp\{nA(s)\}\,\hat{\mu}(s)\int 1(S_n\in B)\exp\{-sS_n\}\frac{r(x_0;s)}{r(x_n;s)}\prod_1^n P_s(\mathrm{d}x_i|x_{i-1})\,\mu_s(\mathrm{d}x_0)$$

$$=\exp\{nA(s)\}\,\hat{\mu}(s)\,\varphi_n(s)\int 1(S_n\in B)\exp\{-sS_n\}\,\mathrm{d}P_{n,s}. \quad \square$$

From Lemma 2.4 we see that the next step is to make expansions for the distribution of S_n under $P_{n,s}$. Expansions for sums of dependent variables have recently been studied in for example Götze and Hipp (1983) and Jensen (1989). In the set up here it is natural to use the ideas in Nagaev (1957), which are explained in Jensen (1987), where the characteristic function of the sum is studied via operators. This is essentially an extension of the above eigenvalue structure to complex values of s. We describe this in the next section.

3 Expansions under the conjugate measure

Let us first introduce a conjugate position and scale by

$$\mu_s=\int f(x)\,v_s(\mathrm{d}x),\qquad \mu_s^0=\int f_0(x)\,\mu_s(\mathrm{d}x)$$

and

$$\sigma_s^2=\int\{f(x)-\mu_s\}^2\,v_s(\mathrm{d}x), \tag{3.1}$$

where $\mu_s(\mathrm{d}x)$ is defined in (2.11). The centering μ_s is used instead of $A'(s)$ because of its natural connection to the assumptions in Theorems 4.1 and 4.2 that deal with the density of $f(X)$ under the measure v. We then let $\hat{P}_s(t)$ and $\hat{\mu}_s(t)$ be an operator and a functional, respectively, with kernels

$$\exp[it\{f(y)-\mu_s\}/\sigma_s]\,P_s(\mathrm{d}y|\cdot)\quad\text{and}\quad r(x;s)\exp[it\{f_0(x)-\mu_s^0\}/\sigma_s]\,\mu_s(\mathrm{d}x). \tag{3.2}$$

Furthermore, let M_s^d, M_{s0} be upper bounds such that

$$\int\left|\frac{f(y)-\mu_s}{\sigma_s}\right|^k P_s(\mathrm{d}y|x)\leq M_s^d\quad\text{for } x\in\mathbb{E}\ \text{ and }\ 1\leq k\leq d \tag{3.3}$$

and

$$\int \left|\frac{f_0(x)-\mu_s^0}{\sigma_s}\right|^k \mu_s(\mathrm{d}x) \leqq M_{s0} \quad \text{for} \quad 1 \leqq k \leqq 3. \tag{3.4}$$

The characteristic function of S_n under the measure $P_{n,s}$ in (2.12) may be expressed through $\hat{P}_s(t)$ and $\hat{\mu}_s(t)$, and we collect a number of results about $\hat{P}_s(t)$ in Lemma 3.1. When writing $c_i(\cdot)$ below we mean a constant dependent only on the quantities specified in the argument. We recall that ρ is defined in Lemma 2.3.

Lemma 3.1. *There exist constants $c_i(\cdot)$ such that,*

(i) *For $|t|<c_1(M_s^1, \rho)$ the operator $\hat{P}_s(t)$ has a maximal eigenvalue $\lambda_s(t)$, with $|1-\lambda_s(t)|<(1-\rho)/3$ and with a one-dimensional eigenspace, and the remaining eigenvalues are bounded by $(1+2\rho)/3$. Letting $P_{s,1}(t)$ be the projection onto the eigenspace corresponding to $\lambda_s(t)$ and $P_{s,2}(t)$ the projection onto the eigenspace for the remaining eigenvalues we have*

$$\begin{aligned}&\int \exp\{it(S_n-n\mu_s-\mu_s^0)/\sigma_s\}\,\mathrm{d}P_{n,s}\\ &\quad=\varphi_n(s)^{-1}\{\lambda_s(t)^n\,\hat{\mu}_s(t)\,P_{s,1}(t)+\hat{\mu}_s(t)\,\hat{P}_s(t)^n\,P_{s,2}(t)\}\{r(\cdot\,;s)^{-1}\}\end{aligned}$$

(ii) *For $|t|<c_2(M_s^l, \rho)<c_1$*

$$\left|\frac{d^l}{dt^l}\hat{\mu}_s(t)\,\hat{P}_s(t)^n\,P_{s,2}(t)\{r(\cdot\,;s)^{-1}\}\right| \leqq \frac{b}{a}c_3(M_s^l, \rho)\left(\frac{1+2\rho}{3}\right)^n$$

(iii) *For $|t|<c_4(M_s^5, \rho)<c_1$,*

$$\log \lambda_s(t)=it\zeta_1(s)-\frac{1}{2}t^2\zeta_2(s)-\frac{i}{6}t^3\zeta_3(s)+\frac{1}{24}t^4\zeta_4(s)+\omega c_5(M_s^5, \rho)|t|^5$$

where $|\omega|\leqq 1$ and also $|\zeta_j(s)|\leqq c_5(M_s^5, \rho)$

(iv) *For $|t|<c_6(M_{s0}, M_s^5, \rho)<c_1$,*

$$\hat{\mu}_s(t)\,P_{s,1}(t)\{r(\cdot\,;s)^{-1}\}=\gamma_0(s)+it\gamma_1(s)-\frac{1}{2}t^2\gamma_2(s)+\omega\frac{b}{a}c_7(M_{s0}, M_s^5, \rho)|t|^3$$

where $|\omega|\leqq 1$ and also $|\gamma_j(s)|\leqq \frac{b}{a}c_7(M_{s0}, M_s^5, \rho)$

(v) *$\zeta_2(s)\geqq \frac{a}{b^2}$, where $\zeta_2(s)$ is given in* (iii). □

Proof. The properties (i)–(iv) are simple generalization of results in Jensen (1987).

Writing $h_s(x)=(f(x)-\mu_s)/\sigma_s$ and E_{π_s} for the mean when the initial distribution is π_s and the transition kernel is P_s, we have $\zeta_1(s)=\pi_s(h_s)$ and

$$\begin{aligned}\zeta_2(s)&=\pi_s(h_s^2)-\{\pi_s(h_s)\}^2+2\sum_1^\infty E_{\pi_s}\{h_s(X_n)-\pi_s(h_s)\}\{h_s(X_0)-\pi_s(h_s)\}\\ &=\lim \frac{1}{n}E_{\pi_s}\left[\sum_0^{n-1}\{h_s(X_i)-\pi_s(h_s)\}\right]^2.\end{aligned} \tag{3.5}$$

Since $dP_s(\cdot|x)/d\nu_s \geq a/b^2 = \alpha$ we may write $X_i = \varepsilon_i \mathscr{U}_i + (1-\varepsilon_i) V_i$, where $P(\varepsilon_i = 1) = 1 - P(\varepsilon_i = 0) = \alpha$, $\mathscr{U}_i$ has the distribution ν_s, ε_i and $\mathscr{U}_i$ are independent of X_{i-1}, and conditionally on X_{i-1} the variable V_i has the distribution $(1-\alpha)^{-1}\{P_s(\cdot|x_{i-1}) - \alpha\nu_s(\cdot)\}$. Then we also have $h_s(X_i) = \varepsilon_i h_s(\mathscr{U}_i) + (1-\varepsilon_i) h_s(V_i)$ and

$$\sum_0^{n-1} h(X_i) = h(X_0) + \sum_1^{n-1} \varepsilon_i h_s(\mathscr{U}_i) + \sum_1^{n-1} (1-\varepsilon_i) h_s(V_i). \tag{3.6}$$

Conditionally on $(\varepsilon_1, \ldots, \varepsilon_{n-1}, V_1, \ldots, V_{n-1})$ the variance of (3.6) is greater than $\varepsilon_1 + \ldots + \varepsilon_{n-1}$ according to (3.1). Thus the variance of (3.6) is greater than $\alpha(n-1)$ and (3.5) is greater than α. □

The relation of $\lambda_s(t)$ to the previous eigenvalue $\lambda(s)$ is $\lambda_s(t) = \lambda(s + it/\sigma_s) \exp(-it\mu_s/\sigma_s)/\lambda(s)$ and therefore

$$\zeta_1(s) = \{\Lambda'(s) - \mu_s\}/\sigma_s \quad \text{and} \quad \zeta_j(s) = \Lambda^{(j)}(s)/\sigma_s^j, \quad j \geq 2 \tag{3.7}$$

where $\Lambda(s) = \log \lambda(s)$. The eigenvector corresponding to $\lambda_s(t)$ is given by

$$r_s(y; t) = r(y; s + it/\sigma_s)/r(y; s),$$

where $r(y; s+iu)$ is the analytic continuation of $r(y; s)$. The projection $P_{s,1}(t)$ can be expressed through $r(\cdot\,; s)$ and $l(\cdot\,; s+it/\sigma_s)$, where the normalization (2.4) is used also for complex arguments, as

$$(P_{s,1}(t)\, g)(x) = \{\int g(y) r(y; s) l(y; s + it/\sigma_s) \nu(dy)\}\, r_s(x; t). \tag{3.8}$$

The formula (3.8) follows from the fact that if h is an eigenfunction with eigenvalue $\lambda_h \neq \lambda_s(t)$ then

$$\begin{aligned} &\lambda_h \int h(x) r(x; s) l(x; s+it/\sigma_s) \nu(dx) \\ &= \int \{\int h(y) \exp\left[it \frac{f(y) - \mu_s}{\sigma_s}\right] \frac{r(y; s)}{r(x, s)} \exp[sf(y)]\, P(dy|x)\}\, r(x; s)\, l\left(x; s + \frac{it}{\sigma_s}\right) \nu(dx) \\ &= \lambda_s(t) \int h(y) r(y; s) l(y, s + it/\sigma_s) \nu(dy), \end{aligned}$$

which shows that $\int h(y) r(y; s) l(y; s+it/\sigma_s) \nu(dy) = 0$. From (3.8) we have in particular that if $f_0 \equiv 0$ then

$$\gamma_0(s) = \hat{\mu}_s(t)\, P_{s,1}(t) \{r(\cdot\,; s)^{-1}\}|_{t=0} = \{\int l(x; s) \nu(dx)\} \{\int r(x; s)\, \mu_0(dx)\}. \tag{3.9}$$

The proof of an Edgeworth expansion usually involves two steps (see Bhattacharya and Rao 1976): a suitable expansion of the characteristic function for small values of the argument and suitable bounds on the characteristic function for large values of the argument. For the distribution of S_n under $P_{n,s}$ the first step is achieved with the results in Lemma 3.1. We now turn to the second step.

We first consider socalled Cramér conditions. Define

$$\hat{h}_s(t) = \int \exp[it\{f(x) - \mu_s\}/\sigma_s]\, \nu_s(dx) \quad \text{and} \quad \delta_s(x) = \sup_{|t| \geq c} |\hat{h}_s(t)|. \tag{3.10}$$

Lemma 3.2. *We have*

$$\sup_{|t|\geqq c}\left|\int \exp\{it(S_n-n_{\mu_s}-\mu_s^0)/\sigma_s\}\,\mathrm{d}P_{n,s}\right|\leqq\left[\left(\frac{a}{b}\right)^6\delta_s(c)+\left\{1-\left(\frac{a}{b}\right)^6\right\}\right]^{[n/2]}. \quad \square$$

Proof. The conditional density w.r.t. v_s of X_1 given (X_0, X_2) under $\dot{P}_s$ can, according to Lemma 2.3, be written as

$$\left(\frac{a}{b}\right)^6+q(x_1)$$

where $q(\cdot)\geqq 0$. Therefore, with E_s denoting mean values under P_s,

$$\text{(3.11)}\qquad |\mathrm{E}_s[\exp\{it(f(X_1)-\mu_s)/\sigma_s\}|x_0,x_2]|\leqq\left(\frac{a}{b}\right)^6|\hat{h}_s(t)|+\left\{1-\left(\frac{a}{b}\right)^6\right\}.$$

When conditioning on X_n, $X_{n-2},\ldots$ under $P_{n,s}$ the variables X_{n-1}, $X_{n-3},\ldots$ become independent with the same conditional density as under P_s. Thus we may use (3.11) $[n/2]$ times to get

$$\left|\int \exp\{it(S_n-n\mu_s-\mu_s^0)/\sigma_s\}\mathrm{d}P_{n,s}\right|\leqq\left[\left(\frac{a}{b}\right)^6|\hat{h}_s(t)|+\left\{1-\left(\frac{a}{b}\right)^6\right\}\right]^{[n/2]}$$

and the result of the lemma follows from (3.10). $\square$

We next turn to integrability properties of the characteristic function. Let $1<\xi<2$ be a fixed number such that $\tau=\xi/(\xi-1)$ is an integer. Let m be Lebesgue measure and define,

$$\text{(3.12)}\qquad h_s(z)=\frac{d\left\{\frac{f(\cdot)-\mu_s}{\sigma_s}\right\}(v_s)}{dm}(z)\quad\text{and}\quad \|h_s\|_\xi=\left\{\int h_s(z)^\xi\,\mathrm{d}z\right\}^{1/\xi},$$

where we implicity have assumed that the density $h_s(\cdot)$ exists.

Lemma 3.3. *We have*

$$\int_{|t>c}\left|\int \exp\{it(S_n-n\mu_s-\mu_s^0)/\sigma_s\}\,\mathrm{d}P_{n,s}\right|\mathrm{d}t$$
$$\leqq\left[\left(\frac{a}{b}\right)^6\delta_s(c)+\left\{1-\left(\frac{a}{b}\right)^6\right\}\right]^{[n/2]-\tau}\left[(2\pi)^{\left(\frac{1}{2}-\frac{1}{\xi}\right)}\left(\frac{b}{a}\right)^6\|h_s\|_\xi\right]^\tau. \quad \square$$

Proof. From the Hausdorff-Young inequality we have

$$\text{(3.13)}\quad \left(\int|E_s[\exp\{it(f(X_1)-\mu_s)/\sigma_s\}|x_0,x_2]|^\tau\,\mathrm{d}t\right)^{1/\tau}$$
$$\leqq(2\pi)^{\left(\frac{1}{2}-\frac{1}{\xi}\right)}\left\|\frac{d\left\{\frac{f(X_1)-\mu_s}{\sigma_s}\right\}(P_s(\cdot|x_0,x_2))}{dm}\right\|_\xi\leqq(2\pi)^{\left(\frac{1}{2}-\frac{1}{\xi}\right)}\left(\frac{a}{b}\right)^6\|h_s\|_\xi,$$

where in the last step we have transformed the result of Lemma 2.3 to the density of $(f(X_1)-\mu_s)/\sigma_s$. Using the same conditional argument as in the proof of Lemma 3.2 we get from (3.11) with $\alpha=\{(a/b)^6\delta_s(c)+\{1-(a/b)^6\}\}^{[n/2]-\tau}$

$$\begin{aligned}
&\int_{|t|>c} \left|\int \exp\{it(S_n-n\mu_s-\mu_s^0)/\sigma_s\}\, dP_{n,s}\right| dt \\
&\leqq \alpha \int_{|t|>c} E\prod_1^{\tau} |E_s[\exp\{it(f(X_{2i})-\mu_s)/\sigma_s\}|x_{2i-1},x_{2i+1}]|\,dt \\
&\leqq \alpha E\prod_1^{\tau} \left(\int |E_s[\exp\{it(f(X_{2i})-\mu_s)/\sigma_s\}|x_{2i-1},x_{2i+1}]|^{\tau} dt\right)^{1/\tau} \\
&\leqq \alpha\left[(2\pi)^{\left(\frac{1}{2}-\frac{1}{\xi}\right)}\left(\frac{a}{b}\right)^6 \|h_s\|_{\xi}\right]^{\tau},
\end{aligned}$$

where we used (3.13) in the last step. □

4 Results

In this section we first write down the main approximation formulae in this paper and then give theorems stating their validity. Let

$$z_s = nA'(s)+\mu_s^0 \tag{4.1}$$

with μ_s^0 given in (3.1). Using the coefficients in Lemma 3.1 the third order Edgeworth expansion for the distribution of S_n under $P_{n,s}$ is

$$\begin{aligned}
P_{n,s}\left(\frac{S_n-z_s}{\sqrt{nA''(s)}}\leqq x\right) &= \varphi_n(s)^{-1}\left\{\alpha_0\,\Phi(x)-\frac{1}{\sqrt{n}}\left\{\alpha_1+\frac{1}{6}\alpha_2(x^2-1)\right\}\varphi(x)\right. \\
&+\frac{1}{n}\left\{-\frac{1}{2}\alpha_3\,x+\frac{1}{24}\alpha_4(-x^3+3x)+\frac{1}{72}\alpha_5(-x^5+10x^3-15x)\right\}\varphi(x) \\
&\left.+\text{Error}\right\},
\end{aligned} \tag{4.2}$$

where

$\alpha_0=\gamma_0(s)\quad \alpha_1=\gamma_1(s)\sigma_s/A''(s)^{1/2}\quad \alpha_2=\gamma_0(s)A^{(3)}(s)/A''(s)^{3/2}$

$\alpha_3=\gamma_2(s)\sigma_s^2/A''(s)\quad \alpha_5=\gamma_0(s)(A^{(3)}(s)/A''(s)^{3/2})^2$

$\alpha_4=\gamma_0(s)A^{(4)}(s)/A''(s)^2+4\gamma_1(s)(\sigma_s/A''(s)^{1/2})A^{(3)}(s)/A''(s)^{3/2}.$

The corresponding expansion for the density is obtained by differentiation.

Combining the expansion (4.2) with the formulae in Lemma 2.4 we get, for $s>0$,

$$P(S_n>z_s)=\exp\left\{nA(s)-z_s\right\}\hat{\mu}(s)\{|s|\sqrt{nA''(s)}\}^{-1}$$
$$\cdot\left\{\alpha_0 B_0(\beta_s)+\frac{1}{\sqrt{n}}\left\{\alpha_1 B_1(\beta_s)+\frac{1}{6}\alpha_2 B_3(\beta_s)\right\}\right.$$
$$\left.+\frac{1}{n}\left\{\frac{1}{2}\alpha_3 B_2(\beta_s)+\frac{1}{24}\alpha_4 B_4(\beta_s)+\frac{1}{72}\alpha_5 B_6(\beta_s)\right\}+\text{Error}\right\} \tag{4.3}$$

where $\beta_s=|s|(nA''(s))^{1/2}$ and the functions $B_0(\cdot), \ldots, B_6(\cdot)$ are given in Jensen (1991). For $s<0$ (4.3) is an approximation to $P(S_n<z_s)$. For the density we get,

$$\frac{dS_n(P)}{dm}(z_s)=\exp\{nA(s)-z_s\}\hat{\mu}(s)\{2\pi nA''(s)\}^{-1/2}$$
$$\cdot\left[\alpha_0+\frac{1}{n}\left\{-\frac{1}{2}\alpha_3+\frac{1}{8}\alpha_4-\frac{5}{24}\alpha_5\right\}+\text{Error}\right]. \tag{4.4}$$

Theorem 4.1. *Assume that the distribution $f(v)$ has a continuous component* w.r.t. *Lebesgue measure, then the error term in* (4.3) *is $O(n^{-3/2})$ uniformly for s in a compact subset of int $\mathscr{D}$.*

Assume that the density $h_0(\cdot)$ exists and $\|h_0\|_\xi<\infty$, see (3.12), *then the error term in* (4.4) *is $O(n^{-2})$ uniformly for s in a compact subset of int $\mathscr{D}$.* □

Proof. We first note that for M_s^d in (3.3) we can, according to Lemma 2.3, take

$$M_s^d=\frac{b}{a^2}\max_{1\leq k\leq d}\int\left|\frac{f(x)-\mu_s}{\sigma_s}\right|^k v_s(\mathrm{d}x). \tag{4.5}$$

It is well known from the theory of Laplace transforms that this quantity is bounded for s in a compact subset of int $\mathscr{D}$. Similarly, we have that M_{s0} is bounded for s in a compact subset of int $\mathscr{D}$ on using (2.10).

Using Lemma 3.1 and Lemma 3.2 we see that to establish that the error term in the Edgeworth expansion (4.2) is $O(n^{-3/2})$, we must show that $\delta_s(c)$ is bounded away from 1, where $\delta_s(c)$ is given in (3.10). To show this we prove that the opposite is wrong. Assume that $\delta_{s_j}(c)\to 1$ for $j\to\infty$, where s_j belongs to a compact subset of int $\mathscr{D}$. This means that $|\hat{v}(s_j+it_j/\sigma_{s_j})/\hat{v}(s_j)|\to 1$ for some $|t_j|>c$. We can assume that $s_j\to s\in$ int $\mathscr{D}$. Since $f(v_s)$ has a continuous component we must have that $|t_j|\to\infty$, but this implies that $\limsup_{|t|\to\infty}|\hat{v}(s+it/\sigma_s)/\hat{v}(s)|$ is not strictly less than 1, which is in conflict with Riemann-Lebesgue Theorem.

To get from the Edgeworth expansion (4.2) to the approximation (4.3) we write the integral in the formula in Lemma 2.4 as

$$s\sqrt{nA''(s)}\int_0^\infty \exp\{-s\sqrt{nA''(s)}u\}\{H_n(u)-H_n(0)\}\,\mathrm{d}u,$$

where $H_n(\cdot)$ is the distribution function of $(S_n - z_s)/\sqrt{n\Lambda''(s)}$ under $P_{n,s}$. To get (4.3) we then substitute (4.2) for $H_n(\cdot)$. This will however give an error term which is $O(n^{-1})$ and not $O(n^{-3/2})$ as stated in the theorem. To get the better error estimate we have to include one more term in the Edgeworth expansion (4.2) so that the error term there becomes $O(n^{-2})$.

For the second statement in the theorem we use Lemma 3.1 and Lemma 3.3, so that we must show that $\|h_s\|_\xi$ is bounded for s in a compact subset of int $\mathscr{D}$. This can be shown by using the argument in the proof of Lemma 7 in Jensen (1991). □

The results in Theorem 4.1 is a generalization of the classical saddlepoint approximation for sums of i.i.d. variables to the setting of Markov dependent variables. We now turn to uniformity results for the case where $s \to \bar{s} = \sup\{s | s \in \mathscr{D}\}$. In the classical setting this has been studied recently in Jensen (1988, 1991). As appears from the proof of Theorem 4.1 above, under the fundamental assumption (2.1) on the Markov chain, validity of the expansion has been reduced to properties of the distribution of f under ν. Intuitively then uniformity of the expansions will hold if the classical saddlepoint approximation holds uniformly for $f(\nu)$.

We say, that a density $q(x)$ on $\mathbb{R}$ has regular right tail, if it is defined for $x < x^*$, say, and for some $x_0 < x^*$ satisfies one of the following three conditions for all $x_0 < x < x^*$,

(i) $q(x) = c(x)\exp(-h(x))$
where $h(x)$ is convex and there exists $0 < c_1 < c_2 < \infty$ such that $c_1 < c(x) < c_2$;
(ii) $x^* = \infty$ and $q(x) = Ax^{\alpha-1} l(x)\exp(-\tau x)$
where α, $\tau > 0$ and $l(x)$ is a slowly varying function at infinity;
(iii) $x^* < \infty$ and $q(x) = a(x^* - x)^{\alpha-1} l(x^* - x)$
where $\alpha > 0$ and $l(x)$ is a slowly varying function at zero.

Theorem 4.2. *Under the assumptions*

(i) $f_0 \equiv 0$, *or* $f_0 = f$ *and* $a_1 < \dfrac{d\mu}{d\nu} < b_1$,

(ii) *the density* $h_0(\cdot)$ *of* $(f(X) - \mu_0)/\sigma_0$ *under* ν *exsists and* $\|h_0\|_\xi < \infty$ *for some* $1 < \xi < 2$,

(iii) $h_0(\cdot)$ *has regular right tail,*

we have that the error terms in (4.3) *and* (4.4) *are uniformly of order* $O(n^{-2})$ *as* $s \to \bar{s}$. □

Proof. We must show that M_s^d, M_{s0} and $\|h_s\|_\xi$ are bounded as $s \to \bar{s}$ and that $\delta_s(c)$ is bounded away from 1. The conditions on M_s^d, $\|h_s\|_\xi$ and $\delta_s(c)$ have been proved in Jensen (1988) for the right tail behaviour (ii) and (iii) above and in Jensen (1991) for the log-concave density. The condition on M_{s0} follows either trivially if $f_0 \equiv 0$ or from the bound on M_s^d if $f_0 = f$. □

We note here that the condition on f_0 is needed because σ_s^2, which is the variance w.r.t. ν_s, appears in the Definition (3.4) of M_{s0}.

Example 4.3. Let us return to example 1.1 and see how the general set up gives the results stated in (1.2) and (1.3). We let the measure ν have density $\dfrac{1}{d}g(w)$

with respect to the product of counting measure and Lebesgue measure, where we assumed $ag(w) \leqq f_{ij}(w) \leqq bg(w)$. Since we are considering the sum ΣW_i where $X_i = (Y_i, W_i)$ we have $f(y, w) = w$. The condition on h_0 in Theorem 4.1 therefore simply becomes $\int g(w)^\xi \mathrm{d}w < \infty$ for some $\xi > 1$, and similarly assumption (iii) of Theorem 4.2 says that $g(\cdot)$ has a regular right tail. The form of $\gamma_0(s)$ follows from (3.9). □

5 Sums of the form $\sum_1^n f(x_i, x_{i-1})$

In this section we will change the point of view from that of Sect. 2–4 and consider instead sums of the form

$$S_n = f_0(X_0) + \sum_1^n f(X_i, X_{i-1}) \tag{5.1}$$

where $f(\cdot,\cdot)$ is a measurable function from $\mathbb{E}^2 \to \mathbb{R}$. Instead of the Definition (2.3) of $\hat{P}(A|x; s)$ we use now

$$\hat{P}(A|x; s) = \int_A \exp\{sf(y, x)\} P(\mathrm{d}y|x),$$

and the eigenfunction and eigenmeasure are defined as in Lemma 2.1. We start by an example to illustrate that the general results of the previous sections do not hold here.

Example 5.1. Let $\mathbb{E} = (0, 1)$, $P(dy|x) = m(dy)$ and

$$f(y, x) = \begin{cases} 2y & \text{for } 0 < x \leqq \frac{1}{2} \\ y & \text{for } \frac{1}{2} < x < 1 \end{cases}.$$

Then the eigenvector $r(x; s)$ can for $s > 0$ be written as

$$r(x; s) = \begin{cases} 1 & 0 < x \leqq \frac{1}{2} \\ r_s & \frac{1}{2} < x < 1 \end{cases}$$

where

$$r_s = (z^2 - z)^{-1} \left[\tfrac{1}{2} z - \sqrt{z} + \tfrac{1}{2} + \{(\tfrac{1}{2} z - \sqrt{z} + \tfrac{1}{2})^2 + 2(z^2 - z)(\sqrt{z} - 1)\}^{1/2} \right]$$
$$\sim 2^{1/2} z^{-3/4} \quad \text{as } s \to \infty$$

with $z = \exp(s)$. Also

$$s\lambda(s) = \tfrac{1}{2}(z-1) + \tfrac{1}{2} r_s (z^2 - z) \sim 2^{-1/2} z^{5/4} \quad \text{as } s \to \infty.$$

This in turn implies that the conjugate kernel asymptotically behaves as

$$\frac{P_s(\cdot|x)}{dm}(y) \approx \begin{cases} 0 & 0<y\leqq\frac{1}{2}, \quad x\leqq\frac{1}{2} \\ s\exp\{2s(y-1)\} & \frac{1}{2}<y<1, \quad x\leqq\frac{1}{2} \\ s\exp\{s(y-\frac{1}{2})\} & 0<y\leqq\frac{1}{2}, \quad x>\frac{1}{2} \\ 0 & \frac{1}{2}<y<1, \quad x>\frac{1}{2} \end{cases}$$

for $s\to\infty$. Thus for $x\leqq\frac{1}{2}$ $P_s(\cdot|x)$ becomes concentrated close to $\frac{1}{2}$ and for $x>\frac{1}{2}$ $P_s(\cdot|x)$ becomes concentrated close to 1. Therefore

$$\sup_{x,y,A}|P_s(A|x)-P_s(A|y)|\to 1 \quad \text{as } s\to\infty$$

in contrast with the previous result in Lemma 2.3.

Also we find that $\lambda'(s)/\lambda(s)\to 5/4$ as $s\to\infty$, whereas the support of $f(X_1, X_0)$ is (0, 2) in contrast to the result of Lemma 2.1 (iv). □

Because of the dependency on x in $f(y,x)$ we shall have to assume boundedness of f and only prove an analogous version of Theorem 4.1. We assume the existence of a constant $M<\infty$ such that

$$(5.2) \qquad |f(y,x)|\leqq M \quad \text{for all } x, y\in\mathbb{E} \quad \text{and} \quad |f_0(x)|\leqq M \quad \text{for all } x\in\mathbb{E}.$$

Having made this assumption we can relax the fundamental assumption (2.1) and assume that for some integer $m\geqq 1$ there exist constants $0<a<b<\infty$ and a probability measure ν such that

$$(5.3) \qquad a\nu(A)\leqq P^m(A|x)\leqq b\nu(A).$$

We also assume that $P(\cdot|x)$ is absolutely continuous w.r.t. ν and write $p(y|x) = \frac{dP(\cdot|x)}{d\nu}(y)$. This is the set up in Kim and David (1979) where large deviation results are considered.

The results of Lemma 2.1 still hold except that in (i) the statement is for $l(\cdot;s)$ and in (iv) only the statement $\Lambda'(\mathbb{R})\subseteq \operatorname{int}\mathcal{S}$ is true, where $\mathcal{S}$ is the convex hull of the support of $f(\nu\otimes P(\cdot|\cdot))$. The conjugate kernel is defined by

$$(5.4) \qquad P_s(A|x)=\int_A \exp\{-\Lambda(s)+sf(y,x)\}\frac{r(y;s)}{r(x;s)}P(dy|x)$$

and the Eq. (2.7) becomes

$$(5.5) \qquad \Lambda'(s)=\iint f(y,x)\,P_s(dy|x)\,\pi_s(dx).$$

From (5.2) and the proof of Lemma 2.3 we easily establish the following bounds.

Lemma 5.2. *Let $r_s>0$ be a continuous function such that $r_s\leqq r(x;s)\leqq 1$ for all $x\in\mathbb{E}$. Then*

(i) $\exp\{-|s|M-\Lambda(s)\}r_s\,p(y|x)\leqq p_s(y|x)\leqq \exp\{|s|M-\Lambda(s)\}r_s^{-1}\,p(y|x)$
(ii) $[\exp\{-|s|M-\Lambda(s)\}r_s]^m a\nu(A)\leqq P_s^m(A|x)\leqq[\exp\{|s|M-\Lambda(s)\}r_s^{-1}]^m b\nu(A)$
(iii) $\{\exp(-|s|M)r_s\}^4 p_0(x_1|x_0,x_2)\leqq p_s(x_1|x_0,x_2)$
$\leqq\{\exp(|s|M)r_s^{-1}\}^4 p_0(x_1|x_0,x_2)$

where $p_s(x_1|x_0, x_2) = \dfrac{p_s(x_2|x_1)p_s(x_1|x_0)}{\int p_s(x_2|x)p_s(x|x_0)\nu(dx)}$

(iv) $\sup_{x,y\in\mathbb{E}, A\in\mathcal{A}} |P_s^m(A|x) - P_s^m(A|y)| \leqq [b^3 - a^3\{\exp(-|s|M)r_s\}^{6m}]/b^3 = \rho_s$. □

The formulae in Lemma 2.4 are still valid, and because of (iv) in Lemma 5.2 we can study the characteristic function of S_n under $P_{n,s}$ in the same way as in Sect. 3. In this section we let μ_s^0 be as in (3.1) and define

$$\mu_s = \Lambda'(s) \quad \text{and} \quad \sigma_s^2 = \Lambda''(s).$$

In (3.3) $f(y)$ is replaced by $f(y, x)$ and M_s^d, M_{s0} are bounded for s in a compact set because of (5.2) and because of the analycity of $\Lambda(s)$. The results of Lemma 3.1 (i)–(iv) are unchanged except that ρ is replaced by ρ_s and b/a is replaced by r_s^{-1}. In particular in order to prove Theorem 4.1 in the set up here we must establish results similar to Lemma 3.2 and Lemma 3.3.

Lemma 5.3. *Assume the existence of constants* $\varepsilon_1, \varepsilon_2 > 0$, $z_1 < z_2$ *and sets* $A, B \in \mathcal{A}$ *such that*

$$\int_A P^2(B|x)\nu(dx) \geqq \varepsilon_1$$

and such that $f(x_2, X_1) + f(X_1, x_0)$ *has a continuous component under* $P(\cdot|x_0, x_2)$ *with density*

$$h(z|x_0, x_2) = \frac{d\{f(x_2, \cdot) + f(\cdot, x_0)\}\{P(\cdot|x_0, x_2)\}}{dm}(z) \geqq \varepsilon_2$$

for $z_1 < z < z_2$ *and for* $(x_0, x_2) \in A \times B$. *We then have that there exist continuous functions* $\varepsilon_1(s)$, $\varepsilon_2(s)$, $c(s) > 0$ *such that*

$$\begin{aligned} &\left|\int \exp\{it(S_n - n\mu_s - \mu_s^0)/\sigma_s\} dP_{n,s}\right| \\ &\leqq \left[\{\varepsilon_2(s) g\left(\frac{t}{\sigma_s}\right) + (1-\varepsilon_2(s))\} c(s)\varepsilon_1(s) + (1 - c(s)\,\varepsilon_1(s))\right]^{[n/(2m+3)]} \end{aligned}$$

where $g(t) = (z_2 - z_1)^{-1} |\{\exp(itz_2) - \exp(itz_1)\}/(it)|$. □

Proof. Let $g(t|x_0, x_2) = |E[\exp\{it(f(x_2, X_1) + f(X_1, x_0))\}|x_0, x_2]|$. Then from the assumptions

$$g(t|x_0, x_2) \leqq \begin{cases} \varepsilon_2\, g(t) + (1-\varepsilon_2) & \text{for } (x_0, x_2) \in A \times B \\ 1 & \text{otherwise} \end{cases}$$

and

$$E\{g(t|X_m, X_{m+2})|X_0, X_{2m+2}\} \leqq \{\varepsilon_2\, g(t) + (1-\varepsilon_2)\}\, \frac{a^2}{b^2}\varepsilon_1 + \left(1 - \frac{a^2}{b^2}\varepsilon_1\right), \tag{5.6}$$

where we have used that the conditional density of (X_m, X_{m+2}) given (X_0, X_{2m+2}) is greater than $(a^2/b^2)p^2(x_{m+2}|x_m)$, which follows from (5.3). Now let $g_s(t|x_0, x_2)$

be the corresponding characteristic function for the kernel P_s. Then in (5.6) the coefficients will depend on s,

$$\varepsilon_1(s)=\varepsilon_1[\exp\{-|s|M-A(s)\}r_s]^2, \quad \varepsilon_2(s)=\varepsilon_2[\exp\{-|s|M-A(s)\}r_s]^4$$

and

$$a(s)=a[\exp\{-|s|M-A(s)\}r_s]^m, \quad b(s)=b[\exp\{|s|M-A(s)\}r_s^{-1}]^m$$

where we have used the bounds in Lemma 5.2.

To get from (5.6) to the result of the lemma we first condition, in the evaluation of the mean value under $P_{n,s}$, on $X^1=(X_n, X_{n-1}, \ldots, X_{n-m}, X_{n-m-2}, \ldots, X_{n-2m-2})$, $X^2=(X_{n-2m-3}, \ldots, X_{n-3m-3}, X_{n-3m-5}, \ldots, X_{n-4m-5})$, and so on. We then get the bound

$$(5.7) \quad E\left\{\prod_{j=0}^{[n/(2m+3)]-1} g_s\left(\frac{t}{\sigma_s}\middle| X_{n-m-j(2m+3)}, X_{n-m-2-j(2m+3)}\right)\right\}$$
$$=E\left[\prod_{j=0}^{[n/(2m+3)]-1} E\left\{g_s\left(\frac{t}{\sigma_s}\middle| X_{n-m-j(2m+3)}, X_{n-m-2-j(2m+3)}\right)\middle|\tilde{X}^{j+1}\right\}\right]$$

where $\tilde{X}^j$ is the first and last variable in the vector X^j. The result of the lemma now follows on combining (5.6) and (5.7) and putting $c(s)=a(s)^2/b(s)^2$. □

Define

$$\delta(c)=\sup_{|t|>c} g(t) \quad\text{and}\quad \delta_s(c)=\{\varepsilon_2(s)\delta\left(\frac{c}{\sigma_s}\right)+1-\varepsilon_2(s)\}c(s)\varepsilon_1(s)+1-c(s)\varepsilon_1(s)$$

with the notation from Lemma 5.2.

Lemma 5.4. *Assume that* $f(x_2, X_1)+f(X_1, x_0)$ *has a density under* $P(\cdot|x_0, x_2)$ *and that the assumptions in Lemma 5.3 hold. Also assume that for some* $1<\xi<2$ *with* $\tau=\xi/(\xi-1)$ *an integer we have*

$$\int_{\mathbb{E}^2}\{\int_{\mathbb{R}} h(z|x_0, x_2)^\xi \mathrm{d}z\}^{1/\xi} P^2(\mathrm{d}x_2|x_0)\nu(\mathrm{d}x_0)=c_1<\infty.$$

Then

$$\int_{|t|>c} |\int \exp\{it(S_n-n\mu_s-\mu_s^0)/\sigma_s\}\,\mathrm{d}P_{n,s}|\,\mathrm{d}t$$
$$\leqq \delta_s(c)^{[n/(2m+3)]-\tau}(2\pi)^{(\frac{1}{2}-\frac{1}{\xi})\tau}\sigma_s\{c_1\, c(s)\}^\tau. \quad □$$

Proof. Using (5.7) we have, with $i_1(j)=n-m-j(2m+3)$ and $i_2(j)=n-m-2-j(2m+3)$, the bound

$$\int_{|t|>c} |\int \exp\{it(S_n-n\mu_s-\mu_s^0)/\sigma_s\}\,\mathrm{d}P_{n,s}|\,\mathrm{d}t$$
$$\leqq \delta_s(c)^{[n/(2m+3)]-\tau}\int_{|t|>c} E\left\{\prod_{j=0}^{\tau-1} g_s\left(\frac{t}{\sigma_s}\middle| X_{i_1(j)}, X_{i_2(j)}\right)\right\}\mathrm{d}t$$
$$\leqq \delta_s(c)^{[n/(2m+3)]-\tau} E\left[\prod_{j=0}^{\tau-1}\left\{\int g_s\left(\frac{t}{\sigma_s}\middle| X_{i_1(j)}, X_{i_2(j)}\right)^\tau \mathrm{d}t\right\}^{1/\tau}\right]$$
$$\leqq \delta_s(c)^{[n/(2m+3)]-\tau}(2\pi)^{(\frac{1}{2}-\frac{1}{\xi})\tau}\sigma_s E\left[\prod_{j=0}^{\tau-1} E\{(\int h(z|X_{i_1(j)}, X_{i_2(j)})^\xi \mathrm{d}z)^{1/\xi}|\tilde{X}^{j+1}\}\right]$$
$$\leqq \sigma_s(c)^{[n/(2m+3)]-\tau}(2\pi)^{(\frac{1}{2}-\frac{1}{\xi})\tau}\sigma_s\{c_1\, c(s)\}^\tau,$$

where in the last step $c(s)$ is the constant from the proof of Lemma 5.3. □

With the bounds established above and in Lemma 5.3 and Lemma 5.4 the proof of Theorem 5.5 below is similar to the proof of Theorem 4.1.

Theorem 5.5. *Assume that the conditions in Lemma 5.3 hold. Then the error term in* (4.3) *is* $O(n^{-3/2})$ *uniformly for s in a compact set.*

Assume that the conditions in Lemma 5.4 *hold. Then the error term in* (4.4) *is* $O(n^{-2})$ *uniformly for s in a compact set.* □

Example 5.6. Let us again return to example 1.1 and relax the conditions on the densities $f_{ij}(w)$. We simply let all the f_{ij}'s be the uniform density on the interval (0, 1), and consider instead the sum $S_n=\prod_{i=1}^{n} g_{Y_i,Y_{i-1}}(W_i)$, where $g_{ij}(w)$ is a real function on (0, 1) for $i,j=1,\dots,d$. The situation in Example 1.1 is then obtained by taking $g_{ij}(w)=F_{ij}^{-1}(w)$, where $F'_{ij}=f_{ij}$. The formulas from Example 1.1 still hold on setting $\varphi_{ij}(s)=\int_0^1 \exp\{s g_{ij}(w)\}\,\mathrm{d}w$.

We now consider the assumptions of Lemma 5.3. The function $f(x_2,x_1)=f((y_2,w_2),(y_1,w_1))$ becomes here $g_{y_1,y_2}(w_2)$, and we take the measure ν to be $\nu(k,dw)=d^{-1}$. We make two assumptions.

Namely first that there exists (k,l), such that if W is uniformly distributed on (0, 1), then $Z=g_{kl}(W)$ has a continuous component with density $h(z)$ greater than ε for $z_1<z<z_2$. Secondly we assume that for some $v\in(0,1)$ and $\delta>0$ we have $|g_{lk}(w)-g_{lk}(v)|<(z_2-z_1)/4$ for $|w-v|<\delta$. Then, with the notation of Lemma 5.3, we take $A=\{k\}\times(0,1)$ and $B=\{k\}\times(v-\delta,\,v+\delta)$. Simple calculations show that

$$\int_A P^2(B|x)\nu(\mathrm{d}x)=\sum_i p_{ik}\,2\delta\, p_{ki}\,d^{-1}$$

and since $f((k,w_2),(I,W))+f((I,W),(k,w_0))=g_{Ik}(w_2)+g_{kI}(W)$ we find

$$\begin{aligned} h(z|(k,w_0),(k,w_2)) &\geqq h(z-g_{lk}(w_2))\,p_{kl}\,p_{lk}\Big\{\sum_i p_{ki}\,p_{ik}\Big\}^{-1}\\ &\geqq \varepsilon\, p_{kl}p_{lk}\Big\{\sum_i p_{ki}p_{ik}\Big\}^{-1}\end{aligned}$$

for $z_1+g_{lk}(v)+(z_2-z_1)/4<z<z_2+g_{lk}(v)-(z_2-z_1)/4$ and $|w_2-v|<\delta$. □

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