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A mean central limit theorem for weakly multiplicative systems and its application to lacunary trigonometric series

Katusi Fukuyama

Institute of Mathematics, University of Tsukuba, Tsukuba-shi, Ibaraki 305, Japan

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Summary. In this paper, we discuss the rate of convergence in the mean central limit theorem for weakly multiplicative systems and apply this result to lacunary trigonometric series under probability measures on $\mathbb{R}$ with Hölder continuous distribution function.

0. Introduction

Let $\{\xi_i\}$ be a sequence of i.i.d. random variables with mean 0, variance 1 and finite absolute third moment. According to Berry-Esseen inequality, we have

$$\|F_n - G_0\|_\infty \leqq Cn^{-1/2}E|\xi_1|^3 \quad n\in\mathbb{N},$$

where F_n is a distribution function of $n^{-1/2}\sum_{i=1}^n \xi_i$, G_m is that of Gaussian distribution with mean m and variance 1 and C is a positive absolute constant.

V.M. Zolotarev [21] obtained the following result under the same conditions on $\{\xi_i\}$.

$$\lim_{n\to\infty} n^{1/2}\|F_n - G_0\|_1 \leqq \tfrac{1}{2}E|\xi_1|^3.$$

Using these theorems, we can estimate the rate of convergence of $\|F_n - G_0\|_p$ to 0 where $p\in[1, \infty]$. Limit theorem of this kind is called a mean central limit theorem (MCLT).

MCLT's for various kinds of sequence of random variables have been obtained. For example, I.A. Ibragimov [8] and T. Nakata [15] obtained the MCLT for a sequence of uniformly bounded martingale differences.

Recently, L. Paditz-Š. Šarachmetov [16] proved the MCLT for a kind of sequence of weakly dependent random variables, so called equinormed strongly multiplicative systems. Before stating this result, we must explain some notions of multiplicative systems.

A sequence $\{\xi_i\}$ of random variables is called a multiplicative system (MS) if

$$E(\xi_{i_1} \ldots \xi_{i_r}) = 0 \quad (r\in\mathbb{N}, i_1 < \ldots < i_r),$$

and is called a strongly multiplicative system (SMS) if

$$E(\xi_{i_1}^{\alpha_1} \dots \xi_{i_r}^{\alpha_r}) = 0 \quad (r \in \mathbb{N}, i_1 < \dots < i_r, \alpha_i = 1, 2)$$

unless all α_i's are equal to 2. $\{\xi_i\}$ is called an equinormed multiplicative system (EMS) if both $\{\xi_i\}$ and $\{\xi_i^2 - 1\}$ are MS, and an equinormed strongly multiplicative system (ESMS) if it is at once an SMS and an EMS. These notions except the EMS are introduced by G. Alexits [1], [2] and have been studied extensively by the Hungarian school. A history of this field is thoroughly summarized in F. Móricz-P. Révész [14].

Of course a sequence of independent random variables with mean 0 and variance 1 is an ESMS and a sequence of martingale differences is an MS whenever all the expectations appearing in the definitions of MS's exist.

Besides, there are important examples of MS's. One of the most important among these is lacunary trigonometric sequence. A sequence $\{\sqrt{2}\cos(2\pi n_j\omega)\}$ on the concrete probability space [0, 1] with Lebesgue measure is an EMS if $\{n_j\} \subset \mathbb{N}$ and $n_{j+1}/n_j \geqq 2 (j \in \mathbb{N})$ and is an ESMS if $\{n_j\} \subset \mathbb{N}$ and $n_{j+1}/n_j \geqq 3 (j \in \mathbb{N})$. Some systems of almost periodic functions under the relative measure can be treated as EMS (Cf. Fukuyama [4]).

The result of L. Paditz-Š. Šarachmetov is as follows.

Theorem A. *Let $\{\lambda_i\}$ be a sequence of real numbers and λ be a positive number satisfying*

$$\sum_{i=1}^{\infty} \lambda_i^2 = 1 \quad \text{and} \quad |\lambda_i| \leqq \lambda \leqq 1 \quad (i \in \mathbb{N}).$$

Let $\{\xi_i\}$ be an ESMS *satisfying $|\xi_i| \leqq K (i \in \mathbb{N})$ for some $K > 0$. Then*

$$\| F - G_0 \|_\infty \leqq LK^{1/4}\lambda^{1/4} \text{ and } \| F - G_0 \|_1 \leqq LK^{4/7}\lambda^{2/7},$$

where F is the distribution function of $\sum_{i=1}^{\infty} \lambda_i \xi_i$ and L is an absolute constant.

Putting $\lambda_i = 1/\sqrt{n} (i \leqq n)$ and $\lambda_i = 0 \, (i > n)$ in this theorem, we get the ordinary MCLT.

The purpose of this paper is to extend this theorem to the case of weakly multiplicative systems (WMS), and to apply it to more general lacunary trigonometric sequences under some singular probability measures on $\mathbb{R}$.

It is, however, rather difficult to explain briefly the notion of WMS. We can find various definitions of WMS in I. Berkes [3], F. Móricz [13], W. Kratz-R. Trautner [20] and K. Fukuyama [5], [6], but they are all different from each other. Only thing they state in common is that expectations appearing in the definitions of MS's are not necessarily exactly equal to 0 but are nearly 0 in some sense.

To state the definition of WMS in our sense, we must prepare some notation. Put

$$b_{i_1, \dots, i_r} = E(\xi_{i_1} \dots \xi_{i_r}), \; \bar{b}_{i_1, \dots, i_r} = E((\xi_{2i_1-1}^2 - 1) \dots (\xi_{2i_r-1}^2 - 1)) \text{ and}$$

$$\bar{\bar{b}}_{i_1, \dots, i_r} = E((\xi_{2i_1}^2 - 1) \dots (\xi_{2i_r}^2 - 1)).$$

We introduce infinite dimensional vectors B_r, $\bar{B}_r$ and $\bar{\bar{B}}_r$ by

$$B_r = (b_{i_1,\ldots,i_r})_{i_1<\cdots<i_r},\ \bar{B}_r = (\bar{b}_{i_1,\ldots,i_r})_{i_1<\cdots<i_r} \text{ and } \bar{\bar{B}}_r = (\bar{\bar{b}}_{i_1,\ldots,i_r})_{i_1<\cdots<i_r}.$$

Let $\|B_r\|_\delta$, $\|\bar{B}_r\|_\delta$ and $\|\bar{\bar{B}}_r\|_\delta$ be l_δ-norms of these vectors, i.e.

$$\|B_r\|_\delta = \left(\sum_{i_1<\cdots<i_r} |b_{i_1,\ldots,i_r}|^\delta\right)^{1/\delta}$$

and so on.

We say that a sequence $\{\xi_i\}$ of random variables is a (δ, B)-WMS if

$$\text{(0.1)}\quad \|B_r\|_\delta^{1/r} \leqq Br^{1-1/\delta},\quad \|\bar{B}_r\|_2^{1/2} \leqq Br^{1/2} \quad\text{and}\quad \|\bar{\bar{B}}_r\|_2^{1/r} \leqq Br^{1/2} \quad (r\in\mathbb{N}).$$

Roughly speaking, (δ, B)-WMS is a sequence of random variables such that $\{\xi_i\}$, $\{\xi_{2i-1}^2 - 1\}$ and $\{\xi_{2i}^2 - 1\}$ are all nearly MS. Of course it seems to be more natural to assume that $\{\xi_i\}$ and $\{\xi_i^2 - 1\}$ are nearly MS because it is a convenient way to extend the notion of EMS, and our definition may be seen an artificial and minor extension of this natural one. We believe, however, that this extension is essential because it enables us to apply this notion to lacunary trigonometric series with more general gap conditons than that stated before.

Anyway, we are in a position to state our first theorem.

Theorem 1. *Let $\{\xi_i\}$ be a (δ, B)-WMS, $\{\lambda_i\}$ be a sequence of real numbers and λ be a positive number satisfying*

$$\text{(0.2)}\qquad \sum_{i=1}^{\infty} \lambda_i^2 = 1,\quad |\lambda_i| \leqq B\lambda \quad \textit{and} \quad \lambda \leqq 1 \quad (i\in\mathbb{N}),$$

$$\text{(0.3)}\qquad |\lambda_i\xi_i| \leqq B\lambda \quad (i\in\mathbb{N})$$

for some $\delta\in[1, 2)$ and $B \geqq 1$. Then the series $\sum_{i=1}^{\infty} \lambda_i\xi_i$ converges in probability and its distribution function F obeys the following estimates.

$$\|F - G_0\|_\infty \leqq LB^2\lambda^{(1/4)\wedge(2\Delta/3)} \quad \textit{and} \quad \|F - G_0\|_1 \leqq LB^3\lambda^{(2/7)\wedge(4\Delta/5)}.$$

Here L is an absolute constant and $\Delta = 2/\delta - 1$.

From this theorem, we can derive the next corollary which plays an important role in the rest of this paper.

Corollary. *Let $B \geqq 1$ and $\delta\in[1, 2)$. Suppose that a double sequence $\{\lambda_{n,i}\}$ of real numbers and a sequence $\{\lambda_n\}$ of positive numbers satisfy*

$$\sum_{i=1}^{\infty} \lambda_{n,i}^2 = 1,\quad |\lambda_{n,i}| \leqq B\lambda_n \quad \textit{and} \quad \lambda_n \leqq 1 \quad (n, i\in\mathbb{N}),$$

and an array $\{\xi_{n,i}\}$ of random variables satisfy the following conditions. For each $n\in\mathbb{N}$, $\{\xi_{n,i}\}_{i\in N}$ is a (δ, B)-WMS and

$$|\lambda_{n,i}\xi_{n,i}| \leqq B\lambda_n \quad (n, i\in\mathbb{N})$$

holds. Then we have

$$\| F_n - G_0 \|_\infty \leqq LB^2 \lambda_n^{(1/4) \wedge (2\Delta/3)} \quad \textit{and} \quad \| F_n - G_0 \|_1 \leqq LB^3 \lambda_n^{(2/7) \wedge (4\Delta/5)} \quad (n \in \mathbb{N}),$$

where F_n is the distribution function of $\sum_{i=1}^\infty \lambda_{n,i}\xi_{n,i}$.

Next we state its applications to lacunary series. Here we assume that a probability measure P on $\Omega = \mathbb{R}^1$ satisfies

$$P\{[\omega, \omega + h]\} \leqq Mh^\rho \quad (\omega \in \Omega, h > 0) \tag{0.4}$$

or

$$|\hat{P}(u)| \leqq M|u|^{-\rho/2} \quad (u \in \mathbb{R}), \tag{0.5}$$

for some $M > 0$ and $\rho \in (0, 1]$ where $\hat{P}$ is the characteristic function of P.

There are important examples satisfying (0.4) or (0.5). According to F. Hausdorff [7] the law of $\sum_{n=1}^\infty a^n r_n$, where $0 < a \leqq 1/2$ and $\{r_n\}$ is a Rademacher sequence, satisfies (0.4) with $\rho = (\log 2)/(\log 1/a)$, and R. Kershner [11] proved that (0.5) does not hold if $a = 1/3, 1/4, \ldots$ N. Wiener-A. Wintner [20] proved that there exists a singular probability measure satisfying (0.5) for each $\rho \in (0, 1)$.

R. Kaufman [10] showed the following theorem.

Theorem B. *Let P satisfy* (0.4). *For $x > 0$, let F_n be the distribution function of $n^{-1/2}\sum_{j=1}^n \sqrt{2}\cos(x^j\omega)$. Then for almost all $x > 2$ with respect to the Lebesgue measure,*

$$F_n \to G_0 \textit{ pointwise as } n \to \infty .$$

After that S. Takahashi [17] obtained the following theorem.

Theorem C. *Let P satisfy* (0.4), *$\{a_j\}$ be a sequence of real numbers satisfying*

$$A_n^2 = a_1^2 + \cdots + a_n^2 \to \infty \quad \textit{and} \quad a_n = o(A_n(\log\log A_n)^{-1}) \quad \textit{as } n \to \infty ,$$

$\{\beta_j\}$ satisfy the following condition which is called Hadamard's gap condition;

$$\beta_1 > 0, \quad \frac{\beta_{j+1}}{\beta_j} > q > 1 \quad (j \in \mathbb{N}),$$

$\{\gamma_j\}$ be an arbitrary sequence of real numbers and F_n be the distribution function of $A_n^{-1}\sum_{j=1}^n a_j\sqrt{2}\cos(\beta_j t\omega + \gamma_j)$. Then, for almost all $t \in \mathbb{R}$ with respect to the Lebesgue measure,

$$F_n \to G_0 \textit{ pointwise as } n \to \infty .$$

Recently S. Takahashi [19] proved the following theorem.

Theorem D. *Let P satisfy* (0.5), *$\{a_j\}$ satisfy*

$$A_n^2 = a_1^2 + \cdots + a_n^2 \to \infty \quad \textit{and} \quad a_n = o(A_n) \quad \textit{as } n \to \infty ,$$

$\{\beta_j\}$ satisfy Hadamard's gap condition, $\{\gamma_j\}$ be an arbitrary sequence and F_n be the distribution function of $A_n^{-1}\sum_{j=1}^n a_j\sqrt{2}\cos(\beta_j\omega + \gamma_j)$. Then,

$$F_n \to G_0 \textit{ pointwise as } n \to \infty .$$

The last three theorems are central limit theorems for lacunary trigonometric sequences with Hadamard's gap under some measures on the real line including a class of singular measures. In this paper, we extend these to the case of Takahashi's gap, that is to say,

$$(0.6)\qquad \beta_1 > 0,\ \frac{\beta_{j+1}}{\beta_j} > 1 + cj^{-\alpha} \quad (j \in \mathbb{N}, \text{ for some } c > 0 \text{ and } 0 \leqq \alpha < \tfrac{1}{2}).$$

As to Theorem B, we extend it to the case that the sequence of frequencies is $\{x^{\phi(j)}\}$ satisfying

$$(0.7)\quad \phi(1) > 0,\ \phi(j+1) - \phi(j) \geqq dj^{-\alpha} \quad (j \in \mathbb{N}, \text{ for some } d > 0 \text{ and } 0 \leqq \alpha < \tfrac{1}{2}),$$

in which $\{x^{\phi(j)}\}$ satisfies Takahashi's gap condition $x^{\phi(j+1)}/x^{\phi(j)} \geqq 1 + (d\log x)j^{-\alpha}$.

In what follows we are concerned with the case in which the sequence $\{a_j\}$ of coefficients satisfy

$$A_n^2 = a_1^2 + \cdots + a_n^2 \to \infty \quad \text{and} \quad a_n = o(A_n n^{-\alpha}(1 + \alpha \log n)^{-1}) \quad \text{as } n \to \infty.$$

Under this condition, we can take a sequence $\{\mu_n\}$ of positive numbers and a positive constant C satisfying

$$(0.8)\qquad \mu_n \to 0,\quad A_n\mu_n \uparrow \infty \quad \text{as } n \to \infty \quad \text{and}$$
$$\mu_n \leqq 1,\quad |a_n| \leqq C\mu_n A_n n^{-\alpha}(1 + \alpha\log n)^{-1} \quad (n \in \mathbb{N}).$$

We have the following three theorems.

Theorem 2. *We assume* (0.4), (0.7) *and* (0.8). *Let F_n be the distribution function of $A_n^{-1}\sum_{j=1}^n a_j\sqrt{2}\cos(x^{\phi(j)}\omega + \gamma_j)$. Then, for almost all $x > 1$ with respect to the Lebesgue measure,*

$$(0.9)\qquad \| F_n - G_0 \|_\infty \leqq L\mu_n^{1/4} \quad \text{and} \quad \| F_n - G_0 \|_1 \leqq L\mu_n^{2/7} \quad (n \in \mathbb{N}).$$

Here L is a positive constant depending only on x, α, d, $\phi(1)$, C, ρ and M.

Theorem 3. *We assume* (0.4), (0.6) *and* (0.8). *Let F_n be the distribution function of $A_n^{-1}\sum_{j=1}^n a_j\sqrt{2}\cos(\beta_j t\omega + \gamma_j)$. Then, for almost all $t \in \mathbb{R}$ with respect to the Lebesgue measure,* (0.9) *holds. Here L is a positive constant depending only on t, α, c, β_1, C, ρ and M.*

Theorem 4. *We assume* (0.5), (0.6) *and* (0.8). *Let F_n be the distribution function of $A_n^{-1}\sum_{j=1}^n a_j\sqrt{2}\cos(\beta_j\omega + \gamma_j)$. Then* (0.9) *holds. Here L is a positive constant depending only on α, c, β_1, C, ρ and M.*

If we put $\alpha = 0$, Takahashi's gap condition is reduced to Hadamard's one, and the condition on $\{a_j\}$ is reduced to that of Theorem D and is weaker than that of Theorem C. Of course Theorem B is a special case of Theorem 2. Hence our results are extensions of original ones, and moreover they assert the MCLT which has the central limit theorem as its corollary.

1. Proof of Theorem 1

First we prepare some lemmas.

Lemma 1. (1) $\sum_{i=1}^{p} \lambda_i \xi_i$ *converges in probability as* $p \to \infty$.
(2) $\sum_{i=1}^{p} \lambda_i^2 \xi_i^2$ *and* $\sum_{i=1}^{p} |\lambda_i \xi_i|^3$ *converge almost surely as* $p \to \infty$.
(3) *Put* $M_p = \sum_{i=1}^{p} \lambda_i b_i$ *where* $p = 1, 2, \ldots, \infty$. *Then we have*

$$|M_p| \leqq B^2 \lambda^{\Delta}, \tag{1.1}$$

$$E\left(\sum_{i=1}^{\infty} \lambda_i^2 (\xi_i^2 - 1)\right)^2 \leqq 24 B^4 \lambda^2. \tag{1.2}$$

(4) $T_p(z) = \prod_{i=1}^{p} (1 + \sqrt{-1} z \lambda_i \xi_i)$ *converges in probability as* $p \to \infty$ *if* $z \in \mathbb{C}$ *and* $|z| \leqq (2B\lambda)^{-1}$.

Proof. (1) We prove that $\sum_{i=1}^{p} \lambda_i \xi_i$ converges in L^2-sense. First we note that $\| B_r \|_2 \leqq \| B_r \|_\delta < B^r$ holds. Put

$$E\left(\sum_{i=l}^{m} \lambda_i \xi_i\right)^2 = \sum_{i=l}^{m} \lambda_i^2 + \sum_{l \leqq 2i-1 \leqq m} \lambda_{2i-1}^2 \bar{b}_i + \sum_{l \leqq 2i \leqq m} \lambda_{2i}^2 \bar{\bar{b}}_i + 2 \sum_{l \leqq i_1 < i_2 \leqq m} \lambda_{i_1} \lambda_{i_2} b_{i_1, i_2}$$
$$= \sum_1 + \sum_2 + \sum_3 + 2 \sum_4 .$$

Using Schwarz's inequality and estimating l_2-norm by l_1-norm, we have

$$\sum_2 \leqq \left(\sum_{l \leqq 2i-1 \leqq m} \lambda_{2i-1}^4\right)^{1/2} \left(\sum_{i=1}^{\infty} \bar{b}_i^2\right)^{1/2} \leqq \left(\sum_{l \leqq 2i-1 \leqq m} \lambda_{2i-1}^2\right) \| \bar{B}_1 \|_2 .$$

$\sum_3$ is estimated in the same way and as to $\sum_4$, we get

$$\sum_4 \leqq \left(\sum_{l \leqq i_1 < i_2 \leqq m} \lambda_{i_1}^2 \lambda_{i_2}^2\right)^{1/2} \left(\sum_{l \leqq i_1 < i_2 \leqq m} b_{i_1, i_2}^2\right)^{1/2} \leqq \left(\sum_{i=l}^{m} \lambda_i^2\right) \| B_2 \|_2 .$$

These estimates imply L^2-convergence of the series.
(2) They are proved in a similar way as (1). Indeed, we have

$$E\left(\sum_{i=1}^{\infty} \lambda_i^2 \xi_i^2\right) \leqq 1 + \sum_{i=1}^{\infty} \lambda_{2i}^2 \bar{\bar{b}}_i + \sum_{i=1}^{\infty} \lambda_{2i-1}^2 \bar{b}_i \leqq 1 + \| \bar{B}_1 \|_2 + \| \bar{\bar{B}}_1 \|_2 < \infty$$

because of (0.1) and $|\lambda_i| \leqq 1$. By (0.3), we have $|\lambda_i \xi_i|^3 \leqq B \lambda \lambda_i^2 \xi_i^2$. From these, almost sure convergence of $\sum_{i=1}^{p} |\lambda_i \xi_i|^3$ is clear.
(3) Using Hölder's inequality, we obtain

$$|M_p| \leqq \left(\sum_{i=1}^{\infty} |\lambda_i|^\varepsilon\right)^{1/\varepsilon} \left(\sum_{i=1}^{\infty} |b_i|^\delta\right)^{1/\delta} \leqq B^{\Delta} \lambda^{\Delta} \| B_1 \|_\delta,$$

where ε is the dual of δ, i.e. $1/\varepsilon + 1/\delta = 1$.

Next we prove (1.2). By Minkowski's inequality,

$$E^{1/2}\left(\sum_{i=1}^{\infty}\lambda_i^2(\xi_i^2-1)\right)^2 \leqq E^{1/2}\left(\sum_{i=1}^{\infty}\lambda_{2i-1}^2(\xi_{2i-1}^2-1)\right)^2+E^{1/2}\left(\sum_{i=1}^{\infty}\lambda_{2i}^2(\xi_{2i}^2-1)\right)^2$$

$$=E_1+E_2\,.$$

Expanding E_1, we have

$$E_1^2\leqq\sum_{i=1}^{\infty}\lambda_{2i-1}^4E\xi_{2i-1}^4+\sum_{i=1}^{\infty}\lambda_{2i-1}^4+2\sum_{i_1<i_2}\lambda_{2i_1-1}^2\lambda_{2i_2-1}^2\bar{b}_{i_1,i_2}\,.$$

Estimating in a similar way as (1) and (2), we get (1.2).
(4) Let $\log w$ denote the principal value, i.e. $|\operatorname{Im}\log w|<\pi$. Since $|\sqrt{-1}z\lambda_i\xi_i|\leqq|z|B\lambda\leqq2^{-1}$, we have $|\log(1+\sqrt{-1}z\lambda_i\xi_i)-\sqrt{-1}z\lambda_i\xi_i|\leqq|\sqrt{-1}z\lambda_i\xi_i|^2$. Since $\sum_{i=1}^p\lambda_i\xi_i$ and $\sum_{i=1}^p\lambda_i^2\xi_i^2$ converge in probability as $p\to\infty$, so does $\sum_{i=1}^p\log(1+\sqrt{-1}z\lambda_i\xi_i)$ also. Thus we can conclude that $T_p(z)=\exp(\sum_{i=1}^p\log(1+\sqrt{-1}z\lambda_i\xi_i))$ converges in probability as $p\to\infty$ if $|z|\leqq(2B\lambda)^{-1}$. □

We denote the limit of $T_p(z)$ by $T_\infty(z)$.

Lemma 2. *If* $|z|\leqq(8B^2\lambda^\Delta)^{-1}$ *and* $p<\infty$, *then*

$$|ET_p(z)-1-\sqrt{-1}zM_p|\leqq10B^4\lambda^{2\Delta}|z|^2, \tag{1.3}$$

$$|ET_p(z)-1|\leqq3B^2\lambda^\Delta|z|\,, \tag{1.4}$$

$$|ET_p(z)|\leqq2\,. \tag{1.5}$$

Proof. First we prove (1.3). We have by Hölder's inequality,

$$\begin{aligned}|ET_p(z)-1-\sqrt{-1}zM_p|&\leqq\sum_{r=2}^{\infty}\sum_{i_1<\cdots<i_r}|z|^r|\lambda_{i_1}\ldots\lambda_{i_r}||b_{i_1,\ldots,i_r}|,\\
&\leqq\left(\sum_{r=2}^{\infty}|2Bzr^{1/\varepsilon}|^{r\varepsilon}\sum_{i_1<\cdots<i_r}|\lambda_{i_1}\ldots\lambda_{i_r}|^\varepsilon\right)^{1/\varepsilon}\\
&\quad\times\left(\sum_{r=2}^{\infty}|2Br^{1/\varepsilon}|^{-r\delta}\sum_{i_1<\cdots<i_r}|b_{i_1,\ldots,i_r}|^\delta\right)^{1/\delta}\\
&\leqq\left(\sum_{r=2}^{\infty}|2B^2zr^{1/\varepsilon}\lambda^\Delta|^{r\varepsilon}\sum_{i_1<\cdots<i_r}\lambda_{i_1}^2\ldots\lambda_{i_r}^2\right)^{1/\varepsilon}\\
&\quad\times\left(\sum_{r=2}^{\infty}(2Br^{1/\varepsilon}\|B_r\|_\delta^{-1/r})^{-r\delta}\right)^{1/\delta}.\end{aligned}$$

Using the idea of W. Kratz-R. Trautner [12];

$$\sum_{i_1<\cdots<i_r}\lambda_{i_1}^2\ldots\lambda_{i_r}^2\leqq\frac{1}{r!}\sum_{\substack{i_1,\ldots,i_r\\\text{are different}\\\text{from each other}}}\lambda_{i_1}^2\ldots\lambda_{i_r}^2\leqq\frac{1}{r!}\leqq\left(\frac{e}{r}\right)^r$$

and (0.1), we have

$$|ET_p(z) - 1 - \sqrt{-1}zM_p| \leqq \left(\sum_{r=2}^{\infty} |4B^2 z\lambda^{\Delta}|^{r\varepsilon}\right)^{1/\varepsilon} \left(\sum_{r=2}^{\infty} 2^{-r\delta}\right)^{1/\delta}.$$

Since $|z| \leqq (8B^2\lambda^{\Delta})^{-1}$, we finally get $|ET_p(z) - 1 - \sqrt{-1}zM_p| \leqq (1/\sqrt{3})|4B^2 z\lambda^{\Delta}|^2$. Because of (1.1) and (1.3), we have $|ET_p(z) - 1| \leqq B^2\lambda^{\Delta}|z| + 10B^4\lambda^{2\Delta}|z|^2 \leqq 3B^2\lambda^{\Delta}|z|$. (1.5) is clear from (1.4). □

Lemma 3. *If* $s \in \mathbb{R}$, $|s| \leqq 3(32B^2\lambda)^{-1}$ *and* $p = 1, 2, \ldots, \infty$, *then*

$$E\exp\left(s\sum_{i=1}^{p} \lambda_i^2(\xi_i^2 - 1)\right) \leqq 4. \tag{1.6}$$

Proof. Suppose $p < \infty$. Let $t \in \mathbb{R}$ and $|t| \leqq (16B^2\lambda)^{-1}$. In a similar way as in the proof of (1.3), we have

$$\begin{aligned} &E\prod_{i=1}^{p}(1 + 4t\lambda_{2i-1}^2(\xi_{2i-1}^2 - 1)) \\ &\leqq \left(\sum_{r=0}^{\infty} \frac{1}{r!}|4\sqrt{2}B^2 t r^{1/2}\lambda|^{2r}\right)^{1/2} \left(\sum_{r=0}^{\infty} (\sqrt{2}Br^{1/2}\|\bar{B}_r\|_2^{-1/r})^{-2r}\right)^{1/2} \\ &\leqq 2. \end{aligned}$$

Since we have $|4t\lambda_{2i-1}^2(\xi_{2i-1}^2 - 1)| \leqq 4|t|\lambda_{2i-1}^2(\|\xi_{2i-1}\|_\infty^2 \vee 1) \leqq 4B^2\lambda^2|t| \leqq 4^{-1}$, applying $\exp(x - x^2) \leqq 1 + x$ ($|x| \leqq 2^{-1}$, $x \in \mathbb{R}$), we get

$$\begin{aligned} &E\exp\left(4t\sum_{i=1}^{p}\lambda_{2i-1}^2(\xi_{2i-1}^2 - 1) - 16t^2\sum_{i=1}^{p}\lambda_{2i-1}^4(\xi_{2i-1}^2 - 1)^2\right) \\ &\leqq E\prod_{i=1}^{p}(1 + 4t\lambda_{2i-1}^2(\xi_{2i-1}^2 - 1)) \\ &\leqq 2. \end{aligned}$$

Because of

$$\begin{aligned} 16t^2\sum_{i=1}^{p}\lambda_{2i-1}^4(\xi_{2i-1}^2 - 1)^2 &\leqq 16t^2\lambda^2B^2\sum_{i=1}^{p}\lambda_{2i-1}^2|\xi_{2i-1}^2 - 1| \\ &\leqq |t|\lambda\sum_{i=1}^{p}\lambda_{2i-1}^2((\xi_{2i-1}^2 - 1) + 2) \\ &\leqq |t|\lambda\sum_{i=1}^{p}\lambda_{2i-1}^2(\xi_{2i-1}^2 - 1) + 8^{-1}, \end{aligned}$$

we have $E\exp((4t - |t|\lambda)\sum_{i=1}^{p}\lambda_{2i-1}^2(\xi_{2i-1}^2 - 1)) \leqq 4$. For given s satisfying $|s| \leqq 3(32B^2\lambda)^{-1}$, let t satisfy $4t - |t|\lambda = 2s$. Then $|t| \leqq (16B^2\lambda)^{-1}$. Thus we have proved $E\exp(2s\sum_{i=1}^{p}\lambda_{2i-1}^2(\xi_{2i-1}^2 - 1)) \leqq 4$ $(p < \infty)$. This inequality is also true if we replace $2i - 1$ by $2i$. Using these inequalities and Schwarz's inequality, we have (1.6) for $p < \infty$. We can write (1.6) in the form $E\exp(s\sum_{i=1}^{p}\lambda_i^2\xi_i^2) \leqq 4\exp(s\sum_{i=1}^{p}\lambda_i^2)$ $(p < \infty)$. By monotone convergence theorem, it remains valid for $p = \infty$. □

Lemma 4. *For* $t \in \mathbb{R}$, $|t| \leqq (16B^2\lambda^{(1/2)\wedge\Delta})^{-1}$ *and* $p = 1, 2, \ldots, \infty$, *we have*

$$E\exp\left(t\sum_{i=1}^{p}\lambda_i\xi_i\right) \leqq 4\exp\left(2t^2\sum_{i=1}^{p}\lambda_i^2\right) \leqq 4e^{2t^2}. \tag{1.7}$$

Proof. Suppose $p < \infty$. By Schwarz's inequality,

$$E\exp\left(t\sum_{i=1}^{p}\lambda_i\xi_i\right) \leqq E^{1/2}\exp\left(2t\sum_{i=1}^{p}\lambda_i\xi_i - 4t^2\sum_{i=1}^{p}\lambda_i^2\xi_i^2\right)E^{1/2}\exp\left(4t^2\sum_{i=1}^{p}\lambda_i^2\xi_i^2\right)$$

$$= E_1 \times E_2.$$

Since $|2t\lambda_i\xi_i| \leqq 2|t|\lambda B < 8^{-1}$, using (1.5), we have $E_1 \leqq E^{1/2}T_p(-\sqrt{-1}2t) \leqq 2$. By (1.6), $E_2 \leqq 2\exp(2t^2\sum_{i=1}^{p}\lambda_i^2) \leqq 2e^{2t^2}$. Thus (1.7) is proved for $p < \infty$. Thanks to Fatou's lemma, we see that (1.7) is also valid in case $p = \infty$. □

Lemma 5. *If* $|z| \leqq (8B^2\lambda^{(1/2)\wedge\Delta})^{-1}$ *and* $|\mathrm{Im}\, z| \leqq (64B^2\lambda^{(1/2)\wedge\Delta})^{-1}$, *then* (1.3), (1.4) *and* (1.5) *hold for* $p = \infty$.

Proof. Since $T_p(z)$ converges to $T_\infty(z)$ in probability, if $\{T_p(z)\}_{p\in\mathbb{N}}$ is uniformly integrable, $\lim_{p\to\infty}ET_p(z) = ET_\infty(z)$ holds and (1.3), (1.4) and (1.5) remain valid for $p = \infty$. Thus it is sufficient to prove L_2-boundedness of $\{T_p(z)\}_{p\in\mathbb{N}}$. Using $1 + x \leqq e^x$ and $|1 + \sqrt{-1}z\lambda_i\xi_i|^2 = 1 - 2(\mathrm{Im}\, z)\lambda_i\xi_i + |z|^2\lambda_i^2\xi_i^2$, we have

$$E|T_p(z)|^2 \leqq E\exp\left(-2(\mathrm{Im}\, z)\sum_{i=1}^{p}\lambda_i\xi_i + |z|^2\sum_{i=1}^{p}\lambda_i^2\xi_i^2\right)$$

$$\leqq E^{1/2}\exp\left(-4(\mathrm{Im}\, z)\sum_{i=1}^{p}\lambda_i\xi_i\right)E^{1/2}\exp\left(2|z|^2\sum_{i=1}^{p}\lambda_i^2\xi_i^2\right).$$

Since $4|\mathrm{Im}\, z| \leqq (16B^2\lambda^{(1/2)\wedge\Delta})^{-1}$ and $2|z|^2 \leqq (32B^4\lambda^{1\wedge 2\Delta})^{-1}$, we can apply (1.6) and (1.7) and conclude $E|T_p(z)|^2 \leqq 4\exp(|z|^2 + 32(\mathrm{Im}\, z)^2)$. □

If $|w| \leqq 2^{-1}$, $e^w = (1 + w)\exp(2^{-1}w^2 + r(w))$ and $|r(w)| \leqq |w|^3$ hold. Thus, in case $|z| \leqq (2B\lambda)^{-1}$, we have

$$\hat{F}_n(z) = ET_\infty(z)\exp\left(-\frac{z^2}{2}\sum_{i=1}^{\infty}\lambda_i^2\xi_i^2 + \sum_{i=1}^{\infty}r(\sqrt{-1}z\lambda_i\xi_i)\right),\ |r(\sqrt{-1}z\lambda_i\xi_i)| \leqq |z\lambda_i\xi_i|^3.$$

Lemma 6. *If* $|z| \leqq (8B^2\lambda^{(1/3)\wedge\Delta})^{-1}$ *and* $|\mathrm{Im}\, z| \leqq (64B^2)^{-1}$, *then*

$$|\hat{F}(z) - \hat{G}M_\infty(z)| \leqq 13B^3\lambda|z|^3 + 22B^4\lambda^{1\wedge(2\Delta)}|z|^2.$$

Proof. Dividing into two parts, we have

$$|\hat{F}(z) - \hat{G}M_\infty(z)| \leqq |\hat{F}(z) - \mathrm{e}^{-z^2/2}ET_\infty(z)| + |\mathrm{e}^{-z^2/2}\|ET_\infty(z) - \exp(\sqrt{-1}zM_\infty)|$$

$$= I_1 + I_2.$$

First we estimate I_2. Obviously, $|e^{-z^2/2}| \leqq e^{(\operatorname{Im} z)^2/2} < 1 + 2^{-10}$. Because of (1.1), (1.3) and $|zM_\infty| \leqq 8^{-1}$, we have

$$\begin{aligned}
&|ET_\infty(z) - \exp(\sqrt{-1}zM_\infty)| \\
&\quad \leqq |ET_\infty(z) - 1 - \sqrt{-1}zM_\infty| \\
&\qquad + |1 + \sqrt{-1}zM_\infty - \exp(\sqrt{-1}zM_\infty)| \\
&\quad \leqq 10B^4\lambda^{2\Delta}|z|^2 + 2^{-1}|zM_\infty|^2 e^{|zM_\infty|} \\
&\quad \leqq (2^3 + 2 + 2^{-1} + 2^{-3})B^4\lambda^{2\Delta}|z|^2.
\end{aligned}$$

From these estimates we get $I_2 \leqq 11B^4\lambda^{2\Delta}|z|^2$. As for I_1, we have

$$\begin{aligned}
I_1 &= \left| ET_\infty(z)\exp\left(-\frac{z^2}{2}\sum_{i=1}^{\infty}\lambda_i^2\xi_i^2 + \sum_{i=1}^{\infty} r(\sqrt{-1}z\lambda_i\xi_i)\right) - e^{-z^2/2}ET_\infty(z)\right| \\
&\leqq E\left|T_\infty(z)\exp\left(-\frac{z^2}{2}\sum_{i=1}^{\infty}\lambda_i^2\xi_i^2\right)\right| \\
&\quad \times \left|\exp\left(\sum_{i=1}^{\infty} r(\sqrt{-1}z\lambda_i\xi_i)\right) - \exp\left(\frac{z^2}{2}\sum_{i=1}^{\infty}\lambda_i^2(\xi_i^2 - 1)\right)\right|.
\end{aligned}$$

Because of $|T_\infty(z)| \leqq \exp(-(\operatorname{Im} z)\sum_{i=1}^{\infty}\lambda_i\xi_i + 2^{-1}|z|^2\sum_{i=1}^{\infty}\lambda_i^2\xi_i^2)$, we get

$$\begin{aligned}
I_1 &\leqq E\exp\left(-(\operatorname{Im} z)\sum_{i=1}^{\infty}\lambda_i\xi_i + (\operatorname{Im} z)^2\sum_{i=1}^{\infty}\lambda_i^2\xi_i^2\right) \\
&\quad \times \left\{\left|\exp\left(\sum_{i=1}^{\infty} r(\sqrt{-1}z\lambda_i\xi_i)\right) - 1\right| + \left|\exp\left(\frac{z^2}{2}\sum_{i=1}^{\infty}\lambda_i^2(\xi_i^2 - 1)\right) - 1\right|\right\}.
\end{aligned}$$

Using $|e^z - 1| \leqq |z|e^{|z|}$, we have

$$\begin{aligned}
I_1 &\leqq e^{(\operatorname{Im} z)^2}E\exp\left(-(\operatorname{Im} z)\sum_{i=1}^{\infty}\lambda_i\xi_i\right)\exp\left((\operatorname{Im} z)^2\sum_{i=1}^{\infty}\lambda_i^2(\xi_i^2 - 1)\right) \\
&\quad \times \left\{\left|\sum_{i=1}^{\infty} r(\sqrt{-1}z\lambda_i\xi_i)\right|\exp\left|\sum_{i=1}^{\infty} r(\sqrt{-1}z\lambda_i\xi_i)\right|\right. \\
&\qquad \left. + \left|\frac{z^2}{2}\sum_{i=1}^{\infty}\lambda_i^2(\xi_i^2 - 1)\right|\exp\left|\frac{z^2}{2}\sum_{i=1}^{\infty}\lambda_i^2(\xi_i^2 - 1)\right|\right\}.
\end{aligned}$$

Noting $\frac{1}{4} + \frac{1}{8} + \frac{1}{2} + \frac{1}{8} = 1$, by Hölder's inequality, we get

$$\begin{aligned}
I_1 &\leqq e^{(\operatorname{Im} z)^2}E^{1/4}\exp\left(-4(\operatorname{Im} z)\sum_{i=1}^{\infty}\lambda_i\xi_i\right)E^{1/8}\exp\left(8(\operatorname{Im} z)^2\sum_{i=1}^{\infty}\lambda_i^2(\xi_i^2 - 1)\right) \\
&\quad \times \left\{E^{1/2}\left|\sum_{i=1}^{\infty} r(\sqrt{-1}z\lambda_i\xi_i)\right|^2 E^{1/8}\exp\left|8\sum_{i=1}^{\infty} r(\sqrt{-1}z\lambda_i\xi_i)\right|\right. \\
&\quad \left. + \frac{|z^2|}{2}E^{1/2}\left(\sum_{i=1}^{\infty}\lambda_i^2(\xi_i^2 - 1)\right)^2 E^{1/8}\exp\left|4z^2\sum_{i=1}^{\infty}\lambda_i^2(\xi_i^2 - 1)\right|\right\} \\
&= e^{(\operatorname{Im} z)^2}E_1E_2(E_3E_4 + 2^{-1}|z|^2E_5E_6).
\end{aligned}$$

We have $e^{(\mathrm{Im}\, z)^2} \leqq 2^{2^{-11}}$. Since $4|\mathrm{Im}\, z| \leqq (16B^2)^{-1}$, using (1.7), $E_1 \leqq (4e^{2(16B^2)^{-2}})^{1/4} < 2^{2^{-1}+2^{-8}}$. Since $8(\mathrm{Im}\, z)^2 \leqq (512B^4)^{-1}$, using (1.6), $E_2 \leqq 2^{2^{-2}}$. Since

$$\left|\sum_{i=1}^{\infty} r(\sqrt{-1}z\lambda_i\xi_i)\right| \leqq \sum_{i=1}^{\infty} |z\lambda_i\xi_i|^3 \leqq B\lambda|z|^3\left(\sum_{i=1}^{\infty} \lambda_i^2(\xi_i^2-1)+1\right),$$

we have $E_3 \leqq B\lambda|z|^3(1+E_5)$. Applying (1.2), we have $E_5 \leqq 5B^2\lambda$ and $E_3 \leqq 6B^3\lambda|z|^3$. Because of the same inequality, we have

$$E_4 \leqq \exp(B\lambda|z|^3)E^{1/8}\exp\left(8B\lambda|z|^3 \sum_{i=1}^{\infty} \lambda_i^2(\xi_i^2-1)\right).$$

Since $8B\lambda|z|^3 \leqq (64B^5)^{-1}$, by (1.6), we have $E_4 \leqq 2^{2^{-2}+2^{-8}}$. Using $e^{|x|} \leqq e^x + e^{-x}$ and Minkowski's inequality, we get

$$\begin{aligned} E_6 &\leqq E^{1/8}\left\{\exp\left(\frac{|z^2|}{2}\sum_{i=1}^{\infty}\lambda_i^2(\xi_i^2-1)\right)+\exp\left(-\frac{|z^2|}{2}\sum_{i=1}^{\infty}\lambda_i^2(\xi_i^2-1)\right)\right\}^8 \\ &\leqq E^{1/8}\exp\left(4|z^2|\sum_{i=1}^{\infty}\lambda_i^2(\xi_i^2-1)\right)+E^{1/8}\exp\left(-4|z^2|\sum_{i=1}^{\infty}\lambda_i^2(\xi_i^2-1)\right). \end{aligned}$$

Since $4|z|^2 \leqq (16B^4\lambda^{(2/3)\wedge(2\Delta)})^{-1}$, using (1.6), we have $E_6 \leqq 2^{1+2^{-2}}$. Combining these estimates, we have $I_1 \leqq 13\lambda B^3|z|^3 + 11\lambda B^2|z|^2$. □

Put $S = \sum_{i=1}^{\infty}\lambda_i\xi_i$ and $D = \{z\in\mathbb{C}; |z| < (8B^2\lambda^{(1/3)\wedge\Delta})^{-1}, |\mathrm{Im}\, z| \leqq (64B^2)^{-1}\}$. Here we prove that $\hat{F}(z)$ is holomorphic on D. By (1.7), if $z\in D$, $E|\exp(\sqrt{-1}zS)| = Ee^{-(\mathrm{Im}\, z)S} \leqq 4e^{2(\mathrm{Im}\, z)^2}$. Thus, for rectifiable closed curve C in D, $\int_C E|\exp(\sqrt{-1}zS)||dz| < \infty$. Hence we can use Fubini's theorem and have $\int_C \hat{F}(z)dz = E\int_C \exp(\sqrt{-1}zS)dz = 0$. Thanks to Morera's theorem, we see that $\hat{F}(z)$ is holomorphic in D. Consequently $H(z) = z^{-1}(\hat{F}(z) - \hat{G}M_\infty(z))$ is holomorphic in D and by lemma 6, we have

$$|H(z)| \leqq 13B^3\lambda|z|^2 + 22B^4\lambda^{1\wedge(2\Delta)}|z| \quad z\in D. \tag{1.8}$$

Moreover, we have the next estimate.

Lemma 7. *If* $t\in\mathbb{R}$ *and* $|t| \leqq (16B^2\lambda^{(1/3)\wedge\Delta})^{-1}$,

$$|H'(t)| \leqq 3328B^5\lambda|t|^2 + 2816B^6\lambda^{1\wedge(2\Delta)}|t| + 44B^4\lambda^{1\wedge(2\Delta)}.$$

Proof. In case $|t| \leqq (64B^2)^{-1}$, a circle $C_t = \{z\in\mathbb{C}; |z-t| = |t|\}$ is in D. By the Cauchy's integral formula,

$$|H'(t)| = \left|\frac{1}{2\pi\sqrt{-1}}\int_{C_t}\frac{H(z)}{(z-t)^2}dz\right| \leqq |t|^{-1}\max_{\substack{|z|<2|t| \\ z\in D}}|H(z)| \leqq 52B^3\lambda|t| + 44B^4\lambda^{1\wedge(2\Delta)}.$$

In case $(64B^2)^{-1} < |t| < (16B^2\lambda^{(1/3)\wedge\Delta})^{-1}$, we have $C_t = \{z\in\mathbb{C};\ |z-t| = (64B^2)^{-1}\} \subset D$ and conclude $|H'(t)| \leqq 64B^2 \max_{\substack{|z|<2|t| \\ z\in D}}|H(z)| \leqq 3328B^5\lambda|t|^2 + 2816B^6\lambda^{1\wedge(2\Delta)}|t|$. □

Applying smoothing lemma of Esseen (Cf. I.A. Ibragimov-Yu. V. Linnik [9]), for all $T > 0$,

$$\| F - G_{M_\infty} \|_\infty \leqq \frac{24}{\sqrt{2\pi^3}\,T} + \frac{1}{\pi} \int_{-T}^{T} |H(t)|\,\mathrm{d}t, \tag{1.9}$$

(1.10)
$$\| F - G_{M_\infty} \|_1 \leqq \frac{8\pi}{T} + \left(\frac{1}{2} + \frac{1}{T^2}\right)^{1/2} \left(\int_{-T}^{T} |H(t)|^2\,\mathrm{d}t \right)^{1/2} + \left(\int_{-T}^{T} |H'(t)|^2\,\mathrm{d}t \right)^{1/2}.$$

Using (1.9) with $T = (8B^2 \lambda^{(1/4) \wedge (2\Delta/3)})^{-1}$, by Lemma 6 we have

$$\| F - G_{M_\infty} \|_\infty \leqq 25 B^2 \lambda^{(1/4) \wedge (2\Delta/3)}.$$

Applying (1.10) with $T = (16 B^2 \lambda^{(2/7) \wedge (4\Delta/5)})^{-1}$, by Lemma 6 and 7 we get

$$\| F - G_{M_\infty} \|_1 \leqq 465 B^3 \lambda^{(2/7) \wedge (4\Delta/5)}.$$

On the other hand, it is easily seen that

$$\| G_{M_\infty} - G_0 \|_\infty < |M_\infty| \leqq B^2 \lambda^\Delta \quad \text{and} \quad \| G_{M_\infty} - G_0 \|_1 = |M_\infty| \leqq B^2 \lambda^\Delta$$

hold. From these estimates we have the conclusion of the theorem.

2. Proof of Theorem 3 and 4

Let us assume (0.6) and (0.8) and prepare some notation and lemma.
We first introduce a sequence $\{p(k)\}$ by

$$\begin{cases} p(0) = 0 \\ p(k) = \max\{ j;\ \beta_j \leqq 2^k \} \quad k \in \mathbb{N}. \end{cases}$$

(We put $\max \phi = 0$.) It is easily seen that $\beta_{p(k)} \geqq 2^{-D_1 + D_2 p^{1-\alpha}(k)}$ $(k \in \mathbb{N})$ holds for some positive constants D_1 and D_2 depending only on α, c and β_1 and it implies that

$$p(k) \leqq \left(\frac{k + D_1}{D_2} \right)^{1/(1-\alpha)} \leqq \left(\frac{k + D_1}{D_2} \right)^2.$$

For $k \in \mathbb{N}$ satisfying $p(k) > 0$, let us put

$$l(k) = \left[\frac{8\alpha}{\rho} \log_2 p(k) + 4 + \log_2 \frac{c+1}{c} \right],$$

where $[x]$ denotes the integer part of x. Since $l(k) \geqq 3$ and $\lim_{k \to \infty}(k - l(k)) = \infty$, we can define a sequence $\{m(i)\}$ by

$$\begin{cases} m(0) = 0, \\ m(1) = \min\{k;\ p(k) > 0\}, \\ m(i) = \max\{k;\ k - l(k) \leqq m(i-1)\} \quad i \geqq 2. \end{cases}$$

For $n, i \in \mathbb{N}$, let us put

$$J_{n,i} = \{j \in \mathbb{N}; p(m(i-1)) < j \leqq p(m(i)) \wedge n\} \quad \text{and} \quad J_i = J_{\infty, i}.$$

For any set $J \subset \mathbb{N}$, we denote its cardinal number by $|J|$.

After now on we denote constant depending only on α, c, β_1, ρ and C by D and it may be different line by line.

Lemma 8. *There exists D satisfying the following inequalities.*

(2.1) $$m(i) - m(i-1) = l(m(i)) \quad i \geqq 2,$$

$$m(i) - m(i-1) \leqq Dl(m(i)) \quad i \in \mathbb{N}.$$

(2.2) $$p(m(i)) - p(m(i-1)) \leqq Dl(m(i))p^{\alpha}(m(i)) \quad i \in \mathbb{N}.$$

(2.3) $$l(m(i)) \leqq Dp^{\alpha}(m(i)) \quad i \in \mathbb{N}.$$

(2.4) $$p(m(i)) \leqq Dp(m(i-1)) \quad i \geqq 2,$$

$$p(m(1)) \leqq D.$$

(2.5) $$1 + \alpha \log_2 p(m(i)) \geqq Dl(m(i)) \quad i \in \mathbb{N}.$$

(2.6) $$|J_{i_1}| \ldots |J_{i_r}| \leqq D^r (p(m(2)) \ldots p(m(i_r - 2)))^{2\alpha}$$

$$\text{if } r \in \mathbb{N},\ i_r > 3 \text{ and } i_1 < \cdots < i_r,$$

$$|J_i| \leqq D \quad \text{if } i = 1, 2, 3.$$

(2.7) $$\max_{j \in J_{n,i}} |a_j| \leqq DA_n p^{-\alpha}(m(i))(1 + \alpha \log_2 p(m(i)))^{-1} \mu_n$$

$$\text{if } n \in J_i,\ i \in \mathbb{N},$$

where $\{\mu_n\}$ *is the sequence appearing in* (0.8).

Proof.

(a) Proof of (2.1). Since each increment of $\{k - l(k)\}$ is below 1 and $p(m(i-1)) > 0$ if $i \geqq 2$, by the definition of $m(i)$, we have $m(i) - l(m(i)) = m(i-1)$ for $i \geqq 2$.

Let $i = 1$. Obviously, $p(m(1) - 1) = 0$ holds and it implies $2^{m(1)-1} < \beta_1$. Thus

$$m(1) < 1 + \log_2 \beta_1 < (1 + \log_2 \beta_1) l(m(1)).$$

(b) Proof of (2.2). If $p(k) > 0$,

$$2 \geqq \frac{\beta_{p(k)}}{\beta_{p(k-1)+1}} \geqq \prod_{j=p(k-1)+1}^{p(k)-1} (1 + cj^{-\alpha}) \geqq 1 + cp^{-\alpha}(k)\{p(k) - p(k-1) - 1\},$$

and hence $p(k) - p(k-1) \leqq 1 + c^{-1}p^{\alpha}(k) \leqq c^{-1}(c+1)p^{\alpha}(k)$ $k \in \mathbb{N}$. Using this and (2.1), we have

$$\begin{aligned} p(m(i)) - p(m(i-1)) &= \sum_{k=m(i-1)+1}^{m(i)} \{p(k) - p(k-1)\} \\ &\leqq \frac{c+1}{c} p^{\alpha}(m(i))\{m(i) - m(i-1)\} \\ &\leqq Dp^{\alpha}(m(i))l(m(i)). \end{aligned}$$

(c) Proof of (2.3). Because of the definition of $l(k)$, it is sufficient to take D as the maximum of $x^{-1}((8/\rho)\log_2 x + 4 + \log((c+1)/c))$ for $x \geqq 1$.

(d) Proof of (2.4). By (2.2) and (2.3), for $i \in \mathbb{N}$, $p(m(i)) - p(m(i-1)) \leqq D^2 p^{2\alpha}(m(i))$. In case $i \geqq 2$, dividing by $p(m(i))$, we have $1 - p(m(i-1))/p(m(i)) \leqq D^2 p^{2\alpha - 1}(m(i)) \to 0$ as $i \to \infty$. In case $i = 1$, using $p(m(0)) = 0$, we get $p(m(1)) \leqq D^{2/1-2\alpha}$.

(e) Proof of (2.5). It is sufficient to take $D \geqq 1$.

(f) Proof of (2.6). Since $|J_i| = p(m(i)) - p(m(i-1))$, by (2.2) and (2.3),

$$|J_{i_1}| \ldots |J_{i_r}| \leqq D^r \{p(m(i_1)) \ldots p(m(i_r))\}^{2\alpha}.$$

First we consider the case $i_r \geqq 2 + r$. Let i^* be the maximal element of

$$\{i \in \mathbb{N}; i \leqq i_r - 2\} \cap \{i_1, \ldots, i_{r-1}\}^c.$$

Then $i_r - r \leqq i^* \leqq i_r - 2$. Using (2.4) $i_r - i^*$ times, we get $p(m(i_r)) \leqq D^{i_r - i^*} p(m(i^*)) \leqq D^r p(m(i^*))$ and hence,

$$|J_{i_1}| \ldots |J_{i_r}| \leqq D^{2r} \{p(m(i^*)) p(m(i_1)) \ldots p(m(i_{r-1}))\}^{2\alpha}.$$

Here $i^*, i_1, \ldots, i_{r-1}$ are different from each other. In case $i_{r-1} = i_r - 1$, in a similar way we can take i^{**} satisfying $i^{**} \neq i^*, i_1, \ldots, i_{r-1}$ and $i_{r-1} - r \leqq i^{**} \leqq i_{r-1} - 1$, and have

$$|J_{i_1}| \ldots |J_{i_r}| \leqq D^{3r} \{p(m(i^{**})) p(m(i^*)) p(m(i_1)) \ldots p(m(i_{r-2}))\}^{2\alpha}.$$

Thus in case $i_r \geqq 2 + r$, using (2.4), we have

$$|J_{i_1}| \ldots |J_{i_r}| \leqq D^{3r+1} \{p(m(2)) \ldots p(m(i_r - 2))\}^{2\alpha}.$$

In the other case, using (2.4), $p(m(i_r)) \leqq D^{r+1}$ and $p(m(i_r - 1)) \leqq D^r$, we have

$$|J_{i_1}| \ldots |J_{i_r}| \leqq D^{5r+2} \{p(m(2)) \ldots p(m(i_r - 2))\}^{2\alpha}.$$

(g) Proof of (2.7). Because of (0.8), using (2.4), we have

$$\max_{j \in J_{n,i}} |a_j| \leqq C\mu_n A_n p^{-\alpha}(m(i-1))(\alpha \log p(m(i-1)) + 1)^{-1}$$

$$\leqq D\mu_n A_n p^{-\alpha}(m(i))(\alpha \log p(m(i)) + 1)^{-1}. \qquad \square$$

Now we proceed to the proof of Theorem 3. Put $\zeta_j(t, \omega) = \sqrt{2}\cos(\beta_j t\omega + \gamma_j)$ and we prepare Rademacher sequence $\{r_j\}$ which is independent of $\{\zeta_j\}$. We define $\{\lambda_{n,i}\}$ and $\{\xi_{n,i}\}$ by

$$\xi_{n,i} = \begin{cases} (A^2_{p(m(i)) \wedge n} - A^2_{p(m(i-1))})^{-1/2} \sum_{j \in J_{n,i}} a_j \zeta_j & \text{if } p(m(i-1)) < n \\ r_i & \text{if } n \leqq p(m(i-1)) \end{cases}$$

$$\lambda_{n,i} = \begin{cases} A_n^{-1}(A^2_{p(m(i)) \wedge n} - A^2_{p(m(i-1))})^{1/2} & \text{if } p(m(i-1)) < n \\ 0 & \text{if } n \leqq p(m(i-1)). \end{cases}$$

We have $\sum_{i=1}^{\infty} \lambda_{n,i}\xi_{n,i} = A_n^{-1}\sum_{j=1}^{n} a_j\zeta_j$, and using (2.2), (2.7), (2.5),

$$|\lambda_{n,i}| \leqq A_n^{-1}|J_i|^{1/2} \max_{j\in J_{n,i}} |a_j| \quad \text{and} \quad |\lambda_{n,i}\xi_{n,i}| \leqq \sqrt{2}A_n^{-1}|J_i| \max_{j\in J_{n,i}} |a_j|,$$

we have

$$(2.8) \qquad |\lambda_{n,i}| \leqq D\mu_n \quad \text{and} \quad |\lambda_{n,i}\xi_{n,i}| \leqq D\mu_n .$$

Put $b_{i_1,\ldots,i_r}^{(n)}(t) = E(\xi_{n,i_1}(t)\ldots\xi_{n,i_r}(t))(i_1 < \cdots < i_r)$. We must estimate $\|B_r^{(n)}(t)\|_1 = \sum_{i_1<\cdots<i_r}|b_{i_1,\ldots,i_r}^{(n)}(t)|$. If $n \leqq p(m(i_r-1))$, $b_{i_1,\ldots,i_r}^{(n)}(t) = 0$ holds. Otherwise, by Schwarz's inequality, we have

$$\begin{aligned}
|b_{i_1,\ldots,i_r}^{(n)}| &= (A^2_{p(m(i_1))} - A^2_{p(m(i_1-1))})^{-1/2}\cdots(A^2_{p(m(i_r))\wedge n} - A^2_{p(m(i_r-1))})^{-1/2} \\
&\quad\times\Bigg|\sum_{\substack{j_q\in J_{n,i_q}\\(q=1,\ldots,r)}} a_{j_1}\ldots a_{j_r}E(\zeta_{j_1}\ldots\zeta_{j_r})\Bigg| \\
&\leqq (A^2_{p(m(i_1))} - A^2_{p(m(i_1-1))})^{-1/2}\cdots(A^2_{p(m(i_r))\wedge n} - A^2_{p(m(i_r-1))})^{-1/2} \\
&\quad\times\Bigg(\sum_{\substack{j_q\in J_{n,i_q}\\(q=1,\ldots,r)}} a^2_{j_1}\ldots a^2_{j_r}\Bigg)^{1/2}\Bigg(\sum_{\substack{j_q\in J_{n,i_q}\\(q=1,\ldots,r)}} E^2(\zeta_{j_1}\ldots\zeta_{j_r})\Bigg)^{1/2} . \\
&\leqq \sum_{\substack{j_q\in J_{i_q}\\(q=1,\ldots,r)}} |E(\zeta_{j_1}\ldots\zeta_{j_r})| .
\end{aligned}$$

Moreover, we get

$$\begin{aligned}
|E(\zeta_{j_1}\ldots\zeta_{j_r})| &\leqq \sqrt{2}^r 2^{1-r} \sum_{\substack{\varepsilon_q=1,-1\\(q=1,\ldots,r-1)}} |E\cos((\beta_{j_r}+\varepsilon_{r-1}\beta_{j_{r-1}}+\cdots+\varepsilon_1\beta_{j_1})\omega t \\
&\quad + (\gamma_{j_r}+\varepsilon_{r-1}\gamma_{j_{r-1}}+\cdots+\varepsilon_1\gamma_{j_1}))| \\
&\leqq \sqrt{2}^r 2^{1-r} \sum_{\substack{\varepsilon_q=1,-1\\(q=1,\ldots,r-1)}} |\hat{P}((\beta_{j_r}+\varepsilon_{r-1}\beta_{j_{r-1}}+\cdots+\varepsilon_1\beta_{j_1})t)| .
\end{aligned}$$

Hence we have proved

$$(2.9) \qquad \sup_{n\in\mathbb{N}} \|B_r^{(n)}(t)\|_1 \leqq D^r \sum_{i_1<\cdots<i_r} \sum_{\substack{j_q\in J_{i_q}\\(q=1,\ldots,r)}} \sum_{\substack{\varepsilon_q=1,-1\\(q=1,\ldots,r-1)}} |\hat{P}((\beta_{j_r}+\varepsilon_{r-1}\beta_{j_{r-1}}+\cdots+\varepsilon_1\beta_{j_1})t)| .$$

Next lemma is due to S. Takahashi [17].

Lemma A. *Let P satisfy* (0.4). *Then there exists a constant D depending only on ρ and M satisfying*

$$\int_v^{v+1} |\hat{P}(ut)|\,\mathrm{d}t \leqq D|u|^{-\rho/2} \quad (u, v\in\mathbb{R}).$$

Applying this lemma to (2.9), we have, for all $v \in \mathbb{R}$,

(2.10)
$$\int_v^{v+1} \sup_{n\in\mathbb{N}} \| B_r^{(n)}(t) \|_1 \, dt \leqq D^r \sum_{i_1 < \cdots < i_r} \sum_{\substack{j_q \in J_{i_q} \\ (q=1,\ldots,r)}} |\beta_{j_r} - \beta_{j_{r-1}} - \cdots - \beta_{j_1}|^{-\rho/2}.$$

To estimate these summands, we prepare the next lemma.

Lemma 9. *Let $i_1 < \cdots < i_r$ and $j_q \in J_{i_q} (q = 1, \ldots, r)$. Then*
$$\beta_{j_r} - \beta_{j_{r-1}} - \cdots - \beta_{j_1} \geqq 2^{3(i_r-2)} \{ p(m(2)) \ldots p(m(i_r - 2)) \}^{8\alpha/\rho}.$$

Proof. If $i_{r-1} < i_r - 1$,
$$\begin{aligned}\beta_{j_r} - \beta_{j_{r-1}} - \cdots - \beta_{j_1} &\geqq 2^{m(i_r-1)} - 2^{m(i_{r-1})} - \cdots - 2^{m(i_1)} \\ &\geqq 2^{m(i_r-1)} - 2^{m(i_{r-1})+1}.\end{aligned}$$

Since $m(i_r - 1) \geqq m(i_r - 2) + 3$,
$$\beta_{j_r} - \beta_{j_{r-1}} - \cdots - \beta_{j_1} \geqq 2^{m(i_r-2)+2}.$$

Using (2.1), we have
$$\begin{aligned}\beta_{j_r} - \beta_{j_{r-1}} - \cdots - \beta_{j_1} &\geqq 2^{l(m(2)) + \cdots + l(m(i_r-1))} \\ &\geqq 2^{3(i_r-2)} \{ p(m(2)) \ldots p(m(i_r-2)) \}^{8\alpha/\rho}.\end{aligned}$$

If $i_{r-1} = i_r - 1$, we obtain
$$\begin{aligned}\beta_{j_r} - \beta_{j_{r-1}} &\geqq \beta_{p(m(i_r-1))+1} \left(1 - \frac{1}{1 + cp^{-\alpha}(m(i_r-1))} \right) \\ &\geqq 2^{m(i_r-1)} \frac{c}{c+1} p^{-\alpha}(m(i_r-1)) \\ &\geqq 2^{m(i_r-1) - l(m(i_r-1)) + 3} \\ &\geqq 2^{m(i_r-2)+3}.\end{aligned}$$

Thus $\beta_{j_r} - \beta_{j_{r-1}} - \cdots - \beta_{j_1} \geqq 2^{m(i_r-2)+3} - 2^{m(i_{r-2})} - \cdots - 2^{m(i_1)} \geqq 2^{m(i_r-2)}$. □

Because of (2.10), Lemma 9 and (2.6), we can conclude
$$\begin{aligned}&\int_v^{v+1} \sup_{n\in\mathbb{N}} \| B_r^{(n)}(t) \|_1 \, dt \\ &\leqq D^r \sum_{i_1 < \cdots < i_r} |J_{i_1}| \ldots |J_{i_r}| 2^{-3(i_r-2)/2} \{ p(m(2)) \ldots p(m(i_r-2)) \}^{-4\alpha} \\ &\leqq D^r \sum_{i_1 < \cdots < i_r} 2^{-3(i_r-2)/2} \\ &\leqq D^r \frac{1}{(r-1)!} \sum_{i=1}^{\infty} i^r 2^{-3i/2} \\ &\leqq D^r.\end{aligned}$$

Next we estimate $\|\bar{B}_r^{(n)}(t)\|_1 = \sum_{i_1<\cdots<i_r} |\bar{b}_{i_1,\ldots,i_r}^{(n)}(t)|$ where

$$\bar{b}_{i_1,\ldots,i_r}^{(n)}(t) = E((\xi_{n,\,2i_1-1}^2(t)-1)\ldots(\xi_{n,\,2i_r-1}^2(t)-1)) \quad (i_1<\cdots<i_r).$$

If $n \leqq p(m(2i_r-2))$, $b_{i_1,\ldots,i_r}^{(n)}(t)=0$. Otherwise, by Schwarz's inequality,

$$|\bar{b}_{i_1,\ldots,i_r}^{(n)}| = \sqrt{2}^r (A_{p(m(2i_1-1))}^2 - A_{p(m(2i_1-2))}^2)^{-1} \ldots (A_{p(m(2i_r-1))\wedge n}^2 - A_{p(m(2i_r-2))}^2)^{-1}$$

$$\times \Bigg| \sum_{\substack{\varepsilon_q=1,-1\\(q=1,\ldots,r)}} \sum_{\substack{j_q,j'_q\in J_{n,\,2i_q-1}\\ j_q\neq j'_q \text{ if } \varepsilon_q=-1\\(q=1,\ldots,r)}} a_{j_1}a_{j'_1}\ldots a_{j_r}a_{j'_r}$$

$$\times E\{\cos((\beta_{j_1}+\varepsilon_1\beta_{j'_1})t\omega + (\gamma_{j_1}+\varepsilon_1\gamma_{j'_1}))\ldots$$

$$\times \cos((\beta_{j_r}+\varepsilon_1\beta_{j'_r})t\omega + (\gamma_{j_r}+\varepsilon_1\gamma_{j'_r}))\}\Bigg|$$

$$\leqq \sqrt{2}^r (A_{p(m(2i_1-1))}^2 - A_{p(m(2i_1-2))}^2)^{-1} \ldots (A_{p(m(2i_r-1))\wedge n}^2 - A_{p(m(2i_r-2))}^2)^{-1}$$

$$\times \Bigg(\sum_{\substack{\varepsilon_q=1,-1\\(q=1,\ldots,r)}} \sum_{\substack{j_q,j'_q\in J_{n,\,2i_q-1}\\ j_q\neq j'_q \text{ if } \varepsilon_q=-1\\(q=1,\ldots,r)}} a_{j_1}^2 a_{j'_1}^2 \ldots a_{j_r}^2 a_{j'_r}^2 \Bigg)^{1/2}$$

$$\times \Bigg(\sum_{\substack{\varepsilon_q=1,-1\\(q=1,\ldots,r)}} \sum_{\substack{j_q,j'_q\in J_{n,\,2i_q-1}\\ j_q\neq j'_q \text{ if } \varepsilon_q=-1\\(q=1,\ldots,r)}} \Bigg| E\Bigg\{ \cos((\beta_{j_1}+\varepsilon_1\beta_{j'_1})t\omega + (\gamma_{j_1}+\varepsilon_1\gamma_{j'_1}))\ldots$$

$$\times \cos((\beta_{j_r}+\varepsilon_1\beta_{j'_r})t\omega + (\gamma_{j_r}+\varepsilon_r\gamma_{j'_r}))\Bigg\}\Bigg|^2\Bigg)^{1/2}$$

$$\leqq 2^r \sum_{\substack{\varepsilon_q=1,-1\\(q=1,\ldots,r)}} \sum_{\substack{j_q,j'_q\in J_{2i_q-1}\\ j_q\neq j'_q \text{ if } \varepsilon_q=-1\\(q=1,\ldots,r)}} |E\{\cos((\beta_{j_1}+\varepsilon_1\beta_{j'_1})t\omega + (\gamma_{j_1}+\varepsilon_1\gamma_{j'_1}))\ldots$$

$$\times \cos((\beta_{j_r}+\varepsilon_r\beta_{j'_r})t\omega + (\gamma_{j_r}+\varepsilon_1\gamma_{j'_r}))\}| .$$

Thus we have

$$\text{(2.11)} \qquad \sup_{n\in\mathbb{N}} \|\bar{B}_r^{(n)}(t)\|_1 \leqq D^r \sum_{i_1<\cdots<i_r} \sum_{\substack{\varepsilon_q=1,-1\\(q=1,\ldots,r)}} \sum_{\substack{j_q,j'_q\in J_{2i_q-1}\\ j_q\neq j'_q \text{ if } \varepsilon_q=-1\\(q=1,\ldots,r)}} \sum_{\substack{\varepsilon'_q=1,-1\\(q=1,\ldots,r-1)}}$$

$$|\hat{P}(t((\beta_{j_r}+\varepsilon_r\beta_{j'_r}) + \varepsilon'_{r-1}(\beta_{j_{r-1}}+\varepsilon_{r-1}\beta_{j'_{r-1}}) + \cdots + \varepsilon'_1(\beta_{j_1}+\varepsilon_1\beta_{j'_1})))| .$$

Thus for all $v\in\mathbb{R}$, it holds that

$$\text{(2.12)} \qquad \int_v^{v+1} \sup_{n\in\mathbb{N}} \|\bar{B}_r^{(n)}(t)\|_1 \mathrm{d}t \leqq D^r \sum_{i_1<\cdots<i_r} \sum_{\substack{j_q,j'_q\in J_{2i_q-1}\\(q=1,\ldots,r)\\ j'_r<j_r}}$$

$$|(\beta_{j_r}-\beta_{j'_r}) - (\beta_{j_{r-1}}+B_{j'_{r-1}}) - \cdots - (\beta_{j_1}+\beta_{j'_1})|^{-\rho/2}.$$

These summands are estimated as follows.

Lemma 10. *If* $j_q, j'_q \in J_{2i_q-1}$, $(q = 1, \ldots, r)$ *and* $j'_r < j_r$,

$$(\beta_{j_r} - \beta_{j'_r}) - (\beta_{j_{r-1}} + \beta_{j'_{r-1}}) - \cdots - (\beta_{j_1} + \beta_{j'_1})$$
$$\geqq 2^{3(2i_r-3)}\{p(m(2)) \ldots p(m(2i_r - 3))\}^{8\alpha/\rho}.$$

Proof. If $j, j' \in J_{2i-1}$, it is easily seen $\beta_{j'} \pm \beta_j \leqq 2^{m(2i-1)+1}$. On the other hand, if $j > j'$ and $j, j' \in J_{2i-1}$, we have $\beta_j \pm \beta_{j'} \geqq \beta_j - \beta_{j-1}$. Let k be an integer satisfying $p(k) < j \leqq p(k+1)$. Then we have $m(2i-2) < k \leqq m(2i-1)$ and $\beta_j - \beta_{j-1} \geqq cp^{-\alpha}(k+1)2^k \geqq 2^{(k+1)-l(k+1)+3} \geqq 2^{m(2i-3)+3}$ by the definition of $\{m(i)\}$. Applying these estimates, we have

$$(\beta_{j_r} - \beta_{j'_r}) - (\beta_{j_{r-1}} + \beta_{j'_{r-1}}) - \cdots - (\beta_{j_1} + \beta_{j'_1})$$
$$\geqq 2^{m(2i_r-3)+3} - 2^{m(2i_{r-1}-1)+1} - \cdots - 2^{m(2i_1-1)+1}$$
$$\geqq 2^{m(2i_r-3)+3} - 2^{m(2i_{r-1}-1)+2}$$
$$\geqq 2^{m(2i_r-3)+2}.$$

Using (2.1), we have the conclusion. □

Because of (2.11), Lemma 10 and (2.6), we have

$$\int_v^{v+1} \sup_{n\in\mathbb{N}} \|\bar{B}_r^{(n)}(t)\|_1 \, dt$$
$$\leqq D^r \sum_{i_1 < \cdots < i_r} |J_{2i_1-1}|^2 \ldots |J_{2i_r-1}|^2 2^{-3(2i_r-2)/2}\{p(m(2)) \ldots p(m(2i_r-3))\}^{-4\alpha}$$
$$\leqq D^r \sum_{i_1 < \cdots < i_r} 2^{-3i_r}$$
$$\leqq D^r.$$

In the same way $\int_v^{v+1} \sup_{n\in\mathbb{N}} \|\bar{\bar{B}}_r^{(n)}(t)\|_1 \, dt \leqq D^r$ can be also proved and consequently we get

$$\int_v^{v+1} \sup_{n\in\mathbb{N}} \left\{ \sum_{r=1}^{\infty} \frac{1}{(2D)^r} (\|B_r^{(n)}(t)\|_1 + \|\bar{B}_r^{(n)}(t)\|_1 + \|\bar{\bar{B}}_r^{(n)}(t)\|_1) \right\} dt \leqq 3.$$

From this we can conclude that for almost all t with respect to the Lebesgue measure, there exists a constant B depending only on t, α, c, β_1, C, ρ and M satisfying

$$\|B_r^{(n)}(t)\|_1^{1/r}, \ \|\bar{B}_r^{(n)}(t)\|_1^{1/r}, \ \|\bar{\bar{B}}_r^{(n)}(t)\|_1^{1/r} \leqq B \quad (n, r \in \mathbb{N}).$$

We can apply the corollary of Theorem 1 because of this inequality and (2.8), and we have the conclusion of Theorem 3. In a similar way, we can prove Theorem 4.

3. Proof of Theorem 2

Theorem 2 can be proved in a similar way as Theorem 3. We assume here that $\phi(1) \geqq 2$ and the other case is proved with non-essential minor change.

First we fix $v > 1$ arbitrarily, put $\beta_n = v^{\phi(n)}$, $c = d \log v$ and define $\{p(k)\}$ in the same way. Put

$$l(k) = \left[\frac{2+4\rho}{\rho} 4\alpha \log_2 p(k) + 4 + \log_2 \frac{c+1}{c} \right]$$

and construct $\{m(i)\}$, $\{J_{n,i}\}$, $\{J_i\}$ and $\{\lambda_{n,i}\}$ as before. Then by Lemma 9 and 10, we have

$$(3.1)\quad v^{\phi(j_r)} - v^{\phi(j_{r-1})} - \cdots - v^{\phi(j_1)} \geqq 2^{3i_r}\{p(m(2)) \ldots p(m(i_r - 1))\}^{4(2+4\rho)/\rho}$$

if $i_1 < \cdots < i_r$ and $j_q \in J_{i_q}(q = 1, \ldots, r)$, and

$$(3.2)\quad (v^{\phi(j_r)} - v^{\phi(j'_r)}) - (v^{\phi(j_{r-1})} + v^{\phi(j'_{r-1})}) - \cdots - (v^{\phi(j_1)} + v^{\phi(j'_1)})$$
$$\geqq 2^{6i_r}\{p(m(2)) \ldots p(m(2i_r - 2))\}^{4(2+4\rho)/\rho}$$

if $i_1 < \cdots < i_r, j_q, j'_q \in J_{2i_q - 1}(q = 1, \ldots, r)$ and $j_r > j'_r$.

Let us define $\{\xi_{n,i}\}$ as before using $\zeta_j(x, \omega) = \sqrt{2}\cos(x^{\phi(j)}\omega + \gamma_j)$ instead of $\zeta_j(t, \omega)$. Obviously (2.8) remains valid, and by (2.9) and (2.11), we have

$$(3.3)\quad \sup_{n\in\mathbb{N}} \| B_r^{(n)}(x) \|_1 \leqq D^r \sum_{i_1 < \cdots < i_r} \sum_{\substack{j_q \in J_{i_q} \\ (q=1,\ldots,r)}} \sum_{\substack{\varepsilon_q = 1,-1 \\ (q=1,\ldots,r-1)}} |\hat{P}(x^{\phi(j_r)} + \varepsilon_{r-1} x^{\phi(j_{r-1})} + \cdots + \varepsilon_1 x^{\phi(j_1)})|$$

and

$$(3.4)\quad \sup_{n\in\mathbb{N}} \| \bar{B}_r^{(n)}(x) \|_1 \leqq D^r \sum_{i_1 < \cdots < i_r} \sum_{\substack{\varepsilon_q = 1,-1 \\ (q=1,\ldots,r)}} \sum_{\substack{j_q, j'_q \in J_{2i_q - 1} \\ j_q \neq j'_q \text{ if } \varepsilon_q = -1 \\ (q=1,\ldots,r)}} \sum_{\substack{\varepsilon'_q = 1,-1 \\ (q=1,\ldots,r-1)}}$$
$$|\hat{P}((x^{\phi(j_r)} + \varepsilon_r x^{\phi(j'_r)}) + \varepsilon'_{r-1}(x^{\phi(j_{r-1})} + \varepsilon_{r-1} x^{\phi(j'_{r-1})}) + \cdots + \varepsilon'_1(x^{\phi(j_1)} + \varepsilon_1 x^{\phi(j'_1)})| .$$

To estimate these summands we use the next lemma due to R. Kaufman, which is a generalization of Van der Corput's lemma.

Lemma B. *Let P satisfy* (0.4). *There exists a constant D depending only on ρ and M, such that for any $f \in C^2[v, v+1]$ satisfying* $\min_{x\in[v,v+1]} f''(x) > 0$,

$$\int_v^{v+1} |\hat{P}(f(x))| dx \leqq D \left(\min_{x\in[v,v+1]} f''(x) \right)^{-\rho/(2+4\rho)} .$$

In case $i_r \geqq 2$, we have $\phi(j_r) \geqq 2$. Thus we have

$$(x^{\phi(j_r)} + \varepsilon_{r-1} x^{\phi(j_{r-1})} + \cdots + \varepsilon_1 x^{\phi(j_1)})''$$
$$\geqq \phi(j_r)(\phi(j_r) - 1)(v+1)^{-2}(x^{\phi(j_r)} - x^{\phi(j_{r-1})} - \cdots - x^{\phi(j_1)}).$$

Here $g(x) = x^{\phi(j_r)} - x^{\phi(j_{r-1})} - \cdots - x^{\phi(j_1)}$ is monotonous increasing because differential inequality $g'(x) \geqq \phi(j_r)x^{-1}g(x)$ and $g(v) > 0$ implies $g'(x) > 0$. Because of (3.1) and this estimate, we get

$$(x^{\phi(j_r)} + \varepsilon_{r-1}x^{\phi(j_{r-1})} + \cdots + \varepsilon_1 x^{\phi(j_1)})'' \geqq (v+1)^{-2}2^{3i_r}\{p(m(2)) \ldots p(m(i_r - 1))\}^{4(2+4\rho)/\rho}.$$

In a similar way, if $i_r \geqq 2$, we can prove

$$((x^{\phi(j_r)} + \varepsilon_r x^{\phi(j_r')}) + \varepsilon'_{r-1}(x^{\phi(j_{r-1})} + \varepsilon_{r-1}x^{\phi(j_{r-1}')}) + \cdots + \varepsilon'_1(x^{\phi(j_1)} + \varepsilon_1 x^{\phi(j_1')}))'' \geqq (v+1)^{-2}2^{6i_r}\{p(m(2)) \ldots p(m(2i_r - 2))\}^{4(2+4\rho)/\rho}.$$

Applying these to (3.3) and (3.4), we can prove in a similar way as a proof of Theorem 3 that for almost all $x > 1$ with respect to the Lebesgue measure, there exists a constant B depending only on x, α, d, $\phi(1)$, ρ, C and M satisfying

$$\|B_r^{(n)}(x)\|_1^{1/r},\ \|\bar{B}_r^{(n)}(x)\|_1^{1/r},\ \|\bar{\bar{B}}_r^{(n)}(x)\|_1^{1/r} \leqq B \quad (n, r \in \mathbb{N}).$$

We can apply Theorem 1 because of this inequality and (2.8).

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