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Probability Theory and Related Fields

A lim inf result in Strassen's law of the iterated logarithm

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Summary. For a function $f(.)$ from Strassen's class, we investigate the lim inf behaviour of its distance from the normalized trajectories of a Wiener process. The lim inf rate is expressed in terms of a certain functional of $f(.)$. In addition, we give a result on the lim inf behaviour of the distance of the normed trajectories from Strassen's class as a whole.

1. Introduction

Let $(W(t), t \geqq 0)$ be a standard Wiener process defined on the probability space $(\Omega, \mathscr{F}, \mathbf{P})$. Furthermore, let

$$\mathscr{S} = \left\{ f(t) = \int_0^t g(u)\,du : t \in [0, 1], \int_0^1 (g)(u))^2\,du \leqq 1 \right\}.$$

Strassen's (1964) law of the iterated logarithm can be considered to consist of two parts: if we let

$$\eta_T(t) = \frac{W(Tt)}{\sqrt{2T \log\log T}}, \; t \in [0, 1]$$

then

$$\lim_{T \to \infty} \inf_{f \in \mathscr{S}} \| f - \eta_T \|_\infty = 0 \tag{1}$$

and

$$\text{for all } f \in \mathscr{S} : \liminf_{T \to \infty} \| f - \eta_T \|_\infty = 0 \,. \tag{2}$$

It seems natural to ask for rates of convergence in (1) and (2). Regarding (1) we may refer to the papers by Bolthausen (1980), Grill (1987), and Goodman and Kuelbs (1991a, b), where some results on the lim sup behaviour are proved. The question of rates of convergence in (2) has been studied by Csáki (1980, 1989) and de Acosta

(1983). In his first paper, Csáki obtains the right rate of convergence for the following cases:

(i) $$\int_0^1 (f'(u))^2\,du < 1 \text{ and } f' \text{ is of bounded variation}$$

and

(ii) $$\int_0^1 (f'(u))^2\,du = 1 \text{ and } f \text{ is piecewise linear.}$$

In his second paper, the case

$$\int_0^1 (f'(u))^2\,du = 1 \text{ and } f(x) = ax^2 + bx$$

is considered.

De Acosta (1983), Theorem 6.1, shows that in (i) the assumption that f' be of bounded variation can be dropped.

In both cases studied where $\int_0^1 (f'(u))^2\,du = 1$, we have

$$\liminf_{T\to\infty} (\log\log T)^{2/3} \|f - \eta_T\|_\infty = \gamma(f)\,,$$

where $\gamma(f)$ can be expressed by the total variation of f' if f is piecewise linear, and in terms of the smallest positive eigenvalue of a certain second order differential operator if f is quadratic.

In general, a result from Goodman and Kuelbs (1991) implies that for any $f \in \mathcal{S}$, $\liminf(\log\log T)^{2/3} \|f - \eta_T\|_\infty$ is finite. On the other hand, by Theorem 1 in Csáki (1980), $\liminf(\log\log T)\|f - \eta_T\|_\infty$ is bounded away from zero.

In the sequel, we shall give the right lim inf rate (up to a factor 2) for any function $f \in \mathcal{S}$. In addition, we shall give a simple lim inf result for (1). Turning now to the first question, let us define:

Definition 1. Let $f(.) \in C[0, 1]$ with $f(0) = 0$. Let

$$I(f) = I(f, 0) = \begin{cases} \int_0^1 (f'(x))^2\,dx & \text{if } f \text{ is absolutely continuous} \\ \infty & \text{otherwise,} \end{cases}$$

$$U(f, \delta) = \{g : g(0) = 0, \|f - g\|_\infty \leqq \delta\}$$

and

$$I(f, \delta) = \inf_{g \in U(f, \delta)} I(g)\,.$$

Let us gather some facts about $I(f, \delta)$:

Lemma 1. *If $I(f) < \infty$ then for $I(f, \delta)$, the following statements are true: For $\delta \leqq \sqrt{I(f, 0)}$*

$$0 \leqq I(f, \delta) \leqq I(f, 0) - \delta\sqrt{I(f, 0)}\,. \tag{3}$$

There is a unique $h_{f,\delta} \in U(f, \delta)$ such that $I(f, \delta) = I(h_{f,\delta})$. (4)

For all $g \in U(f, \delta)$ that are absolutely continuous

$$\int_0^1 g'(x) h'_{f,\delta}(x)\,\mathrm{d}x \geqq I(f, \delta) . \tag{5}$$

For $\rho \in [0, 1]$

$$I(f, 0) - I(f, \delta) \geqq I(f, 0) - I(f, \rho\delta) \geqq \rho(I(f, 0) - I(f, \delta)) \tag{6}$$

$$I(f, \delta) \text{ is continuous in } \delta . \tag{7}$$

Proof. For (3), the lower estimate is trivial. The upper estimate follows from the fact that

$$\| f \|_\infty \leqq \sqrt{I(f, 0)} ,$$

so

$$g = \left(1 - \frac{\delta}{\sqrt{I(f, 0)}}\right) f \in U(f, \delta)$$

and

$$I(f, \delta) \leqq I(g) \leqq I(f, 0) - \delta \sqrt{I(f, 0)} .$$

For (4), let $g_n \in U(f, \delta)$ satisfy

$$I(g_n) \leqq I(f, \delta) + \frac{1}{n} .$$

We clearly have

$$\frac{g_n + g_m}{2} \in U(f, \delta)$$

and so

$$I\left(\frac{g_n + g_m}{2}\right) \geqq I(f, \delta)$$

and

$$I(g_n - g_m) = 2I(g_n) + 2I(g_m) - I(g_n + g_m) \leqq 2\left(\frac{1}{n} + \frac{1}{m}\right) .$$

Thus, (g_n) is a Cauchy sequence with respect to the norm

$$\| g(.) \|_2 = (I(g))^{1/2} .$$

This implies that there is a h such that $g_n \to h$ with respect to the norm $\| . \|_2$. As $\| . \|_\infty \leqq \| . \|_2$, we have $h \in U(f, \delta)$ and $\| h \|_2^2 = \lim \| g_n \|_2^2 = I(f, \delta)$.

In order to prove uniqueness, assume that $\hat{h}$, too, satisfies $\hat{h} \in U(f, \delta)$ and $\| \hat{h} \|_2^2 = I(f, \delta)$. Similar to the above, we obtain

$$\| \hat{h} - h \|_2^2 = 2 \| \hat{h} \|_2^2 + 2 \| h \|_2^2 - 4 \left\| \frac{\hat{h} + h}{2} \right\|_2^2 \leqq 0 ,$$

so $\hat{h} = h$.

Now, in order to prove (5), let $g \in U(f, \delta)$ and $\lambda \in [0, 1]$. Clearly, the function $\lambda g + (1-\lambda) h_{f,\delta}$, is in $U(f, \delta)$, so

$$I(\lambda g + (1-\lambda) h) \geqq I(f, \delta),$$

and some elementary calculus shows that this is only possible if (5) is true. In (6), the left side is clearly trivial. For the right side, observe that

$$\rho h_{f,\delta} + (1-\rho) f \in U(f, \rho\delta).$$

Thus

$$I(f, \rho\delta) \leqq I(\rho h_{f,\delta} + (1-\rho) f) \leqq \rho I(h_{f,\delta}) + (1-\rho) I(f),$$

which implies the desired upper inequality.

Regarding (7), first observe that continuity at $\delta > 0$ already follows from (6), so it remains to prove continuity at $\delta = 0$. To this end, observe that $I(f, \delta)$ increases to a limit if $\delta \to 0$ monotonically. A similar argument as above then shows that $h_{f,\delta}$ converges with respect to the norm $\|.\|_2$ to a limiting function g and $\|g\|_2 = \lim \|h_{f,\delta}\|_2$. Since $\|.\|_\infty \leqq \|.\|_2$, g is also the limit of $h_{f,\delta}$ with respect to $\|.\|_\infty$, so $g = f$ and (7) is proved.

Remark. It is easy to see that (4), (5), and (7) also hold if $I(f) = \infty$. In general, the evaluation of $I(f, \delta)$ is a nontrivial task. In the sequel, we shall give some results for some important special cases.

Lemma 2. *If f' is of bounded variation, then*

$$I(f, \delta) = I(f, 0) - 2\delta(V_0^1(f') + |f'(1)|) + o(\delta)$$

where V_a^b denotes total variation in the interval $[a, b]$, and f' is chosen to be left continuous.

Proof. As f' is of bounded variation, there is a signed measure μ on $[0, 1]$ such that $\mu([0, x)) = f'(x)$. Let g be an absolutely continuous function on $[0, 1]$ with $\|g\|_\infty \leqq \delta$. We have

$$I(f-g) = I(f) + I(g) - 2\int_0^1 f'(x) g'(x)\,dx \geqq$$

$$I(f) - 2\int_0^1 f'(x) g'(x)\,dx = I(f) - 2g(1) f'(1) + 2\int_0^1 g(x)\,df'(x) \geqq$$

$$I(f) - 2\delta(V_0^1(f') + |f'(1)|).$$

On the other hand, let (A, B) be a Hahn decomposition of $[0, 1]$ with respect to μ, where A is a positive set and B is negative. For any $\varepsilon > 0$ we can find open sets $A_\varepsilon \supset A$ and $B_\varepsilon \supset B$ with

$$\mu(A_\varepsilon - A) \geqq -\varepsilon$$

and

$$\mu(B_\varepsilon - B) \leqq \varepsilon.$$

Both A_ε and B_ε are unions of disjoint open intervals. As the union of these intervals covers $[0, 1]$, we can find a finite subcovering. Let A_0 and B_0 be the unions of those intervals in our subcovering that belong to A_ε and B_ε, respectively. Finally, there is a $\theta < 1$ such that $V_\theta^1(f') < \varepsilon$. Let $[0, \theta) \cap A_0 \cap B_0 = \bigcup_{k=1}^n J_k$, where $J_k = (a_k, b_k)$, $k = 1, \ldots, n$ are disjoint open intervals. Let now

$$g(x) = \begin{cases} +\delta & \text{if } x \in A_0 - B_0 \text{ and } x < \theta \\ -\delta & \text{if } x \in B_0 - A_0 \text{ and } x < \theta \\ \dfrac{g(a_k)(b_k - x) + g(b_k)(x - a_k)}{b_k - a_k} & \text{if } x \in J_k \\ \dfrac{g(\theta - 0)(1 - x) + \delta \operatorname{sgn}(f'(1))(x - \theta)}{1 - \theta} & \text{if } \theta \leqq x \leqq 1. \end{cases}$$

With this g, we get, again by integration by parts

$$\begin{aligned} I(f - g) &= I(f) + I(g) - 2\int_0^1 f'(x)g'(x)\,\mathrm{d}x \\ &= I(f) - 2g(1)f'(1) + 2\int_0^1 g(x)\,\mathrm{d}f'(x) + \int_0^1 (g'(x))^2\,\mathrm{d}x \\ &\leqq I(f) - 2\delta(V_0^1(f') + |f'(1)| - 6\varepsilon) + C(\varepsilon)\delta^2\,, \end{aligned}$$

where $C(\varepsilon) < \infty$ only depends on ε. Letting first δ and then ε tend to zero, we finally obtain the assertion of our lemma.

Lemma 3. *If $\|f\|_2 < \infty$ and f is convex from above (i.e., f' is nonincreasing), then if $f'(1) \geqq 0$*

$$I(f, 0) - I(f, \delta) \sim \int_0^{x_\delta} (f'(u))^2\,\mathrm{d}u - x_\delta(f'(x_\delta))^2\,,$$

and if $f'(1) < 0$ then

$$I(f, 0) - I(f, \delta) \sim \int_0^{x_\delta} (f'(u))^2\,\mathrm{d}u - x_\delta(f'(x_\delta))^2 + \int_{y_\delta}^1 (f'(u))^2\,\mathrm{d}u - (1 - y_\delta)(f'(y_\delta))^2\,,$$

where x_δ and y_δ are the solutions of the equations

$$f(x) - xf'(x) = \delta$$

and

$$f(y) + (1 - y)f'(y) = f(1) + 2\delta\,,$$

respectively.

Proof. A little thought shows that $h_{f,\delta}$ consists of an arc of $f - \delta$ and tangent segments to this arc ending in $(0, 0)$ and in $(1, f(1) + z)$, with $|z| \leqq \delta$. Using this and minimizing for z, we obtain the assertion of our lemma.

Corollary 1. *If $f(x) = Cx^{\beta}$ with $C > 0$ and $1/2 < \beta < 1$ then*

$$I(f, 0) - I(f, \delta) \sim C^2 \frac{2(1-\beta)}{2\beta - 1} \beta^2 \left(\frac{\delta}{C(1-\beta)} \right)^{2\beta - 1} .$$

2. Theorems

We are now able to state

Theorem 1. *Let $f \in \mathscr{S}$ and $\eta_T(t) = W(tT)(2 \log\log T)^{-1/2}$. Furthermore, let $d(T)$ be the unique solution of the equation*

$$d^2(1 - I(f, d)) = \frac{\pi^2}{16(\log\log T)^2} .$$

Then

$$1 \leqq \liminf_{T \to \infty} \frac{\| f - \eta_T \|_\infty}{d(T)} \leqq 2 .$$

If, in addition, $1 - I(f, \delta)$ is slowly varying at zero (in particular, if $I(f, 0) < 1$), then

$$\liminf_{T \to \infty} \frac{\| f - \eta_T \|_\infty}{d(T)} = 1 .$$

Remarks. 1. Functions that satisfy the last condition along with $I(f, 0) = 1$ can be easily given. Take, for example, $f(t) = C(\beta) t^{1/2} / \log(\beta/t)$ with some $\beta > \exp(8^{1/2})$. Then Lemma 3 is applicable and after some calculation we have that $I(f, \delta) \approx C(\beta)^2 / (8 \log(1/\delta))$.

2. Csáki's (1980) result on piecewise linear functions shows that in that case the above lim inf equals one, too, although $1 - I(f, \delta)$ is not slowly varying. On the other hand, from his 1989 result on quadratic functions it follows that this is not true in general.

Proof. We concentrate on giving estimates on the probability

$$\mathbf{P}(\| \eta_T - f \|_\infty \leqq \delta) .$$

From there, the rest of the proof proceeds literally as in Csáki's (1980) paper, so this need not be repeated here.

Lemma 4. *For all $\alpha \in [0, 1]$*

$$\exp(-I(f, \alpha\delta) \log\log T) K((1-\alpha)\delta \sqrt{2 \log\log T}) \leqq \mathbf{P}(\| \eta_T - f \|_\infty \leqq \delta)$$
$$\leqq \exp(-I(f, \delta) \log\log T) K(\delta \sqrt{2 \log\log T}) .$$

Here

$$K(z) = \mathbf{P}(\| W \|_\infty \leqq z) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \exp\left(-\frac{\pi^2 (2k+1)^2}{8z^2} \right) .$$

For a proof of this lemma, we use Lemma 2 of Csáki (1980) which states as follows:

Lemma 5. *If* $I(f, 0) < \infty$ *then*

$$\exp(-I(f, 0)/2)K(z) \leqq \mathbf{P}(\| W - f \|_\infty \leqq z) \leqq K(z) .$$

The lower half of our lemma follows directly by setting $\phi = (2\log\log T)^{1/2} h_{f,\alpha\delta}$ and $z = (2\log\log T)^{1/2}(1-\alpha)\delta$. For the upper half, we use the Cameron-Martin translation formula:

$$\mathbf{P}(|W(x) - \phi(x)| \leqq \gamma(x)) =$$

$$\exp(-I(\phi, 0)/2)\mathbf{E}\left(\exp\left(-\int_0^1 \phi'(u)\,\mathrm{d}W(u)\right)\chi(|W(x)| \leqq \gamma(x))\right).$$

We let $\phi = (2\log\log T)^{1/2} h_{f,\delta}$, $z = (2\log\log T)^{1/2}\delta$, and $\gamma = (2\log\log T)^{1/2} \times (f - h_{f,\delta})$. It is an easy consequence of (5) that on the set $\{\| W - \gamma \|_\infty \leqq z\}$ the integral $\int \phi'\,\mathrm{d}W$ is positive. Thus we get the estimate

$$\mathbf{P}(\| W - f\sqrt{2\log\log T} \|_\infty \leqq z) \leqq \exp(-I(f, \delta)\log\log T)\mathbf{P}(\| W - \gamma \|_\infty \leqq z) ,$$

and an application of the upper half of lemma 5 completes the proof of our lemma.
From this lemma, we get the following:

Lemma 6. *If* $f \in \mathscr{S}$ *and* $0 < \alpha < 1$ *then*

$$\exp\left(-I(f, \alpha\delta)\log\log T - \frac{\pi^2}{16(1-\alpha)^2\delta^2\log\log T}(1 + o(1))\right) \leqq \mathbf{P}(\| \eta_T - f \|_\infty \leqq \delta)$$

$$\leqq \exp\left(-I(f, \delta)\log\log T - \frac{\pi^2}{16\delta^2\log\log T}(1 + o(1))\right) \textit{as } T \to \infty .$$

From this point, the proof of theorem 1 is completed by a standard Borel-Cantelli argument that proceeds exactly along the lines of Csáki (1980), so we feel free to omit the details.

We now list some special cases:

Corollary 2 *If f' is of bounded variation and* $\| f \|_2 = 1$

$$1 \leqq \liminf \| \eta_T - f \|_\infty \left(\frac{\pi^2}{32(V_0^1(f') + f'(1))(\log\log T)^2}\right)^{-1/3} \leqq \frac{3}{2^{2/3}} .$$

On the other hand, if f' is not of bounded variation, then

$$\liminf \| \eta_T - f \|_\infty (\log\log T)^{2/3} = 0 .$$

Corollary 3. *If* $\| f \|_2 < 1$ *then*

$$\liminf \| \eta_T - f \|_\infty (\log\log T) \frac{\pi}{4\sqrt{1 - I(f, 0)}} = 1 .$$

Remark. This is a special case of Theorem 6.1 from de Acosta (1983).

Corollary 4. *If* $f(x) = (2\beta - 1)^{1/2}\beta^{-1}x^\beta$ *with* $1/2 < \beta < 1$ *then*

$$1 \leqq \liminf \| \eta_T - f \|_\infty (d(T))^{-1} \leqq \left(\frac{2}{2\beta + 1}\right)^{\frac{2}{2\beta+1}} \left(\frac{2\beta - 1}{2\beta + 1}\right)^{\frac{2\beta-1}{2\beta+1}} ,$$

where $d(T) = C(\beta)(\log\log T)^{-2/(2\beta+1)}$ *and*

$$C(\beta) = \left(\frac{\pi^2}{32}(1-\beta)^{2(1-\beta)}\beta^{(2\beta-1)}(2\beta-1)^{-(2\beta-1)/2}\right)^{1/(2\beta+1)}.$$

Remark. This shows in particular that any rate $(\log\log T)^{-c}$ with $2/3 \leqq c \leqq 1$ can be attained.

Finally, let us give a simple uniform result:

Theorem 2. *There are positive constants* C_1 *and* C_2 *such that*

$$\mathbf{P}(\inf\{\|\eta_T - f\|_\infty : f \in \mathscr{S}\} \leqq C_1/\log\log T \text{ i.o.}) = 1$$

and

$$\mathbf{P}(\inf\{\|\eta_T - f\|_\infty : f \in \mathscr{S}\} \leqq C_2/\log\log T \text{ i.o.}) = 0 .$$

Proof. The first half of this theorem is a trivial consequence of Chung's law of the iterated logarithm. For the second half, let us leave $\gamma > 0$ to be chosen later and let

$$\delta = \gamma(\log\log T)^{-1}$$

and $n = 1/\delta$. We have

$$I(\eta_T, \delta) = \inf\left\{\int_0^1 (g'(x))^2 : \|\eta_T - g\|_\infty \leqq \delta\right\}$$

$$\geqq \inf\left\{\int_0^1 (g'(x))^2 : \left|\eta_T\left(\frac{k}{n}\right) - g\left(\frac{k}{n}\right)\right| \leqq \delta, k = 0, \ldots, n\right\}$$

$$= \inf\left\{\sum_{k=1}^{n} n\left(\eta_T\left(\frac{k}{n}\right) - \eta_T\left(\frac{k-1}{n}\right) - \delta_k + \delta_{k-1}\right)^2 : |\delta_k| \leqq \delta\right\}.$$

Now, let $X_k = (2n\log\log T)^{1/2}\left(\eta_T\left(\frac{k}{n}\right) - \eta_T\left(\frac{k-1}{n}\right)\right)$. (X_k) is a sequence of i.i.d $N(0, 1)$ random variables, and using Cauchy's inequality, we obtain the estimate

$$I(\eta_T, \delta) \geqq \sum_{k=1}^{n} \frac{X_k^2}{2\log\log T} - 4n\delta\left(\sum_{k=1}^{n} \frac{X_k^2}{2\log\log T}\right)^{1/2}.$$

Using this and a little calculation, we get

$$\mathbf{P}(\inf\{\|\eta_T - f\|_\infty : f \in \mathscr{S}\} \leqq \delta) = \mathbf{P}(I(\eta_T, \delta) \leqq 1) \leqq \mathbf{P}\left(\sum_{k=1}^{n} X_k^2 \leqq 50\log\log T\right).$$

Now, $\sum_{k=0}^{n} X_k^2$ is χ_n^2-distributed, and we may use the following estimate: For $x \leqq n$

$$\mathbf{P}\left(\sum_{k=0}^{n} X_k^2 \leqq x\right) \leqq \left(\frac{x}{2}\right)^{n/2} \Gamma\left(\frac{n}{2}\right)^{-1} \mathrm{e}^{-x/2} \sim \sqrt{\frac{4n}{\pi}}\left(\frac{xe}{n}\right)^{n/2} \mathrm{e}^{-x/2}.$$

Thus

$$\mathbf{P}(\inf\{\|\eta_T - f\|_\infty : f \in \mathscr{S}\} \leqq \delta) \leqq (50\gamma e)^{\log\log T/2\gamma} \exp(-25\log\log T).$$

Letting now $\gamma = 1/50e$ and using another Borel-Cantelli argument, the proof of Theorem 2 is completed.

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