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Probability Theory and Related Fields



# A note on superprocesses

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**Summary.** Subject to a mild restriction on $A$, generator of the one-particle motion, we show the $A$-Fleming-Viot superprocess can be obtained from the $A$-Dawson-Watanabe superprocess by conditioning the latter to have constant total mass.

There are two main examples of measure-valued diffusion: the Dawson-Watanabe and the Fleming-Viot superprocesses. Although a greater degree of generality in their definition and construction can be achieved [1] [2], we content ourselves with a characterization via martingales. Thus, let $E$ be a locally compact separable metric space; let $\mathcal{M}_F(E)$ and $\mathcal{M}_1(E)$ be the spaces of finite measures and probability measures respectively, with the topology of weak convergence; and let $\Omega = C([0, \infty), \mathcal{M}_F(E))$ be the space of continuous, measure valued paths with the filtration $\mathcal{F}_t$, $t \geqq 0$ of $\sigma$-algebras generated by the cannonical process $\xi_t$: $\Omega \to \mathcal{M}_F(E)$, $\xi_t(\omega, dx) = \omega(t)(dx)$. Finally let $(A, \mathcal{D}(A))$, $\mathcal{D}(A) \subset C_b(E)$, be the generator of a conservative, Feller process on $E$. This is our restriction on $A$. It is in force so we may be assured that both

a) $1 \in \mathcal{D}(A)$ and $A1 = 0$; and

b) if $\phi \in \mathcal{D}(A)$ then $A\phi \in C_b(E)$, the space of bounded, continuous functions on $E$.

With this notation in hand, recall [5] that for each $\mu \in \mathcal{M}_F$, there is a unique probability $P_\mu$ on $\Omega$ such that

(DW) a) $P_\mu(\xi_0 = \mu) = 1$;

b) $\forall \phi \in \mathcal{D}(A)$, $M_t(\phi) \equiv \xi_t(\phi) - \int_0^t \xi_s(A\phi) ds$ is a $P_\mu$-martingale

such that $\langle M(\phi) \rangle_t = \int_0^t \xi_s(\phi^2) ds$. Similarly, for each $\mu \in \mathcal{M}_1$, there exists a unique

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probability $Q_\mu$ on $\Omega$ such that

(FV) a) $Q_\mu(\xi_0 = \mu) = 1$;

b) $\forall \phi \in \mathscr{D}(A)$, $M_t(\phi) \equiv \xi_t(\phi) - \int_0^t \xi_s(A\phi)\mathrm{d}s$ is a $Q_\mu$-martingale

such that $\langle M(\phi)\rangle_t = \int_0^t \{\xi_s(\phi^2) - \xi_s(\phi)^2\}\mathrm{d}s$. We say that $P.$ and $Q.$ are the laws of the $A$-Dawson-Watanabe and the $A$-Fleming-Viot superprocesses respectively.

These two diffusions are closely connected. It was observed by Konno and Shiga [4] that the normalized and time-changed process

$$\eta_t(\phi) = \xi_{\tau(t)}(1)^{-1}\xi_{\tau(t)}(\phi)$$

where $\tau$ is defined by

$$t = \int_0^{\tau(t)} \xi_s(1)^{-1}\mathrm{d}s$$

has the property that

$$N_t(\phi) \equiv \eta_t(\phi) - \int_0^t \eta_s(\xi_{\tau(s)}(1)\cdot A\phi)\mathrm{d}s$$

is a $P_\mu$-martingale (relative to the time-changed fields $\mathscr{F}_{\tau(t)}$) such that $\langle N(\phi)\rangle_t = \int_0^t \{\eta_s(\phi^2) - \eta_s(\phi)^2\}\mathrm{d}s$. Recently, $R$. Tribe [7] used this "frustratingly close" connection to describe the behavior of the $A$-Dawson-Watanabe superprocess at its time extinction.

The purpose of this note is to display another connection; namely that the Fleming-Viot process can be obtained from the Dawson-Watanabe process by conditioning the latter to have constant total mass. More precisely, let

(1) $$\tau(\varepsilon) = \inf\{t > 0 : |\xi_t(1) - 1| \geqq \varepsilon\}$$

and for each $\varepsilon > 0$, $T > 0$ and $\mu \in \mathscr{M}_F$ such that $|\mu(1) - 1| < \varepsilon$, define a probability on $(\Omega, \mathscr{F}_T)$ by the condition

(2) $$P_\mu^{T,\varepsilon}(\cdot) = P_\mu(\cdot\,|\,\tau(\varepsilon) > T).$$

We prove the following

**Theorem.** *Let $P_{\mu_n}^{T_n,\varepsilon_n}$, $n \geqq 1$ be any sequence of conditional laws of the A-Dawson-Watanabe superprocess such that $\lim_{n\to\infty} \varepsilon_n = 0$ and $\liminf_{n\to\infty} T_n > T$. If $\mu_n$ converges to a probability $\mu$ then $P_{\mu_n}^{T_n,\varepsilon_n}$ converges weakly on $(\Omega, \mathscr{F}_T)$ to $Q_\mu$, the law of the A-Fleming-Viot process up to time T.*

Let's begin with a proposition characterizing $P_\mu^{T,\varepsilon}$ in terms of martingales and then turn to a proof of the theorem. We require an auxilliary function $u_\varepsilon(x, t)$, solution of the initial-boundary value problem

(3) $$\begin{cases} \dfrac{\partial}{\partial t} u_\varepsilon(x,t) = \tfrac{1}{2}x \dfrac{\partial^2}{\partial x^2} u_\varepsilon(x,t); & |x-1| < \varepsilon,\ t > 0 \\ u_\varepsilon(x,0) = 1; & |x-1| < \varepsilon \\ u_\varepsilon(1 \pm \varepsilon, t) = 0; & t > 0. \end{cases}$$

Let us also introduce the notation,

$$v_\varepsilon(x,t) = u_\varepsilon(x,t)^{-1}\frac{\partial}{\partial x}u_\varepsilon(x,t), \quad \dot{u}_\varepsilon = \frac{\partial}{\partial t}u_\varepsilon \quad \text{and} \quad u'_\varepsilon = \frac{\partial}{\partial x}u_\varepsilon.$$

**Proposition.** *The conditional law $P_\mu^{T,\varepsilon}$ is the unique probability on $(\Omega, \mathscr{F}_T)$ satisfying*
a) $P_\mu^{T,\varepsilon}(\xi_0 = \mu) = 1$; *and*
b) $\forall \phi \in \mathscr{D}(A)$, $N_t(\phi) \equiv \xi_{t\wedge T}(\phi) - \int_0^{t\wedge T} \xi_s(A\phi) + v_\varepsilon(\xi_s(1), T-s)\xi_s(\phi)\mathrm{d}s$ *is a $P_\mu^{T,\varepsilon}$-martingale such that* $\langle N(\phi)\rangle_t = \int_0^{t\wedge T}\xi_s(\phi^2)\mathrm{d}s$.

*Proof.* First, for $B\in\mathscr{F}_t$, $t\leqq T$, we may write $P_\mu^{T,\varepsilon}(B) = u_\varepsilon(\mu(1),T)^{-1} \times P_\mu(1_B u_\varepsilon(\xi_t(1), T-t))$. This is a standard application of Itô's formula:

$$\begin{aligned}(4)\qquad & u_\varepsilon(\xi_{t\wedge\tau(\varepsilon)}(1), T - t\wedge\tau(\varepsilon)) - u_\varepsilon(\xi_0(1), T)\\ &= \int_0^{t\wedge\tau(\varepsilon)} -\dot{u}_\varepsilon(\xi_s(1), T-s) + \tfrac{1}{2}\xi_s(1)u''_\varepsilon(\xi_s(1), T-s)\mathrm{d}s\\ &\quad + \int_0^{t\wedge\tau(\varepsilon)} u'_\varepsilon(\xi_s(1), T-s)\mathrm{d}\xi_s(1).\end{aligned}$$

By (3), $u_\varepsilon(\xi_{t\wedge\tau(\varepsilon)}(1), T-t\wedge\tau(\varepsilon))$ is a $P_\mu$-martingale; in particular

$$\begin{aligned}(5)\qquad u_\varepsilon(\mu(1), T) &= P_\mu(u_\varepsilon(\xi_{T\wedge\tau(\varepsilon)}(1), T - T\wedge\tau(\varepsilon)))\\ &= P_\mu(\tau(\varepsilon) > T).\end{aligned}$$

Clearly this yields $P_\mu(\tau(\varepsilon) > T) > 0$. Now

$$\begin{aligned}(6)\qquad P_\mu(B, \tau(\varepsilon) > T) &= P_\mu P_\mu(B, t + \tau(\varepsilon)\circ\theta_t > T|\mathscr{F}_t)\\ &= P_\mu(1_B P_{\xi_t}(\tau(\varepsilon) > T-t))\end{aligned}$$

hence

$$(7)\qquad P_\mu^{T,\varepsilon}(B) = u_\varepsilon(\mu(1),T)^{-1}P_\mu(1_B u_\varepsilon(\xi_t(1), T-t)).$$

If we set

$$(8)\qquad R_t^{T,\varepsilon} = u_\varepsilon(\xi_0(1), T)^{-1}u_\varepsilon(\xi_t(1), T-t)$$

then by the discussion above, $R^{T,\varepsilon}$ is a mean 1, $P_\mu$-martingale up to time $T$. In fact, by (4),

$$(9)\qquad dR_t^{T,\varepsilon} = R_t^{T,\varepsilon}v_\varepsilon(\xi_t(1), T-t)d\xi_t(1).$$

Note that $\lim_{x\to 1\pm\varepsilon} v_\varepsilon(x,t) = \mp\infty$ but that $\sup_{0\leqq t\leqq T}\ \sup_{|x-1|\leqq\delta<\varepsilon} |v_\varepsilon(x,t)| < \infty$. So if $0<\delta<\varepsilon$ then

$$(10)\qquad Z_{t\wedge\tau(\delta)}^{T,\varepsilon} = \int_0^{t\wedge\tau(\delta)} v_\varepsilon(\xi_s(1), T-s)\mathrm{d}\xi_s(1)$$

is a $P_\mu^{T,\varepsilon}$-martingale and by solving Eq. (9) we find

$$(11)\qquad R_{t\wedge\tau(\delta)}^{T,\varepsilon} = \exp\{Z_{t\wedge\tau(\delta)}^{T,\varepsilon} - \tfrac{1}{2}\langle Z^{T,\varepsilon}\rangle_{t\wedge\tau(\delta)}\}.$$

But since $P_\mu^{T,\varepsilon}(\lim_{\delta\uparrow\varepsilon}\tau(\delta)=\tau(\varepsilon)>T)=1$, as a moment's thought reveals, we may dispense with the localizing stopping times and write

(12) $$R_t^{T,\varepsilon}=\exp\{Z_t^{T,\varepsilon}-\tfrac{1}{2}\langle Z^{T,\varepsilon}\rangle_t\} \text{ for all } 0\leqq t\leqq T.$$

We find that $P_\mu^{T,\varepsilon}(B)=P_\mu(1_B R_t^{T,\varepsilon})$, according to Eqs. (7)–(8).

By the transformation-of-drift formula (Cameron-Martin-Maruyama-Motoo-Girsanov) the process

(13) $$M_{t\wedge T}(\phi)=\xi_{t\wedge T}(\phi)-\int_0^{t\wedge T}\xi_s(A\phi)ds$$

which is a martingale under $P_\mu$, has the property that

(14) $$\begin{aligned}N_t(\phi)&\equiv M_{t\wedge T}(\phi)-\langle M(\phi),Z^{T,\varepsilon}\rangle_{t\wedge T}\\&=\xi_{t\wedge T}(\phi)-\int_0^{t\wedge T}\xi_s(A\phi)+v_\varepsilon(\xi_s(1),T-s)\xi_s(\phi)ds\end{aligned}$$

is a $P_\mu^{T,\varepsilon}$-martingale and we have

(15) $$\langle N(\phi)\rangle_t=\langle M(\phi)\rangle_{t\wedge T}=\int_0^{t\wedge T}\xi_s(\phi^2)ds.$$

To show that this condition characterizes $P_\mu^{T,\varepsilon}$, let $W_\mu$ be any law on $(\Omega,\mathscr{F}_T)$ such that $W_\mu(\xi_0=\mu)=1$ and equations (14)–(15) hold. Then under $W_\mu$,

(16)
$$f(\xi_{t\wedge T}(1))=\text{martingale}+\int_0^{t\wedge T}[\xi_s(1)v_\varepsilon(\xi_s(1),T-s)f'(\xi_s(1))+\tfrac{1}{2}\xi_s(1)f''(\xi_s(1))]ds;$$

for any smooth function $f$, compactly supported in $(0,\infty)$. In other words, $\xi_.(1)$ is the inhomogeneous diffusion with generator $\frac{1}{2}x\frac{d^2}{dx^2}+xv_\varepsilon(x,T-t)\frac{d}{dx}$ which we recognize as the motion with generator $\frac{1}{2}x\frac{d^2}{dx^2}$ conditioned on the event $[\tau(\varepsilon)>T]$. So we have $W_\mu(\tau(\varepsilon)>T)=1$ and this implies $u_\varepsilon(\xi_t(1),T-t)^{-1}, 0\leqq t\leqq T$, is a well defined process under $W_\mu$, indeed a semi martingale. For by Itô's formula, and dropping the arguments of $u_\varepsilon$ for conciseness' sake, we find using (3):

(17)
$$\begin{aligned}du_\varepsilon(\xi_t(1),T-t)^{-1}&=u_\varepsilon^{-2}\dot{u}_\varepsilon dt-u_\varepsilon^{-2}u_\varepsilon' d\xi_t(1)+\tfrac{1}{2}u_\varepsilon^{-4}(-u_\varepsilon^2u_\varepsilon''+2u_\varepsilon(u_\varepsilon')^2)d\langle\xi(1)\rangle_t\\&=u_\varepsilon^{-2}(\dot{u}_\varepsilon-\tfrac{1}{2}\xi_t(1)u_\varepsilon'')dt-u_\varepsilon^{-1}v_\varepsilon dN_t(1)\\&=-u_\varepsilon(\xi_t(1),T-t)^{-1}v_\varepsilon(\xi_t(1),T-t)dN_t(1).\end{aligned}$$

Repeating the discussion concerning lines (10)–(12) we find that

(18) $$z_t^{T,\varepsilon}=\int_0^{t\wedge T}-v_\varepsilon(\xi_s(1),T-s)dN_s(1)$$

is a martingale and we may solve (17) as

(19) $$u_\varepsilon(\xi_t(1),T-t)^{-1}=u_\varepsilon(\xi_0(1),T)^{-1}\exp\{z_t^{T,\varepsilon}-\tfrac{1}{2}\langle z^{T,\varepsilon}\rangle_t\}.$$

Another application of the transformation-of-drift formula shows that under the probability $\tilde{W}_\mu$ defined by

(20) $$\tilde{W}_\mu(B) = W_\mu(1_B z_t^{T,\varepsilon}), \qquad B \in \mathscr{F}_t,\ t \leqq T$$
$$= u_\varepsilon(\mu(1), T)\, W_\mu(1_B u_\varepsilon^{-1}(\xi_t(1), T-t))$$

we have

(21) $$M_{t \wedge \tau(\varepsilon) \wedge T}(\phi) = \xi_{t \wedge T(\varepsilon) \wedge T}(\phi) - \int_0^{t \wedge \tau(\varepsilon) \wedge T} \xi_s(A\phi) ds$$

is a $\tilde{W}_\mu$-martingale with

(22) $$\langle M(\phi) \rangle_{t \wedge T \wedge \tau(\varepsilon)} = \int_0^{t \wedge T \wedge \tau(\varepsilon)} \xi_s(\phi^2) ds.$$

It follows from Theorem 6.1.2 of [6] that $\tilde{M}_t(\phi)$ is a martingale with $\langle M(\phi) \rangle_t = \int_0^t \xi_s(\phi^2) ds$ under the probability $\tilde{W}_\mu \otimes_{\tau(\varepsilon) \wedge T} P_{\xi_{\tau(\varepsilon) \wedge T}}$, so by the characterization (DW) this measure must be $P_\mu$ itself. Thus $\tilde{W}_\mu = P_\mu$ on $(\Omega, \mathscr{F}_{T \wedge \tau(\varepsilon)})$. In particular, $\tilde{W}_\mu(\tau(\varepsilon) > T) = u_\varepsilon(\mu(1), T)$, and we may write, for $B \in \mathscr{F}_t$, $t \leqq T$,

(23) $$W_\mu(B) = u_\varepsilon^{-1}(\mu(1), T)\, \tilde{W}_\mu(1_B u_\varepsilon(\xi_t(1), T-t))$$
$$= u_\varepsilon^{-1}(\mu(1), T)\, \tilde{W}_\mu(1_B P_{\xi_t}(\tau(\varepsilon) > T-t))$$
$$= u_\varepsilon^{-1}(\mu(1), T) P_\mu(1_B P_{\xi_t}(\tau(\varepsilon) > T-t))$$
$$= P_\mu^{T,\varepsilon}(B)$$

and this finishes the proof. □

Coming now to a proof of the theorem, we note that by the trite estimations

(24a) $$P_\mu^{T,\varepsilon}\Big(\sup_{0 \leqq t \leqq T} |\xi_t(1) - 1| \leqq \varepsilon\Big) = 1;$$

and

(24b) $$P_\mu^{T,\varepsilon}\left(\sup_{0 \leqq t \leqq T} |\xi_t(\phi) - \xi_t(1)^{-1}\xi_t(\phi)| \leqq \frac{\varepsilon}{1-\varepsilon} \|\phi\|_\infty\right) = 1$$

we need only consider convergence of the normalized process $\eta_t \equiv \xi_t(1)^{-1}\xi_t$. Under $P_\mu^{T,\varepsilon}$, $\eta_t(\phi)$ is a semi martingale and by Itô's formula

(25) $$d\eta_t(\phi) = \xi_t(1)^{-1} d\xi_t(\phi) - \xi_t(1)^{-2}\xi_t(\phi) d\xi_t(1)$$
$$+ \xi_t(1)^{-3}\xi_t(\phi) d\langle \xi(1) \rangle_t - \xi_t(1)^{-2} d\langle \xi(1), \xi(\phi) \rangle_t$$

which, after some simple arithmetic, becomes

(26) $$d\eta_t(\phi) = \eta_t(A\phi)dt + \xi_t(1)^{-1} dM_t(\phi) - \xi_t(1)^{-2}\xi_t(\phi) dM_t(1).$$

Pleasantly enough, the terms involving the conditional drift have cancelled and (26) reveals that

(27) $$G_t(\phi) \equiv \eta_{t \wedge T}(\phi) - \int_0^{t \wedge T} \eta_s(A\phi) ds$$

is a $P_\mu^{T,\varepsilon}$-martingale with

(28) $$\langle G(\phi)\rangle_t = \int_0^{t\wedge T} \xi_s(1)^{-1}[\eta_s(\phi^2) - \eta_s(\phi)^2]\mathrm{d}s.$$

The rest is plain sailing. According to our proposition, equations (23a, b) and Theorem 2.1 of [5], tightness of the laws $P_{\mu_n}^{T_n,\varepsilon_n}$, $n \geqq 1$ follows from tightness of the image laws of the real valued process $\eta.(\phi)$ for $\phi \in \mathscr{D}(A)$. For this we may apply the Aldous-Rebolledo criterion ([5], Theorem 2.3; [3], §2) which is easily verified by the simple inequalities

(29a) $$P_{\mu_n}^{T_n,\varepsilon_n}\left(\int_{\tau_n\wedge T}^{\tau_n+\delta\wedge T} |\eta_s(A\phi)|\mathrm{d}s\right) \leqq \delta\,\|A\phi\|_\infty,$$

and

(29b) $$P_{\mu_n}^{T_n,\varepsilon_n}\left(\int_{\tau_n\wedge T}^{\tau_n+\delta\wedge T} |\xi_s(1)^{-1}[\eta_s(\phi^2) - \eta_s(\phi)^2]|\mathrm{d}s\right) \leqq \frac{\delta}{1-\varepsilon_n}\,\|\phi^2\|_\infty$$

where $\tau_n$ is any sequence of stopping times.

Now let $Q_\mu$ be any limit point of $P_{\mu_n}^{T_n,\varepsilon_n}$, $n \geqq 1$; relabelling the subsequence if necessary let us assume in fact $Q_\mu = \lim_{n\to\infty} P_{\mu_n}^{T_n,\varepsilon_n}$. For $f \in C_c^\infty(\mathbb{R})$ let

(30) $$C_t^{f,\phi}(\omega) = f(\eta_t(\omega,\phi)) - \int_0^{t\wedge T} f'(\eta_s(\omega,\phi))\eta_s(\omega, A\phi)\mathrm{d}s$$
$$- \int_0^{t\wedge T} \tfrac{1}{2}f''(\eta_s(\omega,\phi))[\eta_s(\omega,\phi^2) - \eta_s(\omega,\phi)^2]\mathrm{d}s.$$

Let $0 \leqq s < t \leqq T$, and let $\Phi \in \mathscr{F}_s$ be a bounded continuous function, say $|\Phi| \leqq K$, and let $D = \Phi[C_t^{f,\phi} - C_s^{f,\phi}]$. Since $\omega \to D(\omega)$ is a bounded continuous function on $\Omega$, we have

(31) $$\begin{aligned} Q_\mu(D) &= \lim_{n\to\infty} P_{\mu_n}^{T_n,\varepsilon_n}(D) \\ &= \lim_{n\to\infty} P_{\mu_n}^{T_n,\varepsilon_n}\left(\Phi\int_s^t \tfrac{1}{2}f''(\eta_s(\phi))[\eta_s(\phi^2) - \eta_s(\phi)^2][\xi_s(1)^{-1} - 1]\mathrm{d}s\right) \\ &\leqq \lim_{n\to\infty} \frac{1}{2}KT\|f''\|_\infty\|\phi^2\|_\infty \frac{\varepsilon_n}{1-\varepsilon_n} = 0. \end{aligned}$$

Thus $C_t^{f,\phi}$ is a $Q_\mu$ martingale for all $\phi \in \mathscr{D}(A)$ and smooth compactly supported test functions $f$. This readily implies that under $Q_\mu$ the conditions (FV) are verified, up to time $T$ and this identifies $Q_\mu$ on $(\Omega, \mathscr{F}_T)$ as the law of the $A$-Fleming-Viot process.

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