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Corrigendum and Addendum to my Paper "On the Real Part of an Entire Function, Its Derivative and Its Lower Order"

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It was brought to our notice by Dr. S. JAENISCH that Lemma 2 and the condition that $Re[zf'(z)] > 0$ for entire functions $f(z)$ with $f'(z)$ denoting the derivative of $f(z)$ do not hold good thereby vitiating our argument in a part of the proof of our Theorem B in [1]. This theorem can however be reformulated as given below so as to obtain a counterpart of Rajagopal's theorem ([2], Theorem I).

Theorem. *If $f(z)$ is an entire function of lower order $\lambda (0 \leq \lambda \leq \infty)$ then*

$$\liminf_{r \to \infty} \log [A_1^*(r)/A^*(r)]/\log r \leq \lambda$$

$$\liminf_{r \to \infty} \frac{\log \log A_1^*(r)}{\log r} \geq \lambda$$

where $A_1^*(r) = \underset{|z|=r}{\text{Max.}} \, | Re[zf'(z)] |$ *and* $A^*(r) = \underset{|z|=r}{\text{Max.}} \, | Re\, f(z) |$.

The proof depends on deriving the two inequalities

$$\text{(a)} \ \liminf_{r \to \infty} \frac{\log \lambda(r)}{\log r} \leq \lambda, \ \text{(b)} \ \liminf_{r \to \infty} \frac{\log \log A_1^*(r)}{\log r} \geq \lambda$$

with $\lambda(r) = [A_1^*(r)/A^*(r)]$. The inequality (a) is easily derived as on page 247 in [1] above where we have to replace Max. $Re\, zf'(z)$ by $A_1^*(r)$ with the steps preceding the inequality (5) being replaced by

$$A_1^*(r) \leq 2\, \nu(e_n)\, A^*(e_n) \leq 2\, e_n^{\lambda+\varepsilon}\, A^*(e_n)$$

where ε is arbitrary and the sequence $\{e_n\}$ is derived from the set $E \cap F$ where F denotes the set of points r which lie outside a set of exceptional segments in which for $r > R$, the variation of $\log r$ is less than $K\,\nu(R/k)^{-1/12}$ and E is the set defined as in [1], page 247. Now to derive the inequality (*b*) we need the following lemmas.

Lemma 1. In the preceding notation $A^*(r)$ is an increasing convex function of log r in the interval $r_1 \leq r \leq r_2$.

Proof. This follows easily since $u(z) = | Re\, f(z) |$ is subharmonic in $D: r_1 \leq |z| = r \leq r_2$ and it is well known that $\max\limits_{|z|=r} u(z)$, for any subharmonic function $u \equiv u(z)$, is a convex function of log r.

Lemma 2. For $r > r_0$

$$\operatorname*{Max}_{|z|=r} | Re\, zf'(z) | \geq A^*(r)/(\log r).$$

Proof. With $A^*(r) = | Re\, f(re^{i\theta}) |$,

$$| Re\, z\, f'(z) | = \left| Re \left[r\, e^{i\,\theta} \lim_{\varepsilon \to 0} \frac{f(r\, e^{i\,\theta}) - f(r(1-\varepsilon)\, e^{i\,\theta})}{\varepsilon\, r\, e^{i\,\theta}} \right] \right|$$

$$= \left| \lim_{\varepsilon \to 0} \frac{Re\, f(r\, e^{i\,\theta}) - Re\, f(r(| - \varepsilon)\, e^{i\,\theta})}{\varepsilon} \right|$$

$$\geq \lim_{\varepsilon \to 0} \frac{A^*(r) - A^*(r - r\,\varepsilon)}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{g(r) \log r - g(r - r\,\varepsilon) \log (r - r\,\varepsilon)}{\varepsilon}$$

$$\geq g(r) \lim_{\varepsilon \to 0} \frac{\log r - \log (r - r\,\varepsilon)}{\varepsilon}$$

$$= g(r) \lim_{\varepsilon \to 0} \frac{-\log (1 - \varepsilon)}{\varepsilon}$$

$$= g(r) = A^*(r)/\log r.$$

In the above steps $g(r) = A^*(r)/\log r$, an increasing function of $\log r$ by Lemma 1.

Lemma 3. For the entire function $f(z)$ of order ϱ and lower order λ we have

$$\lim_{r \to \infty} \frac{\sup}{\inf} \frac{\log \log A^*(r)}{\log r} = \frac{\varrho}{\lambda} \qquad (0 \leq \lambda, \varrho \leq \infty).$$

Since

$$| a_n |\, r^n \leq 2\, A^*(r) \leq 2 \operatorname*{Max}_{|z|=r} | f(z) | = 2\, M(r),\ n > 0,$$

we have, if $\mu(r)$ is the maximum term in the power series for $f(z)$ corresponding to $|z| = r$,

$$\mu(r) \leq 2\, A^*(r) \leq 2\, M(r)$$

from which we derive

$$\limsup_{r \to \infty} \frac{\log \log A^*(r)}{\log r} = \varrho$$

and similarly for the lower order λ.

Proof of Theorem. Lemmas (2) and (3) yield

$$\liminf_{r \to \infty} \frac{\log \log A_1^*(r)}{\log r} \geq \lambda$$

which is inequality (b). Since inequality (a) is already established the theorem is completely proved.

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References

(1) LAKSHMINARASIMHAN, T. V.: On the real part of an entire function, its derivative and its lower order. Mh. Math. **70**, 244—247 (1966).

(2) RAJAGOPAL, C. T.: On an asymptotic relation between an entire function, its derivative and their order. Mh. Math. **66**, 339—345 (1962).

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