Probab. Th. Rel. Fields 76, 325–340 (1987)

Probability Theory and Related Fields

Infinite-Dimensional Diffusion Processes as Gibbs Measures on $C[0,1]^{Z^d}$

J. D. Deuschel

ETH, Mathematikdepartement, CH-8092 Zürich, Switzerland

Summary. An infinite lattice system of interacting diffusion processes is characterized as a Gibbs distribution on $C[0,1]^{Z^d}$ with continuous local conditional probabilities. Using estimates for the Vasserstein metric on $C[0,1]$, Dobrushin's contraction technique is applied in order to obtain information about macroscopic properties of the entire diffusion process.

I. Introduction

Consider the infinite-dimensional diffusion process $X=(X_t^i, 0 \leqq t \leqq 1)_{i \in I}$ satisfying

$$dX_t^i = b^i(X_t, t)dt + dW_t^i \qquad (i \in I)$$
$$X_0 \sim \mu$$

where I is some countable set, $(W^i)_{i \in I}$ is a collection of independent real-valued Wiener processes and μ a distribution on $\mathbb{R}^I$. In the time-homogeneous case and under some bounds on the interaction in the drift terms, reversible equilibrium distributions of the process can be described as Gibbs measures on $\mathbb{R}^I$ whose interaction potential is determined by the drifts $(b^i)_{i \in I}$, cf. e.g. Doss-Royer [7], Fritz [10], or Föllmer-Wakolbinger [9] for the non-reversibel case.

Here our approach will be different. We shall view the law Q of the entire process as a Gibbs measure with state space $C[0,1]$ and apply Dobrushin's contraction technique to the system of local conditional probabilities $(Q^i(\ |X))_{i \in I}$ in order to obtain information about the macroscopic properties of Q. In a first paper [5], the infinite-dimensional smoothing problem of computing the conditional distributions is solved in a robust form for local gradient drifts. Using the explicit form of the conditional density given in [5], we shall now estimate the Dobrushin coefficients

$$C^{k,i} := \sup \left\{ \frac{R(Q^i(\ |X), Q^i(\ |\tilde{X}))}{\|X^k - \tilde{X}^k\|} : X^j = \tilde{X}^j, j \neq k \right\}$$

with respect to the Vasserstein metric $R(,)$ on $C[0,1]$. This allows us to give bounds for the interaction of the drifts and initial measure in such a way that Dobrushin's uniqueness condition

$$\sup_i \sum_{k \neq i} C^{k,i} < 1$$

is fulfilled. Applying Dobrushin's contraction technique et al., cf. [6], we shall obtain conditions which guarantee the uniqueness of Q, the exponential decay of correlations and the central limit theorem for Lipschitz continuous functions on $C[0,1]^I$. In particular we can derive the distribution-valued stochastic differential equation associated to the Gaussian field of fluctuation in the sense of Itô [13], and Holley-Stroock [11].

A similar approach has been proposed independently for the pollaron problem by Spohn in [23].

In Sect. 2 we estimate the Dobrushin coefficients assuming that the conditional distributions are given by a smooth exponential family. In Sect. 3 bounds are stated for a gradient system in order to fulfill the uniqueness condition. Finally in Sect. 4 we recall a few applications of Dobrushin's contraction technique and determine the equation of the fluctuation field.

2. The Vasserstein Metric of a Smooth Exponential Family

Let $\Omega = C[0,1]^I$ be the countable product space of continuous real-valued functions on the interval $[0,1]$. Denote by

$$X = (X_t^i, 0 \leq t \leq 1)_{i \in I}$$

the coordinates on Ω and by $\mathscr{F} = (\mathscr{F}_t, 0 \leq t \leq 1)$ the canonical filtration. We shall consider a measure Q on $(\Omega, \mathscr{F})$ as a random field and write $Q^i(\ |X)$ for the conditional probability of the i^{th} coordinate $X^i = (X_t^i, 0 \leq t \leq 1)$ with respect to the σ-field

$$\mathscr{F}^i := \sigma(X_t^k, 0 \leq t \leq 1, k \neq i).$$

Let P^i be the Wiener measure on $C[0,1]$ started off with an initial distribution λ^i where we shall assume, that the function

$$A_{\lambda^i}(p, K) := \inf_{y \in \mathbb{R}} \int |x^i - y|^p e^{K|x^i - y|} \lambda^i(dx^i) / \int e^{-K|x^i - y|} \lambda^i(dx^i) \tag{2.1}$$

is finite for all positive p and K.

We shall say that the law $Q^i(\ |X)$ is given by a *smooth exponential family* with respect to P^i, if there exist a finite neighborhood $V(i)$ of i and a (Fréchet-) differentiable function Ψ^i on $C[0,1] \times C[0,1]^{V(i)}$ with bounded partial derivatives, denoted by $D_k \Psi^i(X^i, X)$, such that

$$Q^i(dX^i | X) = \exp(\Psi^i(X^i, X)) Z^i(X)^{-1} P^i(dX^i) \tag{2.2}$$

with normalizing constant $Z^i(X) = \int \exp(\Psi^i(X^i, X)) P^i(dX^i)$.

Let $\| \ \|$ be the supremum norm on $C[0,1]$, the Vasserstein metric of two probability measures P and $\tilde{P}$ on $C[0,1]$ is then defined by

$$R(P,\tilde{P}) := \sup\{|\int f dP - \int f d\tilde{P}| : \delta(f) \leqq 1\}$$

with $\delta(f) := \sup\left\{\left|\frac{f(X)-f(Y)}{\|X-Y\|}\right| : X \neq Y \in C[0,1]\right\}$.

Assuming that the $Q^i(\ |X)$, $i \in I$, are of the form (2.2), the aim of this section is to compute estimates for the Dobrushin coefficients

$$C^{k,i} := \sup\left\{\frac{R(Q^i(\ |X), Q^i(\ |\tilde{X}))}{\|X^k - \tilde{X}^k\|} : X^j = \tilde{X}^j, j \neq k\right\} \tag{2.4}$$

in terms of the function $A_{\lambda^i}(p,K)$ and of the bounds of the partial derivatives of Ψ^i

$$\|D_k \Psi^i\|_\infty := \sup_{X, X^i} \|D_k \Psi^i(X^i, X)\|_{op} < \infty \qquad (k \in V(i) \cup \{i\})$$

with $\|D_k \Psi^i(X^i, X)\|_{op} := \sup_{\|h\| \leqq 1} |\langle D_k \Psi^i(X^i, X), h\rangle|$.

At first we shall derive an estimate for the moments of the conditional law

$$M_p^i := \sup_X \inf_Y E_{Q^i(\ |X)}[\|X^i - Y\|^p]^{1/p}$$

for $p > 1$. Let $(\mathscr{G}_t^i, 0 \leqq t \leqq 1)$ be the enlarged filtration

$$\mathscr{G}_t^i := \mathscr{F}_t \vee \mathscr{F}^i,$$

then it follows from the theory of the Girsanov transformation that there exist $(\mathscr{G}_t^i, 0 \leqq t \leqq 1)$-adapted drift $(\hat{b}_t^i, 0 \leqq t \leqq 1)$ and Wiener process $(\hat{W}_t^i, 0 \leqq t \leqq 1)$, such that $Q^i(\ |X)$ is the law of

$$\begin{aligned} dX_t^i &= \hat{b}_t^i dt + d\hat{W}_t^i, \\ X_0^i &\sim \hat{\mu}^i(\ |X), \end{aligned} \tag{2.7}$$

cf. Liptser-Shiryayev [18]. Rewriting the initial law of the conditional process in an exponential form with respect to λ^i:

$$\hat{\mu}^i(dX_0^i|X) = \exp(\hat{f}^i(X_0^i, X)) Z^i(X)^{-1} \lambda^i(dX_0^i) \tag{2.8}$$

with $\hat{f}^i(X_0^i, X) := \log(E_{P^i}[\exp(\Psi^i(X^i, X))|\mathscr{G}_0^i])$, one can show by the chain rule that $\hat{f}^i(x^i, X)$ is differentiable in $x^i \in \mathbb{R}$. More precisely we have:

Lemma. (2.9) *$Q^i(\ |X)$ is the law of the stochastic differential equation* (2.7) *where the drift $\hat{b}^i$ and the derivative $\partial_i \hat{f}^i$ are uniformely bounded by*

$$\begin{aligned} \|\hat{b}^i\|_\infty &\leqq \sup_{t, X, X^i} |D_i \Psi^i(X^i, X)((t,1])| =: \|D_i \Psi^i\|_\infty^*, \\ \|\partial_i \hat{f}^i\|_\infty &\leqq \|D_i \Psi^i\|_\infty. \end{aligned} \tag{2.10}$$

Proof. We follow an argument of Bismut and Michel [1]: Let

$$M_t^i := E_{P^i}[\exp(\Psi^i(X^i, X))|\mathscr{G}_t^i],$$

then since the density is locally bounded in $L^p(P^i)$:

$$E_{P^i}[\exp(\Psi^i(X^i, X))^p] \leqq c_p^i \exp\left(\sum_{k\in V(i)} p\|D_k\Psi^i\|_\infty \|X^k\|\right)$$

with c_p^i depending on $\|D_i\Psi^i\|_\infty$, cf. Lemma (2.6) in [5], by Clark's formula we have

$$M_t^i = M_0^i + \int_0^t h_s^i dX_s^i$$

with $h_t^i = E_{P^i}[\exp(\Psi^i(X^i, X))D_i\Psi^i(X^i, X)((t,1])|\mathcal{G}_t^i]$.

Hence the Girsanov transformation implies that

$$d\hat{W}_t^i := dX_t^i - \hat{b}_t^i dt$$

is a Wiener process under $(\mathcal{G}_t^i) - Q^i(\ |X)$ with

$$\hat{b}_t^i = h_t^i/M_t^i = E_{P^i}[\exp(\Psi^i(X^i, X))D_i\Psi^i(X^i, X)((t,1])|\mathcal{G}_t^i]/E_{P^i}[\exp(\Psi^i)|\mathcal{G}_t^i].$$

This gives us the first bound. For the second, it suffices to apply the chain rule and to differentiate inside the expectation:

$$\partial_i \hat{f}^i(X_0^i, X) = E_{P^i}[\exp(\Psi^i(X^i, X))D_i\Psi^i(X^i, X)([0,1])|\mathcal{G}_0^i]/E_{P^i}[\exp(\Psi^i)|\mathcal{G}_0^i]. \quad \square$$

Remark. (2.11) If the function Ψ^i is additive on the interval $[0,1]$, i.e. if for all $t \in [0,1]$

$$\Psi^i(X^i, X) = \Psi_t^i(X^i, X) + \Psi^{i,t}(X^i, X)$$

with $\mathcal{G}_t^i$-measurable Ψ_t^i and $\mathcal{G}^{i,t} := \sigma(X_s^i, t < s \leqq 1) \vee \mathcal{F}^i$-measurable $\Psi^{i,t}$, then the drift of the conditional process is of the form

$$\hat{b}^i(X_t^i, X, t) = E_{P^i}[\exp(\Psi^{i,t})D_i\Psi^{i,t}((t,1])|\sigma(X_t^i) \vee \mathcal{F}^i]/E_{P^i}[\exp(\Psi^{i,t})|\sigma(X_t^i) \vee \mathcal{F}^i]$$

by Markov property of P^i. This together with (2.7) imply that X^i is a Markov process under $Q^i(\ |X)$.

Corollary. (2.12) *The moments of the conditional law are uniformly bounded in X with*

$$M_p^i \langle A_{\lambda^i}(p, \|D_i\Psi^i\|_\infty)^{1/p} + \|D_i\Psi^i\|_\infty^* + qc_p \qquad (p > 1). \tag{2.13}$$

Proof. $Q^i(\ |X)$ being the law of the equation (2.7), we have by Doob's inequality

$$E_{Q^i(\ |X)}[\|X^i - Y_0\|^p]^{1/p} \leqq E_{\hat{\mu}^i(\ |X)}[|X_0^i - Y_0|^p]^{1/p} + \|\hat{b}^i\|_\infty + qE_{Q^i(\ |X)}[|\hat{W}_1^i|^p]^{1/p}.$$

Hence by (2.10) it suffices to show that

$$\sup_X \inf_{Y_0} \int |X_0^i - Y_0|^p \exp(\hat{f}^i(X_0^i, X))\lambda^i(dX_0^i)/\int \exp(\hat{f}^i(X_0^i, X))\lambda^i(dX_0^i)$$
$$< A_{\lambda^i}(p, \|D_i\Psi^i\|_\infty).$$

But this follows from the Lipschitz continuity of $\hat{f}^i(x^i, X)$ in x^i and the definition of A_{λ^i}. $\square$

The coefficients $C^{k,i}$ will be now estimated by the mean value theorem, cf. Simon [22], Levin [16]. For fixed X and $\tilde{X}$ in $C[0,1]^{V(i)}$ let

$$X^\tau := X + \tau(\tilde{X} - X) \qquad (0 \leqq \tau \leqq 1)$$

be a path from X to $\tilde{X}$ in $C[0,1]^{V(i)}$. By the chain rule $\Psi^i(\tau) := \Psi^i(X^i, X^\tau)$ is differentiable in $\tau \in [0,1]$ with

$$\frac{d}{d\tau} \Psi^i(\tau) = \sum_{k \in V(i)} \langle D_k \Psi^i(\tau), \tilde{X}^k - X^k \rangle$$

and

$$\left| \frac{d}{d\tau} \Psi^i(\tau) \right| \leqq \sum_{k \in V(i)} \| D_k \Psi^i \|_\infty \| \tilde{X}^k - X^k \| .$$

For a function f on $C[0,1]$ such that $\delta(f) < \infty$ put

$$Q^i(f | X^\tau) := E_{Q^i(\ |X^\tau)}[f(X^i)] .$$

Lemma. (2.15) *The function $Q^i(f|X^\tau)$ is differentiable in $\tau \in [0,1]$ with*

$$\frac{d}{d\tau} Q^i(f|X^\tau) = E_{Q^i(\ |X^\tau)} \left[\{ f(X^i) - f(Y) \} \left\{ \frac{d}{d\tau} \Psi^i(\tau) - E_{Q^i(\ |X^\tau)} \left[\frac{d}{d\tau} \Psi^i(\tau) \right] \right\} \right] \qquad (2.16)$$

for all Y in $C[0,1]$.

Proof. Since the density $\exp(\Psi^i(X^i, X^\tau))$ is locally bounded in $L^p(P^i)$ we can differentiate in the expectation and obtain successively

$$\frac{d}{d\tau} Z^i(X^\tau) = \frac{d}{d\tau} E_{P^i}[\exp(\Psi^i(\tau))] = E_{P^i}\left[\exp(\Psi^i(\tau)) \frac{d}{d\tau} \Psi^i(\tau) \right]$$

and

$$\frac{d}{d\tau} Q^i(f|X^\tau) = \frac{d}{d\tau} E_{P^i}[f(X^i) \exp(\Psi^i(\tau)) Z^i(X^\tau)^{-1}]$$

$$= E_{P^i}\left[f(X^i) \exp(\Psi^i(\tau)) Z^i(X^\tau)^{-1} \left\{ \frac{d}{d\tau} \Psi^i(\tau) - E_{Q^i(\ |X^\tau)} \left[\frac{d}{d\tau} \Psi^i(\tau) \right] \right\} \right]. \qquad \square$$

We have now the tools to prove the following estimates:

Theorem. (2.17) *Under assumption* (2.2), *the Vasserstein metric of any two conditional probabilities $Q^i(\ |X)$ and $Q^i(\ |\tilde{X})$, $X, \tilde{X}$ in $C[0,1]^{V(i)}$, satisfies*

$$R(Q^i(\ |X), Q^i(\ |\tilde{X})) \leqq \sum_{k \in V(i)} C^{k,i} \| X^k - \tilde{X}^k \| \qquad (2.18)$$

with

$$C^{k,i} < M_2^i \| D_k \Psi^i \|_\infty .$$

Proof. By the definition (2.3) of the Vasserstein metric, the mean-value theorem and preceding lemma we have

$$R(Q^i(\ |X), Q^i(\ |\tilde{X})) \leqq \sup_{\delta(f)\leqq 1} \sup_{0\leqq\tau\leqq 1} \left|\frac{d}{d\tau} Q^i(f|X^\tau)\right|$$

$$\leqq \sup_{0\leqq\tau\leqq 1} \inf_Y E_{Q^i(\ |X^\tau)}\left[\|X^i - Y\| \left|\frac{d}{d\tau}\Psi^i(\tau) - E_{Q^i(\ |X^\tau)}\left[\frac{d}{d\tau}\Psi^i(\tau)\right]\right|\right]$$

Hence (2.14) and Schwarz's inequality imply (2.18) with

$$C^{k,i} \leqq \sup_{0\leqq\tau\leqq 1} \inf_Y \{E_{Q^i(\ |X^\tau)}[\|X^i - Y\|^2]^{1/2}$$

$$\times E_{Q^i(\ |X^\tau)}[\|D_k\Psi^i(\tau) - E_{Q^i(\ |X^\tau)}[D_k\Psi^i(\tau)]\|_{\mathrm{op}}^2]^{1/2}\}$$

$$\leqq M_2^i \|D_k\Psi^i\|_\infty. \quad \square$$

Remark. (2.20) i) From the proof one sees that $\|D_k\Psi^i\|_\infty$ can be replaced in (2.19) by a bound of the conditional variance of $D_k\Psi^i(X^i, X)$:

$$\sigma^2(D_k\Psi^i) := \sup_X E_{Q^i(\ |X)}[\|D_k\Psi^i(X^i, X) - E_{Q^i(\ |X)}[D_k\Psi^i(X^i, X)]\|_{\mathrm{op}}^2],$$

cf. Sect. III.3 of [3].

ii) If the conditional probabilities are given directly in terms of the Eq. (2.7) with smooth drift $\hat{b}^i(X^i, X, t)$ and function $\hat{f}^i(X_0^i, X)$, then one can derive the following estimate for the Dobrushin coefficients

$$C^{k,i} \leqq \exp(\|D_i\hat{b}^i\|_\infty) A^i(2, \|\partial_i\hat{f}^i\|_\infty)^{1/2} \|D_k\hat{f}^i\|_\infty + \|D_k\hat{b}^i\|_\infty,$$

cf. [5].

iii) Usually the $Q^i(\ |X)$, $i\in I$, are determined only for Q almost all X. By inequality (2.18) one can use the completeness of $\mathcal{M}_1(C[0,1])$ with respect to the Vasserstein metric and extend the definition of the $Q^i(\ |X)$ to all X.

Let $L(C[0,1]^I)$ be the class of Lipschitz continuous functions f on $C[0,1]^I$ satisfying

$$|f(X) - f(Y)| \leqq \sum_{i\in I} \delta_i(f) \|X^i - Y^i\|, \qquad \sum_{i\in I} \delta_i(f) < \infty$$

where $\delta_i(f) := \sup\left\{\left|\frac{f(X) - f(Y)}{\|X^i - Y^i\|}\right| : X^k = Y^k,\ k \neq i\right\}$ is the oscillation of f at i. If for each $f\in L(C[0,1]^I)$ and $i\in I$, the function

$$X \to \int f dQ^i(\ |X)$$

is again in $L(C[0,1]^I)$, then Q is called Gibbs measure with conditional probabilities $(Q^i(\ |X))_{i\in I}$. Since the inequality (2.18) implies this continuity condition we obtain:

Corollary. (2.21) *Under assumption* (2.2), *Q is a Gibbs measure with conditional probabilities* $(Q^i(\ |X))_{i\in I}$.

In the next section we shall apply our result to smooth gradient systems of finite range. The following example, taken from Föllmer-Walkolbinger [9], provides an other illustration of the Gibbsian approach to infinite-dimensional diffusion processes:

Example. (2.22) Let Q be the distribution of an infinite-dimensional Wiener process conditioned to have terminal distribution ν at time 1:

$$Q = \int_{\mathbb{R}^I} P^y \nu(dy)$$

with $P^y := \prod_{i \in I} P^{y^i}$, where P^{y^i} is the law of a one-dimensional Brownian bridge leading from 0 at time 0 to y^i at time 1. Using time reversal, one can determine the stochastic differential equation satisfied by $(X_t, 0 \leqq t \leqq 1)$, cf. [9]. The conditional probability $Q^i(\ |X)$ is simply the law of a Brownian motion constrained to have terminal distribution $\nu^i(\ |X_1)$. If the $\nu^i(\ |X_1)$ are given by

$$\nu^i(dX_1^i|X_1) = \exp(f^i(X_1^i, X_1)) Z^i(X_1)^{-1} d\varrho^i$$

with smooth f^i on $\mathbb{R} \times \mathbb{R}^{N(i)}$, $|N(i)| < \infty$, and $\varrho^i := N(0,1)$, the standard normal law, then $Q^i(\ |X)$ depends only on finitely many X_1^k, $k \in N(i)$, and is of the form (2.1) with $\Psi^i(X^i, X) = f^i(X_1^i, X_1)$. Hence the $Q^i(\ |X)$, $i \in I$, fulfill a (spatial) Markov property whereas the drifts of the global forward description are not local.

3. Dobrushin's Uniqueness Condition for Gradient Systems

Consider the probability measure Q on $(C[0,1]^I, \mathscr{F})$ of a diffusion process $(X_t^i, 0 \leqq t \leqq 1)_{i \in I}$ satisfying

$$\begin{aligned} dX_t^i &= b^i(X_t, t)dt + dW_t^i \quad (i \in I), \\ X_0 &\sim \mu \end{aligned} \tag{3.1}$$

where $(W_t^i, 0 \leqq t \leqq 1)_{i \in I}$ is a collection of independent Wiener processes and μ is a distribution on $\mathbb{R}^I$ which is tempered in the following sense

$$\sup_i E_\mu[|X_0^i - y^i|^2] < \infty$$

for a fixed $(y^i)_{i \in I}$ with $\sum_i (1+|i|)^{-2p} |y^i|^2 < \infty$ for some $p \geqq 1$. Let $\mathcal{O}$ be the class of finite subsets of I. We assume the following conditions:

[A.1] The drifts $(b^i)_{i \in I}$ are given by a smooth gradient system of the form

$$b^i(x,t) = \partial_i H^i(x,t), \qquad H^i(x,t) = \sum_{M: i \in M} B^M(x,t) \qquad (i \in I)$$

where $(B^M : M \in \mathcal{O})$ is a potential of finite range, i.e. a family of smooth functions with bounded derivatives on $\mathbb{R}^M \times [0,1]$, such that for each $i \in I$ the set

$$N(i) := \{k \neq i | \partial_i B^M(x,t) \not\equiv 0 \text{ and } \partial_k B^M(x,t) \not\equiv 0 \text{ for some } M \in \mathcal{O}\}$$

is finite. Moreover we have

$$\sup_i \sum_{k \in N(i) \cup \{i\}} \|\partial_i \partial_k B^M\|_\infty < \infty$$

[A.2] The conditional laws of the initial distribution $(\mu^i(\ |X_0))_{i\in I}$ are given by a smooth exponential family with respect to a measure λ^i satisfying (2.1):

$$\mu^i(dX_0^i|X_0) = \exp(f^i(X_0^i, X_0)) Z^i(X_0)^{-1} \lambda^i(dX_0^i) \qquad (i \in I)$$

where f^i is a real smooth function on $\mathbb{R} \times \mathbb{R}^{N(i)}$ with bounded derivatives.

Under these conditions the system (3.1) has a unique tempered solution satisfying

$$\sup_i E_Q[\|X^i - Y^i\|^2] < \infty$$

for a fixed tempered $(Y^i)_{i\in I}$ with $\sum_i (1+|i|)^{-2P} \|Y^i\|^2 < \infty$ for some $p \geqq 1$, *cf. Shiga-Shimizu* [20].

Let $N^2(i)$ denote the second order neighborhood of i

$$N^2(i) := N(i) \bigcup_{k \in N(i)} N(k) - \{i\}$$

and define a smooth function g^i with bounded derivatives on $\mathbb{R} \times \mathbb{R}^{N^2(i)} \times [0,1]$ by

$$\begin{aligned} g^i(x,t) := \partial_t H^i(x,t) + \sum_{k \in N(i) \cup \{i\}} &\{\tilde{b}^k(x,t) \partial_k H^i(x,t) \\ &+ 1/2 \partial_k^2 H^i(x,t) + 1/2 (\partial_k H^i(x,t))^2\} \end{aligned} \tag{3.3}$$

$$\text{with } \tilde{b}^k(x,t) := b^k(x,t) - \partial_k H^i(x,t) = \sum_{M: k \in M, i \notin M} \partial_k B^M(x,t).$$

Then for each $i \in I$ the conditional probability $Q^i(\ |X)$ is given by a smooth exponential family of the form (2.1) with respect to the Wiener measure P^i on $C[0,1]$ with additive Ψ^i

$$\Psi^i(X^i, X) := H^i(X_1^i, X_1, 1) - \int_0^1 g^i(X_t^i, X_t, t) dt + f^i(X_0^i, X_0) - H^i(X_0^i, X_0, 0) \tag{3.4}$$

cf. Proposition (2.3) of [5].

By the preceding section, Q is a Gibbs measure with conditional probabilities $(Q^i(\ |X))_{i\in I}$ depending on finitely many X^k, $k \in N^2(i)$. Hence a local interaction for the drifts and initial measure in Eq. (3.1) implies the (spatial) Markov property of Q. Note that this is not a necessary condition, cf. example (2.12).

Remark. (3.5) i) We need here a gradient system for the drifts in order to obtain the continuity of the conditional density $\exp(\Psi^i(X^i, X))$ in X with respect to the norm $\|\ \|$, cf. Remark (2.5) of [5]. Otherwise, for $f \in L(C[0,1]^I)$, the function

$$X \to \int f dQ^i(\ |X)$$

is not in $L(C[0,1]^I)$ and we cannot apply Dobrushin's technique.

ii) More generally the finite-dimensional conditional probabilities $Q^M(\ |X^{I-M})$, $M \subset I$ finite, have also a smooth exponential form with respect to the product Wiener measure P^M on $C[0,1]^M$, cf. Sect. II.3 of [3].

iii) If we introduce diffusion coefficients $\sigma^i(x^i)$ to our system (3.1):

$$dX_t^i = b^i(X_t, t)dt + \sigma^i(X_t^i)dW_t^i \qquad (i \in I)$$

then the $Q^i(\ |X)$, $i \in I$, are still of the form (2.1). The Wiener measure P^i has to be replaced by the law of the martingale X^i satisfying

$$dX_t^i = \sigma^i(X_t^i)dW_t^i,$$

cf. [5].

Using the estimates of the Dobrushin coefficients $C^{k,i}$ computed in Sect. 2, we shall now bound the interaction of the drifts and conditional initial measures in such a way that Dobrushin's uniqueness condition

$$\sup_i \sum_{k \neq i} C^{k,i} < 1 \tag{3.6}$$

is fulfilled. We restrict ourself to pair interaction, i.e.

$$B^M \equiv 0 \quad \text{for} \quad |M| > 2$$

and

$$f^i(x^i, x) := \sum_{k \in N(i)} f^{i,k}(x^i, x^k)$$

in order to simplify the computation. The method can be applied to the general gradient case as well.

Let us associate the bounds γ, β, σ^2, and $N \in \mathbb{R}^+$ to our systems $(b^i)_{i \in I}$ and $(\mu^i(\ |X_0))_{i \in I}$:

[B.1] γ for the self-interaction:

$$\sup_i \|\partial_t \partial_i B^i\|_\infty < \gamma \quad \text{and} \quad \sup_i \|\partial_i^{(n)} B^i\|_\infty < \gamma \quad \text{for} \quad n = 1, 2, 3$$

[B.2] β for the pair-interaction:

$$\sup_{i,k} \|\partial_i f^{i,k}\|_\infty \vee \|\partial_k f^{i,k}\|_\infty < \beta, \qquad \sup_{i,k} \|\partial_t \partial_i B^{i,k}\|_\infty \vee \|\partial_t \partial_k B^{i,k}\|_\infty < \beta$$

and

$$\sup_{i,k} \|\partial_i^{(n)} \partial_k^{(m)} B^{i,k}\|_\infty < \beta \quad \text{for} \quad n + m = 1, 2, 3$$

[B.3] σ^2 for the variance of λ^i:

$$\sup_i \operatorname{var}(\lambda^i) < \sigma^2$$

[B.4] N for the range:

$$\sup_i |N(i)| \leqq N.$$

Moreover let ν be the symmetric exponential distribution with variance σ^2:

$$\nu(dx)=1/(\sqrt{2\sigma})\exp(-\sqrt{2/\sigma}|x|)dx,$$

then we shall assume that

$$\begin{aligned}\sup_i A_{\lambda^i}(2,K) &\leqq A_\nu(2,K)\\ &=\int x^2\exp((K-\sqrt{2/\sigma})|x|)dx/\int\exp(-(K+\sqrt{2/\sigma})|x|)dx\\ &=\sigma^2\frac{1+K\sigma/\sqrt{2}}{(1-K\sigma/\sqrt{2})^3}.\end{aligned}$$

In terms of these bounds we have:

Lemma. (3.7) *The following estimates hold*

$$\sup_i \|\partial_i H^i\|_\infty<\gamma+N\beta$$

$$\sup_i\sum_{k\neq i}\|\partial_k H^i\|_\infty<N\beta$$

$$\sup_i\|\partial_i g^i\|_\infty<2(N\beta)^2+2N\beta+3N\beta\gamma+3/2\gamma+\gamma^2=:p_1(N\beta,\gamma)$$

$$\sup_i\sum_{k\neq i}\|\partial_k g^i\|_\infty<N\beta(4N\beta+2+3\gamma)=:p_2(N\beta,\gamma).$$

Proof. In the pair interactive case we have

$$H^i(x^i,x,t)=B^i(x^i,t)+\sum_{k\in N(i)}B^{i,k}(x^i,x^k,t)$$

thus

$$\partial_i H^i=\partial_i B^i+\sum_{k\in N(i)}\partial_i B^{i,k}\quad\text{and}\quad \partial_k H^i=\partial_k B^{i,k}$$

which gives the two first bounds. Rewriting g^i we have

$$\begin{aligned}g^i=&\partial_t B^i+\sum_{k\in N(i)}\partial_t B^{i,k}+\sum_{k\in N(i)}(\tilde{b}^k\cdot\partial_k B^{i,k}+1/2\partial_k^2B^{i,k}+1/2(\partial_k B^{i,k})^2)\\ &+1/2\partial_i^2B^i+1/2\sum_{k\in N(i)}\partial_i^2B^{i,k}+1/2(\partial_i B^i+\sum_{k\in N(i)}\partial_i B^{i,k})^2\end{aligned}$$

with $\tilde{b}^k=\partial_k B^k+\sum_{j\in N(k),j\neq i}\partial_k B^{k,j}$ independent of x^i. Hence

$$\begin{aligned}\partial_i g^i=&\partial_i\partial_t B^i+\sum_{k\in N(i)}\partial_i\partial_t B^{i,k}+\sum_{k\in N(i)}(\tilde{b}^k\partial_i\partial_k B^{i,k}+1/2\partial_i\partial_k^2B^{i,k}+\partial_i\partial_k B^{i,k}\partial_k B^{i,k})\\ &+1/2\partial_i^3B^i+1/2\sum_{k\in N(i)}\partial_i^3B^{i,k}+\left(\partial_i^2B^i+\sum_{k\in N(i)}\partial_i^2B^{i,k}\right)\left(\partial_i B^i+\sum_{k\in N(i)}\partial_i B^{i,k}\right)\end{aligned}$$

implying the third bound. On the other hand we have

$$\begin{aligned}\partial_k g^i=&\partial_k\partial_t B^{i,k}+\partial_k\tilde{b}^k\partial_k B^{i,k}+\tilde{b}^k\partial_k^2B^{i,k}+1/2\partial_k^3B^{i,k}+\partial_k^2B^{i,k}\partial_k B^{i,k}+1/2\partial_k\partial_i B^{i,k}\\ &+\partial_i\partial_k B^{i,k}\left(\partial_i B^i+\sum_{k\in N(i)}\partial_i B^{i,k}\right)+\sum_{j\in N(k)\cap N(i)}\partial_k\partial_j B^{j,k}\partial_j B^{i,j}\end{aligned}$$

$$\text{with } \partial_k\tilde{b}^k=\partial_k^2B^k+\sum_{j\in N(k),j\neq i}\partial_k^2B^{k,j}\quad\text{for}\quad k\in N(i)$$

and

$$\partial_k g^i = \sum_{j \in N(k) \cap N(i)} \partial_k \partial_j B^{k,j} \partial_j B^{i,j} \quad \text{for} \quad k \in N^2(i) - N(i) \cup \{i\}$$

Hence we obtain the last bound with

$$\sup_i \sum_{k \neq i} \|\partial_k g^i\|_\infty < (4N^2 - 2N)\beta^2 + 2N\beta + 3N\gamma < p_2(N\beta, \gamma). \quad \square$$

Now by the explicit form (3.4) of the function Ψ^i we have

$$D_k \Psi^i(X^i, X)(ds) = \partial_k H^i(X_1^i, X_1, 1)\delta_1(ds) - \partial_k g^i(X_s^i, X_s, s)ds$$
$$+ (\partial_k f^i(X_0^i, X_0) - \partial_k H^i(X_0^i, X_0, 0))\delta_0(ds) \quad \text{for} \quad k \in N(i) \cup \{i\}$$

and

$$D_k \Psi^i(X^i, X)(ds) = -\partial_k g^i(X_s^i, X_s, s)ds \quad \text{for} \quad k \in N^2(i) - N(i) \cup \{i\}.$$

This together with (3.8) imply the bounds

$$\|D_i \Psi^i\|_\infty < 2\|\partial_i H^i\|_\infty + \|\partial_i g^i\|_\infty + \sum_{k \in N(i)} \|\partial_k f^{i,k}\|_\infty < 3N\beta + 2\gamma + p_1(N\beta, \gamma),$$

$$\|D_i \Psi^i\|_\infty^* < \|\partial_i H^i\|_\infty + \|\partial_i g^i\|_\infty < N\beta + \gamma + p_1(N\beta, \gamma), \tag{3.9}$$

$$\sum_{k \neq i} \|D_k \Psi^i\|_\infty < \sum_{k \neq i} 2\|\partial_k H^i\|_\infty + \|\partial_k g^i\|_\infty + \|\partial_k f^{i,k}\|_\infty < 3N\beta + p_2(N\beta, \gamma).$$

Using the estimates (2.19) and (2.13) for $p=2$ we finally obtain

$$\sup_i \sum_{k \neq i} C^{k,i} < r(N\beta, \gamma, \sigma) \tag{3.10}$$

with

$$r(N\beta, \gamma, \sigma) := \left[\sigma \frac{(1 + (3N\beta + 2\gamma + p_1(N\beta, \gamma))\sigma/\sqrt{2})^{1/2}}{(1 - (3N\beta + 2\gamma + p_1(N\beta, \gamma))\sigma/\sqrt{2})^{3/2}} + N\beta + \gamma + p_1(N\beta, \gamma) + 2 \right] \cdot [3N\beta + p_2(N\beta, \gamma)]$$

For fixed γ and $\sigma < \sqrt{2}/(7/2\gamma + \gamma^2)$, let $\beta^* = \beta^*(\gamma, \sigma)$ be the smallest positive root of the equation

$$r(\beta^*, \gamma, \sigma) - 1 = 0. \tag{3.11}$$

Since $r(x, \gamma, \sigma)$ is continuous and monoton increasing in x with

$$r(0, \gamma, \sigma) \equiv 0,$$

β^* is strictly positive. Therefore we have shown:

Theorem. (3.12) *Suppose we can associate the bounds* [B.1]–[B.4] *to our gradient system* (3.1), *then for all* γ *and* $\sigma < \sqrt{2}/(7/2\gamma + \gamma^2)$ *we can find a* $\beta^* = \beta^*(\gamma, \sigma) > 0$ *solution of the equation* (3.11) *such that the condition* (3.6) *holds if* $\beta < \beta^*(\gamma, \sigma)/N$.

Remark. (3.13) If the process $(X_t^i, 0 \leqq t \leqq 1)_{i \in I}$ starts off with a deterministic law, then $r(x, y)$ reduces to a polynomial of 4th grade:

$$r(x,\gamma) = (x+\gamma+p_1(x,\gamma)+2)(x+p_2(x,\gamma))$$
$$= 8x^4 + 18(1+\gamma)x^3 + (13\gamma^2 + 18\gamma + 17)x^2 + 3(1+\gamma)(\gamma^2 + 5/2\gamma + 2)x$$

and Eq. (3.11) can be explicitely solved.

4. Exponential Decay of Correlations and Central Limit Theorem

In the preceding sections we have shown that, under conditions [A.1] and [A.2], the law Q of the diffusion (3.1) can be viewed as a Gibbs measure, tempered in the sense of (3.2).

Moreover we have stated conditions on the bounds [B.1]–[B.4] which guarantee Dobrushin's uniqueness condition:

$$\sup_i \sum_{k \neq i} C^{k,i} < 1. \tag{3.6}$$

We can now apply Dobrushin's contraction technique in order to derive the exponential decay of correlations and the central limit theorem for functions of the class $L(C[0,1]^I)$.

In this section we shall suppose that the interaction of the drifts $(b^i)_{i \in I}$ and initial measure μ satisfies the condition of Theorem (3.12) such that (3.6) holds. The first result due to Dobrushin [6] is the unicity of Q. More precisely, let $G((Q^i(\ |X))_{i \in I})$ be the set of *tempered* distributions $\tilde{Q}$ on $(C[0,1]^I, \mathscr{F})$ such that, for all $i \in I$

$$\tilde{Q}^i(\ |X) = Q^i(\ |X) \quad \tilde{Q} \text{ almost surely}.$$

Proposition. (4.2) *(Dobrushin) Under conditions* (3.2) *and* (3.6), *the measure Q is uniquely determined by its conditional probabilities $(Q^i(\ |X))_{i \in I}$, i.e. $|G((Q^i(\ |X))_{i \in I})| = 1$.*

Remark. (4.3) i) It is important to restrict one self to tempered distributions. In general there exist non-tempered distributions satisfying (4.1).

ii) In the Gaussian case conditions (3.6) is sharp, i.e. one can construct a diffusion process with linear interaction with $\sup_i \sum_{k \neq i} C^{k,i} = 1$ such that $|G((Q^i(\ |X))_{i \in I})| = \infty$: In example (2.22) take

$$v^i(\ |X_1) = N\Big(\sum_{k \neq 0} a^k X^{i+k}, 1\Big) \quad \text{for} \quad I = Z^d.$$

In this case the Dobrushin coefficients $C^{k,i}$ reduce to $|a^{k-i}|$. Choosing the a^k, $k \neq 0$, such that

$$\sum_{k \neq 0} |a^k| = 1$$

for $d \geqq 3$ implies a phase transition for v, cf. Künsch [14], which is equivalent to a phase transition for Q, cf. also Sect. III.7 of [3].

Let $D=(D^{k,i})$ denotes the sum $\sum_{n=0}^{\infty} C^n$ of the powers of C and define the constant

$$s^2:=\sup_i \int \left[\inf_Y \int \|X^i - Y\|^2 Q^i(dX^i|X)\right] Q(dX).$$

Then by Theorem (3.7) of Föllmer [8], the following estimate holds

$$|\mathrm{cov}_Q(f,g)| \leqq s^2 \sum_{i,k} \delta_i(f) D^{i,k} \delta_k(g) \qquad (f,g \in L(C[0,1]^I)),$$

see also Künsch [15] and Gross [12]. Let $I=Z^d$ be the d-dimensional lattice and let θ^i be the usual shift on $C[0,1]^I$, i.e. $(\theta^i X)^k = X^{i+k}$. Consider a translation invariant metric $d(\,,)$ on I such that

$$\zeta:=\sup_i \sum_{k\neq i} e^{d(k,i)} C^{k,i} < 1. \tag{4.5}$$

Then by Corollary (1.7) of [8] the correlations decay exponentially:

Proposition. (4.6) *If* $\zeta<1$ *then*

$$\sum_i |\mathrm{cov}_Q(f, g\cdot\theta^i)| e^{d(i,0)} \leqq s^2(1-\zeta)^{-1} \|f\|_0 \|g\|_0 \qquad (f,g \in L(C[0,1]^I))$$

with $\|f\|_0 := \sum_i e^{d(i,0)} \delta_i(f)$.

The rest of this section is devoted to the derivation of the central limit theorem and the equation of the fluctuation field. We shall assume from now on that the drifts $(b^i)_{i\in I}$, $I=Z^d$, and the initial distribution μ are translation invariant under the shift $(\theta^k)_{k\in I}$, i.e. $b^{i+k}=b^i\cdot\theta^k$ and $\mu=\mu\cdot\theta^{-k}$ for all i, $k\in I$, which implies under uniqueness the translation invariance of the measure Q and its conditional probabilities $(Q^i(\;|X))_{i\in I}$. Moreover $C^{k,i}$ depends only on the difference $k-i$ and condition (3.6) becomes

$$\sum_{k\neq 0} C^k < 1.$$

For a function f in $L(C[0,1]^I)$ let

$$S_n^*(f) := |V_n|^{-1/2} \sum_{i\in V_n} \{f\cdot\theta^i - E_Q[f]\}$$

where V_n is the cube $[-n,n]^d$ in Z^d. Since the $Q^i(\;|X)$ are of finite range we have

$$\sum_{k\neq 0} C^k |k|^d < \infty \tag{4.7}$$

and we can apply the central limit theorem of Künsch [16], see also Bolthausen [2]:

Theorem. (4.8) *Under conditions* (3.2), (3.6), *and* (4.7), *the distribution of* $S_n^*(f)$ *converges for* $f\in L(C[0,1]^I)$ *to the centered normal law of variance*

$$\sigma^2(f) := \sum_{k\in I} \mathrm{cov}_Q(f, f\cdot\theta^k) \geqq 0. \tag{4.9}$$

Let $\mathscr{S}(\mathbb{R}^{V_n})$ be the Schwarz space of all rapidly decreasing smooth functions on $\mathbb{R}^{V_n}$ and $\mathscr{S}'(\mathbb{R}^{V_n})$ be the space of tempered distributions. Put $\mathscr{S}_\infty := \bigoplus_{n \geqq 0} \mathscr{S}(\mathbb{R}^{V_n})$ for the direct sum viewed as a nuclear space, cf. Yamazaki [24], and $\mathscr{S}'_\infty := \bigotimes_{n \geqq 0} \mathscr{S}'(\mathbb{R}^{V_n})$ for the dual. We define the $\mathscr{S}'_\infty$-valued process

$$Y^{(n)} := (Y_t^{(n)}(\phi), 0 \leqq t \leqq 1)_{\phi \in \mathscr{S}_\infty}$$

by

$$Y_t^{(n)}(\phi) := S_n^*(\phi(X_t)). \tag{4.10}$$

Since $\mathscr{S}_\infty \subset L(C[0,1]^I)$ with

$$|\phi(x) - \phi(y)| \leqq \sum_{k \in V_m} \|\partial_k \phi\|_\infty |x^k - y^k| \quad \text{for} \quad \phi \in \mathscr{S}(\mathbb{R}^{V_m}),$$

a multi-dimensional version of Theorem (4.8) implies that, for all $t_1, \ldots, t_k \in [0,1]$ and $\phi_1, \ldots, \phi_k \in \mathscr{S}_\infty$, the vector $(Y_{t_1}^{(n)}, \ldots, Y_{t_k}^{(n)})$ converges in law to a centered Gaussian vector. More precisely:

Proposition. (4.12) *The process $Y^{(n)}$ converges in law to a continuous $\mathscr{S}'_\infty$-valued Gaussian process $Y = (Y_t(\phi), 0 \leqq t \leqq 1)_{\phi \in \mathscr{S}_\infty}$ with variance*

$$\sigma_t^2(\phi) = \sum_{k \in I} \mathrm{cov}_Q(\phi(X_t), \phi \cdot \theta^k(X_t)) \geqq 0.$$

Proof. Since all finite-dimensional marginal distributions converge it suffices to show the tightness of the laws of $Y^{(n)}(\phi)$, $n \in \mathbb{N}$, for each $\phi \in \mathscr{S}_\infty$, cf. Theorem 5.3 of Mitoma [19]. But this follows from inequality (4.11), cf. Proposition 5.4 of [19]. □

From now on suppose that $b^i(t, \cdot) \in \mathscr{S}(\mathbb{R} \times \mathbb{R}^{N(i)})$, $|N(i)| < \infty$. Let L_t and D be the linear operators from $\mathscr{S}_\infty$ to $\mathscr{S}_\infty$ defined by

$$\begin{aligned} L_t\phi(x) &:= \sum_{k \in V_m} b^k(x,t)\partial_k\phi(x) + 1/2\partial_k^2\phi(x) \\ D\phi(x) &:= \sum_{k \in V_m} (\partial_k\phi \cdot \theta^{-k})(x) \quad \text{for} \quad \phi \in \mathscr{S}(\mathbb{R}^{V_m}). \end{aligned} \tag{4.13}$$

We can derive the following $\mathscr{S}'_\infty$-valued stochastic differential equation for the fluctuation field $(Y_t, 0 \leqq t \leqq 1)$, cf. Itô [13]:

Proposition. (4.14) *The process $(Y_t, 0 \leqq t \leqq 1)$ satisfies the linear stochastic differential equation*

$$\begin{aligned} dY_t(\phi) &= Y_t(L_t\phi)dt + dB_t(D\phi), \\ Y_0(\phi) &\sim N(0, \sigma_0^2(\phi)) \qquad (\phi \in \mathscr{S}_\infty) \end{aligned} \tag{4.15}$$

where $B = (B_t(\phi), 0 \leqq t \leqq 1)_{\phi \in \mathscr{S}_\infty}$ is a $\mathscr{S}'_\infty$-valued Wiener process with quadratic variation

$$\langle B(\phi) \rangle_t = \int_0^t E_Q[(\phi(X_s))^2] ds.$$

Proof. Let $\phi \in \mathcal{S}(\mathbb{R}^{V_m})$, then by Itô's formula we have

$$Y_t^{(n)}(\phi) - Y_0^{(n)}(\phi) - \int_0^t Y_s^{(n)}(L_s\phi)ds = M_t^{(n)}(\phi)$$

where $(M_t^{(n)}(\phi), 0 \leqq t \leqq 1)$ is the martingale

$$M_t^{(n)}(\phi) := |V_n|^{-1/2} \sum_{i \in V_n} \sum_{k \in V_m} \int_0^t (\partial_k \phi \cdot \theta^i)(X_s) dW_s^{k+i}.$$

Since $L_t\phi$ is in $\mathcal{S}_\infty$ the L.H.S. of (4.16) converges in law to

$$Y_t(\phi) - Y_0(\phi) - \int_0^t Y_s(L_s\phi)ds$$

by Proposition (4.12). On the other hand, by ergodic theorem we have

$$\lim_{n\to\infty} \langle M^{(n)}(\phi)\rangle_t = \lim_{n\to\infty} |V_n|^{-1} \sum_{i \in V_n} \int_0^t (D\phi)^2 \cdot \theta^i(X_s) ds$$

$$= \int_0^t E_Q[(D\phi(X_s))^2] ds \quad \text{Q.a.s.}$$

which implies the convergence in law of the R.H.S. of (4.16) to $B_t(D\phi)$, cf. Shiryayev [20]. □

Remark. (4.17) i) If we simply take ϕ in $\mathcal{S}(\mathbb{R})$, then we have

$$dY_t(\phi) = Y_t(L_t\phi)dt + dB_t(\phi')$$

with $\langle B(\phi)\rangle_t = \int_0^t E_Q[(\phi(X_s^0))^2]ds = \int_0^t \int_{\mathbb{R}} (\phi(x))^2 p_s(x) dx\, ds$ where $p_t(x)$ is the density of the law of X_t^0. In this case the Wiener process B can be represented as a stochastic integral with respect to a Brownian sheet, cf. [4].

ii) Let $(T_{s,t}, 0 \leqq s \leqq t \leqq 1)$ be the semigroup associated to $(X_t, 0 \leqq t \leqq 1)$:

$$T_{s,t}\phi(X_s) := E_Q[\phi(X_t) | \mathcal{F}_s]$$

then the solution of the equation (4.15) is given by

$$Y_t(\phi) = Y_0(T_{0,t}\phi) + \int_0^t dB_s(D(T_{s,t}\phi))$$

cf. Holley-Stroock [11]. From this explicit form one can easily see that the variance

$$\sigma_t^2(\phi) = \sigma_0^2(T_{0,t}\phi) + \int_0^t E_Q[(D(T_{s,t}\phi))^2] ds$$

is strictly positive whenever $\|\partial_k \phi\|_\infty \neq 0$ for some $k \in I$.

Acknowledgement. The author wishes to express his gratitude to H. Föllmer for suggesting the problem and for his constant encouragement and valuable advice. Thanks are also due to H. Künsch for numerous and fruitful discussions.

References

1. Bismut, J.M., Michel, D.: Diffusions conditionnelles II. J. Funct. Anal. **45**, 274–292 (1982)
2. Bolthausen, E.: On the central limit theorem for stationary mixing random fields. Ann. Probab. **10**, 1047–1050 (1982)
3. Deuschel, J.D.: Lissage de diffusions à dimension infinie et leurs propriétés en tant que mesure de Gibbs, Thèse ETH Zürich 7823, 1985
4. Deuschel, J.D.: Représentation du champ de fluctuation de diffusions indépendantes par le drap brownien. "Séminaire de Probabilités XXI", Lecture Notes in Mathematics, vol. 1247, pp. 428–433. Berlin, Heidelberg, New York: Springer 1986
5. Deuschel, J.D.: Non-linear smoothing of infinite-dimensional diffusion processes. Stochastics **19**, 237–261 (1986)
6. Dobrushin, R.L.: Prescribing a system of random variables by conditional distributions. Theor. Probab. Appl. **15**, 458–486 (1970)
7. Doss, H., Royer, G.: Processus de diffusion associés aux mesures de Gibbs. Z. Wahrscheinlichkeitstheor. Verw. Geb. **46**, 125–158 (1979)
8. Föllmer, H.: A covaraince estimate for Gibbs measures. J. Funct. Anal. **46**, 387–395 (1982)
9. Föllmer, H.: Wakolbinger, A.: Time reversal of infinite-dimensional diffusions. Stochastic Processes Appl. **22**, 59–77 (1986)
10. Fritz, J.: Stationary measures of stochastic gradient systems, Infinite lattice models. Z. Wahrscheinlichkeitstheor. Verw. Geb. **59**, 479–490 (1982)
11. Holley, R., Stroock, D.: Generalized Ornstein-Uhlenbeck processes and infinite particle branching Brownian motions. Pub. of R.I.M.S. Kyoto Univ. **14**, 741–788 (1978)
12. Gross, L.: Decay of correlations in classical lattice models at high temperatures. Commun. Math. Phys. **68**, 9–27 (1979)
13. Itô, K.: Distribution-valued processes arising from independent Brownian motions. Math. Z. **182**, 17–33 (1983)
14. Künsch, H.: Thermodynamics and statistical analysis of Gaussian random fields. Z. Wahrscheinlichkeitstheor. Verw. Geb. **58**, 407–421 (1981)
15. Künsch, H.: Decay of correlations under Dobrushin's uniqueness condition and its applications. Commun. Math. Phys. **84**, 207–222 (1982)
16. Künsch, H.: The central limit theorem for Gibbs field with weak interaction. Unpublished note, 1982
17. Levin, S.L.: Application of Dobrushin's uniqueness theorem to N-vector models. Commun. Math. Phys. **78**, 65–74 (1980)
18. Liptser, R.S., Shiryayev, A.N.: Statistics of Random processes I. Berlin, Heidelberg, New York: Springer 1977
19. Mitoma, I.: Tightness of probabilities on $C([0,1],\mathscr{S}')$ and $D([0,1],\mathscr{S}')$, Ann. Probab. **11**, 989–999 (1983)
20. Shiga, T., Shimizu, A.: Infinite dimensional stochastic differential equations and their applications. J. Math. Kyoto Univ. **20**, 395–416 (1980)
21. Shiryayev, A.N.: Martingales: recent developments, results and applications. Int. Stat. Rev. **49**, 199–233 (1981)
22. Simon, B.: A remark on Dobrushin's uniqueness theorem. Commun. Math. Phys. **68**, 183–185 (1975)
23. Spohn, H.: Models of statistical mechanics in one dimension originating from quantum ground states. To appear in Proceedings of "Statistical Mechanics and Field Theory", Groningen 1985
23. Yamazaki, Y.: Measures on infinite-dimensional spaces. Series in Pure Mathematics 5, World Scientific, Singapore, 1985

Received March 25, 1986