Probab. Th. Rel. Fields 76, 257–289 (1987)

Probability Theory and Related Fields

Uniform Measure Results for the Image of Subsets Under Brownian Motion

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Summary. The paper obtains bounds on the Hausdorff and packing measures of the image $X(E)$ of a Borel set E by a transient strictly stable process X_t which a.s. hold for all E and for every measure function $h_{\beta,\gamma}(s)=s^\beta|\log s|^\gamma$. In some cases examples are constructed to show that the bounds are sharp.

Introduction

In this paper we consider not only Brownian motion in $\mathbb{R}^d$ $(d \geqq 2)$, but also strictly stable processes $X(t)$ of index α in $\mathbb{R}^d$ $(\alpha \leqq d)$. Our object is to explore connections between the measure properties of a Borel set E contained in a compact interval and those of its image $X(E)$ on the sample trajectory. Whenever E is a fixed subset such results were obtained first by McKean [10] if X is a Brownian motion and then by Blumenthal and Getoor [1] for strictly stable X. These early methods are not valid whenever E is a random set depending on the process. The only effective way of attacking the problem for such sets is to seek results which are a.s. valid simultaneously for every Borel set $E \subset [0, M]$. Hausdorff dimension is the most studied fractal index, so the first natural result is to show that for a fixed stable process of index α in $\mathbb{R}^d$

$$\text{a.s. } \dim X(E)=\alpha \dim E \quad \text{for every Borel } E, \tag{0.1}$$

where $\dim E$ stands for the Hausdorff dimension of E.

When $\alpha=2$ and X is Brownian motion, (0.1) was first proved by Kaufman [8]: it was extended to all strictly stable X by Hawkes and Pruitt [6]. We note that not all independent increment processes in $\mathbb{R}^d (d \geqq 2)$ have an index α which makes (0.1) valid: the first counterexample is due to Hendricks [7].

* While preparing this paper, the author was partially supported by NSERC and by NSF on contract # DMS-8317815

Within a given Hausdorff dimension β we can obtain more precise information about the size of a set by considering its Hausdorff measure with respect to

$$h_{\beta,\gamma}(s)=s^{\beta}\left(\log\frac{1}{s}\right)^{\gamma}$$

for real numbers γ. More precise results of this kind for a *fixed* set E and a subordinator X were obtained by Hawkes [21]. Our object in this paper is to search for the best possible uniform comparison theorems which will imply that for a fixed strictly stable X of index α, a.s.

$$h_{\alpha\beta,\gamma_1}-m(X(E))\leqq c_{0.1}h_{\beta,\gamma}-m(E) \quad \text{for all} \quad \gamma\in\mathbb{R}, \beta\in(0,1), \text{Borel } E \tag{0.2}$$

$$h_{\alpha\beta,\gamma_2}-m(X(E))\geqq c_{0.2}h_{\beta,\gamma}-m(E) \quad \text{for all} \quad \gamma\in\mathbb{R}, \beta\in(0,1), \text{Borel } E \tag{0.3}$$

for some known functions $\gamma_i=\gamma_i(\beta,\gamma)$, $i=1,2$.

Such results were obtained by Kaufman [8] for planar Brownian motion. Our Theorem 3.4 improves his lower bound while example 4.2 shows that his upper bound is best possible. The proof of (0.1) given in Hawkes and Pruitt [6] does yield some estimates for γ_i which satisfy (0.2) and (0.3), but we modify their arguments to get better bounds. The lower bound result (0.3) takes a different form for the critical cases of planar Brownian motion ($\alpha=2=d$) and linear symmetric Cauchy $X(\alpha=1=d)$ and the noncritical cases ($\alpha<d$). Whenever $0<\alpha<1$ we recall that Taylor [15] distinguishes two types of stable process X of index α: the type A process has a positive transition density at 0 and none of its linear projections is monotone, while the type B process has zero density at 0 and projects on a suitable line to give a subordinator. The lower bound for type B processes is given by Theorem 3.1, and for type A processes by Theorem 3.4. Examples are given to show that Theorem 3.1 is best possible (Corollary 3.3), while Theorem 3.4 is sharp at least up to a factor $\log\log\frac{1}{s}$ (see Examples 3.7, 3.8).

The key idea needed to obtain uniform lower bounds is to consider the asymptotic behaviour as $a\downarrow 0$ of the maximum sojourn time in any ball of radius a. Results of this kind are stated without proof in [16] but only for noncritical Brownian motion in $\mathbb{R}^d (d\geqq 3)$. In Lemma 2.3 precise results are proved for $d=2$, the symmetric Cauchy X in $\mathbb{R}$, and strictly stable process with $\alpha<d$.

Uniform upper bounds leading to (0.2) are easy for Brownian motion. The corresponding result for the stable processes is contained in Theorem 4.1. We cannot decide whether this theorem is best possible.

In a recent paper of Taylor and Tricot [20], a new fractal measure $\phi-p$ is defined for each growth function ϕ. Packing measure $\phi-p(E)\geqq\phi-m(E)$, the Hausdorff ϕ-measure, and the two fractal measures can be of a different order of magnitude. Using the scale $\phi(s)=s^{\beta}$, $\beta>0$ yields a new fractal index which we denoted by $\operatorname{Dim} E$ and call the packing dimension of E. Our fundamental lemmas proved in Sect. 2 are good enough to give uniform results which yield the analogues of (0.2), (0.3) for packing measure. This is the main content of Sect. 5. For technical reasons we need a result (Lemma 1.2) which estimates the effect of requiring packing to be done with equal balls, rather than allowing unequal balls –

we believe this lemma to have independent usefulness. This means that our uniform packing measure results may not be sharp – constructing interesting examples to test the theorems seems to be hard. However, the theorems in Sect. 5 certainly imply that for each X stable of index $\alpha \leqq d$,

$$\text{a.s. } \operatorname{Dim} X(E) = \alpha \operatorname{Dim} E \quad \text{for all Borel } E. \tag{0.4}$$

In [17] Taylor defines a set $E \subset \mathbb{R}^d$ to be a fractal if $\dim E = \operatorname{Dim} E$ but is different from the topological dimension of E. Thus (0.1) and (0.4) together tell us that, for any stable process X,

$$\text{a.s. the image } X(E) \text{ of each fractal } E \text{ is a fractal.} \tag{0.5}$$

We remark that the results in Taylor [18] show that, for more general Lévy processes (0.5) may fail.

We start by collecting, in Sect. 1, definitions and probability estimates which are essential to our calculations. Finite positive constants whose values are unimportant, and often unknown, will be numbered consecutively in each section.

1. Preliminaries

In this paper we are interested in the study of subsets of $\mathbb{R}$ or $\mathbb{R}^d$ of zero Lebesgue measure. In order to get information about their size we use either Hausdorff measure or packing measure. We restrict attention to the class Φ of functions $\varphi:(0,\delta)\to(0,1)$ which are monotone increasing, right continuous with $\varphi(0+)=0$, and smooth in the sense that there is a finite constant $c_{1.1}$ with

$$\varphi(2s)/\varphi(s) \leqq c_{1.1} \quad \text{for} \quad 0<s<\tfrac{1}{2}\delta. \tag{1.1}$$

The Hausdorff measure of a set E is defined by

$$\varphi - m(E) = \lim_{\eta \downarrow 0} \inf\{\Sigma \varphi(\operatorname{diam} E_i) : E \subset \bigcup E_i, \operatorname{diam} E_i < \eta\} \tag{1.2}$$

and may be zero, finite and positive, or infinite. If we replace arbitrary coverings by coverings from the class of cubes of side 2^{-k} and vertices whose coordinates are of the form $j2^{-k}$ (j, k are integers) we get a new "dyadic" Hausdorff measure $\varphi - m_D(E)$ which satisfies

$$\varphi - m(E) \leqq \varphi - m_D(E) \leqq c_{1.2} \varphi - m(E) \tag{1.3}$$

for all sets E, using (1.1) and replacing an arbitrary cover by a dyadic cover.

When we need to obtain an upper bound for $\varphi - m(E)$ we only require to produce economical covers of E by sets (or dyadic cubes) of small diameter. To obtain a lower bound, the definition (1.2) requires us to consider all possible covers by sets of small diameter. In the present paper we will repeatedly use the following result, which is useful for Cantor-like sets E.

Lemma 1.1. *Suppose $\Psi \in \Phi$, $\eta > 0$, $c > 0$. Let K be a compact set with the representation*

$$K = \bigcap_{m=1}^{\infty} E_m, E_{m+1} \subset E_m, E_m = \bigcup_{i=1}^{M_m} I_{m,i},$$

where the $I_{m,i}(1 \leqq i \leqq M_m)$ *are disjoint closed subintervals of* $[0,1]$ *contained in* E_{m-1}. *Then*

$$\Psi - m(K) \geqq c^{-1}$$

if, for every interval $J \subset [0,1]$ *with* $|J| < \eta$, *there is a finite integer* $m(J)$ *such that*

$$N_m(J) \leqq c\Psi(|J|)M_m \quad \textit{for} \quad m \geqq m(J),$$

where $N_m(J)$ *denotes the number of the* $I_{m,i}$ *which are contained in* J.

This is Lemma 2.2 of [11].

In a recent paper [19] we defined a new set function $\varphi - P(E)$ in which economical coverings are replaced by disjoint packings. For $\varphi \in \Phi$,

$$\varphi - P(E) = \lim_{\eta \downarrow 0} \sup\{\Sigma \varphi(2r_i) : S(x_i, r_i) \text{ disjoint, } x_i \in E,\ r_i < \eta\}, \tag{1.4}$$

where $S(x_i, r_i)$ denotes the open ball of radius r_i centered at x_i. Again, it is helpful to have a restricted class of sets to pack, which are almost nested. We show that if we replace $S(x_i, r_i)$ by semidyadic cubes of side 2^{-k}, and vertices of the form either $j2^{-k}$ or $(j+\frac{1}{2})2^{-k}$, which contain a point of E in the concentric cube of side 2^{-k-2}, we obtain a set function $\varphi - P_D(E)$ which satisfies

$$c_{1.3}\varphi - P(E) \leqq \varphi - P_D(E) \leqq c_{1.4}\varphi - P(E). \tag{1.5}$$

The set function $\varphi - m$ is a Carathéodory outer measure, but $\varphi - P$ is not because it fails to be countably subadditive. To obtain an outer measure, which is called φ-packing measure, we need a final step in the construction

$$\varphi - p(E) = \inf\{\Sigma \varphi - P(E_i) : E \subset \bigcup E_i\}.$$

There is enough structure in the set function $\varphi - P$ to deduce that, for each Borel E, $\varphi - p(E)$ has good approximations of the form $\varphi - P(E_n)$ with $E_n \subset E$. In fact, we show that

$$\varphi - p(E) = \inf\{\lim \varphi - P(E_n) : E_n \uparrow E\}. \tag{1.6}$$

In general we have $\varphi - m(E) \leqq \varphi - p(E)$, so there are two definitions of dimension coming from these measures:

$$\begin{aligned} \dim E &= \inf\{\alpha > 0 : s^\alpha - m(E) = 0\} \\ \operatorname{Dim} E &= \inf\{\alpha > 0 : s^\alpha - p(E) = 0\} \end{aligned} \tag{1.7}$$

which are called, respectively, the Hausdorff dimension and the packing dimension of E. For $E \subset \mathbb{R}^d$

$$0 \leqq \dim E \leqq \operatorname{Dim} E \leqq d$$

and all values allowed by these inequalities are attainable for a compact set E. Packing measures are less easy to calculate because of the disjointness condition in (1.4). This time it is the upper bound for $\varphi - P(E)$ which is hard to determine. However, we can get some (imprecise) information by considering packings of E by balls all of the same radius. Let $N_r(E)$ be the maximum number of disjoint open balls of radius r which can be centered in E.

Lemma 1.2. *Suppose $\varphi, \Psi \in \Phi$, $h(s)=\varphi(s)\Psi(s)$, and $\int_{0+} \frac{\Psi(s)}{s} ds < \infty$. Then for any E,*

$$\limsup_{r \downarrow 0} N_r(E)\varphi(2r) < \infty \Rightarrow h-P(E)=0.$$

Proof. For $\eta > 0$, $K < \infty$, suppose

$$0 < r \leqq \eta \Rightarrow N_r(E)\varphi(2r) \leqq K$$

but $h-P(E) > c_{1.5} > 0$. Then, for each $\delta \in (0, \eta)$ we can find a packing of E by disjoint open balls $S(x_i, r_i)$ with $r_i < \delta$ and $\Sigma h(2r_i) > c_{1.5}$. The balls remain disjoint if we replace each r_i by 2^{-k_i} where $2^{-k_i+1} > r_i \geqq 2^{-k_i}$. By (1.1) we now have disjoint balls with dyadic radii such that $\Sigma h(2^{1-k_i}) > c_{1.6}$. Group these according to the value of k_i and note that $N_{2^{-k}}(E)$ is not less than the number of these balls for which $k_i = k$. Hence,

$$c_{1.6} < \sum_{k=r}^{\infty} \sum_{k_i=k} h(2.2^{-k_i}) \leqq \sum_{k=r}^{\infty} N_{2^{-k}}(E) h(2^{1-k}) \leqq K \sum_{k=r}^{\infty} \Psi(2^{1-k}).$$

This final estimate converges to zero as $r \to \infty$, or as $\delta \downarrow 0$, since $\int_{0+} \frac{\Psi(s)}{s} ds < \infty$, so we have obtained a contradiction, establishing the lemma. □

Remark. For Hausdorff measure, it follows from some unpublished work of Claude Tricot that economical covers by balls of the same radii, lead to a very different measure which can give a different value to the dimension using (1.7). The above lemma shows that we could define packing dimension using balls of equal radii and get the same result, $\operatorname{Dim} E$. The maximum effect of using equal balls is less than a factor $\left(\log \frac{1}{s}\right)^{1+\varepsilon}$, $\varepsilon > 0$ in the measure function.

In the present paper we will consider only the strictly stable processes of index $\alpha (0 < \alpha \leqq 2)$ in $\mathbb{R}^d$. These have the important property that, for each $\lambda > 0$,

$$\lambda^{-1/\alpha} X(\lambda t) \text{ is another version of } X(t). \tag{1.8}$$

This scaling property is used repeatedly, often without mention. Whenever $d=1$ we will usually assume that a.s. $X(t)$ does not hit a singleton $\{x_0\}$ for $t > 0$, which is equivalent to $0 < \alpha \leqq 1$; as $1 < \alpha$, $d=1$ implies that $X(t)$ fills intervals and no uniform measure results of the form (0.2), (0.3) are then possible. For $0 < \alpha \leqq d$, the trajectory of a stable process has zero Lebesgue measure, but complete information about both Hausdorff measure and packing measure is available, see [19] for a survey of results.

The class of strictly stable processes is defined and discussed in [16]. We always assume that $X(t)$ is genuinely a process in $\mathbb{R}^d$, that is, there is no subspace of lower Euclidean dimension in which the process takes its values. We also assume that we have a nice version which has cadlag paths and satisfies the strong Markov property. When $\alpha = 2$, the paths are continuous but, for $0 < \alpha < 2$, the discontinuities are a.s. everywhere dense. For $\alpha = 2$ there is only a Gaussian component in the Lévy-Khintchine formula. This means that there is no loss in generality in assuming that $X(t)$ is a standard Brownian motion. Since this is often the most interesting case of our results, we will state many theorems with Brownian motion singled out as a specific case. We use $B(t)$ to denote the Brownian motion case. The natural potential theory for this case comes from the kernel

$$k(x,y)=|x-y|^{2-d}, \quad d\geqq 3.$$

In the critical planar case, $B(t)$ is neighborhood recurrent, so the potential theory has to be carried out relative to the first exit from some disc. We summarise the results for hitting probabilities which we need. If $\tau_E=\inf\{t>0: X(t)\in E\}$, $\Gamma_1=S(0,r_1)$, $\Gamma_2=S(0,r_2)^c$, $r_1\leqq\varrho\leqq r_2$ and $x\in\mathbb{R}^d$ with $|x|=\varrho$. Then, for a standard Brownian motion process in $\mathbb{R}^d$;

$$d=2, P^x\{\tau_{\Gamma_1}<\tau_{\Gamma_2}\}=\frac{\log\frac{r_2}{\varrho}}{\log\frac{r_2}{r_1}}, \tag{1.9}$$

$$d\geqq 3, P^x\{\tau_{\Gamma_1}<\tau_{\Gamma_2}\}=\frac{\varrho^{2-d}-r_2^{2-d}}{r_1^{2-d}-r_2^{2-d}}, \tag{1.10}$$

$$P^x\{\tau_{\Gamma_1}<\infty\}=\left(\frac{r_1}{\varrho}\right)^{d-2}. \tag{1.11}$$

Whenever $X(t)$ is symmetric stable of index $\alpha<2$, that is, the characteristic function is

$$E\{\exp i(X(t+h)-X(t),u)\}=e^{-ch|u|^\alpha},$$

the relevant potential theory comes from the symmetric kernel $k(x,y)=|x-y|^{\alpha-d}$ when $d>\alpha$, or the logarithmic kernel for the symmetric Cauchy process. As proved by Takeuchi [15], this leads to bounds for the hitting probabilities:

$$d=1=\alpha, P^x(\tau_{\Gamma_1}<\tau_{\Gamma_2})\leqq\frac{\log\frac{r_2}{\varrho}}{\log\frac{r_2}{r_1}}=1-\frac{\log\frac{\varrho}{r_1}}{\log\frac{r_2}{r_1}}, \tag{1.12}$$

$$d>\alpha, P^x(\tau_{\Gamma_1}<\infty)=\left(\frac{r_1}{\varrho}\right)^{d-\alpha}. \tag{1.13}$$

It is quite easy to get a lower bound in (1.13), but we do not need to use it. However, we will require a sharp lower bound for (1.12) which needs some argument. Spitzer [14] pointed out that the symmetric Cauchy process in $\mathbb{R}^1$ can be recovered from planar Brownian motion by observing it on one of the axes. To be precise, if $B_1(t)$, $B_2(t)$ are independent Brownian motions on $\mathbb{R}$, $L(s)$ is the local time of $B_2(t)$ at $x=0$ and $Y(t)$ is the function inverse to $L(s)$, then $X(t)=B_1(Y(t))$ is symmetric Cauchy on $\mathbb{R}$. We could use this fact with an explicit conformal mapping of the complement in $\mathbb{R}^2$ of $(-\infty,1]\cup[-r,r]\cup[1,\infty)$ into an annulus to obtain the required estimate. However, we prefer to obtain a proof by probabilistic reasoning.

Lemma 1.3. *For a symmetric Cauchy process in* $\mathbb{R}$

$$P^x(\tau_{\Gamma_2}<\tau_{\Gamma_1})\leqq c_{1.7}\frac{\log\dfrac{\varrho}{r_1}}{\log\dfrac{r_2}{r_1}}$$

for $er_1\leqq\varrho\leqq r_2$.

Proof. $B(t)$ now denotes a planar Brownian motion, the probability we have to estimate is

$$P^x\{\tau_{A_2}<\tau_{A_1}\}\quad\text{for } x \text{ real, } er_1\leqq x\leqq r_2,$$

where $A_1=\{(x,0):|x|<r_1\}$, $A_2=\{(x,0):|x|\geqq r_2\}$. If we put

$$q(r_1,r_2)=\sup\{P^y\{\tau_{\Gamma_2}<\tau_{A_1}\}:|y|=r_1\},$$

then the strong Markov property implies that

$$P^x\{\tau_{A_2}<\tau_{A_1}\}\leqq P^x\{\tau_{A_2}<\tau_{\Gamma_1}\}+q(r_1,r_2),$$

so that

$$P^x\{\tau_{A_2}<\tau_{A_1}\}\leqq P^x\{\tau_{\Gamma_2}<\tau_{\Gamma_1}\}+q(r_1,r_2).\tag{1.14}$$

All the probabilities now relate to planar Brownian motion. Now choose M large enough to ensure that, for all $y\in\mathbb{R}^2$ with $|y|=r_1$, we have

$$P^y\{\tau_{\Gamma_3}<\tau_{A_1}\}\leqq\tfrac{1}{2},\tag{1.15}$$

where $\Gamma_3=S(0,r_1/M)$. Then, for $|y|=r_1$,

$$\begin{aligned}P^y\{\tau_{\Gamma_2}<\tau_{A_1}\}&\leqq P^y\{\tau_{\Gamma_3}<\tau_{\Gamma_2}<\tau_{A_1}\}+P^y\{\tau_{\Gamma_2}<\tau_{\Gamma_3}\}\\&\leqq\tfrac{1}{2}q(r_1,r_2)+P^y\{\tau_{\Gamma_2}<\tau_{\Gamma_3}\},\text{ by (1.15)}.\end{aligned}$$

The right hand side does not depend on y with $|y|=r_1$, so we can take the supremum over y to give

$$q(r_1,r_2)\leqq\tfrac{1}{2}q(r_1,r_2)+P^y\{\tau_{\Gamma_2}<\tau_{\Gamma_3}\}.$$

Thus

$$q(r_1,r_2)\leqq\frac{2\log M}{\log r_2/r_1}\leqq 2\log M\frac{\log\dfrac{\varrho}{r_1}}{\log\dfrac{r_2}{r_1}},$$

since $\varrho\geqq er_1$. Since M does not depend on r_1, r_2 (1.14) now completes the proof with $c_{1.7}=(1+2\log M)$. □

In [16], Taylor divided transient ($\alpha<d$) strictly stable processes into two classes:

Type A, if the transition density $p(t,x)$ is positive at $x=0$;
type B, if $p(t,0)=0$.

Type B processes can only occur for $0<\alpha<1$ and are such that their projection on some line through the origin is a subordinator. Simpler arguments work for type B processes because the first passage time of a subordinator out of $S(0,r)$ is the same as the total sojourn time, if the process starts at 0. Difficulties arise with type A processes because the potential kernel need not be of the same order as that for the symmetric case. In studying sample path properties we overcome this problem by considering delayed hitting probabilities,

$$Q(x,a,T)=P^x\{X(t)\in S(0,a) \text{ for some } t\geqq T\}.$$

In Pruitt and Taylor [13, Theorem 4] it is proved that

$$Q(x,a,T)\leqq c_{1.8}(aT^{-1/\alpha})^{d-\alpha} \tag{1.16}$$

for any strictly stable X with index $\alpha<d$. No lower bound of the form (1.13) is valid in the general nonsymmetric case, if $d\geqq 2$.

It was proved by Erdös and Taylor [4] that planar Brownian motion satisfies a strong law in respect of the number q_k of dyadic squares with side 2^{-k} entered by $B(t)$ for $0\leqq t\leqq M$. There is a constant $c_{1.9}$ such that

$$\text{a.s. } q_k k 2^{-2k}\to Mc_{1.9}. \tag{1.17}$$

For $d\geqq 3$, the corresponding strong law is

$$q_k 2^{-2k}\to Mc_{1.10}. \tag{1.18}$$

We believe the corresponding strong laws hold for any strictly stable $X(t)$ of index $\alpha\leqq d$, but a rigorous proof will be tedious. It will be sufficient to use the fact that given $\varepsilon>0$,

$$\text{a.s. } q_k 2^{-\alpha k-\varepsilon}<1<q_k 2^{-\alpha k+\varepsilon} \quad \text{for} \quad k\geqq k_0. \tag{1.19}$$

The upper bound in (1.19) follows from a simple first moment argument, while the lower bound comes from considering a sublattice of larger dyadic cubes and an inclusion/exclusion estimate.

If $\varrho>0$, $\psi(\varrho)$ will denote the collection of intervals I of the form $[i\varrho,(i+1)\varrho]$.

2. Uniform Bounds for the Sojourn Time

For a stable process of index α in $\mathbb{R}^d(\alpha<d)$ which hits a ball $S(x,r)=J$, we expect the sojourn time to be of the order r^α. If $\alpha=1=d$ or $\alpha=2=d$, the expected sojourn time is of order $r^\alpha\log 1/r$. Our main objective in this section is to obtain information about the a.s. asymptotic behaviour of $\sup\{|X^{-1}(S(x,r))|:x\in\mathbb{R}^d\}$ as $r\downarrow 0$. We have two methods of doing this, both using a decomposition of the sojourn time. If we use a decomposition given by successive exits from a larger ball, the proof can be completed for all the symmetric processes while using blocks of sojourn time of a fixed length works for all the ($\alpha\leqq d$) strictly stable processes of type A. For planar Brownian motion the first method gives more precise information, but we will use the second method as it works for all cases.

We start with a preliminary lemma.

Lemma 2.1. *For fixed $r \in (0,1)$, let Γ_1 be $\{x : |x| < r\}$, $\Gamma_2 = \{x : |x| \geqq r^\beta\}$, $\beta \in (-1,1)$ and suppose $c_{2.1} > 0$, $A = \{X(t)$ hits Γ_1 before Γ_2 for values of $t \geqq c_{2.1} r^\alpha\}$. Then if $X(t)$ is either planar Brownian motion ($\alpha = 2$) or a symmetric Cauchy process ($\alpha = 1$) on $\mathbb{R}$,*

$$1 - \frac{c_{2.2}}{1-\beta}\left(\log\frac{1}{r}\right)^{-1} \geqq P^x(A) \geqq 1 - \frac{c_{2.3}}{1-\beta}\left(\log\frac{1}{r}\right)^{-1}$$

uniformly for $x \in \Gamma_1$.

Proof. We do the case where $X(t)$ is Cauchy: a simpler version of the same argument works for planar Brownian motion. For $t = c_{2.1} r$, $x \in \Gamma_1$,

$$E_s = \{(e^{s-1} - 1) r < |X(t) - X(0)| \leqq (e^s - 1) r\}$$

satisfies $P^x(E_s) \leqq ce^{-s}$ for $s = 1, 2, \ldots$ and $\omega \in E_s \Rightarrow |X(t)| \leqq e^s r$ so that, by Lemma 1.3, for $s \geqq 3$,

$$P^{X(t)}\{\tau_{\Gamma_2} < \tau_{\Gamma_1}\} \leqq c_{1.7} \frac{s}{(1-\beta)}\left(\log\frac{1}{r}\right)^{-1},$$

and for $s = 1, 2$ a similar inequality holds with $c_{1.7}$ replaced by $3c_{1.7}$. Hence,

$$\begin{aligned} P^x(A^c) &\leqq \sum_{s=1}^{\infty} P(E_s) P^{X(t)}\{\tau_{\Gamma_2} < \tau_{\Gamma_1} | E_s\} \\ &\leqq \sum_{s=1}^{\infty} ce^{-s} \frac{s}{1-\beta}\left(\log\frac{1}{r}\right)^{-1} \\ &= \frac{c_{2.2}}{1-\beta}\left(\log\frac{1}{r}\right)^{-1}. \end{aligned}$$

Now, for $F = \{|X(t) - X(0)| < (e+1) r\}$ we have $P(F^c) = c_{2.3} < 1$. Now, $\omega \in F^c \Rightarrow |X(t)| > er$ so that, for all $x \in \Gamma_1$,

$$\begin{aligned} P^x(A) &\leqq P(F) + P(F^c) P^{er}\{\tau_{\Gamma_1} < \tau_{\Gamma_2}\} \\ &\leqq 1 - \frac{c_{2.3}}{1-\beta}\left(\log\frac{1}{r}\right)^{-1}, \text{ by (1.12).} \quad \square \end{aligned}$$

Given any strictly stable process with $|X(0)| \leqq r$ and $c > 0$ a fixed constant, we set up a sequence of stopping times $\tau_0 = 0$

$$\tau_i = \inf\{t \geqq \tau_{i-1} + cr^\alpha : |X(t)| < r\}, \qquad i = 1, 2, \ldots.$$

This sequence a.s. terminates whenever $\alpha < d$. For fixed $\beta \in (-1,1)$ we can also define

$$\sigma(\beta) = \inf\{t > 0 : |X(t)| \geqq r^\beta\},$$

and finally denote by $M_\beta(\omega)$ the number of the $\tau_i < \sigma(\beta)$: thus the random integer M_β is given by

$$M_\beta = 1 + \max\{i : \tau_i < \sigma(\beta)\}.$$

We now want to estimate the probability of M_β being large.

Lemma 2.2. *Suppose $X(t)$ is strictly stable of type A and index $\alpha \leqq d$ in $\mathbb{R}^d$. Fix $c>0$, $\gamma>0$, $\beta \in (-1,1)$ and define M_β as above in terms of successive hitting times of $S(0,r)$ after a delay of cr^α. Then*

(a) *If X is planar Brownian motion or a linear symmetric Cauchy process and*

$$P\left\{M_\beta(\omega) \geqq \gamma\left(\log\frac{1}{r}\right)^2\right\} = r^{\delta(r)},$$

then

$$\frac{\gamma}{1-\beta} c_{2.4} \leqq \delta(r) \leqq \frac{\gamma}{1-\beta} c_{2.5} \quad \text{for} \quad 0<r\leqq r_0.$$

(b) *If $\alpha<d$ and c is large enough,*

$$P\left\{M_\beta(\omega) \geqq \gamma\left(\log\frac{1}{r}\right)\right\} = r^{\delta(r)},$$

then

$$\gamma c_{2.6} \leqq \delta(r) \leqq \gamma c_{2.7} \quad \text{for} \quad 0<r\leqq r_0.$$

Proof. (a) We decompose the path using the stopping times τ_i. Since $|X(\tau_i)| \leqq r$ we can use Lemma 2.1 to give, for $i=1,2,\ldots$

$$1-\frac{c_{2.2}}{(1-\beta)}\left(\log\frac{1}{r}\right)^{-1} \geqq P\{\tau_i<\sigma(\beta)|\tau_{i-1}<\sigma(\beta)\} \geqq 1-\frac{c_{2.3}}{1-\beta}\left(\log\frac{1}{r}\right)^{-1}.$$

Hence, by iteration,

$$\left[1-\frac{c_{2.2}}{1-\beta}\left(\log\frac{1}{r}\right)^{-1}\right]^{\gamma\left(\log\frac{1}{r}\right)^2} \geqq P\left\{M_\beta \geqq \gamma\left(\log\frac{1}{r}\right)^2\right\}$$
$$\geqq \left[1-\frac{c_{2.3}}{1-\beta}\left(\log\frac{1}{r}\right)^{-1}\right]^{\left[\gamma\left(\log\frac{1}{r}\right)^2\right]}.$$

If we replace both extremes by exponentials and let $r\downarrow 0$, we obtain the required estimate.

(b) Since $\alpha<d$ and the process is of type A, the potential kernel $k(x,y)$ satisfies

$$k(x,y) \geqq \frac{c_{2.8}}{|x-y|} d-\alpha \qquad \text{(see [13])}$$

for all $x, y \in \mathbb{R}^d$. Hence, if $|x| \leqq 2r$

$$P^x\{|X(t)| \leqq r \text{ for some } t>0\} \geqq c_{2.9}>0.$$

Take $\beta_1 = \frac{1}{2}(\beta+1)$, then $\beta<\beta_1<1$. By taking r small, we can arrange that

$$P\{\sigma(\beta) \leqq r^{\alpha\beta_1}\} = P\left\{\sup_{0<s\leqq r^{\alpha\beta_1}} |X(s)|>r^\beta\right\}$$
$$< c_{2.10} r^{\alpha(\beta_1-\beta)}$$
$$< \tfrac{1}{4} c_{2.9}.$$

Now

$$P^x\{X(t) \text{ hits } S(0,r) \quad \text{for} \quad t \geqq r^{\alpha\beta_1}\}$$
$$< c_{1.8}(r^{1-\beta_1})^{d-\alpha} \text{ by (1.16)}$$
$$< \tfrac{1}{4}c_{2.9}$$

when r is small. Hence, for all x satisfying $|x| \leqq 2r$,

$$P^x\{X(t) \text{ hits } S(0,r) \text{ before } \sigma(\beta)\} > \tfrac{1}{2}c_{2.9} = c_{2.11} > 0.$$

Now choose c big enough to make the estimate

$$c_{1.8}(aT^{-1/\alpha})^{d-\alpha} \leqq \tfrac{1}{2}$$

whenever $a = r$, $T = cr^\alpha$. By (1.16) we now have

$$P\{\tau_i < +\infty | \tau_{i-1} < +\infty\} \leqq \tfrac{1}{2},$$

and iteration yields

$$P\left\{M_\beta \geqq \gamma \log \frac{1}{r}\right\} \leqq \tfrac{1}{2}^{\left[\gamma \log \frac{1}{r}\right]} \leqq r^{\gamma c_{2.6}} \text{ as required.}$$

On the other hand

$$P\{\tau_i < \sigma(\beta) | \tau_{i-1} < \sigma(\beta)\} \geqq P\{|X(cr^\alpha) - X(0)| < r\} c_{2.11}$$
$$= P\{|X(c)| < 1\} c_{2.11} = c_{2.12}$$

so, by iteration, we get

$$P\left\{M_\beta \geqq \gamma \log \frac{1}{r}\right\} \geqq c_{2.12}^{\left[\gamma \log \frac{1}{r}\right]} \geqq r^{\gamma c_{2.7}}$$

whenever $0 < r \leqq r_0$. □

We now come to the basic lemma which estimates the maximum concentration of the sample path in a ball. There are three distinct cases to consider.

Lemma 2.3. *Throughout $X(t)$ is a strictly stable process of index α in $\mathbb{R}^d$ and $\alpha \leqq d$. For fixed M, let*

$$E_{z,r} = \{t \in [0, M] : |X(t) - z| \leqq r\}.$$

For $k \in N$, let $\lambda = [\alpha k]$ and let $N_k(z)$ be the number of dyadic intervals $[j2^{-\lambda}, (j+1)2^{-\lambda}]$ which intersect $E_{z,r}$ for $r = 2^{-k}$. Then

(a) *in the critical cases, $\alpha = 2 = d$ of planar Brownian motion and $\alpha = 1 = d$ of linear symmetric Cauchy process,*

(i) a.s. $\exists k_0 = k_0(\omega)$ *such that* $k \geqq k_0 \Rightarrow \forall z$, $N_k(z) \leqq c_{2.13}k^2$;

(ii) $\forall k \geqq k_0(\omega)$, $\exists z_k$ *such that* $N_k(z_k) \geqq c_{2.14}k^2$;

(iii) a.s. $\exists r_0 = r_0(\omega) > 0$ *such that* $0 < r \leqq r_0 \Rightarrow \forall z$,
$$|E_{z,r}| \leqq c_{2.15} r^\alpha \left(\log \frac{1}{r}\right)^2;$$

(iv) $\forall r\in(0,r_0)\,\exists z_r$ *such that* $|E_{z_r,r}|\geqq c_{2.16}r^\alpha\left(\log\frac{1}{r}\right)^2$.

(b) *For* $\alpha<d$, $X(t)$ *of type A.*

(i) a.s. $\exists k_0$ *such that* $k\geqq k_0 \Rightarrow \forall z,\ N_k(z)\leqq c_{2.17}k$;

(ii) a.s. $\exists r_0$ *such that* $0<r\leqq r_0 \Rightarrow |E_{z,r}|\leqq c_{2.18}r^\alpha\log\frac{1}{r}$;

(iii) $\forall r\in(0,r_0)$, $\exists z=z(r,\omega)$ *such that* $E_{z,r}$ *contains an interval* J *with* $|J|\geqq c_{2.19}r^\alpha\log\frac{1}{r}$.

(c) *For* $0<\alpha<1$, $X(t)$ *of type B and such that its projection on the first axis is a subordinator,*

(i) a.s. $\exists k_0$ *such that* $k\geqq k_0 \Rightarrow \forall z,\ N_k(z)\leqq c_{2.20}k^{1-\alpha}$;

(ii) a.s. $\exists r_0>0$ *such that* $0<r\leqq r_0 \Rightarrow \forall z,\ |E_{z,r}|\leqq c_{2.21}r^\alpha\left(\log\frac{1}{r}\right)^{1-\alpha}$;

(iii) $\forall r\in(0,r_0)$, $\exists z=z(r,\omega)$ *such that* $E_{z,r}$ *contains an interval* J *with* $|J|\geqq c_{2.22}r^\alpha\left(\log\frac{1}{r}\right)^{1-\alpha}$.

Proof. (a) We give details for planar Brownian motion, but write the proof so that only trivial and obvious changes are needed to deal with the symmetric Cauchy case on $\mathbb{R}$. Start by considering $E_{z,r}$ for the points $z_{i,k}=(i_1 2^{-k-1}, i_2 2^{-k-1})$, $i_1, i_2\in\mathbb{Z}$ and $r=2^{-k}$. The disk

$$D_{i,k}=\{z\in\mathbb{R}^2: |z-z_{i,k}|\leqq 2^{-k}\}$$

may be hit by $B(t)$, $0\leqq t\leqq M$. Let Q_k be the number of such $D_{i,k}$ which are hit. If $0<\varepsilon<\frac{1}{8}$, we can find $k_1=k_1(\omega)$ such that, for $k\geqq k_1$,

$$2^{2k(1-\varepsilon)}<Q_k<2^{2k(1+\varepsilon)}. \tag{2.1}$$

Now fix attention on one disk $D_{i,k}$. Let $\tau_0=\inf\{t>0: B(t)\in D_{i,k}\}$ and form the sequence of stopping times used in the proof of Lemma 2.2(a), with $c=1$. If $E_{i,k}$ is the corresponding set of sojourn times in this disc, we can write $E_{i,k}\subset\bigcup_{j=0}^{\infty}F_{j,i,k}$, where

$$F_{j,i,k}\subset[\tau_j, \tau_j+2^{-2k}].$$

We can clearly find $k_2=k_2(\omega)$ such that all of $B[0,M]$ is within $2^{\frac{1}{2}k-1}$ of the origin, for $k\geqq k_2$. For $k\geqq k_2$, all of $B[0,M]$ will be included in $L_1=M_{-1/2}(\omega)$ pieces $F_{j,i,k}$. Hence

$$|E_{i,k}|\leqq\sum_{j=0}^{L_1}Y_j, \tag{2.2}$$

where $Y_j=\int_{\tau_j}^{\tau_j+2^{-2k}}1_D(B(t))\,dt$ are the successive sojourn times in D. The random variables Y_j have a distribution depending on the starting point $B(\tau_j)$, which will sometimes be inside the disc. However, by scaling, the random variables $2^{2k}Y_j=V_j$

are uniformly bounded above by 1 and below by

$$V = \int_0^1 1_D(B(t))\,dt,$$

where D is the unit disc $\{|z| \leqq 1\}$ and $B(t)$ is a planar Brownian motion starting at 1.

On the other hand, provided $\tau_0(D) < \frac{1}{2}M$, there is a $k_3 = k_3(\omega)$ such that, for $k \geqq k_3$

$$|E_{i,k}| \geqq \sum_{j=0}^{L_2} Y_j, \tag{2.3}$$

where $L_2 = M_{1/2}(\omega)$ is defined in Lemma 2.2(a). The bound

$$N_k(z_{i,k}) \geqq 2^{2k} |E_{i,k}| \tag{2.4}$$

is immediate. Since each interval $(\tau_j, \tau_j + 2^{-2k})$ intersects at least one, and at most two, dyadics of length 2^{-2k}, we also have, for $k \geqq \max(k_2, k_3)$,

$$L_2 \leqq N_k(z_{i,k}) \leqq 2L_1. \tag{2.5}$$

By Lemma 2.2, for one disk $D_{i,k}$

$$P\{L_1 \geqq \gamma(\log 2^k)^2\} \leqq 2^{-k\gamma\frac{2}{3}c_{2.4}}$$

so that the probability that at least one of the $Q_k D_{i,k}$ hit by $B[0, M]$ satisfies $\left\{M_{-1/2} \geqq \gamma\left(\log\frac{1}{r}\right)^2\right\}$ is bounded by $2^{2k(1+\varepsilon)} \cdot 2^{-k\gamma\frac{2}{3}c_{2.4}}$. A suitable choice of γ makes this the general term of a summable series. By Borel Cantelli, there is a $k_4 = k_4(\omega)$ such that, for $k \geqq k_4$ every one of the $D_{i,k}$ hit by $B(t)$ satisfies

$$L_1 \leqq \gamma(\log 2^k)^2.$$

With (2.5) this establishes (a) (i) and (2.4) now gives (a) (iii) for these special discs. But any disk of radius r satisfying $2^{-k-2} < r \leqq 2^{-k-1}$ is contained in at least one of the disks $D_{i,k}$, so we have proved a(i) and a(iii) for all disks in $\mathbb{R}^2$ which are sufficiently small.

To show that our bounds are of the right order of magnitude, we use a large deviation argument to give, for large enough K,

$$P\left\{\sum_{j=0}^{K-1} V_j < \tfrac{1}{2}KE(V)\right\} < e^{-c_{2.23}K}. \tag{2.6}$$

In order to avoid independence problems, we use only a small number of the disks $D_{i,k}$, and use only that part of the path which is traversed before the next disc is hit. In fact, we only consider the L_2 returns before the first exit from a concentric disc of radius $2^{-\frac{1}{2}k}$. By (2.1) this gives us more than $2^{k(1-\varepsilon)}$ fresh starts in $[0, M]$ and the strong Markov property ensures that each of these pieces are independent.

In Lemma 2.2(a) we now choose γ small enough to ensure

$$P\left\{M_{1/2}(\omega) \geqq \gamma\left(\log\frac{1}{r}\right)^2\right\} \geqq r^{1/2}$$

so that the event $A_0=\left\{\omega: M_{1/2}(\omega)<\gamma\left(\log\frac{1}{r}\right)^2\right\}$ satisfies

$$P(A_0)\leqq 1-r^{1/2}. \tag{2.7}$$

Now set up another sequence of stopping times $\xi_0=0$,

$$\xi_{j+1}=\inf\{t>\xi_j: |B(t)-B(\xi_j)|\geqq 2^{-\frac{1}{2}k}\}.$$

If A_i is the event corresponding to A_0 for the piece of path starting at $t=\xi_i$, using the disk $D_{i,k}$ whose centre is nearest to $B(\xi_i)$, then the events A_i are independent and (2.7) implies

$$P\left(\bigcap_{i=0}^{s} A_i\right)\leqq(1-r^{1/2})^s.$$

When k is large enough we can take $s=2^{k(1-\varepsilon)}$ and deduce that the event that $\{M_{1/2}(\omega)<\gamma(\log 2^k)^2\}$ happens for every one of the discs $D_{i,k}$ whose centre is nearest to $B(\xi_j)$ has probability bounded by $[1-2^{-\frac{1}{2}k}]^{2^{k(1-\varepsilon)}}<e^{-2^{\frac{1}{4}k}}$ by taking $\varepsilon<\frac{1}{4}$. Since this is the general term of a summable series, at least one of this particular set of discs of radius 2^{-k}, each $k\geqq k_5(\omega)$ must satisfy

$$L_2=M_{1/2}(\omega)\geqq\gamma\left(\log\frac{1}{r}\right)^2.$$

An application of (2.5) now completes the proof of (a) (ii).

Now assume k is large enough to ensure that (2.6) is valid with $K=\gamma(\log 2^k)^2=c_{2.24}k^2$.

Using (2.1) again, the probability that at least one of the discs $D_{i,k}$ hit by $B[0,M]$ satisfies

$$\left\{\sum_{j=0}^{K-1} V_j<\tfrac{1}{2}KE(V)\right\}$$

is now bounded by $2^{2k(1+\varepsilon)}e^{-c_{2.25}k^2}$. By Borel-Cantelli, this will never happen for $k\geqq k_6(\omega)$. By (2.3), the disc which we just found to satisfy $N_k(z_k)\geqq c_{2.14}k^2$, must when $k\geqq\max(k_5,k_6)$ have $|E_{z_k,2^{-k}}|\geqq\frac{1}{2}KE(V)2^{-2k}\geqq c_{2.26}2^{-2k}(\log 2^k)^2$. This is still valid for $2^{-k}\leqq r<2^{-k+1}$, so we have established (a) (iv).

(b) The uniform upper bounds (i) and (iii) come from the same arguments as used in (a) using a value of c large enough to make Lemma 2.2(b) valid with the stopping times $\tau_{j+1}=\inf\{t>\tau_j+cr^\alpha: X(t)\in S(z,r)\}$. However, the arguments used to establish (a) (ii) and (iv) will not work now, since the bound given by the large deviation estimate is too large when multiplied by the number of balls hit. Results corresponding to (a) (ii) and (iv) are true since they follow from the stronger result (b) (iii) which we now establish. Thus (b) (iii) shows us that in this transient case, the largest sojourn time in a ball is not more than a constant multiple of the largest first exit time. This is not the case for the critical processes discussed in (a).

Divide $[0,M]$ into Mn pieces of the form $I_j=[jn^{-1},(j+1)n^{-1}]$, and put

$$R_j(n)=\sup\{|X(t)-X(jn^{-1})|: t\in I_j\}.$$

Then the $R_j(n)$ are independent and, by scaling,

$$P\{R_j(n) < \lambda n^{-1/\alpha}\} \geqq \exp(-c_{2.27}\lambda^{-\alpha})$$

using the corollary to Lemma 5 of Taylor [16]. Hence,

$$P\left[\bigcap_{j=1}^{Mn} \{R_j(n) \geqq \lambda n^{-1/\alpha}\}\right] \leqq [1-\exp(-c_{2.27}\lambda^{-\alpha})]^{Mn}.$$

Now take $n_k = c2^{k\alpha}k^{-1}$, $\lambda = 2^{-k}n_k^{+1/\alpha} = c^{1/\alpha}k^{-1/\alpha}$, and we have obtained an upper bound estimate for the probability of the event B_k that there is no I_j such that $X(I_j)$ is contained in a ball of radius 2^{-k}, this is

$$P(B_k) \leqq [1-\exp(-c \cdot c_{2.27}k)]^{Mc2^{k\alpha}k^{-1}},$$

which is the term of a summable series provided $\alpha \log 2 > c \cdot c_{2.27}$. This will be true for a suitable choice of $c = c_{2.28} > 0$. By Borel-Cantelli we can now find $k_0(\omega)$ such that, for $k \leqq k_0$, there is always a ball $S(z, 2^{-k})$ such that $X^{-1}(S)$ contains an interval of size $\varrho_k = c_{2.28}{}^{-1}k2^{-k\alpha}$. Hence, for $2^{-k+1} \geqq r \geqq 2^{-k}$, $k \geqq k_0$, there will be a ball of radius r such that $X^{-1}(\dot{S})$ contains an interval J of length at least $c_{2.19}r^\alpha\left(\log\frac{1}{r}\right)$.

(c) This case is quite different. Let $Y(t)$ be the projection of $X(t)$ on the first axis. Now $Y(t)$ is a stable subordinator of index α, so the uniform lower growth rate established in Hawkes [5] gives, for $t \in [0, M]$,

$$Y(t+h) - Y(t) \geqq c_{2.29}h^{1/\alpha}\left(\log\frac{1}{h}\right)^{\frac{-1+\alpha}{\alpha}} \tag{2.8}$$

for all $h \in (0, h_0)$. Clearly (2.8) implies

$$|X(t+h) - X(t)| \geqq c_{2.29}h^{1/\alpha}\left(\log\frac{1}{h}\right)^{\frac{-1+\alpha}{\alpha}}$$

so that $X^{-1}(S(z,r))$ has to be contained in an interval of length $c_{2.30}r^\alpha\left(\log\frac{1}{r}\right)^{1-\alpha}$ for $r < r_0(\omega)$. We now easily get (c) (i) and (ii).

If the Lévy measure generating the distribution of $X(1)$ is supported within a cone of semi-angle $\theta < \pi/2$ and axis the first axis, we can now deduce c(iii) from the Hawkes result showing that (2.8) is best possible. However, the support can be a complete hemi-sphere, and in this case we have to adopt the same method used above for (b) (iii) together with the estimate in Lemma 6 of [16] that

$$P\{S(a) \geqq \lambda a^\alpha\} \geqq \exp(-c_{2.31}\lambda^{1/(1-\alpha)}),$$

where $S(a)$ denotes the first exit by $X(t)$ from a ball of radius a. □

Remark. Throughout the above arguments we have not tried to find the best constants. In our first proof for planar Brownian motion, we decomposed the sojourn time in $S(z,r)$ in terms of successive exits from $S(z, er)$ and entries to $S(z,r)$.

If one takes care the method of proof we used in (a) yields that a.s.

$$\tfrac{1}{4}c \leqq \liminf_{r\downarrow 0}\left\{\sup_{z\in\mathbb{R}^2}\frac{|E_{z,r}|}{\left(r\log\frac{1}{r}\right)^2}\right\} \leqq \limsup_{r\downarrow 0}\left\{\sup_{z\in\mathbb{R}^2}\frac{|E_{z,r}|}{\left(r\log\frac{1}{r}\right)^2}\right\} \leqq c,$$

where $c=2E(U)$ and U is the time spent in $\{|z|\leqq 1\}$ before exiting from $\{|z|=e\}$ by a planar Brownian motion starting at 1. We believe the following

Conjecture 2.4. *If $B(t)$ is a planar Brownian motion and $E_{z,r}$ is as defined in Lemma 2.3,*

$$\text{a.s. } \lim_{r\downarrow 0}\left\{\sup_{z\in\mathbb{R}^2}\frac{|E_{z,r}|}{\left(r\log\frac{1}{r}\right)^2}\right\}=2E(U).$$

We note that the corresponding results for the transient case were announced in [17].

The uniform upper bound estimates for Brownian motion will follow easily from Lévy's modulus of continuity. For $0<\alpha<2$ paths are discontinuous so we replace continuity by a uniform covering principle. The idea is due to Hawkes and Pruitt [6], but we obtain a sharper version of their Lemma 5. This time the same result is valid for all strictly stable processes of index α in $\mathbb{R}^d$.

Lemma 2.5. *Suppose $X(t)$ is strictly stable of index α. Then for a suitable $c_{2.32}$, a.s. there is a $\delta_0(\omega)>0$ such that, if the interval $J\subset[0,M]$ and $|J|=\eta\leqq\delta_0$, then $X(J)$ can be covered by $c_{2.32}\log(1/\eta)$ balls of diameter $\eta^{1/\alpha}(\log 1/\eta)^{-1/\alpha}$.*

Proof. It is clearly sufficient to prove the result for every semi-dyadic $J=[u,v]$ where $k\in\mathbb{N}$, $v-u=2^{-k}$, $u=i2^{-k}$ or $(i+\frac{1}{2})2^{-k}$, $0\leqq i<M2^k$. The number of such J is M^{2k+1}. For a fixed $c>0$, we call J bad if $X(J)$ cannot be covered by ck balls of diameter $k^{-1/\alpha}2^{k/\alpha}$. Set up the sequence of stopping times

$$\tau_0(J)=u,\ \tau_{i+1}(J)=\inf\{t>\tau_i: |X(t)-X(\tau_i)|>k^{-1/\alpha}2^{k/\alpha}\}.$$

Then, if J is bad, $\sum_{i=1}^{ck-1}(\tau_{i+1}-\tau_i)<2^{-k}$. Putting

$$T(s)=\inf\{t>0: |X(t)|>s\},$$

writing $m_k=ck-1$ and using the scaling property, gives

$$\begin{aligned}
P(J \text{ is bad}) &\leqq P\left\{\sum_{i=1}^{m_k} T_i(k^{-1/\alpha})<1\right\}\\
&= P\{e^{-\lambda\Sigma T_i(k^{-1/\alpha})}>e^{-\lambda}\}, \quad \text{for } \lambda>0\\
&\leqq e^{\lambda}[E(e^{-\lambda T(k^{-1/\alpha})})]^{m_k}
\end{aligned}$$

since $T_i(k^{-1/\alpha})$ are independent, identically distributed,

$$=e^{\lambda}[E(e^{-\lambda T(1)k^{-1}})]^{m_k}.$$

Now choose $\lambda = c_{2.33}k$ where $c_{2.33}$ is such that $E(e^{-c_{2.33}T(1)}) < \frac{1}{2e}$ and choose the initial $c = (1 + c_{2.33})$ to give

$$P(J \text{ is bad}) \leqq e^{c_{2.33}k} \left(\frac{1}{2e}\right)^{c_{2.33}k+k-1}$$
$$< 2e \cdot 2^{-k} e^{-k}.$$

Hence, $P\{\text{at least one } J \text{ is bad}\} < 4eMe^{-k}$. An application of Borel-Cantelli now tells us that a.s. $\exists k_0 = k_0(\omega)$ such that, for $k \geqq k_0$, every semi-dyadic interval of length 2^{-k} is good. □

Corollary 2.6. *If $X(t)$ is strictly stable of index α, there is a constant $c_{2.32}$ such that a.s. there is a $\delta_0(\omega) > 0$ with the property that no $J \subset [0, M]$ with $|J| = \lambda \leqq \delta_0$ has an image $X(J)$ containing more than $c_{2.32} \log \frac{1}{\lambda}$ points which are mutually separated by at least $\left(\lambda / \log \frac{1}{\lambda}\right)^{1/\alpha}$.*

We remark that, because the sample paths are continuous for Brownian motion $B(t)$, it would be more efficient to cover $B(J)$ with a single ball of size $c(\eta \log 1/\eta)^{1/2}$, so Lemma 2.5 merely recovers Lévy's uniform modulus when $\alpha = 2$.

3. Lower Bounds for Hausdorff Measure

We start with the easiest case – when $0 < \alpha < 1$ and $X(t)$ is strictly stable of index α and type B. Choose an axis on which the projection $Y(t)$ of $X(t)$ is a subordinator of index α, and use the fact that any covering of a subset of $X[0, M]$ projects onto a covering of the corresponding subset of $Y[0, M]$.

Theorem 3.1. *Suppose $X(t)$ is strictly stable of index $\alpha < 1$ and type B. There is a constant $c_{3.1}$ such that a.s.*

$$\textit{for all Borel } E \subset (0, \infty), \textit{ for all } \Psi \in \Phi, \textit{ if}$$
$$\varphi(s) = \Psi(c_{3.1} s^\alpha (\log^+ 1/s)^{1-\alpha}), \textit{ then } \varphi - m(X(E)) \geqq \Psi - m(E).$$

Proof. For fixed $M > 0$, if $\eta(h) = h^{1/\alpha} (\log 1/h)^{1-\frac{1}{\alpha}}$, Theorem 1 of Hawkes [5] tells us that a.s. there is an $\varepsilon_0 = \varepsilon_0(\omega) > 0$ and $c_{3.2}$ such that

$$Y(t+h) - Y(t) \geqq c_{3.2} \eta(h) \quad \text{for} \quad h \in (0, \varepsilon_0], \quad t \in [0, M]. \tag{3.1}$$

Take $c_{3.1} = 2 c_{3.2}^{-\alpha} \alpha^{1-\alpha}$, so that for small enough y

$$\eta^{-1}(y / c_{3.2}) \leqq 2 c_{3.2}^{-\alpha} \alpha^{1-\alpha} y^\alpha (\log 1/y)^{1-\alpha}. \tag{3.2}$$

Start with a fixed ω for which $\varepsilon_0(\omega) > 0$ and measure functions φ, Ψ related as in the hypotheses. Choose a covering $\{J_i\}$ of $Y(E)$ with $|J_i| < c_{3.2} \eta(\varepsilon_0)$ and

$$\Sigma \varphi(|J_i|) < \varphi - m(Y(E)) + \delta$$

for a fixed $\delta>0$. We may assume that $J_i=Y(I_i)$ for a collection of open intervals $\{I_i\}$ covering E, and of length less than ε_0, by (3.1). We may also assume that all intervals are short enough to make (3.2) valid. Hence, by (3.1) and (3.2) with $y=Y(t+h)-Y(t)$,

$$\Sigma\Psi(|I_i|)\leqq\Sigma\Psi\circ g(|J_i|)=\Sigma\varphi(|J_i|)<\varphi-m(Y(E))+\delta,$$

where $g(s)=c_{3.1}s^{\alpha}(\log 1/s)^{1-\alpha}$. Since δ is arbitrary, we have shown that $\varphi-m(Y(E))\geqq\Psi-m(E)$ for bounded Borel E. This extends trivially to all E, and the theorem follows from

$$\varphi-m(X(E))\geqq\varphi-m(Y(E)).\quad\square$$

Lemma 1.1 will allow us to show Theorem 3.1 is best possible up to a constant factor (Corollary 3.3 below). We first obtain a general result which is also needed for an example in the next section.

Lemma 3.2. *Let $X(t)$ be an $\mathbb{R}^d$-valued process with independent increments and cadlag paths. Fix $\eta\in\Phi$ which is strictly monotone and continuous with inverse function η^{-1}. Assume that for each $\delta>0$, there is a $K>0$ and $r_0>0$ such that*

$$P\left\{\sup_{0\leqq h\leqq r}|X(t+h)-X(t)|\leqq K\eta^{-1}(r)\right\}\geqq r^{\delta},\tag{3.3}$$

for $0<r\leqq r_0$, $t\in[0,M]$. If $\beta\in(0,1)$ and $\varphi_\beta(s)=(\eta(s))^\beta$, there is a random closed set $E\subset[0,1]$ and a constant $c_{3.3}$, depending on η,β and the law of $X(1)$, such that

$$\tfrac{1}{4}\leqq x^{\beta}-m(K)\leqq 1\quad\text{and}\quad\varphi_\beta-m(X(K))\leqq c_{3.3}.$$

Proof. Given β, fix $\delta>0$ such that $\beta+\delta<1$ and choose K, r_0 such that (3.3) holds. Choose inductively a decreasing sequence r_m such that $r_1<r_0$, $r_m^{-\beta}\in\mathbb{N}$, and $\varrho_m=\prod_{j=1}^{m}r_j$, then

$$\sum_{m=1}^{\infty}\varrho_m^{-1}r_{m+1}^{-\beta}\exp(-\varrho_m^{\delta}r_{m+1}^{\delta+\beta-1}/4)<\infty.\tag{3.4}$$

Let $\mathscr{I}_m=\mathscr{I}(\varrho_m)$ (see the end of Sect. 1). For any $I\in\mathscr{I}_m$ we can partition I into $r_{m+1}^{-\beta}$ subintervals J_i each of length $\frac{1}{2}r_{m+1}^{\beta}\varrho_m$ and separated by intervals of the same length. Within each of these J_i there are at least $[\frac{1}{2}r_{m+1}^{\beta-1}]-2\geqq\frac{1}{4}r_{m+1}^{\beta-1}$ subintervals of $\mathscr{I}_{m+1}$ (make r_0 smaller if needed to ensure this). Therefore, using (3.3) and independent increments,

$$\begin{aligned}P\Big\{&\sup_{s,t\in J}|X(s)-X(t)|>2K\eta^{-1}(\varrho_{m+1})\quad\text{for all}\quad J\subset J_i,\,J\in\mathscr{I}_{m+1}\\&\text{and at least one } J_i\subset I\in\mathscr{I}_m\Big\}\\&\leqq\varrho_m^{-1}r_{m+1}^{-\beta}(1-\varrho_{m+1}^{\delta})^{\frac{1}{4}r_{m+1}^{\beta-1}}\\&\leqq\varrho_m^{-1}r_{m+1}^{-\beta}\exp(-\varrho_{m+1}^{\delta}r_{m+1}^{\beta-1}/4).\end{aligned}$$

Since (3.4) makes this the term of a convergent series, Borel-Cantelli allows us a.s. to find $m_0=m_0(\omega)<\infty$ such that, if $m\geqq m_0$, $I\in\mathscr{I}_m$ and J_i is any one of the subintervals of the construction, there is a $J \subset J_i$, $J \in \mathscr{I}_{m+1}$ such that

$$|X(s)-X(t)| \leqq 2K\eta^{-1}(\varrho_{m+1}) \ \forall s,t \in J\,. \tag{3.5}$$

We can now inductively construct a Cantor-like random closed set K. Let $E_0=[0,1]$. Assume $E_m=\bigcup_{j=1}^{M_m} I_j^m$ where $I_j^m \in \mathscr{I}_m$, $M_m=\varrho_m^{-\beta}$ and the distance between neighbouring I_j^m's is at least $\frac{1}{2}r_m^\beta \varrho_{m-1}$. Divide each I_j^m into $r_{m+1}^{-\beta}$ subintervals of length $\frac{1}{2}r_{m+1}^\beta \varrho_m$, separated by intervals of the same length. Choose one interval from $\mathscr{I}_{m+1}$ contained in each of these subintervals. This will give us a collection of M_{m+1} intervals in $\mathscr{I}_{m+1}\{I_j^{m+1}: j \leqq M_{m+1}\}$. By (3.5), for $m \geqq m_0(\omega)$ we may always choose I_j^{m+1} so that

$$\sup_{s,t \in I_j^{m+1}} |X(s)-X(t)| \leqq 2K\eta^{-1}(\varrho_{m+1})\,. \tag{3.6}$$

Then $E_{m+1}=\bigcup_{j=1}^{M_{m+1}} I_j^{m+1} \subset E_m$. Put $K=\bigcap_{m=1}^{\infty} E_m$.

Fix ω such that $m_0(\omega)<\infty$; we will use Lemma 1.1 to show that $s^\beta-m(K)>0$. If $m_m(J)$ denotes the number of the intervals $\{I_j^m\}$ which intersect J and $|J|<\varrho_{m_0}$, we can choose $n \geqq m_0$ such that $\varrho_{n+1} \leqq |J| < \varrho_n$. By dividing J into two subintervals if necessary we may assume $J \subset I_j^n$ for some j. The spacing of the subintervals now ensures that

$$M_{n+1}(J) \leqq 2|J|\varrho_n^{-1} r_{n+1}^{-\beta}+2\,,$$

so that, for $m>n$

$$\begin{aligned} M_m(J)/M_m &\leqq (2|J|\varrho_n^{-1}r_{n+1}^{-\beta}+2)\left(\prod_{j=n+2}^{m} r_j^{-\beta}\right)\varrho_m^\beta \\ &\leqq 2|J|\varrho_n^{\beta-1}+2\varrho_{n+1}^\beta \leqq 4|J|^\beta\,. \end{aligned}$$

Lemma 1.1 now gives $s^\beta-m(K) \geqq \frac{1}{4}$ and the obvious cover gives $s^\beta-m(K) \leqq 1$.

Now (3.6) implies that, for $m \geqq m_0$, $X(K)$ is covered by a union of $\varrho_{m+1}^{-\beta}$ sets each of diameter $2K\eta^{-1}(\varrho_{m+1})$. The growth condition (1.1) on $\eta \in \Phi$ now tells us there has to be a constant $c_{3.3}<\infty$ such that

$$\varphi_\beta(2Ks) \leqq c_{3.3}\varphi_\beta(s)$$

so that

$$\varrho_{m+1}^{-\beta}\varphi_\beta(2K\eta^{-1}(\varrho_{m+1})) \leqq c_{3.3}\varrho_{m+1}^{-\beta}\cdot\varrho_{m+1}^\beta=c_{3.3}\,,$$

and this establishes

$$\varphi_\beta-m(X(K)) \leqq c_{3.3}\,. \quad \square$$

Corollary 3.3. *Let $T(t)$ be a stable subordinator of index α, and $\beta \in (0,1)$, $\varphi_\beta(s)=[s^\alpha(\log^+ 1/s)^{1-\alpha}]^\beta$. Then a.s. there is a random set $K \subset [0,1]$ such that $\frac{1}{4} \leqq s^\beta-m(K) \leqq 1$, but $\varphi_\beta-m(T(K)) \leqq c_{3.3}$.*

Proof. Note that this shows that Theorem 3.1 is best possible up to constant factors, for sets of all dimensions. We apply Lemma 3.2 with $X=T$ and $\eta^{-1}(y) = y^{1/\alpha}(\log^+ 1/y)^{1-1/\alpha}$. The required estimate (3.2) follows from Lemma 1 of Hawkes [5]. $\square$

The result for type A stable processes is more complicated, instead of a uniform growth rate on $T(t)$ we use Lemma 2.3(a) or (b).

Theorem 3.4. *Let $X(t)$ be a strictly stable process of index α in $\mathbb{R}^d (\alpha \leqq d)$ which is of type A. Then there is a constant $c_{3.4}$ depending only on the process such that a.s., for each $\Psi \in \Phi$, if*

$$\varphi(s) = \begin{cases} c_{3.4}\Psi(s^\alpha)(\log^+ 1/s)^2 & \text{if} \quad d=\alpha \\ c_{3.4}\Psi(s^\alpha)(\log^+ 1/s) & \text{if} \quad d>\alpha \end{cases}$$

then $\varphi - m(X(E)) \geqq \Psi - m(E)$ for all Borel $E \subset [0, \infty)$.

Proof. All the cases of this theorem have the same proof, based on Lemma 2.3(a) or (b), so we will write the detailed argument only for the critical case $\alpha = 2 = d$ of planar Brownian motion. Fix ω so that the conclusion of Lemma 2.3(a) (i) holds with the constant $c_{2.13}$ and the integer $k_0(\omega) < \infty$. For fixed $M > 0$, and E Borel $\subset [0, \infty)$, cover $X(E)$ by balls $S(z_i, r_i)$ with $r_i < \frac{1}{4} \wedge 2^{-k_0}$. Choose k_i such that $2^{-k_i - 1} < r_i \leqq 2^{-k_i}$, and let $\{I_j^i : j = 1, \ldots, N_i\}$ be the set of intervals in $\mathscr{I}(2^{-2k_i})$ that intersect $X^{-1}(S(z_i, r_i)) \cap [0, M]$. Then, by Lemma 2.3(a) (i), $N_i \leqq c_{2.13} k_i^2$. Take $c_{3.4} = 4c_{2.13}(\log 2)^{-2}$ and define $\varphi(s)$ by the formula in the statement. Then

$$\begin{aligned} \sum_{i=1}^{\infty} \varphi(2r_j) &\geqq c_{3.4} \sum_{i=1}^{\infty} (\log 2^{k_i - 1})^2 \Psi(2^{-2k_i}) \\ &\geqq c_{3.4} (\log 2)^2 \frac{1}{4} \sum_{i=1}^{\infty} k_i^2 \Psi(2^{-2k_i}) \\ &\geqq \sum_{i=1}^{\infty} \sum_{j=1}^{N_i} \Psi(|I_j^i|) \geqq \Psi - m(E) - \varepsilon \end{aligned}$$

whenever all the balls $r_i \leqq \delta$. The theorem now follows for $E \subset [0, M]$ and extends to $E \subset [0, \infty)$ by monotone convergence. □

We now try to show that Theorem 3.4 is close to best possible as a uniform bound. For this purpose Lemma 3.2 is a useful construction tool. We can obtain the hypothesis (3.3) by using estimates on exit times from a ball (Ciesielski, Taylor [3] for Brownian motion or Taylor [16] for a type A process). The appropriate functions are

$$\eta^{-1}(r) = r^{1/\alpha}(\log 1/r)^{-1/\alpha}, \qquad \eta(y) \sim \alpha y^\alpha \log 1/y \quad \text{as} \quad y \downarrow 0.$$

Lemma 3.2 now shows that for $\beta \in (0, 1)$ there is a random closed set $K \subset [0, 1]$ such that

$$\tfrac{1}{4} \leqq s^\beta - m(K) \leqq 1, \; \varphi_\beta - m[X(K)] \leqq c_{3.3},$$

where $\varphi_\beta(s) = [s^\alpha(\log^+ 1/x)]^\beta$. There is a substantial gap between this and the results of Theorem 3.4, particularly for the critical cases $d = \alpha$, but also in the transient case when β is small. This means we need a more subtle construction to get an example. All our processes will now be symmetric stable, and we give the details for the

Brownian motion case. In the transient case $\alpha<d$, there is still a gap of a factor log $\log 1/s$ between the theorem and our Example 3.7, but in the critical cases $d=\alpha$, we can do considerably better (Example 3.8).

We first introduce the notation

$$U_X(t,a,r)=\inf\{s\geqq t: X(s)\notin S(a,r)\}$$
$$V_X(t,a,r)=\inf\{s\geqq t: X(s)\in \overline{S(a,r)}\}.$$

$B(t)$ is a d-dimensional Brownian motion, $B(0)=0$. g, h are decreasing functions on $(0,\delta]$ such that $2<g<h$, $h(0+)=\infty$ and

$$\frac{\log g(s)}{\log h(s)}\to 0 \quad \text{if} \quad d=2, \tag{3.7}$$

$$\frac{\log g(s)}{\log 1/s}\to 0, \quad \text{and} \quad \frac{g(s)}{h(s)}\to 0 \quad \text{if} \quad d\geqq 3. \tag{3.8}$$

For a fixed $r\in(0,\delta]$, define inductively a sequence of stopping times by $V=U_B(0,0,rh(r))$, $S_1=U_B(0,0,\frac{1}{2}r)$, $T_i=U_B(S_i,0,r)$, $U_i=U_B(T_i,0,rg(r))$, $S_{i+1}=V_B(U_i,0,\frac{1}{2}r)$. We call $[S_i,U_i]$ a good pass from $S(0,\frac{1}{2}r)$ to $S(0,rg(r))^c$ if

$$T_i-S_i\geqq 2r^2 \quad \text{and} \quad U_i-T_i\geqq r^2(g(r))^2.$$

Let $N(r,g,h)$ be the number of good passes completed by time V, that is

$$N(r,g,h)=\sum_{i=1}^{\infty} I(S_i<V, T_i-S_i\geqq 2r^2, U_i-T_i\geqq r^2(g(r))^2).$$

Lemma 3.5. *Given $\eta>0$, there are positive constants $c_{3.5}=c_{3.5}(\eta)$ and $r_0=r_0(g,h)$ such that, if $0<r<r_0$*

$$P\left\{N(r,g,h)\geqq c_{3.5}\frac{(\log 1/r)\log h(r)}{\log g(r)}\right\}\geqq r^\eta, \quad \textit{if} \quad d=2;$$
$$P\left\{N(r,g,h)\geqq c_{3.5}\frac{(\log 1/r)}{\log g(r)}\right\}\geqq r^\eta, \quad \textit{if} \quad d\geqq 3.$$

Proof. We now suppress dependence on r, g, h where possible and let $N'=\sum_{i=1}^{\infty} I(S_i<V)$. N' is a geometric random variable by the strong Markov property. Moreover, the well-known hitting probabilities for $B(t)$, (1.9), (1.10), give

(i) $d=2$,

$$\begin{aligned}P(N'\geqq n)&=\left[1-\left(\log\frac{2}{r}-\log\frac{1}{rg(r)}\right)\left(\log\frac{2}{r}-\log\frac{1}{rh(r)}\right)^{-1}\right]^n\\&\geqq \exp\{-3n\log g(r)/\log h(r)\},\end{aligned} \tag{3.9}$$

for $0<r<r_0(g,h)$, where $r_0(g,h)>0$ and we have used (3.7) and $g>2$.

(ii) $d\geqq 3$,

$$P(N'\geqq n)=\left[1-\left(\left(\frac{r}{2}\right)^{-(d-2)}-(rg(r))^{-(d-2)}\right)\left(\left(\frac{r}{2}\right)^{-(d-2)}-(rh(r))^{-(d-2)}\right)^{-1}\right]^n$$
$$\geqq[1-(1-(2h(r))^{-(d-2)})^{-1}+(2g(r))^{-(d-2)}]^n$$
$$\geqq(4g(r))^{-n(d-2)} \tag{3.10}$$

for $0<r<r_0(g,h)$, where $r_0(g,h)>0$, and we have used (3.8).

The excursions of $|B|$ on $[S_i, U_i]$ are independent of the excursions of $|B|$ on $[U_i, S_{i+1}]$ by the strong Markov property. Therefore, conditional on $N'=n$, N is binomial (n,p), where, if $T(r)$ is the total time spent in $S(0,r)$,

$$p=P\left(T(r)\geqq 2r^2/|B_0|=\frac{r}{2}\right)P(T(rg(r))\geqq r^2g(r)^2/|B_0|=r)$$
$$=P(T(1)\geqq 2/|B_0|=1/2)P(T(1)\geqq 1/|B_0|=g(r)^{-1})$$
$$(\text{scaling})\geqq p_0>0,$$

where we have used $g(r)\geqq 2$ in the last line. Therefore there is a universal constant $c_1>0$ such that for $0<r<r_0(g,h)$ and $n\in\mathbb{N}$,

$$P(N\geqq np_0/2)\geqq E(P(N\geqq p_0N'/2|N')I(N'\geqq n))$$
$$\geqq c_{3.6}\begin{cases}\exp\{-3n\log g(r)/\log h(r)\} & d=2\\ (4g(r))^{-n/(d-2)} & d\geqq 3\end{cases} \tag{3.11}$$

(by (3.9), (3.10)).

(i) $d=2$

For $c>0$, and $r\in(0,r_0)$, let

$$n=[(2c/p_0)\log 1/r\,\log h(r)/\log g(r)]+1$$

in (3.11) to see that for some $c_{3.6}>0$,

$$P(N\geqq c(\log 1/r)\log h(r)/\log g(r))$$
$$\geqq c_{3.6}\exp\{-(6c/p_0)\log 1/r-3\log g(r)/\log h(r)\}$$
$$\geqq c_{3.7}r^{6c/p_0}\quad(\text{by } 3.7).$$

(ii) $d\geqq 3$

For $c>0$ and $r\in(0,r_0)$, let $n=[((6c/p_0)\log 1/r)/\log g(r)]+1$ in (3.11) to see that

$$P(N\geqq(c\log 1/r)/\log g(r))$$
$$\geqq c_1(4g(r))^{-((2c/p_0)(d-2)\log 1/r)/\log g(r)-(d-2)}$$
$$\geqq c_1r^{6c(d-2)/p_0}(g(r))^{-3(d-2)}\quad(\text{recall that } g>2).$$

The lemma now follows from the above two estimates and (3.8). □

In the next lemma, we use the notation:

$$f(s)=\begin{cases}(\log^+ 1/s)^2/\log^+\log^+ 1/s, & \text{if } d=2\\ (\log^+ 1/s)/\log^+\log^+ 1/s, & \text{if } d\geqq 3.\end{cases}$$

Lemma 3.6. *Let* $\beta\in(0,2)$, $\gamma>0$, $\delta>0$, *and* $\eta\in(0,1)$ *satisfy* $\beta+\delta<2\eta$, *and let* $l,r\in(0,1)$, $r^{-\beta}\in\mathbb{N}$ *and* $s=lr$. *There are constants* $r_0'=r_0'(\gamma,\eta)$, $c_{3.8}$ *and* $c_{3.9}=c_{3.9}(\gamma,\delta,\eta)$ *such that if* $s<r_0'$, *then except for a set of probability at most*

$$3l^{-2}r^{-\beta}\exp\{-c_{3.8}r^{\beta+\delta-2\eta}l^{2(1-\eta)+\delta}\}, \tag{3.12}$$

for any $I\in\mathscr{I}(l^2)$, $I\subset[0,1]$, *I contains* $r^{-\beta}$ *closed intervals of length* $l^2r^\beta/2$ *separated by* $l^2r^\beta/2$, *such that each interval contains at least* $c_{3.9}f(s)$ *subintervals in* $\mathscr{I}(s^2)$ *that are separated by a distance* $s^2(\log 1/s)^{2\gamma}$ *and are all mapped into a single ball of radius s by B.*

Proof. Consider $I=[0,l^2]$. Inductively define stopping times by

$$T_0^j=jr^\beta l^2,\quad T_{i+1}^j=U_B(T_i^j,B(T_i^j),s^\eta),\quad j=0,\ldots,r^{-\beta}-1,\quad i\in\mathbb{N}_0.$$

Let

$$N_j=\min\{i: T_i^j\geqq(j+1/2)r^\beta l^2\}.$$

If $c_1=(4E(T(1)))^{-1}$ and $\{T_i: i\in\mathbb{N}\}$ is an i.i.d. sequence with each T_i equal in law to $T(1)$, then

$$\begin{aligned}P(N_j\leqq c_1s^{-2\eta}r^\beta l^2)&=P[T^0_{[c_1s^{-2\eta}r^\beta l^2]}\geqq r^\beta l^2/2]\\&=P\left[\sum_{i=1}^{[c_1s^{-2\eta}r^\beta l^2]}T_i\geqq r^\beta l^2s^{-2\eta}/2\right]\\&\leqq P\left[\sum_{i=1}^{[c_1s^{-2\eta}r^\beta l^2]}T_i-E(T_i))/[c_1s^{-2\eta}r^\beta l^2]\geqq E(T)\right]\\&\leqq\exp\{-\theta r^{\beta-2\eta}l^{2(1-\eta)}\}\end{aligned}$$

for some universal constant $\theta>0$. Here we have used the fact that T_i has an exponentially bounded tail (see Ciesielski-Taylor [3] for its exact law) and the well-known exponential bounds of Cramer. We have shown

$$\begin{aligned}&P(N_j\leqq c_1r^{\beta-2\eta}l^{2(1-\eta)}\,\forall 0\leqq j\leqq r^{-\beta}-1)\\&\quad\leqq r^{-\beta}\exp\{-\theta r^{\beta-2\eta}l^{2(1-\eta)}\}.\end{aligned}\tag{3.13}$$

Let N_i^j denote the number of good passes completed by $B_i^j(t)=B(T_i^j+t)-B(T_i^j)$, from $S(0,s/2)$ to $S(0,s(\log 1/s)^\gamma)^c$ before leaving $S(0,s^\eta)$. (Here a good pass is as defined before Lemma 3.5 with $g(s)=(\log 1/s)^\gamma$.) Let $c_{3.9}=c_{3.5}\delta(1-\eta)/\gamma$. Call $[T_i^j,T_{i+1}^j]$ a good interval if $i<N_j$ and $N_i^j\geqq{}_{3.9}f(s)$. Apply Lemma 3.5, with $g(s)=(\log 1/s)^\gamma$ and $h(s)=s^{\eta-1}$, and (3.13) to see that if $s<r_0(g,h)\equiv r_0'(\gamma,\eta)$, then

P (there is no good interval in $[T_0^j, T_{N_j}^j]$ for some $j \leqq r^{-\beta}-1$)

$$\leqq r^{-\beta} \exp\{-\theta r^{\beta-2\eta} l^{2(1-\eta)}\} + \sum_{j=0}^{r^{-\beta}-1} P(N_i^j < c_{3.9} f(s)$$

$$\text{for all } 1 \leqq i \leqq c_1 r^{\beta-2\eta} l^{2(1-\eta)})$$

$$\leqq r^{-\beta}[\exp\{-\theta r^{\beta-\eta} l^{2(1-\eta)}\} + (1-s^{\delta})^{[c_1 r^{\beta-2\eta} l^{2(1-\eta)}]}]$$

$$\leqq r^{-\beta}[\exp\{-\theta r^{\beta-\eta} l^{2(1-\eta)}\} + 2\exp\{-c_1 r^{\beta+\delta-2\eta} l^{2(1-\eta)+\delta}\}]$$

(make r_0 smaller, if necessary)

$$\leqq 3r^{-\beta} \exp\{-c_{3.8} r^{\beta+\delta-2\eta} l^{2(1-\eta)+\delta}\},$$

where $c_{3.8} = \theta \wedge c_1$. The above estimate is valid for any $I \in \mathcal{I}(l^2)$, so except for a set whose probability is bounded by (3.12) for each $[(k-1)l^2, kl^2] \subset [0,1]$ in $\mathcal{I}(l^2)$ and each $0 \leqq j \leqq r^{-\beta}-1$ there is a good interval, $[T_{i-1}^j, T_i^j]$, in $[(k-1)l^2 + jr^\beta l^2, (k-1)l^2 + (j+1/2)r^\beta l^2]$. Each such good interval contains at least $c_{3.9} f(s)$ subintervals in $\mathcal{I}(s^2)$ that are separated by at least $s^2(\log 1/s)^{2\gamma}$ and are all mapped into a single ball of radius s by B. This follows from the definition of a good pass. The proof is complete. □

Recall the definition of f given before Lemma 3.6.

Example 3.7. Suppose $0 < \beta < 2$, $\Psi_\beta(s) = s^{\beta/2}(f(s^{1/2}))^{-1}$, $\varphi_\beta(s) = \Psi_\beta(s^2) f(s) = s^\beta$ and $g \in \Phi$. Then there is a random closed subset E of $[0,1]$, and a constant $c_{3.9} > 0$ such that a.s.

$$\Psi_\beta - m(E) \geqq c_{3.9}, (g\varphi_\beta) - m(B(E)) = 0.$$

Proof. Fix $\beta \in (0,2)$ and $g \in \Phi$. Let $\gamma = 2/\beta$ and choose $\delta > 0$ and $\eta \in (0,1)$ so that $\beta + \delta < 2\eta$. Let $F(r) = c_{3.8} f(r)$. Choose $r_m \downarrow 0$ $(r_m > 0)$ fast enough so that if $\varrho_m = \prod_{i=1}^m r_i$ then $r_m^{-\beta} \in \mathbb{N}$,

$$\sum_{m=1}^{\infty} \varrho_m^{-2} r_{m+1}^{-\beta} \exp\{-c_{3.7} r_{m+1}^{\beta+\delta-2\eta} \varrho_m^{2(1-\eta)+\delta}\} < \infty, \tag{3.14}$$

$$\prod_{i=1}^{m-1} F(\varrho_i) \leqq g(\varrho_m)^{-1/2} \tag{3.15}$$

$$r_1 \leqq r_0'(\gamma, \eta), F(r_1) \geqq 1. \tag{3.16}$$

We now use Lemma 3.6 inductively to construct a Cantor-like random set, E, with the desired properties. Let $E_0 = I_1^0 = [0,1]$ and $M_0 = 1$. At the m^{th} stage suppose E_m is a union of disjoint closed intervals $\{I_j^m : j = 1 \dots M_m\}$ in $\mathcal{I}(\varrho_m^2)$ contained in $[0,1]$, where $M_m = \prod_{i=1}^m r_i^{-\beta}[F(\varrho_i)]$. Each I_j^m contains $r_{m+1}^{-\beta}$ intervals of length $\varrho_m^2 r_{m+1}^\beta/2$ separated by intervals of the same length. In each such interval choose $[F(\varrho_{m+1})]$ subintervals in $\mathcal{I}(\varrho_{m+1}^2)$ separated by a distance of at least $\varrho_{m+1}^2(\log(1/\varrho_{m+1}))^{2\gamma}$ and, if possible, so that all of these $[F(\varrho_{m+1})]$ intervals are mapped by B into the same ball of radius ϱ_{m+1}. By (3.14), Lemma 3.6 and the Borel-Cantelli Lemma, there is an $m_0(\omega) < \infty$ a.s. such that if $m \geqq m_0(\omega)$, this last condition holds for each I_j^m and each of the $r_{m+1}^{-\beta}$ subintervals of I_j^m. Let $\left\{I_j^{m+1}: j \leqq M_{m+1}=\varrho_{m+1}^{-\beta} \prod_{i=1}^{m+1}[F(\varrho_i)]\right\}$ denote the collection of closed intervals constructed from $\{I_j^m: j \leqq M_m\}$ as above, let E_{m+1} denote their union and finally let $E=\bigcap_{m=1}^{\infty} E_m$, a closed random set.

Fix ω such that $m_0(\omega)<\infty$. Again we use Lemma 1.1 to get a lower bound on $\Psi_\alpha-m(E)$. Let J be an interval of length $|J|<\varrho_{m_0}^2$ and choose $n \geqq m_0$ so that $\varrho_{n+1}^2 \leqq |J| < \varrho_n^2$. $M_m(J)$ denotes the number of intervals in $\{I_j^m: j \leqq M_m\}$ that intersect J. As in the proof of Lemma 3.2, we may assume that $J \subset I_j^n$ for some j. The spacing of the $r_{n+1}^{-\beta}$ subintervals of I_j^n implies that the number of such subintervals that intersect J is at most $2|J|\varrho_n^{-2} r_{n+1}^{-\beta}+2$. Therefore, if $m>n$, then

$$\begin{aligned} M_m(J)/M_m &\leqq (2|J|\varrho_n^{-2} r_{n+1}^{-\beta}+2)[F(\varrho_{n+1})] \prod_{i=n+2}^{m} r_i^{-\beta}[F(\varrho_i)] \Big/ \prod_{i=1}^{m} r_i^{-\beta}[F(\varrho_i)] \\ &\leqq 4(|J|\varrho_n^{-2} r_{n+1}^{-\beta}+1)\varrho_{n+1}^{\beta} F(\varrho_n)^{-1} \quad \text{(by (3.16))} \\ &\leqq 4|J|\varrho_n^{\beta-2} F(\varrho_n)^{-1}+4\varrho_{n+1}^{\beta} F(\varrho_n)^{-1} \\ &\leqq 4|J|^{\beta/2} F(|J|^{1/2})^{-1}+4\varrho_{n+1}^{\beta} F(\varrho_n)^{-1} \end{aligned} \tag{3.17}$$

by the monotonicity of $x \to x^{2-\beta} F(x)$.

Case 1. $\varrho_{n+1}^{\beta} F(\varrho_n)^{-1} \leqq |J|^{\beta/2} F(|J|^{1/2})^{-1}$.

From the above we get

$$M_m(J)/M_m \leqq 8 c_{3.8}^{-1} |J|^{\beta/2} f(|J|^{1/2})^{-1}.$$

Case 2. $\varrho_{n+1}^{\beta} F(\varrho_n)^{-1} > |J|^{\beta/2} F(|J|^{1/2})^{-1}$.

The spacing of the I_j^{n+1}'s shows that

$$M_{n+1}(J) \leqq |J|\varrho_{n+1}^{-2}(\log 1/\varrho_{n+1})^{-2\gamma}+2$$

and therefore if $m>n+1$, then

$$\begin{aligned} M_m(J)/M_m &\leqq (|J|\varrho_{n+1}^{-2}(\log 1/\varrho_{n+1})^{-2\gamma}+2) \prod_{i=n+2}^{m} r_i^{-\beta}[F(\varrho_i)] \Big/ \left(\varrho_m^{-\beta} \prod_{i=1}^{m}[F(\varrho_i)]\right) \\ &\leqq 2\,|J|\varrho_{n+1}^{\beta-2}(\log 1/\varrho_{n+1})^{-2\gamma} F(\varrho_{n+1})^{-1}+4\varrho_{n+1}^{\beta} F(\varrho_{n+1})^{-1} \\ &\leqq 2|J|^{\beta/2}(\log 1/\varrho_{n+1})^{-2\gamma} F(|J|^{1/2})^{2/\beta-1} F(\varrho_{n+1})^{-1} F(\varrho_n)^{1-2/\beta} \\ &\quad +4|J|^{\beta/2} F(|J|^{1/2})^{-1} \end{aligned}$$

(using the assumed lower bound on ϱ_{n+1} to handle the first term)

$$\begin{aligned} &\leqq |J|^{\beta/2} F(|J|^{1/2})^{-1}(2F(|J|^{1/2})^{2/\beta}(\log 1/\varrho_{n+1})^{-2\gamma}+4) \\ &\leqq c_1 |J|^{\beta/2} F(|J|^{1/2})^{-1} \end{aligned} \tag{3.18}$$

for some $c_1<\infty$ by the choice of γ and definition of F.

Lemma 1.1 now shows that for some $c_{3.9}>0$ we have $\Psi_\beta - m(E) \geqq c_{3.9}$. Finally for $m \geqq m_0(\omega)$, $B(E_m)$ is contained in a union of $\varrho_m^{-\beta} \prod_{i=1}^{m-1} [F(\varrho_i)] \leqq \varrho_m^{-\beta} g(\varrho_m)^{-1/2}$ (by (3.15)) balls of radius ϱ_m. Therefore $(g\varphi_\beta) - m(B(E)) = 0$ and the proof is complete. □

There are several possible improvements and extensions of the above example. We only sketch the arguments.

It is not hard to drop the arbitrary $g \in \Phi$ in Example 3.7, at the cost of replacing Ψ_β by a slightly more complicated measure function, $\tilde{\Psi}_\beta$, which is still of the form $s^{\beta/2}L(s)$ for some slowly varying L. The conclusion then remains valid with $\tilde{\varphi}_\beta(s) = \tilde{\Psi}_\beta(s^2) f(s)$ in place of $g\varphi_\beta$, and $\tilde{\Psi}_\beta$ in place of Ψ_β. This is done by taking advantage of the product of the $F(\varrho_i)$'s which were dropped in (3.17) and (3.18), and constructing $g \in \Phi$ such that $g(\varrho_n)^{-1} = \prod_{i=1}^{n} F(\varrho_i)$. Given the gap of a factor of $\log\log 1/s$ that still exists between this example and Theorem 3.4, a more interesting refinement (if $d=2$) is

Example 3.8. Let $\beta \in (0,2)$, $n \in \mathbb{N}$, $f_n(s) = (\log^+ 1/s)^2 (\log^{+(n)}(1/s))^{-1}$ ($\log^{+(n)}$ denotes the composition of $\log^+$), $\Psi_{\beta,n}(s) = s^{\beta/2} f_n(s^{1/2})^{-1}$, and $\varphi_{\beta,n}(s) = \Psi_{\beta,n}(s^2) f_n(s)$. If $d=2$, there is a random closed subset, E, of $[0,1]$ and a positive constant $c_{3.10}$ such that $\Psi_{\beta,n}(E) \geqq c_{3.10}$ but $\varphi_{\beta,n}(B(E)) = 0$ a.s. □

This effectively shows that Theorem 3.4 is best possible, at least if $d=2$. If $n=3$, say, the key idea in the proof is to define a very good pass from $S(0,s/2)$ to $S(0,s(\log 1/s)^\gamma)^c$ as a good pass that contains $\log^{(2)}(1/s)/\log^{(3)}(1/s)$ good passes from $S(0,s/2)$ to $S(0,s(\log^{(2)} 1/s)^\gamma)^c$. (3.9) shows that if $d=2$ an appreciable portion of good passes are very good passes, and hence the fundamental estimate obtained in Lemma 3.5 remains valid for the number of very good passes. (This is not the case if $d \geqq 3$ as an application of (3.10) suggests.) Now proceed as before. In the $(m+1)^{st}$ stage of the inductive construction of E we find $cr_{m+1}^{-\beta}(\log(1/\varrho_{m+1}))^2/\log^{(3)}(1/\varrho_{m+1})$ subintervals of each I_j^m, that belong to $\mathscr{I}(\varrho_{m+1}^2)$. They are divided into $r_{m+1}^{-\beta}$ "spaced" blocks, each containing at least $c(\log(1/\varrho_{m+1}))^2/\log^{(2)}(1/\varrho_{m+1})$ (appropriately spaced) very good passes. Each of these very good passes in turn contains at least $\log^{(2)}(1/\varrho_{m+1})/\log^{(3)}(1/\varrho_{m+1})$ (appropriately spaced) good passes, each of which contains one of the I_j^{m+1}'s. The appropriate spacing at each level gives $\Psi_{\beta,3} - m(E) > 0$ and the fact that groups of $c(\log(1/\varrho_{m+1}))^2/\log^{(3)}(1/\varrho_{m+1})$ of these intervals are all mapped into a single ball of radius ϱ_{m+1} leads to $\varphi_{\beta,3} - m(B(E)) < \infty$. ($\varphi_{\beta,3}(B(E)) = 0$ follows from the $n=4$ case.) To handle a general n we can iterate this scheme.

Finally, with a bit more work similar examples can also be found in the symmetric stable case. In this case define

$$f(s) = \begin{cases} \log^+(1/s)^2/\log^+\log^+ 1/s & \text{if } \ d=\alpha \\ \log^+(1/s)/\log^+\log^+ 1/s & \quad d>\alpha. \end{cases}$$

Example 3.9. Let $0<\beta<\alpha\leqq 2$, $\Psi_{\alpha,\beta}(s) = s^{\beta/\alpha} f(s^{1/\alpha})^{-1}$, $\varphi_{\alpha,\beta}(s) = \Psi_{\alpha,\beta}(s^\alpha) f(s)$ and $g \in \Phi$. If X is a symmetric stable process of index α, there is a random closed set, $E(\omega)$, and a constant $c_{3.11} > 0$ such that $\Psi_{\alpha,\beta} - m(E) \geqq c_{3.11}$ but $(g\varphi_{\alpha,\beta}) - m(X(E)) = 0$ a.s.

The proof is essentially the same but it is now complicated by the "overshoots" of X when it exits from or hits from a ball. This means that in the proof of Lemma 3.5 the random variable N' is no longer geometric. By using the distribution of these overshoots, given in Blumenthal-Getoor-Ray [2] Theorems A and B, one can stochastically bound N' below by an appropriate geometric distribution. Similarly, although the law of N conditioned on $N'=n$ is no longer binomial, it is not hard to bound this conditional distribution below by an appropriate binomial law. Once the analogue of Lemma 3.5 is established the proof proceeds with only minor alterations (the required estimates for the passage time of X out of a sphere may be found in Taylor [16], Lemma 5.

If $d=\alpha=1$, one can construct a better example, analogous to Example 3.8.

Finally we note an interesting corollary of Theorem 3.4 and its proof for planar Brownian motion or linear Cauchy processes. It is known that these processes have some points of multiplicity c. How big a time set can map into a singleton?

Corollary 3.10. *If $X(t)$ is a strictly stable d-dimensional process, with $\alpha=d$, then a.s. $h-m(X^{-1}(\{z\}))<\infty$ $\forall z\in\mathbb{R}^d$, where $h(s)=(\log^+ 1/s)^{-2}$.*

This leads to a natural question.

Problem 3.11. For which $\alpha\in(0,2]$ can one find $z\in\mathbb{R}^2$ such that $h_\alpha-m(B^{-1}\{z\})>0$, where $h_\alpha(s)=(\log^+ 1/s)^{-\alpha}$.

4. Upper Bounds for Hausdorff Measure

Consider now the problem of finding a uniform (in E) upper bound on the Hausdorff measure of $X(E)$, where X is a strictly stable d-dimensional process of index α. For Brownian motion one need only use Lévy's modulus of continuity (see Kaufman [8]) and for $\alpha<2$, the result is a simple consequence of Lemma 2.5.

Theorem 4.1. *Let X be a strictly stable process of index α. There is a positive constant, $c_{4.1}$, depending only on the law of X, such that for a.a. ω and any $\Psi\in\Phi$, if*

$$\varphi(s)=\begin{cases}\Psi(c_{4.1}s^2\log^+(1/s)^{-1}) & \text{if } \alpha=2\\ c_{4.1}\log^+(1/s)^{-1}\,\Psi(c_{4.1}s^\alpha\log^+(1/s)) & \text{if } \alpha<2,\end{cases}$$

then $\varphi-m(X(E))\leqq\Psi-m(E)$ for all $E\in\mathscr{B}([0,\infty))$.

Proof. We omit the well-known (and trivial) proof in the Brownian case and fix $\alpha<2$.

By Lemma 2.5 we may choose ω outside a null set so that for each $M\in\mathbb{N}$ there is an $\delta_M(\omega)>0$ such that if $I\subset[0,M]$ and $|I|\leqq\delta_M(\omega)$, then $X(I)$ is contained in a union of at most $c_{2.32}\log(|I|^{-1})$ balls of diameter $(\log(|I|^{-1}))^{-1/\alpha}|I|^{1/\alpha}$. If $\delta>0$, $E\in\mathscr{B}([0,M])$ and $\Psi\in\Phi$, choose $\{J_i:i\in\mathbb{N}\}$ so that $|J_i|\leqq\delta_M(\omega)\wedge\delta$ and $\sum_{i=1}^{\infty}\Psi(|J_i|)\leqq\Psi-m(B)+\delta$. Then $X(J_i)\subset\bigcup_{k=1}^{N_i}B_k^i$, where $N_i\leqq c_{2.32}\log(1/|J_i|)$ and B_k^i is a ball of diameter $(\log 1/|J_i|)^{-1/\alpha}|J_i|^{1/\alpha}$. If φ is defined as above for some $c_{4.1}$, then by taking

$\sup_i |J_i|$ smaller, if necessary, we have

$$\begin{aligned}
&\sum_{i=1}^{\infty} \sum_{k=1}^{N_i} \varphi((\log 1/|J_i|)^{-1/\alpha} |J_i|^{1/\alpha}) \\
&\leqq \sum_{i=1}^{\infty} c_{2.32} \log(1/|J_i|) c_{4.1} (\alpha^{-1} \log 1/|J_i| + \alpha^{-1} \log\log 1/|J_i|)^{-1} \\
&\quad \times \Psi(c_{4.1} (\log 1/|J_i|)^{-1} |J_i| (\alpha^{-1} \log 1/|J_i| + \alpha^{-1} \log\log 1/|J_i|)) \\
&\leqq \sum_{i=1}^{\infty} c_{2.32} c_{4.1} 2\alpha^{-1} \Psi(c_{4.1} 2\alpha^{-1} |J_i|) \\
&\leqq \sum_{i=1}^{\infty} \Psi(|J_i|) \leqq \Psi - m(B) + \delta,
\end{aligned}$$

providing $c_{4.1}$ is chosen small enough (independent of δ). Let $\delta \downarrow 0$ to complete the proof for bounded B and hence in general by monotone convergence. □

The following example shows the theorem is best possible at least for Brownian motion.

Example 4.2. Let $\beta \in (0,1)$, $\tilde{\Psi}_\beta(s) = s^{\beta/2} (\log^+ 1/s)^{\beta/2}$, $\tilde{\varphi}_\beta(s) = \tilde{\Psi}_\beta(s^2 (\log^+ 1/s)^{-1})$ and B be a d-dimensional Brownian motion starting at zero ($d \geqq 1$). There is a random compact set $E(\omega) \subset [0, \infty)$ such that $\tilde{\Psi}_\beta - m(E) \leqq c_{3.2} < \infty$ and $\frac{1}{4} \leqq \tilde{\varphi}_\beta - m(B(E)) < \infty$.

Proof. We will apply Lemma 3.2 to the passage time process, $\tau_B(s)$, of B. It is well known that $|B(t)| = W(t) + A(t)$ where W is a one-dimensional Brownian motion and A is a nondecreasing, nonnegative process. Therefore, if $s \geqq 0$, $h > 0$ and $T(u) = \inf\{t : W(t) \geqq u\}$, then

$$P(\tau_B(s+h) - \tau_B(s) \leqq K\eta(h)) \geqq P(T(h) \leqq K\eta(h)).$$

As in the proof of Corollary 3.3, one sees that (3.2) holds with $\eta^{-1}(y) = y^2/\log^+ 1/y)$ and τ_B in place of X (recall T is a stable subordinator of index 1/2). Lemma 3.2 now shows there is a random closed set $\tilde{E}(\omega) \subset [0,1]$ such that $1/4 \leqq x^\beta - m(\tilde{E}) \leqq 1$ but $\tilde{\Psi}_\beta - m(\tau_B(\tilde{E})) \leqq c_{3.3}$. Let $E = \tau_B(\tilde{E})$. E is closed because τ_B^{-1} is continuous, and hence E is a compact set a.s. Moreover, one has

$$|B|(E) = |B|(\tau_B(\tilde{E})) = \tilde{E}.$$

An elementary covering argument shows that

$$s^\beta - m(B(E)) \geqq s^\beta - m(|B|(E)) = s^\beta - m(\tilde{E}) \geqq 1/4$$

[note that any cover of $B(E)$ trivially produces a cover of $|B|(E)$]. The fact that $\tilde{\varphi}_\beta(y) y^{-\beta} \to 2^{\beta/2}$ as $y \downarrow 0$ implies

$$\tilde{\varphi}_\beta - m(B(E)) \geqq 2^{\beta/2}/4 \geqq 1/4.$$

The finiteness of $\tilde{\varphi}_\beta - m(B(E))$ follows from Theorem 4.1. □

Our conjecture is that Theorem 4.1 is sharp for $\alpha<2$, but we have not been able to produce an example.

5. Uniform Packing Measure Results

There are technical difficulties in the construction resulting from the condition that the packing sets have to be disjoint. This introduces an uncertainty factor in the relevant measure functions of order $\log 1/s$, but less than $(\log 1/s)^{1+\varepsilon}$, see Lemma 1.2. For general nonrandom sets one can give examples to show this factor is really there, but our techniques are not sharp enough to produce such examples on the trajectory of a stable process. We start with a lower bound result:

Theorem 5.1. *Suppose $X(t)$ is a strictly stable process of type A of index $\alpha \leqq d$ in $\mathbb{R}^d$, a.s. if $h \in \Phi$ is such that $\int_{0+} \frac{h(s)}{s} ds < \infty$, $\Psi \in \Phi$, and if $E \subset [0, M]$,*

(a) *For $\alpha = 2 = d$, $\varphi(s) = \Psi(s^2)(\log 1/s)^{-2} h(s)$,*

$$\varphi - p(E) > 0 \Rightarrow \Psi - p(B(E)) = +\infty;$$

(b) *For $\alpha = 1 = d$, $\varphi(s) = \Psi(s)(\log 1/s)^{-2} h(s)$,*

$$\varphi - p(E) > 0 \Rightarrow \Psi - p(X(E)) = +\infty;$$

(c) *For $\alpha < d$, $\varphi(s) = \Psi(s^\alpha)(\log 1/s)^{-1} h(s)$,*

$$\varphi - p(E) > 0 \Rightarrow \Psi - p(X(E)) = +\infty.$$

Proof. (a) It is sufficient to prove the corresponding result for the premeasures $\varphi - P$, $\Psi - P$. Using (1.6), if $\Psi - p(B(E)) \leqq K < \infty$, we can find $A_n \uparrow B(E)$ such that $\Psi - P(A_n) \uparrow c < K+1$ and so $B^{-1}(A_n) \uparrow F \supset E$. Then $\varphi - p(E) \leqq \varphi - p(F) = \lim_{n\to\infty} \varphi - p(B^{-1}(A_n))$; thus, for large n we have $\varphi - P(B^{-1}(A_n)) \geqq \varphi - p(B^{-1}(A_n)) > 0$, but $\Psi - P(A_n) < K+1 < +\infty$.

Now take any set $A \subset \mathbb{R}^2$ with $\Psi - P(A) \leqq K < \infty$. This implies that, for suitable $\delta > 0$, if $0 < r < \delta$, then

$$M_r(A)\,\Psi(2r) \leqq K+1, \tag{5.1}$$

where $M_r(A)$ is the maximum number of disjoint balls of radius r with centres in A.

If $2^{-k} \geqq r > 2^{-k-1}$, then Lemma 2.3(a) (i) tells us that for $k \geqq k_0$, and all $x \in \mathbb{R}^2$, we have not more than $c_{5.1} k^2$ semidyadic intervals of length 2^{-2k} which intersect $B^{-1}(S(x,r))$. Hence, if $E = B^{-1}(A) \cap [0, M]$, $r = 2^{-k} < \delta$, $k \geqq k_0$,

$$M_{2^{-2k}}(E) \leqq c_{5.1} k^2 M_{2^{-k}}(A).$$

Using (5.1) now yields

$$\frac{M_{2^{-2k}}(E)\,\Psi(2^{-2k})}{k^2} \leqq c_{5.2},$$

which implies $M_r(E)\Psi r^2)(\log 1/r)^{-2} \leqq c_{5.3} < +\infty$, and an application of Lemma 1.2 now gives $\varphi - P(E) = 0$.

(b) and (c) can be proved by identical arguments substituting the other cases of Lemma 2.3. □

Much sharper results can be obtained for subordinators, as in this case Lemma 1.2 is not needed.

Theorem 5.2. *Suppose $X(t)$ is a stable process of type B and index α, $0<\alpha<1$ in $\mathbb{R}^d$, $\varphi \in \Phi$ and $\Psi(s) = \varphi(s^\alpha)(\log 1/s)^{1-\alpha}$. Then there exists a constant $c_{5.4}$ such that, for all Borel sets E*

$$\Psi - p(X(E)) \geqq c_{5.4}\varphi - p(E).$$

Proof. As explained at the beginning of the proof of Theorem 5.1, it is sufficient to show that

$$\Psi - P(X(E)) \geqq c_{5.4}\varphi - P(E). \tag{5.2}$$

Without loss of generality we may assume that the projection $Y(t)$ of $X(t)$ on the first coordinate axis is a subordinator, and for $h>0$, the increment

$$|X(t+h) - Y(t)| \geqq \{Y(t+h) - Y(t)\}. \tag{5.3}$$

By Hawkes [5], Lemma 2, since $Y(t)$ is a stable subordinator of index α, there exists $c_{5.5}$ such that if $0 \leqq h \leqq h_0 = h_0(\omega) > 0$, $0 \leqq t \leqq M$

$$Y(t+h) - Y(t) \geqq c_{5.5} h^{1/\alpha}(\log 1/h)^{1-1/\alpha}.$$

Hence, if $t_0 < t_1 < \ldots < t_n$ are a finite set of centres in E of disjoint intervals of lengths $2r_i \leqq \delta$, the intervals with centres $Y(t_i)$ and radii $\varrho_i = c_{5.5}(r_i/2)^{1/\alpha}(\log 2/r_i)^{1-1/\alpha}$ will be disjoint. By projection this means that the balls centred at $X(t_i)$ with these radii are also disjoint. Thus each packing of E with

$$\sum_{i=0}^{n} \varphi(2r_i) \geqq (1-\varepsilon)\varphi - P(E)$$

yields a packing of $X(E)$ such that

$$\Sigma\Psi(2\varrho_i) \geqq c_{5.6}\Sigma\varphi(2r_i),$$

using the fact that $s^\alpha(\log 1/s)^{1-\alpha}$ and $ch^{1/\alpha}(\log 1/h)^{1-1/\alpha}$ are asymptotic inverses as $s \downarrow 0$. This establishes (5.2). □

Remark. Theorem 5.2 relies on the fact that a stable subordinator in $\mathbb{R}^d$ escapes at a uniform rate from each point that it hits. We had originally hoped to use this simple method of proof to obtain uniform dimension results for packing measure of the subsets of transient Brownian motion and symmetric stable processes. The method fails completely, because even when there are no double ponts ($2\alpha < d$), this uniform escape rate is a power strictly greater than $1/\alpha$, so that it gives just a very crude estimate with the wrong power if we use the method of Theorem 5.2. These escape rate problems are discussed in detail in [12].

We now turn to the upper bound results.

Theorem 5.3. *If $B(t)$ is a standard Brownian motion in $\mathbb{R}^d$ and $M>0, \eta>2$; then a.s. for all Borel sets $A\subset[0,M]$, all $\Psi\in\Phi$, if we define $\varphi\in\Phi$ by $\varphi(2s)=\Psi\left[\frac{s^2}{\eta\log 1/s}\right]$, then*

$$\varphi-p(B(A))\leqq\Psi-p(A).$$

Proof. It is again sufficient to show that, for all Borel sets $A\subset[0,M]$ we have

$$\varphi-P(B(A))\leqq\Psi-P(A). \tag{5.4}$$

Pick $\eta>\eta'>2$. Then the Lévy modulus of continuity tells us that a.s. $\exists\delta_0=\delta_0(\omega)>0$ such that

$$0\leqq t\leqq M, 0<h\leqq\delta_0 \Rightarrow |B(t+h)-B(t)|\leqq(\eta' h\log 1/h)^{1/2}. \tag{5.5}$$

There is nothing to prove if $\varphi-P(B(A))=0$. First assume $0<\varphi-P(B(A))<+\infty$. For $\varepsilon>0$, $\varrho_0=(\delta_0\log 1/\delta_0)^{1/2}$ and any ϱ such that $0<\varrho\leqq\varrho_0$, we can find a packing of $B(A)$ by a finite collection of balls $B(x_i,\varrho_i)$ with $\varrho_i\leqq\varrho$, $x_i=X(t_i)$ and $0\leqq t_0<t_i<t_{i+1}\leqq M$, $t_i\in A$ and

$$\Sigma\varphi(2\varrho_i)\geqq(1-\varepsilon)\varphi-P(B(A)).$$

Now let J_i be the interval centre t_i and length $2r_i$ where $r_i=\frac{\varrho_i^2}{2\eta\log 1/\varrho_i}$. Since $(\eta h\log 1/h)^{1/2}$ and $s^2(2\eta\log 1/s)^{-1}$ are asymptotic inverses and $s^2(\log 1/s)^{-1}$ is convex, (5.5) now ensures that these intervals J_i are disjoint. We have therefore constructed a packing of A by small disjoint intervals and, by definition,

$$\begin{aligned}\Sigma\Psi(2r_i)&=\Sigma\Psi\left(\frac{\varrho_i^2}{\eta\log 1/\varrho_i}\right)=\Sigma\varphi(2\varrho_i)\\&\geqq(1-\varepsilon)\varphi-P(B(A)).\end{aligned}$$

Hence, $\Psi-P(A)\geqq(1-\varepsilon)\varphi-P(B(A))$. Since ε is arbitrary, (5.4) is established in this case. The same argument shows that, if $\varphi-P(B(A))=+\infty$, then so is $\Psi-P(A)$. □

Corollary 5.4. *A.s. for each $\beta\in(0,1)$, $\gamma\in\mathbb{R}$ every Borel $A\subset\mathbb{R}$,*

$$2^{\gamma-\beta}h_{2\beta,\gamma-\beta}-p(B(A))\leqq h_{\beta,\gamma}-p(A).$$

This follows by taking $\Psi(s)=h_{\beta,\gamma}(s)$ in the theorem and using a countable sequence of $\eta\downarrow 2$. This is a result of type (0.2) for packing measure, which we believe to be close to best possible. For the strictly stable processes of index α we have no modulus of continuity and our replacement forces us to consider balls of equal size. As we saw earlier this necessarily results in a possible error factor of the order $(\log 1/s)$. The precise result we can prove is

Theorem 5.5. *Suppose $X(t)$ is strictly stable of index $\alpha\in(0,2)$ in $\mathbb{R}^d$. Then a.s. if $\varphi\in\Phi$ and $\Psi\in\Phi$ is such that $\int_{0+}\frac{\Psi(s)}{s}ds<\infty$, then for every Borel set $A\subset[0,M]$, if*

$$\varphi_1(s)=\Psi(s)(\log 1/s)^{-1}\varphi(s^\alpha\log 1/s),$$

then

$$\varphi - p(A) < \infty \Rightarrow \varphi_1 - p(X(A)) = 0 .$$

Proof. As usual, it will be sufficient to show that

$$\varphi - P(A) < \infty \Rightarrow \varphi_1 - P(X(A)) = 0 . \tag{5.6}$$

Suppose A is such that $\varphi_1 - P(X(A)) > 0$. By Lemma 2.2, if $M_r(E)$ denotes the maximum number of disjoint balls of radius r and centres in E, we must have

$$\limsup_{r \downarrow 0} M_r \varphi_2(r) = +\infty ,$$

where $\varphi_2(s) = \varphi_1(s)/\Psi(s)$. If we put $r_k = 2^{-k/\alpha}(\log 2^{k/\alpha})^{1/\alpha}$, then $r_k/r_{k+1} \sim 2^{1/\alpha}$ is bounded, so using (1.1) for $\varphi_2(s)$ gives

$$\limsup_{r = r_k, k \to \infty} M_r \varphi_2(r) = +\infty .$$

For each K, however large, we can find a sequence $k_i \to \infty$ such that

$$M_{r_{k_i}} > K/\varphi_2(r_{k_i}) .$$

If $I \subset [0, M]$ and $|I| = 2^{-k_i}$, Corollary 2.6 tells us that not more than $c_{5.7} k_i$ of these centres of balls can lie in $X(I)$. Hence, the number of dyadic intervals $[j2^{-k_i}, (j+1)2^{-k_i}]$ which intersect A must be at least $\dfrac{K}{c_{5.7} k_i} \varphi_2(r_{k_i})$ which is greater than $c_{5.8} K/\varphi(2^{-k_i})$. This gives $\varphi - P(A) \geqq c_{5.8} K$. Since K is arbitrary, we must have $\varphi - P(A) = +\infty$ which establishes (5.6). □

Corollary 5.6. *For any* $\varepsilon > 0$, *a.s. if* $0 < \beta < 1$, $\gamma \in \mathbb{R}$ *and* $\gamma_1 = \beta + \gamma - 2 - \varepsilon$, *then* $\forall$ *Borel* A

$$h_{\beta,\gamma} - p(A) < \infty \Rightarrow h_{\alpha\beta,\gamma_1} - p(X(A)) = 0 .$$

Example 5.7. Take $\varphi(s) = s$ in the theorem and we see that $h_{\alpha,-1-\varepsilon} - p(X[0,1]) = 0$. In fact, this is not too far from the truth, for

$$h_{\alpha,-1/2} - p(X[0,1]) = +\infty .$$

A proof of this is contained in [18].

We state a final

Corollary 5.8. *If* $X(t)$ *is strictly stable of index* $\alpha \leqq d$ *in* $\mathbb{R}^d$, *then a.s. for all Borel sets* E

$$\operatorname{Dim} X(E) = \alpha \operatorname{Dim} E .$$

Together with the corresponding result for Hausdorff measure this means that for any stable $X(t)$ of index $\alpha \leqq d$, a.s. every fractal A maps into a fractal $X(A)$.

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Received December 19, 1986