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A new approach to the single point catalytic super-Brownian motion

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Summary. A new approach is provided to the super-Brownian motion X with a single point-catalyst δ_c as branching rate. We start from a superprocess U with constant branching rate and spatial motion given by the $\frac{1}{2}$-stable subordinator. We prove that the occupation density measure λ^c of X at the catalyst c is distributed as the total occupation time measure of U. Furthermore, we show that X_t is determined from λ^c by an explicit representation formula. Heuristically, a mass $\lambda^c(ds)$ of "particles" leaves the catalyst at time s and then evolves according to Itô's Brownian excursion measure. As a consequence of our representation formula, the density field x of X satisfies the heat equation outside of c, with a noisy boundary condition at c given by the singularly continuous random measure λ^c. In particular, x is $\mathscr{C}^\infty$ outside the catalyst. We also provide a new derivation of the singularity of the measure λ^c.

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1 Introduction and main results

1.1 Basic model and motivation

The (critical continuous) *single point catalytic super-Brownian motion X_t*, $t \geq 0$ on the real line R has been introduced by Dawson and Fleischmann [5] (see also [4]). Intuitively, the process X describes the evolution of a large population of small branching particles on the real line, in the case when the branching phenomenon occurs only at a fixed point $c \in R$ called the *catalyst*. When particles are away from c, they move according to independent linear Brownian motions. On the other hand, when the particles arrive at c, they are subject to a critical branching mechanism, heuristically with an infinite branching rate (to compensate for the smallness of the "branching area"). In Dynkin's formulation [8], the local time at c of the Brownian particles governs the branching phenomenon.

A rigorous description of X can be given in terms of Laplace functionals. To this purpose, we denote by $\mathcal{M}_f$ the set of all finite measures on R equipped with the topology of weak convergence, by $\mathscr{C}(R,R_+)$ the set of all bounded nonnegative continuous functions on R, and by $\mathscr{B}(R,R_+)$ the set of all bounded nonnegative Borel measurable functions on R. Once and for all we fix a constant $\varrho > 0$, representing a multiplicative weight of the branching intensity δ_c at c. By definition, X is the time-homogeneous continuous Markov process on $\mathcal{M}_f$ such that, for every $\mu \in \mathcal{M}_f$, $t > 0$ and $h \in \mathscr{C}(R,R_+)$,

$$E\Big[\exp - < X_t, h > \;\Big|\; X_0 = \mu\Big] = \exp - \int \mu(\mathrm{d}b)\, v(0,b), \tag{1}$$

where the function $v(s,b)$, $s \geq 0$, $b \in R$ is the unique nonnegative solution of the integral equation

$$v(s,b) + \varrho \int_s^\infty \mathrm{d}r\, p(r-s, c-b)\, v^2(r,c) = 1_{\{s<t\}} \int \mathrm{d}y\, p(t-s, y-b)\, h(y). \tag{2}$$

Here $p(s,b)$ denotes the *Brownian transition density*

$$p(s,b) := \frac{1}{\sqrt{2\pi s}} \exp -\frac{b^2}{2s}, \qquad s > 0,\ b \in R. \tag{3}$$

Formulas (1) and (2) can be extended to get the Laplace functionals associated with the *finite-dimensional marginals* of X. In fact, using the Markov property, for $0 < t_1 < t_2 < \ldots < t_n$ and $h_1, \ldots, h_n \in \mathscr{C}(R,R_+)$,

$$E\Big[\exp - \sum_{i=1}^n < X_{t_i}, h_i > \;\Big|\; X_0 = \mu\Big] = \exp - \int \mu(\mathrm{d}b)\, v(0,b), \tag{4}$$

where the function $v(s,b)$, $s \geq 0$, $b \in R$ is the unique nonnegative solution of

$$v(s,b) + \varrho \int_s^\infty \mathrm{d}r\, p(r-s, c-b) v^2(r,c) = \sum_{i=1}^n 1_{\{s<t_i\}} \int \mathrm{d}a\, p(t_i - s, a-b) h_i(a). \tag{5}$$

For this formula, see Lemma 3.1.1 of [5], with a slightly different formulation, and Lemma 4.1 in Dynkin [8] (in a much more general setting).

Let us note some important features of equation (5). For $b \neq c$, the function $v(s,b)$, $s \geq 0$ is given by an explicit formula in terms of $v(s,c)$, $s \geq 0$ (this corresponds to the degenerate branching rate $\varrho\, \delta_c$). Then the function $v(s,c)$, $s \geq 0$ solves an integral equation for which the uniqueness of the nonnegative solution follows from an easy extension of the classical Gronwall lemma.

Let Y denote the *occupation time process* related to X. That is, Y_t for $t \geq 0$ fixed is the random element in $\mathcal{M}_f$ defined by

$$< Y_t, h > := \int_0^t \mathrm{d}s\, < X_s, h >, \qquad h \in \mathscr{C}(R,R_+).$$

It was proved in [5] that the measure Y_t has a density $y_t(b)$, $b \in R$ where y can be chosen to be jointly continuous in (t,b). For every b fixed, $y_t(b)$ is a monotone increasing function of t and one can consider the associated measure $\lambda^b(\mathrm{d}t)$ defined by $\lambda^b([u,v)) = y_v(b) - y_u(b)$. Henceforth λ^b is called the *occupation density measure* at b. When $b \neq c$, the measure λ^b is absolutely continuous with respect to Lebesgue measure. On the other hand, it was recently proved in [6] that λ^c is *singular*, although its carrying dimension is one ([5]).

A *heuristic explanation* can be given as follows. Mass arriving at c by the heat flow will "normally" be killed by the infinite branching rate, leading to a vanishing "density" $\mathrm{d}y_t(c)/\mathrm{d}t$ at c for Lebesgue almost all t. But again by this infinite rate, "occasionally" mass will be created at c. This density of mass at c will not occur at a fixed time, but nevertheless on a time set of "full dimension".

Our *goal in this work is* to provide a new approach to the process X. The motivation was to get a better understanding of the role of the occupation time measure λ^c at the catalyst. We will present a self-contained construction of X, which makes it clear that λ^c is the basic object in the model. Indeed, the process X is given in terms of λ^c by a deterministic formula. Our construction will allow us to rederive several known properties of X, such as the existence of the density field x of X and the singularity of the measure λ^c. We will also obtain some interesting new properties concerning the smoothness of x and the long-time behavior of X.

1.2 The super-stable subordinator

A key ingredient of our construction is the (critical continuous) superprocess U with constant branching rate $\varrho > 0$ whose spatial motion is the (one-dimensional) stable subordinator with index $1/2$. For convenience we call this process U the *super-stable subordinator.* Recall that the *stable subordinator* with index $1/2$ is the Lévy process on the real line whose transition probabilities are given by

$$q(s,b) := 1_{\{b>0\}} \frac{s}{\sqrt{2\pi b^3}} \exp -\frac{s^2}{2b}, \qquad s > 0,\ b \in R. \tag{6}$$

Notice that $q(s,\cdot)$ can also be interpreted as the density function of the (first) *hitting time* of the point s by a linear Brownian motion started at the origin.

The associated superprocess U with constant branching rate $\varrho > 0$ is by definition the $\mathcal{M}_f$-valued time-homogeneous continuous Markov process characterized as follows. For $\nu \in \mathcal{M}_f$, $0 < t_1 < \ldots < t_n$ and $h_1, \ldots, h_n \in \mathscr{C}(R, R_+)$,

$$E\Big[\exp - \sum_{i=1}^{n} < U_{t_i}, h_i > \ \Big|\ U_0 = \nu\Big] = \exp - \int \nu(\mathrm{d}b)\, u(0,b) \tag{7}$$

where $u(t,b)$, $t \geq 0$, $b \in R$ is the unique nonnegative solution to the integral equation

$$\begin{aligned} u(s,b) + \varrho \int_s^\infty \mathrm{d}r \int \mathrm{d}a\, q(r-s, a-b)\, u^2(r,a) \\ = \sum_{i=1}^{n} 1_{\{s<t_i\}} \int \mathrm{d}a\, q(t_i - s, a-b)\, h_i(a). \end{aligned} \tag{8}$$

Observe the analogy between equations (5) and (8): The Brownian transition density p is replaced by the stable transition density q, and the catalytic point measure $\varrho\,\delta_c(a)\,\mathrm{d}a$ is replaced by the measure $\varrho\,\mathrm{d}a$ corresponding to a constant branching rate ϱ. Both formulas are special cases of a formula valid for more general superprocesses (recall again Lemma 4.1 in Dynkin [8]).

Consider the *total occupation measure* $V := \int_0^\infty \mathrm{d}s\, U_s$ of U. This is a random *finite* measure on R. The finiteness of V follows from the well-known property that $U_s = 0$ for s sufficiently large, a.s. Moreover, the total mass process $U_s(R)$, $s \geq 0$ is a critical Feller diffusion (i.e. a zero-dimensional squared Bessel process). Hence its integral $V(R)$ is a stable random variable with index $1/2$.

Later, we will consider the situation when the initial value $U_0 = \nu$ of U is a (deterministic) measure supported on R_+. Then V is also supported on R_+ and we will interpret it as a measure on R_+.

We will need the Laplace functional of V. Such functionals were first computed by Dawson [2] and Iscoe [11] in special cases and later generalized; see Dynkin [8] or Section 7.4 in Dawson [3]. Here, we may simply start from the Laplace functional (7) for the finite-dimensional marginals of U, take $t_i = i/k$, $h_i = \varphi/k$, and, by a suitable passage to the limit, we arrive at the following result. Let $g(b)$ denote the *Green function* of the stable subordinator with index $1/2$:

$$g(b) := \int_0^\infty \mathrm{d}t\, q(t,b) = 1_{\{b>0\}}\,\frac{1}{\sqrt{2\pi b}}\,. \tag{9}$$

Then V has the following *Laplace functional:* For $\nu \in \mathscr{M}_f$ and $\varphi \in \mathscr{C}(R,R_+)$ with compact support,

$$E\Big[\exp - \langle V,\varphi\rangle \;\Big|\; U_0 = \nu\Big] = \exp - \langle \nu, w\rangle \tag{10}$$

where $w(b)$, $b \in R$ is the unique nonnegative solution to the equation

$$w(b) + \varrho\int \mathrm{d}a\, g(a-b)\,w^2(a) = \int \mathrm{d}a\, g(a-b)\,\varphi(a). \tag{11}$$

The uniqueness of the solution to (11) is easily established using arguments similar to the classical Gronwall lemma. Formulas (10) and (11) can be extended to any $\varphi \in \mathscr{B}(R,R_+)$ with compact support, via the monotone class theorem, and more generally, by a monotonicity argument, to any nonnegative Borel measurable function φ with compact support satisfying the following *finiteness condition:*

$$\sup_{b\in R} \int \mathrm{d}a\, g(a-b)\,\varphi(a) < \infty. \tag{12}$$

It is again easy to verify that equation (11) still has a unique nonnegative solution under this more general assumption.

1.3 Main result

We keep the notation introduced in the previous subsections. In addition *from now on we fix a nonzero measure $\mu \in \mathscr{M}_f$ and assume that $X_0 = \mu$ and $U_0 = \nu_\mu$, where the measure ν_μ is defined by*

$$< \nu_\mu , \varphi > := \int \mu(\mathrm{d}b) \int_0^\infty \mathrm{d}s\; q\big(|c-b|,s\big)\,\varphi(s), \qquad \varphi \in \mathscr{C}(R_+,R_+), \quad (13)$$

with the convention that $\int_0^\infty \mathrm{d}s\, q(0,s)\,\varphi(s) = \varphi(0)$ (that is $q(0,\cdot) = \delta_0$). The measure ν_μ corresponds to the "law" of the hitting time of c by a Brownian motion "distributed" according to $\mu(\mathrm{d}b)$ at time 0. In particular, $\nu_\mu = \delta_0$ if $\mu = \delta_c$.

It was proved in [5] that the point catalytic super-Brownian motion X lives on the set of all absolutely continuous measures, i.e. it can a.s. be represented as

$$X_t(\mathrm{d}b) = x_t(b)\,\mathrm{d}b, \qquad t > 0.$$

Moreover, the *density field* $x_t(b)$ can be chosen to be *jointly continuous* on the set $\{t>0\}\times\{b\neq c\}$.

We finally introduce the transition density $p^*(t,a,b)$ of *Brownian motion killed at* c. Obviously, $p^*(t,a,b) = 0$ if $(a-c)(b-c) \le 0$, and the reflection principle gives

$$p^*(t,a,b) = p(t,b-a) - p(t,b+a-2c) \quad \text{if} \quad (a-c)(b-c) > 0. \qquad (14)$$

We are now ready to state our main result.

Theorem 1 (representation formulas)

(a) **(the catalyst's occupation density measure** λ^c**)** *The random measures V and λ^c are identically distributed. In particular, the topological support of λ^c is R_+ a.s.*

(b) **(the mass density field** x**)** *With probability one, the density field $x_t(b)$ of X can be represented as*

$$x_t(b) = \int \mu(\mathrm{d}a)\, p^*(t,a,b) + \int_{[0,t)} \lambda^c(\mathrm{d}s)\; q\big(|b-c|,t-s\big), \qquad (15)$$

for $t > 0$, $b \neq c$. In particular, $x_t(b) > 0$ for every $t > 0$, $b \neq 0$, a.s.

Part (a) of this theorem is in a sense a *superprocess analogue* of the classical result saying that the local time measure at 0 of a linear Brownian motion is the occupation measure of a stable subordinator of index $1/2$ (equivalently, the inverse local time at 0 of a linear Brownian motion is a stable subordinator of index $1/2$, see e.g. [16], p. 223).

Note that the deterministic first term of the right hand side of (15) vanishes when $\mu = \delta_c$ (so that $\nu_\mu = \delta_0$). Let us briefly explain this representation formula.

Clearly, the first term in the right hand side corresponds to the contribution of (approximating) particles that have not yet reached the catalyst by time t. In other words, it results from the heat flow with absorption at c. To understand the second term, notice that, for a fixed $s > 0$, the function $q\big(|b|, s\big)$, $b \in R$ is the density of the *Brownian excursion* at time s, under the Itô measure denoted by $\mathbf{n}(\mathrm{d}e)$:

$$q\big(|b|, s\big)\, \mathrm{d}b = \mathbf{n}\{\ell(e) > s,\ e(s) \in \mathrm{d}b\}$$

where $\ell(e)$ is the duration of the excursion e, and $e(s)$ its location at time s (see e.g. [16], p. 456). This allows us to give the following *intuitive interpretation* for the second term in the representation formula (15). The mass present at time s in the vicinity of the catalyst can be measured by $\lambda^c(\mathrm{d}s)$. In terms of an approximating particle system, particles then move away from c according to Brownian excursions (recall that no branching is allowed outside of c). Such an excursion gives a contribution to $x_t(b)$ if it has not yet returned to the catalyst and has position b at time t.

We can combine both parts of Theorem 1 to get a *complete construction of the process* X. Indeed, starting from the total occupation measure V of the super-stable subordinator U, we define $Z_0 = \mu$ and for every $t > 0$

$$Z_t(\mathrm{d}b) := \Big(\int \mu(\mathrm{d}a)\, p^*(t, a, b) + \int_{[0,t)} V(\mathrm{d}s)\, q\big(|b - c|, t - s\big) \Big)\, \mathrm{d}b. \tag{16}$$

By (a) and (b), the measure-valued process Z is then a single point catalytic super-Brownian motion started at μ, and the occupation density measure of Z at the catalyst is V.

1.4 Proof of the main result

We will prove Theorem 1 by checking that the process Z defined from V via formula (16) is a single point catalytic super-Brownian motion, and then that the occupation density measure of Z at c is $V(\mathrm{d}s)$. Since the density of the random measure Z_t, as given in formula (16), is clearly a continuous function on $\{t > 0\} \times \{b \neq c\}$, we will thus obtain that this function coincides with the density field of X, completing the proof of formula (15).

We will make use of the following *identities* (recall the notations introduced in (3,6,9,14)): For $0 \le s < t$ and $a, b \in R$,

$$g(t - s) = p(t - s, 0), \tag{17}$$

$$p(t, b - a) = p^*(t, a, b) + \int_0^t \mathrm{d}r\, q\big(|c - a|, r\big)\, p(t - r, b - c). \tag{18}$$

The first equality is trivial. Formula (18) is easy to prove by a probabilistic argument. Indeed, the function $p(t, \cdot - a)$, which is the density at time t of a linear *Brownian motion* B started at a, is the sum of two contributions: The first one coming from those paths that do not hit c by time t, and the second one from the remaining paths. To obtain the contribution of the latter, notice that $q(|c-a|,\cdot)$ is the density function of the hitting time of c by B, and apply the strong Markov property at that hitting time.

Note the special case of (18) when $b = c$:

$$p(t, c-a) = \int_0^t \mathrm{d}r\, q\big(|c-a|, r\big)\, p(t-r, 0). \tag{19}$$

We will compute now the *Laplace functionals* of Z. Let $0 < t_1 < \ldots < t_n$ and $h_1, \ldots, h_n \in \mathscr{C}(R, R_+)$. Then, by the definition (16) of Z,

$$\sum_{i=1}^n < Z_{t_i}, h_i > \;=\; \sum_{i=1}^n \int \mu(\mathrm{d}a) \int \mathrm{d}b\; p^*(t_i, a, b)\, h_i(b) + < V, \varphi > \quad \text{with}$$

$$\varphi(s) := \sum_{i=1}^n 1_{\{0 \le s < t_i\}} \int \mathrm{d}b\, q\big(|b-c|, t_i - s\big) h_i(b), \qquad s \geq 0. \tag{20}$$

It is easily checked that φ satisfies the finiteness condition (12). So we can use (10), (11) to calculate the Laplace functional of Z:

$$\begin{aligned} &E \exp - \sum_{i=1}^n < Z_{t_i}, h_i > \qquad\qquad (21)\\ &\quad = \exp\Big[- \sum_{i=1}^n \int \mu(\mathrm{d}a) \int \mathrm{d}b\, p^*(t_i, a, b)\, h_i(b) \;-\; < \nu_\mu, w > \Big] \end{aligned}$$

where, for $s \geq 0$,

$$w(s) + \varrho \int_s^\infty \mathrm{d}r\, g(r-s)\, w^2(r) \;=\; \int_s^\infty \mathrm{d}r\, g(r-s)\, \varphi(r).$$

By the definition (20) of φ and using the identities (17) and (19), we get that w satisfies the equation

$$w(s) + \varrho \int_s^\infty \mathrm{d}r\, p(r-s, 0)\, w^2(r) \;=\; \sum_{i=1}^n 1_{\{s < t_i\}} \int \mathrm{d}a\, p(t_i - s, a - c)\, h_i(a).$$

Comparing this with (5) in the special case $b = c$ we obtain $w(s) = v(s, c)$ where v is the unique nonnegative solution of (5). We then use this integral equation and the definition (13) of ν_μ to get

$$\begin{aligned} < \nu_\mu, w > \;=\; &\sum_{i=1}^n \int \mu(\mathrm{d}b) \int_0^{t_i} \mathrm{d}s\, q\big(|c-b|, s\big) \int \mathrm{d}a\, p(t_i - s, a - c)\, h_i(a) \\ &- \varrho \int \mu(\mathrm{d}b) \int_0^\infty \mathrm{d}s\, q\big(|c-b|, s\big) \int_s^\infty \mathrm{d}r\, p(r-s, 0)\, v^2(r, c). \end{aligned}$$

Next we exploit (19) to arrive at

$$
\begin{aligned}
< \nu_\mu, w > \;=\; & \sum_{i=1}^{n} \int \mu(\mathrm{d}b) \int_0^{t_i} \mathrm{d}s \; q\big(|c-b|, s\big) \int \mathrm{d}a \; p(t_i - s, a - c) \, h_i(a) \\
& - \varrho \int \mu(\mathrm{d}b) \int_0^\infty \mathrm{d}r \; p(r, c-b) \, v^2(r,c).
\end{aligned}
$$

Finally, using the identity (18) the exponent in the right hand side of (21) becomes

$$
\begin{aligned}
& \sum_{i=1}^{n} \int \mu(\mathrm{d}b) \int \mathrm{d}a \; p^*(t_i, b, a) \, h_i(a) \; + \; < \nu_\mu, w > \\
& = \int \mu(\mathrm{d}b) \sum_{i=1}^{n} \int \mathrm{d}a \; p(t_i, b-a) \, h_i(a) - \varrho \int \mu(\mathrm{d}b) \int_0^\infty \mathrm{d}r \; p(r, c-b) v^2(r,c).
\end{aligned}
$$

But by (5) with $s = 0$, this is nothing else than $\int \mu(\mathrm{d}b) \, v(0,b)$. Inserting into (21) and comparing with (4), we get that Z has the same finite-dimensional marginals as X. Hence ([5]), a version $\overline{Z}$ of Z must be a (continuous) single point catalytic super-Brownian motion. However, it is immediate on the defining formula (16) of Z that $< Z_t, \varphi >$ is a.s. continuous whenever $\varphi \in \mathscr{C}(R, R_+)$ has a compact support not containing c. For such functions φ we have thus $< Z_t, \varphi > = < \overline{Z}_t, \varphi >$ for every $t \geq 0$ a.s. Moreover, we know from [5] that $\overline{Z}_t(\{c\}) = 0$, for every $t > 0$, a.s. Since the same property holds for Z by definition, we conclude that $\overline{Z}$ and Z are indistinguishable, so that Z itself is a single point catalytic super-Brownian motion, what had to be proved.

Next we want to calculate the occupation density measures related to Z. Set

$$
g^*(t,a,b) := \int_0^t \mathrm{d}s \; p^*(s,a,b), \qquad F(s) := \int_0^s dr \; q(1,r). \tag{22}
$$

Observe that by the definition (16) of Z, for $h \in \mathscr{C}(R, R_+)$,

$$
\begin{aligned}
& \int_0^t \mathrm{d}s \int Z_s(\mathrm{d}b) \, h(b) \\
& = \int \mathrm{d}b \; h(b) \int \mu(\mathrm{d}a) \, g^*(t,a,b) + \int \mathrm{d}b \; h(b) \int_{[0,t)} V(\mathrm{d}s) \int_s^t \mathrm{d}r \; q\big(|b-c|, r-s\big).
\end{aligned}
$$

Using the scaling property

$$
K^2 q(Ks, K^2 b) = q(s,b)
$$

of the stable subordinator and the definition (22) of F, we conclude that the measure $\int_0^t ds \, Z_s$ has a density given by

$$
b \mapsto \int \mu(\mathrm{d}a) \, g^*(t,a,b) + \int_{[0,t)} V(\mathrm{d}s) \, F\Big(\tfrac{t-s}{(b-c)^2}\Big), \qquad b \neq c. \tag{23}
$$

Since F is a distribution function, by setting $F(\infty) = 1$ we see that the previous formula defines a continuous function of $b \in R$, which we can therefore identify with the (jointly continuous) occupation density field of Z (denoted again by y), taken at time $t > 0$. Hence $\lambda^c([0,t)) = y_t(c) = V([0,t))$ a.s., for every fixed t. But both functions are left-continuous in $t > 0$, and we get $\lambda^c = V$ a.s., as wanted.

We still have to prove the remaining assertions of Theorem 1. If μ is not concentrated at c, then by the definition (13) of the initial measure ν_μ, the topological support of ν_μ is R_+. On the other hand, $\mu = \alpha\,\delta_c$ implies $\nu_\mu = \alpha\,\delta_0$. In both cases, the topological support of the total occupation measure V is R_+ by Theorem 1.5 of Perkins [14]. In particular, $V([0,\varepsilon]) > 0$ for every $\varepsilon > 0$, a.s. Then (a) and the representation formula (15) show that $x_t(b) > 0$ for every $t > 0$, $b \neq c$, a.s. □

Remark Let us briefly discuss the relationship between our approach and the results of [5]. Without any reference to [5], the previous proof shows that the process Z defined by (16) has the marginal distributions of the single point catalytic super-Brownian motion, as given in (4) and (5). We mainly needed to refer to [5] for the almost sure continuity of Z (a proof not depending on [5] would require some information on the local behavior of the random measure V). Our construction clearly gives the existence and joint continuity of the mass density field of Z on $\{t > 0\} \times \{b \neq c\}$. The joint continuity of the associated occupation density field on $R_+ \times R$ also follows from the explicit formula (23), provided we know a priori that the random measure V has no atoms (this fact can be easily deduced from the second moment formulas for V, see (37) and (46) below). Finally, we can also remark that the Laplace functionals of the density field $x_t(b)$ or the occupation density field $y_t(b)$ follow immediately from the expression of these quantities in terms of λ^c and the formulas of § 1.2. ◊

2 Some applications

In this section, we develop a few simple applications of our main result.

2.1 Measurability of the past with respect to the present

We will use Theorem 1 (b) to obtain a somewhat surprising measurability property of X. We assume that this process is defined on a probability space $(\Omega, \mathscr{F}, P)$.

Corollary 2 ("backward" measurability) *Fix $t > 0$. Denote by $\sigma(X_t)$ the σ-field generated by X_t, augmented by the $\mathscr{F}$-measurable sets of P-probability zero. The random measure $1_{[0,t)}(s)\,\lambda^c(\mathrm{d}s)$ is $\sigma(X_t)$-measurable. Thus, $(X_s\,,\ 0 < s \le t)$ is $\sigma(X_t)$-measurable.*

Proof . Clearly, the densities $x_t(b)$, $b \neq c$ are $\sigma(X_t)$-measurable. By the representation Theorem 1 (b), this implies that, for every $\alpha > 0$,

$$\int_{[0,t)} \lambda^c(\mathrm{d}s)\, q(\alpha, t-s) = \int_{[0,t)} \lambda^c(\mathrm{d}s)\, \frac{\alpha}{\sqrt{2\pi\,(t-s)^3}}\, \exp -\frac{\alpha^2}{2(t-s)}$$

is measurable with respect to $\sigma(X_t)$. Let $\eta(\mathrm{d}s)$ denote the image measure of $1_{[0,t)}(s)\,(t-s)^{-3/2}\lambda^c(\mathrm{d}s)$ by the mapping $s \mapsto (t-s)^{-1}$. We get that

$$\int_{[0,t)} \eta(\mathrm{d}r)\,\exp - \frac{\alpha^2}{2}\,r$$

is $\sigma(X_t)$-measurable for every $\alpha > 0$. It follows that η, hence $1_{[0,t)}(s)\,\lambda^c(\mathrm{d}s)$ is $\sigma(X_t)$-measurable. Finally, if r belongs to $[0,t]$, then by the representation formula (15), X_r is a measurable function of $1_{[0,t)}(s)\,\lambda^c(\mathrm{d}s)$, finishing the proof. $\square$

2.2 *Smoothness of the mass density field*

The next result shows that (outside the catalyst) $x_t(b)$ is much smoother than in the constant branching rate case (recall that the one-dimensional super-Brownian motion density field is commonly believed to have a critical Hölder index $1/2$, with respect to regularity in the space variable; see Reimers [15] for a one-sided estimate). This in particular answers a question of Adler [1, p. 14].

Corollary 3 (smoothness of the mass density field) *With probability one, the density field $x_t(b)$ is a $\mathscr{C}^\infty$-function of (t,b) on the set $\{t > 0,\ b \neq c\}$ and satisfies the heat equation:*

$$\frac{\partial x_t(b)}{\partial t} = \frac{1}{2}\,\frac{\partial^2 x_t(b)}{\partial b^2}, \qquad t > 0, \quad b \neq c.$$

Proof . By symmetry, we may restrict our attention to the domain $D := \{(t,b);\ t > 0,\ b > c\}$. We start from the representation formula (15) of x and first observe that the function $b \mapsto \int \mu(\mathrm{d}a)\,p^*(t,a,b)$ is of class $\mathscr{C}^\infty$ on D and solves the heat equation in this domain. Therefore, we only need to consider the term

$$\overline{x}_t(b) := \int_{[0,t)} \lambda^c(\mathrm{d}s)\,q(b-c,t-s), \qquad (t,b) \in D. \tag{24}$$

Note that, for every choice of integers $k,\ell \geq 0$, the partial derivative

$$q_{k,\ell}(b,t) := \frac{\partial^{k+\ell}}{\partial t^k \partial b^\ell}\,q(b,t)$$

can be written as a finite linear combination of terms of the type

$$b^i\,t^{-3/2-j}\exp\big(-b^2/2t\big)$$

where i,j are nonnegative integers such that $i \leq j+1$. However $b^i\,t^{-3/2-j} = (b^2/t)^{3/2+j}\,b^{-3-2j+i}$ where $-3-2j+i \leq -2-j < 0$. Hence, for $k,\ell \geq 0$ and $\varepsilon > 0$ fixed,

$$\sup_{t>0,\ b\geq\varepsilon} |q_{k,\ell}(b,t)| < \infty. \tag{25}$$

We can then use this bound to verify by induction that the function $\overline{x}_t(b)$ defined in (24) is $\mathscr{C}^\infty$ in D and that

$$\frac{\partial^{k+\ell}}{\partial t^k \partial b^\ell}\,\overline{x}_t(b) = \int_{[0,t)} \lambda^c(\mathrm{d}s)\, q_{k,\ell}(b-c, t-s). \tag{26}$$

Since $q(b,t)$ satisfies the heat equation in $\{t > 0,\, b > 0\}$, it follows readily from (26) that $\overline{x}_t(b)$ solves the heat equation in D. □

Remark The function $x_t(b)$ solves the heat equation in D with the *generalized boundary conditions* $\mu(\mathrm{d}b)$ on $\{t = 0\}$ and $\lambda^c(\mathrm{d}t)$ on $\{b = c\}$. These conditions should be understood as the a.s. statements

$$\lim_{t\downarrow 0} \int \mathrm{d}b\; \varphi(b) x_t(b) = \int \mu(\mathrm{d}b)\, \varphi(b),$$

$$\lim_{b\to c} \int \mathrm{d}t\; \psi(t) x_t(b) = \int \lambda^c(\mathrm{d}t)\, \psi(t),$$

for $\varphi \in \mathscr{C}(R, R_+)$, $\psi \in \mathscr{C}(R_+, R_+)$ (the second convergence follows easily from the joint continuity of the occupation density field $y_t(b)$ on $R_+ \times R$). ◇

2.3 Asymptotic behavior

Recall that, for $a \geq 0$, $F(a) = \int_0^a \mathrm{d}b\, q(1,b)$, and for $a, b \in R$ set

$$g^*(a,b) := g^*(\infty, a, b) = \int_0^\infty \mathrm{d}t\, p^*(t,a,b)$$

so that, from (14),

$$g^*(a,b) = 1_{\{(a-c)(b-c)>0\}}\, 2\left(|a-c| \wedge |b-c|\right). \tag{27}$$

The next result is a refinement of Theorem 1.3.2 in [5]. (See also Dynkin [9]).

Corollary 4 (total occupation density) *With probability one, for every b in R,*

$$y_t(b) \underset{t\to\infty}{\longrightarrow} y_\infty(b) := \int \mu(\mathrm{d}a)\, g^*(a,b) + \lambda^c(R_+),$$

where the random variable $\lambda^c(R_+)$ has a stable distribution with index $1/2$.
Proof . We simply pass to the limit as $t \to \infty$ in the formula

$$y_t(b) = \int \mu(\mathrm{d}a)\, g^*(t,a,b) + \int_{[0,t)} \lambda^c(\mathrm{d}s)\, F\left(\tfrac{t-s}{(b-c)^2}\right),$$

(recall (23)) using that $F(t) \uparrow 1$ as $t \to \infty$. In § 1.2 we have already noticed that $\lambda^c(R_+)$ or equivalently $V(R_+)$ has a stable distribution of index $1/2$. □

Now we want to complement the local extinction Proposition 1.3.1 of [5] (where X starts according to the Lebesgue measure) by a total extinction property:

Corollary 5 (total mass process) *The total mass of X at time t is*

$$X_t(R) = \int \mu(\mathrm{d}a)\Big(1 - F\Big(\tfrac{t}{(a-c)^2}\Big)\Big) + \int_0^t \lambda^c(\mathrm{d}s)\sqrt{\tfrac{2}{\pi\,(t-s)}}\,. \tag{28}$$

This total mass is (strictly) positive for every $t \geq 0$ a.s. and converges to 0 a.s. as $t \to \infty$.

Proof . To get (28), integrate (15) with respect to db by using the identity (18). The positivity of $X_t(R)$ follows from that of $x_t(b)$ (recall Theorem 1 (b)). Then, by dominated convergence,

$$\lim_{t\to\infty} \int \mu(\mathrm{d}a)\Big(1 - F\Big(\tfrac{t}{(a-c)^2}\Big)\Big) = 0.$$

To complete the proof, first notice that

$$\lim_{t\to\infty} \int_0^{t-1} \lambda^c(\mathrm{d}s)\sqrt{\tfrac{2}{\pi\,(t-s)}} = 0 \qquad \text{a.s.}$$

because the measure λ^c is finite. To calculate the first moment of the remaining part of the integral we may replace λ^c by V and get by the well-known superprocess first-moment formula,

$$\begin{aligned} E\int_{t-1}^t \frac{V(\mathrm{d}s)}{\sqrt{t-s}} &= \int_0^t \nu_\mu(\mathrm{d}r)\int_r^\infty \mathrm{d}s\, g(s-r)\,\frac{1_{\{t-1\leq s<t\}}}{\sqrt{t-s}} \\ &= \int_0^t \nu_\mu(\mathrm{d}r)\int_{(t-1)\vee r}^t \frac{\mathrm{d}s}{\sqrt{2\pi\,(t-s)(s-r)}}\,. \end{aligned}$$

But this obviously converges to 0 as $t \to \infty$ by dominated convergence. Hence $X_t(R)$ converges to 0 in probability as $t \to \infty$. However, $t \mapsto X_t(R)$ is a continuous nonnegative martingale (by the Markov property and the first-moment formula of [5], Theorem 1.2.1 for instance). Thus the a.s. convergence follows from the martingale convergence theorem. □

3 Singularity of the occupation density measure at the catalyst

3.1 Statement of the result

By Theorem 1 (a) the topological support of the measure λ^c coincides a.s. with R_+. We also already mentioned that λ^c is a continuous measure. Nonetheless, in the case when $X_0 = \delta_c$, Dawson, Fleischmann, Li and Mueller [6] have recently proved that λ^c is a.s. singular. We will propose an *alternative approach* to that result, which applies to the case of a general initial state $\mu \in \mathcal{M}_f$ and also provides some additional information.

Recall that by Theorem 1 (a) the random measures λ^c and V are identically distributed provided that X and U are started with μ and ν_μ, respectively. We will now consider the super-stable subordinator U with a general initial state $\nu \in \mathcal{M}_f$, and its total occupation measure $V = \int_0^\infty \mathrm{d}s\, U_s$.

Set $h(\varepsilon) := \varepsilon \log\log \frac{1}{\varepsilon}$ and denote by h-m the associated *Hausdorff measure.*

Theorem 6 (singularity of V) *We can find two positive constants C, C' such that, with probability one, for the set*

$$H := \left\{ b \in R;\ \limsup_{\varepsilon \downarrow 0} \frac{V\big([b-\varepsilon, b+\varepsilon]\big)}{\varepsilon \log\log(1/\varepsilon)} \geq C \right\}$$

we have $V(R \backslash H) = 0$ *and* $h\text{-m}(H) \leq C' V(R) < \infty$.

Roughly speaking, the theorem states that for V-almost all b one can find a sequence $\varepsilon_n = \varepsilon_n(b) \to 0$ as $n \to \infty$ such that V has mass of order $\geq \varepsilon_n \log\log(1/\varepsilon_n)$ in the ε_n-vicinity of b. In particular, the "upper density" of the measure V in these points must be $+\infty$.

Note that the bound $h\text{-m}(H) \leq C' V(R)$ follows from the well-known density theorems of Rogers and Taylor [17] so that we only have to check that almost surely V is concentrated on H.

3.2 Canonical measures and Campbell measure formula

Before proceeding to the proof of Theorem 6, we recall, in the special case of the process U, a few basic facts about canonical measures of superprocesses. We refer to Section 4 in El Karoui and Roelly [10] (see also Dawson and Perkins [7] and Le Gall [12] for related results).

We assume that the process U is the *canonical* process on the space $\Omega := \mathcal{C}(R_+, \mathcal{M}_f)$ of all continuous functions from R_+ into $\mathcal{M}_f$. Denote by $\mathbf{P}_\nu$ the probability on Ω under which U is a super-stable subordinator (with index $1/2$) started at $\nu \in \mathcal{M}_f$. We also set $\sigma := \sup\{s;\ U_s \neq 0\}$ which represents the *extinction time* of U. Then, for every $b \in R$, the limit

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \mathbf{P}_{\varepsilon\delta_b} =: \mathbf{Q}_b \tag{29}$$

exists in the following "weak" sense. $\mathbf{Q}_b$ is a σ-finite measure on Ω that does not charge the zero trajectory and satisfies:

(i) $\mathbf{Q}_b(\sigma \geq t) < \infty$ for every $t > 0$,

(ii) for every $0 < t_1 < \ldots < t_n$, for every function φ continuous and bounded on $\mathcal{M}_f^n$ such that $\varphi(0, \ldots, 0) = 0$,

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mathbf{E}_{\varepsilon\delta_b}\, \varphi(U_{t_1}, \ldots, U_{t_n}) = \mathbf{Q}_b\, \varphi(U_{t_1}, \ldots, U_{t_n}).$$

The measures $\mathbf{Q}_b$ are called the *canonical measures* of the super-stable subordinator U.

Conversely, we can recover the laws $\mathbf{P}_\nu$ from the collection $\mathbf{Q}_b$ in the following way. If $\mathcal{N}(\mathrm{d}\omega)$ denotes a Poisson point measure on Ω with intensity $\int \nu(\mathrm{d}b)\, \mathbf{Q}_b(\cdot)$, then the process

$$\mathscr{U}_t := \int \mathcal{N}(\mathrm{d}\omega)\, U_t(\omega),\ t \geq 0$$

has distribution $\mathbf{P}_\nu$ (see Théorème 17 in [10]). In other words, the measures $\mathbf{Q}_b$ describe the "cluster processes" of the infinitely divisible law $\mathbf{P}_\nu$. Notice that $\mathscr{V} := \int \mathcal{N}(\mathrm{d}\omega)\, V(\omega)$ is the total occupation measure of $\mathscr{U}$.

We now observe that it is enough to check that the statement of Theorem 6 holds a.e. under every measure $\mathbf{Q}_b$ (by translation invariance it even suffices to consider $\mathbf{Q}_0$). Indeed, we can then apply this result to each atom in the (countable) support of $\mathcal{N}(\mathrm{d}\omega)$ and easily conclude that the same property holds for the total occupation measure $\mathscr{V}$ of $\mathscr{U}$.

We need to derive a few properties of the canonical measures $\mathbf{Q}_b$. Using the classical *first-moment formula* for superprocesses and the connection between measures $\mathbf{P}_\nu$ and $\mathbf{Q}_b$, one immediately obtains the formula

$$\mathbf{Q}_b < V, \varphi > = \int \mathrm{d}a\, g(a-b)\, \varphi(a) \tag{30}$$

for any nonnegative measurable function φ on R (g is defined in (9)).

Another key ingredient is the *Campbell measure formula* we will now establish. On the Skorohod space $\mathscr{D}\big([0,t], R_+\big)$, let $P_b^t(\mathrm{d}f)$ denote the law of the stable subordinator with index $1/2$ started at b (and running on the time interval $[0,t]$). Let Θ denote the space of all *point measures* on $R \times \Omega$ (precisely, the set of all counting measures on $R \times \Omega$ that are finite on sets of the type $[0,t] \times \{\sigma \geq \varepsilon\}$) and write θ for the generic element of Θ. For $f \in \mathscr{D}\big([0,t], R_+\big)$, we let $\mathbf{P}^f(\mathrm{d}\theta)$ denote the unique probability measure on Θ under which $\theta(\mathrm{d}s\, \mathrm{d}\omega)$ is a *Poisson point measure on* $R \times \Omega$ *with intensity*

$$2\, \varrho\, 1_{[0,t]}(s)\, \mathrm{d}s\, \mathbf{Q}_{f(s)}(\mathrm{d}\omega). \tag{31}$$

The announced formula now reads as follows:

Proposition 7 (Campbell measure formula) *For $b \in R$ and every nonnegative measurable function Φ on $R \times \mathscr{M}_f$,*

$$\mathbf{Q}_b \int V(\mathrm{d}a)\, \Phi(a, V) = \int_0^\infty \mathrm{d}t \int P_b^t(\mathrm{d}f)\, \mathbf{E}^f\, \Phi\Big(f(t), \int \theta(\mathrm{d}s\, \mathrm{d}\omega)\, V(\omega)\Big). \tag{32}$$

Formulas such as (32) are part of the folklore of the subject (see [7], Section 4, and [12] for closely related facts). For the sake of completeness, we will provide a short proof. But first let us briefly give a *heuristic interpretation* of (32). The left-hand side describes the "law" of the pair (a, V) where V is the total occupation measure $\int_0^\infty \mathrm{d}t\, U_t$ under the canonical measure $\mathbf{Q}_b$, and a is a point "chosen" according to $V(\mathrm{d}a)$. In the right-hand side one selects "at random" a point t from R_+, and considers the value $f(t)$ at time t of a stable subordinator with index $1/2$ starting at b. Moreover, at Poissonian time points s along the path $f(s)$, one starts "cluster processes" ω "distributed" according to $\mathbf{Q}_{f(s)}(\mathrm{d}\omega)$, and then one superimposes their total occupation measures $V(\omega)$ to arrive at the quantity $\int\theta(\mathrm{d}s\,\mathrm{d}\omega)\,V(\omega)$. The formula (32) states that the pair $\Big(f(t),\ \int\theta(\mathrm{d}s\,\mathrm{d}\omega)\,V(\omega)\Big)$ constructed in this way has the same "law" as the previously defined pair (a, V).

Proof . It is enough to consider the case $\Phi(y, V) = \varphi(y)\exp - < V, \psi >$ where both φ and ψ are nonnegative continuous functions on R with compact support. Using the Poisson exponential formula, the right-hand side of (32) can then be written as

$$\int_0^\infty \mathrm{d}t \int P_b^t(\mathrm{d}f)\,\varphi\big(f(t)\big)\,\exp -2\varrho\int_0^t \mathrm{d}s\ \mathbf{Q}_{f(s)}\Big(1-\exp - < V,\psi >\Big)$$
$$= E_b\int_0^\infty \mathrm{d}t\ \varphi(\xi_t)\,\exp -2\varrho\int_0^t \mathrm{d}s\,w(\xi_s) =: \widetilde{u}(b) \qquad (33)$$

where $w(a) := \mathbf{Q}_a\big(1 - \exp - < V, \psi >\big)$ and the process $(\xi_s,\ s \geq 0)$ is a stable subordinator with index $1/2$ that starts at b under the probability measure P_b.

On the other hand, the left-hand side of (32) is

$$u(b) := \mathbf{Q}_b < V, \varphi > \exp - < V, \psi > .$$

For $\lambda \in [0, 1]$ and $b \in R$, set

$$w_\lambda(b) := \mathbf{Q}_b\Big(1 - \exp - < V,\ \lambda\varphi + \psi >\Big). \qquad (34)$$

Then $w_\lambda(b)\ \leq\ \mathbf{Q}_b < V,\ \lambda\varphi + \psi >$, and the first-moment formula (30) shows that the functions w_λ are uniformly bounded over R. Now recall the way the measures $\mathbf{P}_\nu$ can be reconstructed from the canonical measures $\mathbf{Q}_b$. Using the exponential formula for Poisson measures and comparing with formulas (10) and (11), we immediately get that w_λ is the unique nonnegative solution of

$$w_\lambda(b) + \varrho\int \mathrm{d}a\ g(a-b)\,w_\lambda^2(a) = \int \mathrm{d}a\ g(a-b)\,(\lambda\varphi+\psi)(a). \qquad (35)$$

Moreover, by differentiating (34),

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\,w_\lambda(b)\Big|_{\lambda=0+} = \mathbf{Q}_b < V, \varphi > \exp - < V, \psi > = u(b).$$

The justification is easy thanks to the finiteness of first moments (30) and bounded convergence. We can then differentiate (35) with respect to λ at $\lambda = 0+$ to get

$$u(b) + 2\varrho\int \mathrm{d}a\ g(a-b)\,u(a)\,w(a) = \int \mathrm{d}a\ g(a-b)\,\varphi(a). \qquad (36)$$

The justification is again easy because both the functions w_λ and their derivatives with respect to $\lambda \in [0, 1]$ are uniformly bounded and vanish outside some common bounded region. A Gronwall lemma-type argument shows that u is uniquely determined by equation (36) when w and φ are given. It is then a simple exercise to check that the function ("Feynman-Kac solution") $\widetilde{u}$ of (33) solves (36). Hence $u = \widetilde{u}$, which completes the proof. □

We shall finally need the *second-moment formula* for the total occupation measure V under $\mathbf{Q}_b$:

$$\mathbf{Q}_b < V, \varphi >^2 \;=\; 2\varrho \int \mathrm{d}a\, \varphi(a) \int \mathrm{d}a' \varphi(a') \int \mathrm{d}z\, g(z-b) g(a-z) g(a'-z) \quad (37)$$

for every nonnegative measurable function φ on R. This formula can for instance be obtained by taking $\Phi(a, V) = \varphi(a) < V, \varphi >$ in (32).

3.3 *Proof of the singularity theorem*

As already explained, the proof of Theorem 6 reduces to finding a constant $C > 0$ such that, $\mathbf{Q}_0$ a.e., $V(\mathrm{d}a)$ a.e.,

$$\limsup_{\varepsilon \downarrow 0} \frac{V\big([a-\varepsilon, a+\varepsilon]\big)}{h(\varepsilon)} \geq C.$$

By Proposition 7, we see that this claim is in turn equivalent to checking that $\mathrm{d}t$ a.e., $P_0^t(\mathrm{d}f)$ a.s., $\mathbf{P}^f(\mathrm{d}\theta)$ a.s.,

$$\limsup_{\varepsilon \downarrow 0} \frac{1}{\varepsilon \log\log(1/\varepsilon)} \int_{[0,t]\times\Omega} \theta(\mathrm{d}s\, \mathrm{d}\omega)\, V(\omega)\Big(\big[f_t - \varepsilon, f_t + \varepsilon\big]\Big) \geq C \qquad (38)$$

where we now write $f_t := f(t)$ for convenience. We can in fact even verify the existence of a constant C such that (38) holds for all $t > 0$, $P_0^t(\mathrm{d}f)$ a.s., $\mathbf{P}^f(\mathrm{d}\theta)$ a.s. We fix $t > 0$ and consider the unique integer N such that $2^{-N} \leq t < 2^{-N+1}$. For every integer $n \geq N$, we consider the following subset of $\mathscr{D}\big([0,t], R\big)$:

$$H_n := \Big\{ f \in \mathscr{D}\big([0,t], R\big);\; f_s \in \big[f_t - 2^{-2n+1}, f_t - 2^{-2n}\big], \forall s \in [t-2^{-n},\, t-2^{-n-1}] \Big\}.$$

Next introduce the *scaling transformation*

$$\mathsf{T}:\; \big(f(s),\; 0 \leq s \leq t\big) \longmapsto \Big(4\big[f(\tfrac{t}{2} + \tfrac{s}{2}) - f(\tfrac{t}{2})\big],\; 0 \leq s \leq t\Big).$$

Note that $\mathsf{T}^{-1} H_n = H_{n+1}$ and that the law P_0^t of the stable subordinator with index $1/2$ is invariant with respect to T. We can then apply Birkhoff's individual ergodic theorem, together with Blumenthal's zero-one law for the stable subordinator reversed at time t, to obtain $P_0^t(\mathrm{d}f)$ a.s.

$$\lim_{m\to\infty} \frac{1}{m} \sum_{n=N}^{m} 1_{H_n}(f) \;=\; c_0 \;>\; 0, \qquad (39)$$

where $c_0 = P_0^t(H_N)$ does not depend on t, again by a scaling argument.

We now fix $f \in \mathscr{D}\big([0,t],R\big)$ such that (39) holds, and we will prove for this function f that (38) is true $\mathbf{P}^f(\mathrm{d}\theta)$ a.s. for a suitable constant C independent of t, f and θ.

For $\varepsilon > 0$ and $0 \le r \le u \le t$, set

$$W_\varepsilon[r,u] := \int_{[r,u]\times\Omega} \theta(\mathrm{d}s\,\mathrm{d}\omega)\, V(\omega)\Big(\big[f_t - \varepsilon, f_t + \varepsilon\big]\Big). \tag{40}$$

Let $\varepsilon > 0$ and $n \ge N$ be such that $2^{-2n} > 3\varepsilon$ and $f \in H_n$. By the definition of the law $\mathbf{P}^f$, we have

$$\begin{aligned}\mathbf{E}^f W_\varepsilon\big[t-2^{-n},\, t-2^{-n-1}\big] &= 2\varrho\int_{t-2^{-n}}^{t-2^{-n-1}} \mathrm{d}s\ \mathbf{Q}_{f_s} V\Big(\big[f_t-\varepsilon, f_t+\varepsilon\big]\Big)\\ &= 2\varrho\int_{t-2^{-n}}^{t-2^{-n-1}} \mathrm{d}s \int_{f_t-\varepsilon}^{f_t+\varepsilon} \frac{\mathrm{d}r}{\sqrt{2\pi\,(r-f_s)}}\,,\end{aligned}$$

using the first-moment formula (30) together with the explicit expression (9) for the Green function g and also observing that $r - f_s \ge 2\varepsilon > 0$ in the range of integration, by our assumption $f \in H_n$. More precisely, the bound $2^{-2n-1} \le r - f_s \le 2^{-2n+2}$ for $r \in [f_t - \varepsilon, f_t + \varepsilon]$ leads to the following estimate

$$c_1\,\varepsilon \le \mathbf{E}^f W_\varepsilon\big[t-2^{-n},\, t-2^{-n-1}\big] \le C_1\,\varepsilon \tag{41}$$

where $c_1 := 2^{-1/2}\pi^{-1/2}\varrho,\ C_1 := 2\,\pi^{-1/2}\varrho.$

Keeping the same assumptions on ε, n, f, we turn to the variance with respect to $\mathbf{P}^f$:

$$\mathrm{var}_{\mathbf{P}^f}\, W_\varepsilon\big[t-2^{-n},\, t-2^{-n-1}\big] = 2\varrho\int_{t-2^{-n}}^{t-2^{-n-1}} \mathrm{d}s\ \mathbf{Q}_{f_s} V^2\Big(\big[f_t-\varepsilon, f_t+\varepsilon\big]\Big).$$

By the second-moment formula (37) and (9), for $f_s \in \big[f_t - 2^{-2n+1}, f_t - 2^{-2n}\big]$, we have

$$\begin{aligned}&\mathbf{Q}_{f_s} V^2\Big(\big[f_t-\varepsilon, f_t+\varepsilon\big]\Big)\\ &= 2\varrho\,(2\pi)^{-3/2}\int_{[f_t-\varepsilon,\, f_t+\varepsilon]^2} \mathrm{d}a\,\mathrm{d}a' \int_{f_s}^{a\wedge a'} \frac{\mathrm{d}z}{\sqrt{(z-f_s)(a-z)(a'-z)}}\\ &= 8\varrho\,(2\pi)^{-3/2}\int_{f_s}^{f_t+\varepsilon} \frac{\mathrm{d}z}{\sqrt{z-f_s}}\Big(\sqrt{f_t+\varepsilon-z} - \sqrt{(f_t-\varepsilon)\vee z - z}\Big)^2.\end{aligned}$$

Consider first the part of the integral corresponding to $z \ge f_t - 2\varepsilon$. On this set, $z - f_s \ge 2^{-2n-2}$, and omitting the second square root, we get the upper bound $2^{n+1}(3\varepsilon)^2$ for this part of the integral. By a linear approximation, the other part can be estimated from above by

$$\varepsilon^2 \int_{f_s}^{f_t-2\varepsilon} \frac{\mathrm{d}z}{\sqrt{z-f_s}\,(f_t-\varepsilon-z)}\,.$$

If $z \leq (f_s + f_t - 2\varepsilon)/2$ then $f_t - \varepsilon - z \geq (f_t - f_s)/2 \geq 2^{-2n-1}$, whereas

$$\int_{f_s}^{(f_s+f_t-2\varepsilon)/2} \frac{dz}{\sqrt{z - f_s}} \leq 2\sqrt{(f_t - f_s)/2} \leq 2^{-n+1},$$

resulting in a term $\varepsilon^2 2^{n+2}$. In the remaining case $z > (f_s + f_t - 2\varepsilon)/2$ we use $z - f_s \geq (f_t - f_s)/2 - \varepsilon \geq 2^{-2n-4}$, whereas

$$\int_{(f_s+f_t-2\varepsilon)/2}^{f_t-2\varepsilon} \frac{dz}{f_t - \varepsilon - z} = \log\frac{f_t - f_s}{2\varepsilon} \leq \log\frac{2^{-2n}}{\varepsilon},$$

giving a term $\varepsilon^2 2^{n+2} \log\frac{2^{-2n}}{\varepsilon}$. Putting all these terms together, we get

$$\operatorname{var}_{\mathbf{P}^f} W_\varepsilon[t - 2^{-n}, t - 2^{-n-1}] \leq c_2\,\varepsilon^2 \log\frac{2^{-2n}}{\varepsilon}.$$

Combining with the first-moment upper bound in (41) we conclude that

$$\mathbf{E}^f W_\varepsilon^2[t - 2^{-n}, t - 2^{-n-1}] \leq c_3\,\varepsilon^2 \log\frac{2^{-2n}}{\varepsilon}. \tag{42}$$

Clearly, the constants c_2 and c_3 do not depend on t, f, ε, n, under our assumptions $f \in H_n$, $2^{-2n} > 3\varepsilon$.

Next we use the elementary inequality

$$P\Big(\xi \geq \frac{\delta}{2}\Big) \geq \frac{(E\xi)^2}{4E\xi^2} \quad \text{if} \quad E\xi \geq \delta \geq 0.$$

Then from the lower bound in (41) and (42) we obtain

$$\mathbf{P}^f\Big(W_\varepsilon[t - 2^{-n}, t - 2^{-n-1}] \geq \frac{c_1}{2}\varepsilon\Big) \geq \frac{c_1^2}{4c_3}\Big(\log\frac{2^{-2n}}{\varepsilon}\Big)^{-1}.$$

Now we specialize to $\varepsilon := 2^{-2m}$ where $m > n+1, : n \geq N$ and let the (measurable) subset A_n^m of Θ be defined by

$$A_n^m := \Big\{W_{2^{-2m}}\big[t - 2^{-n}, t - 2^{-n-1}\big] \geq \frac{c_1}{2} 2^{-2m}\Big\}. \tag{43}$$

Then our inequality gives

$$\mathbf{P}^f\big(A_n^m\big) \geq \frac{c_1^2}{4c_3}\left(\log\frac{2^{-2n}}{2^{-2m}}\right)^{-1} = c_4 (m-n)^{-1} \tag{44}$$

where $c_4 := c_1^2/(8c_3 \log 2) > 0$. Recall that we only considered n such that $f \in H_n$. Summing up over all these values of n yields

$$\sum_{\{n; f \in H_n,\ N \leq n < m-1\}} \mathbf{P}^f\big(A_n^m\big) \geq c_4 \sum_{\{n; f \in H_n,\ N \leq n < m-1\}} (m-n)^{-1}. \tag{45}$$

Now recall that by assumption f satisfies (39). Let c_5 be any constant with $0 < c_5 < c_0 c_4$. An elementary reasoning shows that there must exist a sequence $m_k \uparrow \infty$ (depending on the fixed function f) such that

$$\sum_{\{n;f\in H_n,\ N\le n<m_k-1\}} \mathbf{P}^f\left(A_n^{m_k}\right) \ge c_5 \log m_k$$

for every k. In fact, if we assume that there exist no such sequence m_k, we easily arrive at a contradiction with (39) and (45).

We have in particular

$$\lim_{k\to\infty} \sum_{\{n;f\in H_n,\ N\le n<m_k-1\}} \mathbf{P}^f(A_n^{m_k}) = \infty$$

which implies, using the independence of the events A_n^m (for m fixed),

$$\lim_{k\to\infty} \frac{\sum_{\{n;f\in H_n,\ N\le n<m_k-1\}} 1_{A_n^{m_k}}}{\mathbf{E}^f \sum_{\{n;f\in H_n,\ N\le n<m_k-1\}} 1_{A_n^{m_k}}} = 1$$

in $\mathscr{L}^2(\mathbf{P}^f)$. By extracting a subsequence converging a.s., we get

$$\limsup_{k\to\infty} \frac{1}{\log m_k} \sum_{n=N}^{m_k} 1_{A_n^{m_k}} \ge c_5\,,$$

$\mathbf{P}^f$ a.s. From the definition (43) of the sets A_n^m and the additivity of W_ε (recall (40)), we conclude

$$\limsup_{k\to\infty} \frac{1}{2^{-2m_k}\log m_k}\, W_{2^{-2m_k}}[0,t] \ge \frac{c_5\, c_1}{2},$$

$\mathbf{P}^f$ a.s. Consequently,

$$\limsup_{k\to\infty} \frac{1}{2^{-2m_k}\log m_k} \int \theta(\mathrm{d}s\, \mathrm{d}\omega)\, V(\omega)\Big(\big[f_t - 2^{-2m_k}, f_t + 2^{-2m_k}\big]\Big) \ge C$$

completing the proof of (38) and of Theorem 6. □

3.4 Remarks

1. As we already mentioned, the carrying dimension of λ^c is 1 a.s. This result can be easily recovered from the second moment formula (37), using essentially the same argument as in [5]. Formula (37) and some easy calculations imply that, for $0<\delta<K$ and $0<\gamma<1$,

$$E \int_{[\delta,K]^2} \frac{V(\mathrm{d}a)V(\mathrm{d}b)}{|a-b|^\gamma} < \infty. \tag{46}$$

Then, the classical connection between Hausdorff dimension and capacity shows that a.s. for any Borel set H supporting $V(\mathrm{d}a)$ we have $\dim H = 1$. ◊

2. Since the measure λ^c is a.s. singular,

$$\lim_{\varepsilon\to 0} \frac{\lambda^c\big([t-\varepsilon, t+\varepsilon]\big)}{\varepsilon} = 0, \qquad \mathrm{d}t \text{ a.e., a.s.}$$

The representation formula (15) and some easy estimates then give

$$\lim_{b\to c} x_t(b) = 0, \qquad \mathrm{d}t \text{ a.e., a.s.} \tag{47}$$

(compare with Theorem 1.2.3 of [5]). On the other hand, a minor modification of the proof of Theorem 6 gives

$$\limsup_{\varepsilon \to 0} \frac{\lambda^c([t-\varepsilon, t-\varepsilon/2])}{\varepsilon \log\log(1/\varepsilon)} \geq C > 0, \qquad \lambda^c(dt) \text{ a.e., a.s.}$$

This result combined with the representation (15) shows that

$$\limsup_{b \to c} \frac{x_t(b)}{\log\log(1/|b-c|)} \geq C' > 0, \qquad \lambda^c(dt) \text{ a.e., a.s.}$$

which is in contrast to (47). (Recall also the heuristic explanations given in § 1.1.) ◊

3. The method of proof of Theorem 6 is inspired from Le Gall and Perkins [13]. The function h in Theorem 6 is however certainly not the best possible. ◊

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