Probab. Theory Relat. Fields 102, 57–61 (1995)

Probability Theory and Related Fields

Free energy for Brownian and geodesic homology

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Received: 10 March 1994/21 November 1994

Summary. Existence and analyticity of the free energies associated with the asymptotic homology of Brownian paths and geodesics are proved and a simple relation is found between them.

Mathematics Subject Classification: 60F10, 58F17

1 Presentation of the results

Let M be a compact Riemannian manifold with constant curvature -1 and $\mathscr{H}$ the space of harmonic 1-forms on M. Let F be the unit tangent bundle of M and p the projection of F onto M.

We denote by m the Liouville measure on F and by $\ell = p(m)$ the volume element on M. We also consider the law $\mathbb{P}_x$ of the Brownian motion on M starting at x, and the geodesic flow on F, denoted θ_t. Given an arc of curve γ in M, and ω in $\mathscr{H}$ let us denote $\omega(\gamma)$ the integral of ω along this arc γ. (It is invariant under homotopic deformation.)
We show that the free energies of the Brownian and geodesic homology respectively denoted λ and π can be defined as positive analytic functions on $\mathscr{H}$ by the formulas

$$\lambda(\omega) = \lim_{t\uparrow\infty} t^{-1}\log\left(\int \exp((\omega(\xi_s,\ 0 \leqq s \leqq t))P_\ell(d\xi)\right) \tag{1}$$

with $P_\ell = \int \ell(dx)P_x$.

$$\pi(\omega) = \lim_{t\uparrow\infty} t^{-1}\log\left(\int \exp(\omega(p(\theta_s x),\ 0 \leqq s \leqq t))m(dx)\right). \tag{2}$$

$\lambda(\omega)$ can also be defined as the principal eigenvalue of the operator

$$L^\omega = \Delta/2 + \langle\omega, d\rangle + \|\omega\|^2/2.$$

We show that π and λ are related by the formulas

$$2\lambda = (d-1)\pi + \pi^2 , \qquad 2\pi = \sqrt{(d-1)^2 + 8\lambda} + 1 - d . \tag{3}$$

The existence of λ and π are already known in a broader context, cf. [D-S, Ki, B-D, M, T]. To be self-contained, we provide for λ an argument which yields the analyticity easily and works also in variable curvature. For π, the existence follows by an easy argument which does not rely upon coding theory.

The differentiability of λ and π implies that the related entropy functionals are strictly convex and that a *large deviation principle* holds (cf. [E, Chap. VII]).

The method should apply in the finite volume case also. More general functionals should also be treated by using the Brownian motion on the leaves as in [L2]. The method might also be adapted to non constant negative curvature using the harmonic invariant measure for the geodesic flow (cf. [Led]).

2 The Brownian free energy

Define $A(t) = E_\ell(\exp\, \omega(\xi_s, 0 \leqq s \leqq t))$. By Girsanov and Feynman–Kac formulas, it is clear that

$$A(t) = \int Q_t^\omega 1(x)\ell(dx), \tag{4}$$

where Q_t^ω is the semigroup of generator L^ω. Q_t^ω is a semigroup of compact positive operators on $C(M)$ which map non negative functions into positive functions. By the Krein–Rutman theorem, cf. [Kr, Chap. 2], there is a positive function h in $C(M)$, a projector π of $C(M)$ into $h\mathbb{R}$ and a real number $\lambda(\omega)$ such that

$$- Q_t h = e^{\lambda t} h,$$
$$- Q_t \pi = \pi Q_t$$

and

$$- \lim_{t \uparrow \infty} \frac{1}{t} \log \| Q_t(I-\pi) \| < \lambda .$$

The analyticity in ω of $\lambda(\omega)$ follows from standard perturbation theory. One checks very easily that the generators of Q_t form a holomorphic family of sectorial operators of type B_0 is Kato's terminology (cf. [Ka, VII-4]).

We have to show that $\lim_{t\uparrow\infty} (1/t) \log \int Q_t 1 d\ell = \lambda$.

The majoration by λ follows from the fact that $e^{\lambda t}$ is the spectral radius of Q_t: for every t, $(nt)^{-1} \log \int Q_{nt} 1 d\ell \leqq (nt)^{-1} \log \|Q_t^n\|$ which converges towards λ.

We get the minoration from the existence of $\varepsilon > 0$ such that $1 \geqq \varepsilon h$. Finally, we give a lower bound for λ which ensures the finiteness of the entropy functional:

$$\lambda(\omega) \geqq \frac{1}{2} \int \langle \omega, \omega \rangle \, d\ell . \tag{5}$$

Note first that since the generator of Q_t is elliptic, h is C^∞. h can be normalized in order to have $\int h^2 \, d\ell = 1$.

We have

$$2\lambda = \int[-\|dh\|^2 + 2\langle\omega, dh\rangle h + \|\omega\|^2 h^2]\, d\ell$$
$$= Z_h(h,h),$$

where Z_h is the symmetric bilinear form defined on $C^1(M)$ as follows :

$$Z_h(f,f) = \int[-\|df\|^2 + 2\langle\omega, \tfrac{dh}{h}\rangle f^2 + \|\omega\|^2 f^2]\, d\ell.$$

It is easy to check that Z_h takes its maximal value on $\left\{f \in C^1(M), \quad \int f^2 d\ell = 1\right\}$ for $f = h$. Therefore $2\lambda \geqq Z_h(1,1) = \int \|\omega\|^2\, d\ell$.

Remark

- The constant negative curvature assumption was not used in this section.
- This argument can be generalized to yield large deviation principles in various cases.

3 Existence of π

We shall use the same representation as in our note [L1] on windings in cusps. (We can also refer to this note for the central limit theorem in the present situation.)

We define a stopping time τ_t by inf $(s,\, d(\hat{\xi}_0, \hat{\xi}_s) = t)$ where $\hat{\xi}$ can be any lift of the Brownian motion ξ into the hyperbolic space (i.e. the universal covering of M).

Set $C_t = \int \exp(\omega(p(\theta_s x),\, 0 \leqq s \leqq t))\, m(dx)$.

Then $C_t = E_\ell\ (\exp(\omega(\xi_s, 0 \leqq s \leqq \tau_t)))$.

By the Girsanov theorem we have immediately

$$C_t = E_\ell^\omega\left(\exp\left(\tfrac{1}{2}\int_0^{\tau_t} \|\omega\|^2(\xi_s)\, ds\right)\right),$$

where E_x^ω is the law of the diffusion with generator $\Delta/2 + \langle\omega, d\rangle$ starting at x. By definition of τ_t we have

$$\tau_{t+s} \geqq \tau_t \circ \theta_{\tau s} + \tau_s. \qquad (6)$$

Set $\tilde{C}_t = \inf_{x \in M} E_x^\omega\left(\exp\tfrac{1}{2}\int_0^{\tau_t} \|\omega\|^2(\xi_s)\, ds\right)$. Then $\tilde{C}_{t+s} \geqq \tilde{C}_t \tilde{C}_s$ by (6) and the strong Markov property. Besides $\tilde{C}_t$ is obviously bounded by exp $(\|\omega\|_\infty t)$ as C_t is by definition and larger than 1. Hence $(1/t)$ log $\tilde{C}_t$ converges as $t \uparrow \infty$ towards a constant $C \geqq 0$ and we have lim inf t^{-1} log $C_t \geqq C$.

Let R be the diameter of M, and y a point such that

$$\tilde{C}_{t+R} = E_y^\omega\left(\exp\left(\tfrac{1}{2}\int_0^{\tau_{t+R}} \|\omega\|^2(\xi_s)\, ds\right)\right).$$

From (6) we obtain easily the inequality

$$\tilde{C}_{t+R} \geqq \tfrac{1}{R}\int_0^R du\, E_y^{\omega}\left(\exp\left(\tfrac{1}{2}\int_{\tau_u}^{\tau_t\circ\theta_{\tau u}+\tau_u} \|\omega\|^2(\xi_s)\,ds\right)\right)$$
$$= \int \nu(dx) E_x^{\omega}\left(\exp\left(\tfrac{1}{2}\int_0^{\tau_t}\|\omega\|^2(\xi_s)\,ds\right)\right)$$

with $\nu(\cdot) = (1/R)\int_0^R du\, P_y^{\omega}(X_{\tau_u} \in \cdot)$.

Since the diffusion of law P_y^{ω} is elliptic, ν dominates $\varepsilon\ell$ for some $\varepsilon > 0$, and therefore $\tilde{C}_{t+R}$ dominates εC_t.

Hence $\lim t^{-1}\log C_t = C = \pi(\omega)$ (by definition).

4 The relation between λ and π

If r_t is the distance to the origin of the hyperbolic Brownian motion, we have

$$A_t = E(C_{r_t}). \tag{7}$$

There is a 1-dimensional Wiener process W_t such that

$$r_t = W_t + \tfrac{d-1}{2}\int_0^t \coth(r_s)\,ds \geqq W_t + \tfrac{(d-1)t}{2}.$$

For every $\varepsilon > 0$, there is R_ε such that when $r > R_\varepsilon$, $|r^{-1}\log C_r - C| < \varepsilon$. Hence

$$A_t \geqq E\left(e^{(C-\varepsilon)r_t} 1_{\{r_t > R_\varepsilon\}}\right) \geqq E\left(e^{(C-\varepsilon)(W_t+t(d-1)/2)} 1_{\{W_t+t(d-1)/2 > R_\varepsilon\}}\right),$$

which for $t > 2R_\varepsilon$ is larger than

$$E\left(e^{(C-\varepsilon)[W_t+t(d-1)/2]} 1_{W_t} > 0\right) = \frac{1}{2} e^{\left((C-\varepsilon)(d-1)/2+(C-\varepsilon)^2/2\right)^t}.$$

On the other hand, since $\coth r \leqq 1 + 1/r$ for every $r > 0$, by a comparison theorem (cf. [I-W, Chap. VI]) r_t is always smaller than ρ_t where ρ_t solves

$$\rho_t = W_t + \tfrac{d-1}{2}\left[\int_0^t \tfrac{1}{\rho_s}\,ds + t\right].$$

If u_t solves the equation

$$u_t = W_t + \tfrac{d-1}{2}\int_0^t \frac{1}{u_s}\,ds,$$

we have also by comparison $\rho_t \geqq u_t$ and therefore

$$\rho_t \leqq W_t + \tfrac{(d-1)}{2}\left(\int_0^t \tfrac{1}{u_s}\,ds + t\right) \leqq u_t + \tfrac{(d-1)}{2}t.$$

u_t is a Bessel diffusion. Its density at time t is $N_d r^{d-1} t^{-d/2} e^{-r^2/2t}\, dr$. By (7)

$$A_t \leqq C_{R_\varepsilon} + E\left(e^{(C+\varepsilon)r_t}\right) \leqq C_{R_\varepsilon} + E\left(e^{(C+\varepsilon)((d-1)t/2+u_t)}\right)$$
$$= C_{R_\varepsilon} + e^{(C+\varepsilon)t(d-1)/2} N_d t^{-d/2} \int_0^\infty e^{(C+\varepsilon)r} r^{d-1} e^{-r^2/2t}\, dr$$
$$\leqq C_{R_\varepsilon} + e^{(C+\varepsilon)t(d-1)/2+(C+\varepsilon)^2 t/2} \int_{-\infty}^{\infty} N^d t^{-d/2} r^{d-1} e^{-(r-(C+\varepsilon)t)^2/2t}\, dr.$$

The last integral is a rational function of t. Hence

$$\limsup \frac{1}{t} \log A_t \leqq (C+\varepsilon)(d-1)/2 + (C+\varepsilon)^2/2.$$

Letting $\varepsilon \downarrow 0$ we get that

$$(d-1)\pi(\omega) + \pi^2(\omega) = 2\lambda(\omega)$$

or equivalently

$$\pi(\omega) = \frac{1}{2}\left(\sqrt{8\lambda + (d-1)^2} - d + 1\right).$$

Acknowledgements. I am indebted to G. Ben Arous for stimulating conversations and useful references.

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