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Probability Theory and Related Fields

Ergodicity of stochastic plates

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Summary. We prove the stabilization of plates subject to a stochastic evolution of Ginzburg Landau type. We study the asymptotic behaviour of the rate of stabilization when the intensity of the noise vanishes. An approximate simulated annealing property is also given.

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1 Introduction

Let us consider a regular bounded open subset G of $\mathbb{R}^N$ and the self-adjoint operator on $L^2(G)$ defined by $A = (-\Delta)^s$, where Δ denotes the Laplacian with Dirichlet boundary conditions, and s is greater than $N/2$. Let X_t be a $\mathscr{C}_0(G)$-valued process formally defined by the following stochastic differential equation:

$$dX_t = \varepsilon(t)\,\alpha(\cdot, dt) - AX_t\,dt - \mathscr{V}'(X_t)dt\ , \tag{1.1}$$

where α is a space-time white noise, and $\mathscr{V}$ is a real function on $\mathbb{R}$ with bounded derivatives of any order . We are interested in a simulated annealing problem: what is the asymptotic behaviour of the law of X_t as $t \longrightarrow \infty$, if $\varepsilon(t)$ is a function slowly vanishing at infinity? As a matter of fact, this problem is not solved in the present article, although we give an approximate answer. We obtain a preliminary result that we will now describe. Let $\underline{L}_\varepsilon$ be the infinitesimal generator of the homogeneous process defined by the previous equation, when $\varepsilon(t)$ is frozen at a constant value ε. More precisely, we have for smooth functionals F:

$$\underline{L}_\varepsilon F(w) = \frac{\varepsilon^2}{2}\underline{\Delta} F(w) - \big(w, A[\nabla F(w)]\big) - \big(\nabla F(w), \mathscr{V}'(w)\big), \tag{1.2}$$

where ∇ and $\underline{\Delta}$ are respectively the Gâteaux's gradient and the infinite dimensional Laplacian with respect to the $L^2(G)$ scalar product $(\cdot,\cdot)$. As in [HKS], we investigate the asymptotics of the constant in the log-Sobolev inequality associated with $\underline{L}_\varepsilon$ as ε tends to 0. This extends the work of the first author [Ja2] to more natural generators and to dimension N. In the same way and with the help of infinite dimensional processes we give an approximate simulated annealing method. Given d, it allows us to calculate exactly in a suitable wavelet basis the first d coordinates of the ground states of the following energy function:

$$S(w) = \int_G \frac{1}{2}\left[(-\Delta)^{s/2}w(x)\right]^2 + \mathscr{V}(w(x))\,dx \ , \tag{1.3}$$

This energy has some physical meaning at least for $N = 2$ and $s = 2$, when it can be considered as the energy of an elastic clamped plate with an extra interaction potential $\mathscr{V}$.

A great part of the present work is devoted to the case of a constant ε. We establish a stabilization (strong ergodicity) result for the process (1.1) and we study the rate of stabilization as $\varepsilon \to 0$. For $N = 1$ and $s = 1$, this ergodicity was proved by a different method, apparently for the first time, in [Ja3] .

2 Preliminaries

Wavelets

It appears that the wavelet basis in $L^2(G)$ built by Jaffard and Meyer [JM] is suitable for an extension of the estimates of [Ja2], but we shall rather use the orthonormal basis of $\mathscr{H}$, supplied by Benassi et al., to go a little farther. These constructions require some regularity condition. We assume like in [JM] that G satisfies an uniform external ball condition .

Let m be a positive integer. We set $\Lambda = \bigcup_{j\in\mathbb{Z}, j\geq j_0} \Gamma_j$, where the elements of Γ_j are the points of the lattice $2^{-j}\mathbb{Z}^N$ which are at a distance greater than $(m+2)2^{-j}$ from $\mathbb{R}^N \setminus G$ with respect to the norm $|x| = \sup|x_i|, \quad i \leq N$, and $j_0 = \inf\{j \mid \Gamma_j \neq \emptyset\}$. Note that, for any λ, there is a unique scale parameter j determined by $\lambda \in \Lambda_j$, where $\Lambda_j = \Gamma_{j+1} \setminus \Gamma_j$, and λ and j are assumed to be coupled according to this rule in formulae. The wavelets of Jaffard and Meyer are the elements of an orthonormal basis $\Psi = \{\psi_\lambda,\ \lambda \in \Lambda\}$ of $L^2(G)$ with the following properties:

1. The wavelets are functions of class $\mathscr{C}^{2m}$, vanishing outside G.
2. There exist positive constants C and γ, such that

$$|\partial_\alpha \psi_\lambda(x)| \leq C 2^{j|\alpha|} 2^{Nj/2} \exp(-\gamma 2^j |x-\lambda|) \qquad \lambda \in \Lambda_j, \quad |\alpha| \leq 2m \,.$$

3. By a scaling of the wavelets, namely if we consider $\widetilde{\psi}_\lambda = 2^{-js}\psi_\lambda$, $\lambda \in \Lambda$, we get a Riesz basis of $\mathscr{H}$.

Recall that a family $f_i, i \in I$ of vectors constitute a Riesz basis of a Hilbert space H, if the space of finite linear combinations $\sum_{i\in F} \alpha_i f_i$ is dense in H and there exists a constant k such that for any such linear combination we have:

$$k^{-1} \sum_{i\in F} \alpha_i^2 \leq \left\| \sum_{i\in F} \alpha_i f_i \right\|^2 \leq k \sum_{i\in F} \alpha_i^2 . \tag{2.1}$$

In the sequel, we will keep the integer m fixed and such that $2m > s$. One define the wavelet coefficients of a distribution w on G as $w_\lambda = (w, \psi_\lambda)$, provided w is regular enough to be tested on $\mathscr{C}^{2m}$ functions. We shall mainly use the following fundamental result, which extends the property 3 above:

Theorem 2.1 *For any real σ, $-2m < \sigma < 2m$, there exist some constants C_1, C_2 such that:*

$$C_1 \left(\sum_{\lambda\in\Lambda} 4^{\sigma j} w_\lambda^2 \right)^{1/2} \leq \|w\|_{H_0^\sigma} \leq C_2 \left(\sum_{\lambda\in\Lambda} 4^{\sigma j} w_\lambda^2 \right)^{1/2} .$$

The wavelets of Benassi, Jaffard and Roux are defined by $\theta_\lambda = A^{-1/2}\psi_\lambda$ and they thus are in $H_0^{2m+s} \subset H_0^{2s}$. Their collection forms an orthonormal basis Θ of $\mathscr{H}$ and, using the fact that $A^{1/2}$ is an isomorphism of L^2 onto $H^{-s} = H_0^{-s}$, and Theorem 2.1, we get :

Proposition 2.2 *The functions $\widetilde{\theta}_\lambda = 2^{js}\theta_\lambda$ yield a Riesz basis $\widetilde{\Theta}$ of $L^2(G)$.*

We need these wavelets for the basic convexity argument in Lemma 3.9, and apparently the eigenfunctions of Δ cannot serve that purpose (moreover, they are not well known in general).

Boltzmann-Gibbs measures

For a given ε we shall call free measure the centered Gaussian measure Π_ε on $L^2(G)$, associated with the nuclear covariance operator $\frac{1}{2}\varepsilon^2 A^{-1} = \frac{1}{2}\varepsilon^2(-\Delta)^{-s}$:

$$\int_{\mathscr{C}_0(G)} (w, \varphi)^2 \, d\Pi_\varepsilon(w) = \frac{1}{2}\varepsilon^2 (\varphi, A^{-1}\varphi) , \qquad \varphi \in \mathscr{D}(G) . \tag{2.2}$$

We remark that Theorem 2.1 readily implies that A^{-1} is nuclear, as required. Moreover, an expansion of the corresponding random field with respect to the wavelet basis Ψ shows that the free measure is supported by the space $\mathscr{C}_0(G)$ of continuous functions on G, which tend to 0 at its boundary: using estimates similar to those of [BJR], this expansion can be shown to converge, almost surely and uniformly.

Let $\mathscr{H}$ be the Sobolev space $H_0^s(G)$. We may define the topology of $\mathscr{H}$ with the help of the scalar product $\langle \cdot, \cdot \rangle = (A\cdot, \cdot)$. The measure Π_ε is the transform of

Π_1 under an ε-dilation and its reproducing kernel Hilbert space is $\mathscr{H}$, endowed with the scalar product $\langle\cdot,\cdot\rangle_\varepsilon = 2\varepsilon^{-2}(A\cdot,\cdot)$.

In our context, the Boltzmann measure μ_ε is the probability on $\mathscr{C}_0(G)$ with the density with respect to Π_ε equal to:

$$(1/Z_\varepsilon)\exp(-2\varepsilon^{-2}V)\,, \tag{2.3}$$

$$V(w) = \int_G \mathscr{V}(w(x)\,dx\,, \quad Z_\varepsilon = \int \exp(-\varepsilon^{-2}V)\,d\Pi_\varepsilon\,. \tag{2.4}$$

The parameter $\varepsilon^2/2$ is called temperature.

Gâteaux's derivatives

A functional F on $W = \mathscr{C}_0(G)$ is said to have a Gâteaux's derivative at a point w_0: if $\partial_\tau F(w) = \lim_{h\to 0} h^{-1}\big(F(w_0 + h\tau) - F(w_0)\big)$ exists for any function τ in W and it is a linear continuous function of τ with respect to the $L^2(G)$ norm. One denotes this limit by $\big(\nabla F(w_0),\tau\big)$ and ∇ is called the Gâteaux's gradient. For example S admits a gradient at every smooth point w_0, and we may usefully understand the equation (1.1) at an even more formal level, if we do not pay attention to this smoothness:

$$dX_t = \varepsilon(t)\,\alpha(\cdot,dt) - \nabla S(X_t)\,dt\,.$$

Consider the so-called class $\mathscr{S}$ of cylindrical smooth functionals, i.e. the class of functions F of the configuration w, which may be written under the form

$$F(w) = f((w,\varphi_1),\ldots,(w,\varphi_n))$$

for some $f \in \mathscr{C}_b^\infty(\mathbb{R}^n)$ and $\varphi_1,\ldots,\varphi_n \in \mathscr{D}(G)$. The gradient ∇F of such a functional is an element of $L^2(W, L^2(G))$. Taking the derivative of this vectorial function we define at any point w a Hessian operator of finite rank acting on $L^2(G)$ by $h \longmapsto \partial_h\nabla F$. It is thus possible to define the Laplacian $\underline{\Delta} F$ for functions in S as the trace of the Hessian: $\underline{\Delta} F = \sum_n \big(\partial_{e_n}\nabla F, e_n\big)$, if (e_n) is an orthonormal basis of $L^2(G)$, and to define $\underline{L}_\varepsilon F$ as a functional in $L^2(W)$. One can check, for example by using a quasi-invariance property, that $\underline{L}_\varepsilon$ is a symmetric operator on the domain $\mathscr{S} \subset L^2(\mu_\varepsilon)$ in the sense that

$$\int_W \underline{L}_\varepsilon F\; G\,d\mu_\varepsilon = \int_W F\;\underline{L}_\varepsilon G\,d\mu_\varepsilon = -\frac{\varepsilon^2}{2}\mathscr{E}_{\mu_\varepsilon}(F,G)\,,$$

where $\mathscr{E}_{\mu_\varepsilon}$ denotes the Dirichlet form

$$\mathscr{E}_{\mu_\varepsilon}(F,F) = \int_W \big(\nabla F,\nabla F\big)_{L^2(G)}\,d\mu_\varepsilon\,. \tag{2.5}$$

Since the positive quadratic form $\mathscr{E}_{\mu_\varepsilon}(F,F)$ is generated by a symmetric operator, it is closable, and we shall denote by $\mathbb{D}(\mu_\varepsilon)$ the domain of its closure, which will be still denoted by $\mathscr{E}_{\mu_\varepsilon}$.

3 Small noise behaviour in the logarithmic Sobolev inequality

Due to the strong assumptions that we impose, it will be quickly seen, by comparison to the Gaussian case, that a logarithmic Sobolev inequality of the form

$$\int_W F^2 \log\left(\frac{|F|}{\|F\|_2}\right) d\mu_\varepsilon \leq c(\varepsilon)\mathscr{E}_{\mu_\varepsilon}(F,F) \tag{3.1}$$

does hold. But in order to study the stabilization rate of our processes when ε is small, we will estimate the best constant $c(\mu_\varepsilon)$ in this inequality (Theorem 3.4 below). Let us precise the meaning of this kind of inequalities: the norm $\|\cdot\|_2$ is always the L^2-norm of the measure appearing in the left side (here μ_ε). Moreover it has to be true for any F in L^2 if we allow the value ∞ for its both sides: for the left-hand integral if it doesn't converge and for the right-hand side if F is not in the domain of the quadratic form $\mathscr{E}$.

The Gaussian case

We start from the following Bakry and Emery's result which will be also useful in the sequel:

Theorem 3.1 *Let S be a function of class $\mathscr{C}^2$ on $\mathbb{R}^d$ such that $S''(x) \geq CI$ uniformly in x and define $\nu(dx) = \exp(-S(x))\,dx$. Then we have:*

$$\int \psi^2 \log\left(\frac{|\psi|}{\|\psi\|_2}\right) d\nu \leq \frac{1}{C}\int \sum_{i=1}^{d} \partial_i\psi^2\, d\nu\,, \qquad \psi \in \mathscr{C}_b^\infty(\mathbb{R}^d)\,.$$

Let γ be the lower bound of the operator A on $L^2(G)$.

Theorem 3.2 *For any F in $\mathbb{D}_{\Pi_\varepsilon}$*

$$\gamma \int_W F^2 \log\left(\frac{|F|}{\|F\|_2}\right) d\Pi_\varepsilon(w) \leq \frac{1}{2}\varepsilon^2 \mathscr{E}_{\Pi_\varepsilon}(F,F)\,.$$

It is sufficient to prove this theorem, when $F \in \mathscr{S}$ and this leads directly to the study of a finite-dimensional Gaussian law. The constant C in Theorem 3.1 is bounded from below by $2\varepsilon^{-2}\gamma$. To go from the Gaussian case to the non-linear one, we may use the following general result:

Lemma 3.3 *Let P and Q be probability measures, such that $Q = \dfrac{1}{Z}\exp(-V)P$ for a bounded function V. A logarithmic Sobolev inequality*

$$\int F^2 \log\left(\frac{|F|}{\|F\|_2}\right) dP \leq C\, \mathscr{E}_P(F,F) \qquad F \in \mathbb{D}(P)$$

for P implies an analogous one for Q, with a new constant $C \exp(\sup(V) - \inf(V))$.

This result, proved for example in [S], is valid for any logarithmic Sobolev inequality, provided that the inequality between measures $P \leq kQ$ implies $\mathscr{E}_P \leq k\mathscr{E}_Q$ on a domain $\mathbb{D}(P) \subset \mathbb{D}(Q)$. In our particular case it implies the rough estimate $c(\mu_\varepsilon) \leq (2\gamma)^{-1}\, \varepsilon^2 \exp(2\, \mathrm{v}(G) M \varepsilon^{-2})$, where M is a majorant of $\mathscr{V}$, and $\mathrm{v}(G)$ is the volume of G.

Hajek's constant

Generally speaking, let S be a map defined on a set W, taking its values in $[a, +\infty]$ for some finite a, and let $\mathscr{H} = \{S < \infty\}$. We denote by $\Gamma_S(\varphi, \psi)$ the set of all paths: $\gamma : [0,1] \longmapsto \mathscr{H}$ joining two elements φ and ψ of $\mathscr{H}$, such that $S \circ \gamma$ is finite and continuous. We also put:

$$\begin{aligned} h(\gamma) &= \sup_{0 \leq t \leq 1} S(\gamma(t)) \\ m_S &= \sup_{\varphi,\psi} \inf_{\gamma \in \Gamma(\varphi,\psi)} (h(\gamma) - S(\varphi) - S(\psi) + \inf(S)) \end{aligned} \tag{3.2}$$

We apply this definition to the functional S defined by (1.3) (1.5), which is set by convention to equal $+\infty$ outside $\mathscr{H}$. Our aim is to prove the following :

Theorem 3.4 *We have* $\lim(\varepsilon^2 \log(c(\mu_\varepsilon)) = 2\, m_S$*, as* $\varepsilon \to 0$*, where* $c(\mu_\varepsilon)$ *stands for the best constant in the the logarithmic Sobolev inequality (3.1).*

We will rely on a finite dimensional result, proven in [Ja1], extending to $\mathbb{R}^d$ a work of R.Holley et al. ([HKS]). Let v be a $\mathscr{C}^2$ function on $\mathbb{R}^d$, which satisfies the following growth condition:

$$\begin{cases} |\nabla v|(x) \to \infty \quad \text{and} \quad v(x) \to +\infty, \quad \text{when} \quad |x| \to \infty \\ |\nabla v|^2 - \Delta v \text{ is bounded from below} \\ \text{there exist constants } k,\, k', \text{ such that } v \leq k|\nabla v|^2 + k' \end{cases} \tag{3.3}$$

We consider the measure:

$$\nu_\varepsilon(dx) = Z_\varepsilon^{-1} \exp(-2\varepsilon^{-2} v(x)) dx \tag{3.4}$$

where the factor Z_ε is determined by the requirement that $\nu_\varepsilon(\mathbb{R}^d) = 1$. This probability is called the Boltzmann measure associated with v at temperature $\varepsilon^2/2$. The symmetric operator of $L^2(\nu_\varepsilon)$, defined by

$$L_\varepsilon \psi = \frac{\varepsilon^2}{2} \Delta\psi - \nabla\psi \cdot \nabla v \tag{3.5}$$

is under these hypothesis essentially self-adjoint on $\mathscr{D}(\mathbb{R}^d)$, and the domain of the closure of the quadratic form $\mathscr{E}_{\nu_\varepsilon} = -2\varepsilon^{-2}(\,.\,, L_\varepsilon .)_{L^2(\nu_\varepsilon)}$ is the space $H^1(\nu_\varepsilon)$ of all functions in $L^2(\nu_\varepsilon)$ whose gradients in the distributional sense are again in $L^2(\nu_\varepsilon)$ (see [DSi], Proposition 4.4). On this domain: $\mathscr{E}_{\nu_\varepsilon}(\psi,\psi) = \int \sum_{i=1}^d \partial_i \psi^2 \, d\nu_\varepsilon$.

Theorem 3.5 *There exists a constant c so that*

$$\int \psi^2 \log\left(\frac{|\psi|}{\|\psi\|_2}\right) d\nu_\varepsilon \leq c\, \mathscr{E}_{\nu_\varepsilon}(\psi,\psi) \qquad \psi \in H^1(\nu_\varepsilon) \tag{3.6}$$

and for the best possible constant $c(\nu_\varepsilon)$ in this inequality we have:

$$\lim_{\varepsilon \to 0} (\varepsilon^2 \log(c(\nu_\varepsilon)) = 2m_\upsilon$$

Proof. The main step in the proof is accomplished in [Ja1]. More precisely, it is shown that a result similar to that we want is valid for the best constant $\tilde{c}(\nu_\varepsilon)$ in the Poincaré inequality, which is the inequality that we get when we replace the left member of (3.5) by $\mathrm{var}_{\nu_\varepsilon}(\psi)$. On the other hand, as an application of a method by Carmona [C] that we will now describe, we can prove that ν_ε satisfies a logarithmic Sobolev inequality in some wider sense (see also [R2]). The symmetric operator $-\varepsilon^{-2}L_\varepsilon$ on $L^2(\nu_\varepsilon)$ is transformed into

$$H_\varepsilon = -\frac{1}{2}\Delta + Q_\varepsilon \quad \text{with} \quad Q_\varepsilon = \frac{1}{2}\left(\varepsilon^{-4}|\nabla \upsilon|^2 - \varepsilon^{-2}\Delta\upsilon\right)$$

by the Hilbert space isomorphism $\psi \longmapsto \psi \exp(-\varepsilon^{-2}\upsilon)$ on $L^2(dx)$. If we take care of the constants, Proposition 5.3 in [C] says that there exists a constant a, which depends only on the dimension d, such that the dominations $-b \leq Q$ and $\upsilon \leq b'H + b''$ with respect to the order of operators in $L^2(dx)$ imply:

$$\int \varphi^2 \log\left(\frac{|\varphi|}{\|\varphi\|_2}\right) d\nu \leq \frac{1}{2}(a+b')\mathscr{E}_\nu + (b''+ab)\|\varphi\|^2_{L^2(\nu)}$$

In addition, there is a way to mix such an inequality and the Poincaré inequality like it is done in Deuschel and Stroock [DS], Exercise 6.1.31, using

$$\int \psi^2 \log\left(\frac{|\psi|}{\|\psi\|_2}\right) d\nu \leq \int \overline{\psi}^2 \log\left(\frac{|\overline{\psi}|}{\|\overline{\psi}\|_2}\right) d\nu + \|\overline{\psi}\|^2_{L^2(\nu)},$$

where $\overline{\psi} = \psi - \int \psi \, d\nu$. By making use of Carmona's result, we leave aside the operator $-\Delta$, since it is positive in $L^2(dx)$, and we have to check inequalities of the following form:

$$\varepsilon^{-2}\upsilon \leq b'_\varepsilon(\varepsilon^{-4}|\nabla\upsilon|^2) + b''_\varepsilon \qquad Q_\varepsilon \geq -b_\varepsilon\,.$$

We may take $b'_\varepsilon = \varepsilon^{-2}$, $b''_\varepsilon = \tilde{b}''\varepsilon^{-2}$ and $b_\varepsilon = \varepsilon^{-2}\tilde{b}$ for ε small enough. So if we apply first the Deuschel and Stroock inequality and then Carmona's inequality to $\varphi = \overline{\psi}$ we obtain the logarithmic Sobolev inequality

$$\int \psi^2 \log\left(\frac{|\psi|}{\|\psi\|_2}\right) d\nu \leq C(\varepsilon)\mathscr{E}_{\nu_\varepsilon}(\psi,\psi)$$
$$C(\varepsilon) = \left(1 + (\widetilde{b}'' + a\widetilde{b}')\varepsilon^{-2}\right)\widetilde{c}(\nu_\varepsilon) + \frac{1}{2}(a + \varepsilon^{-2}).$$

We thus find $\overline{\lim}_{\varepsilon\to 0}(\varepsilon^2 \log(c(\nu_\varepsilon)) \leq \overline{\lim}_{\varepsilon\to 0} \varepsilon^2 \log(\widetilde{c}(\nu_\varepsilon)) = 2m_v$.

The lower bound reduces to the analogous result for Poincaré inequality, since this inequality is implied by the logarithmic Sobolev inequality, with the same constant(see for example [S] 2.6). □

We will split W into the finite dimensional subspace W_J of W generated by the wavelets θ_λ for $\lambda \in \Gamma_J$ and its complementary closed subspace W_J^* spanned by θ_λ for $\lambda \in \Gamma \setminus \Gamma_J$. Since the $2^{js}\theta_\lambda, \lambda \in \Lambda$ form a Riesz basis of $L^2(G)$, we have a sequence of continuous projections E_J from $L^2(G)$ on W_J which converges pointwise to the identity map when J tends to infinity. We denote by m_J the Hajek's constant associated with the restriction S_J of S to W_J; we shall omit the proof of the following result, which is strictly parallel to that used in [Ja2]:

Lemma 3.6 *When $J \longrightarrow \infty$, m_J converges to m_S.*

We need an estimate to bound the probability of large fluctuations of w_J along the wavelets of high resolution . Let us denote by w_J and w_J^* the orthogonal projection in $\mathscr{H}$ of w on W_J and W_J^* respectively.

Lemma 3.7 *Let us choose s' such that $N < 2s' < 2s$. Then there exists a constant C so that for any δ and J:*

$$\Pi_1(\|w_J^*\|_{L^2(G)} \geq \delta) \leq C \exp(-C\, 2^{(2s'-N)J}\delta^2)\,. \tag{3.7}$$

Proof. We have a Wiener's expansion of the free measure $\Pi_1(dw)$. It is identical to the law of the stochastic field defined by: $w = 2^{-1/2}\sum_{\lambda\in\Lambda} X_\lambda\theta_\lambda$, where X_λ are mutually independent standard gaussian variables. We write the tail field as: $w_J^* = \sum_{\lambda\notin\Gamma_J} 2^{-js-\frac{1}{2}}X_\lambda\widetilde{\theta}_\lambda$ and we use the fact that $\widetilde{\Theta}$ is a Riesz basis of $L^2(G)$ to get the estimate

$$\left(\|w_J^*\|_{L^2(G)}\right)^2 \leq C_1 \sum_{\lambda\notin\Gamma_J} 2^{-2js}(X_\lambda)^2\,.$$

Since $c = \sum_{j>j_0} 2^{-2j(s-s')}$ is finite, we have for any J:

$$\Pi_1\left(\|w_J^*\|_{L^2(G)} > \delta\right) \leq \sum_{j>J}\Pi_1\left(\sum_{\lambda\in\Lambda_j} X_\lambda^2 > c^{-1}\delta^2 2^{2js'}\right).$$

For a standard gaussian variable X we have $\mathbb{E}(e^{(1/4d)X^2}) = \left(1-(1/4d)\right)^{-1/2}$, so for a χ^2 distributed variable Y with d degrees of freedom $\mathbb{E}\left[\exp((1/4d)Y)\right]$ is

bounded independently of d. Thus, by Markov inequality,
$\Pi_1(|Y| > h) \leq C_2 \exp(-(1/4d)h)$.

We note that the number of points in Λ_j is less than $v(G)2^{Nj}$ and we thus get:

$$\Pi_1\left(\|w_j^*\|_{L^2(G)} > \delta\right) \leq C_2 \sum_{j \notin \Gamma_J} \exp\left(-\left(\frac{1}{4vc}\right) 2^{(2s'-N)j} \delta^2\right) .$$

It is easy to see that the ratio between two consecutive terms of this series is bounded by a constant which is less than one. The remainder is thus bounded from above by a constant multiple of its first term and this gives precisely the wanted result. □

We shall identify W_j and $\mathbb{R}^d$ where $d = \#\Gamma_j$ using the basis $\widetilde{\theta}_\lambda, \quad \lambda \in \Gamma_j$. Let $\mu_{\varepsilon,j}$ and $\Pi_{\varepsilon,j}$ be the projections on W_j of μ_ε and Π_ε respectively. Later, we shall consider the conditional law $\mu^*_{\varepsilon,j}(dw \mid v)$ of w under the condition that its projection on W_j is equal to v. We note that by orthogonality in the reproducing kernel Hilbert space, the conditional free measure $\Pi^*_{\varepsilon,j}(dw \mid v)$ does not depend on v, and it equals the law of the tail $w^*_{\varepsilon,j}$. From the preceding lemma, taking into account the Lipschitz property of $\mathscr{V}$, it is easily seen that the Boltzmann measure $\nu_{\varepsilon,j}$ (with temperature $\varepsilon^2/2$) associated with S on the finite-dimensional space W_j, endowed with the Lebesgue measure $\otimes dw_j$, is not far from $\mu_{\varepsilon,j}$:

Lemma 3.8

1 *There exists a constant C, such that for any ε and j:*

$$\frac{1}{C} \exp(-C\varepsilon^{-2}) \leq \frac{d\mu_{\varepsilon,j}}{d\nu_{\varepsilon,j}} \leq C \exp(C\varepsilon^{-2}) \tag{3.8}$$

2 *For any $\eta > 0$, one can find J and ε, $\varepsilon_0 > 0$, so that:*

$$\exp(-\eta\varepsilon^{-2}) \leq \frac{d\mu_{\varepsilon,j}}{d\nu_{\varepsilon,j}} \leq \exp(\eta\varepsilon^{-2}), \qquad 0 < \varepsilon \leq \varepsilon_0, \quad J \leq j \, . \tag{3.9}$$

We will prove the most important for applications half of Theorem 3.4, the upper bound: $\overline{\lim_{\varepsilon \to 0}}(\varepsilon^2 \log(c(\mu_\varepsilon)) \leq 2m_S$. The lower bound can be checked as in the original work of R.Holley et al. with the help of suitable functions of a finite number of coordinates and not depending on ε (see [Ja2]).

It is convenient to work with the space $\mathscr{S}_0$ of functions F depending on a finite number of coordinates with respect to the basis $\widetilde{\theta}_\lambda$: $F(w) = f(w_{\lambda_1}, \cdots, w_{\lambda_n})$ where $w_\lambda = 4^{-sj}(w, A\widetilde{\theta}_\lambda)$ and $f \in \mathscr{C}_b^\infty(\mathbb{R}^n)$. Since the functions $A\widetilde{\theta}_\lambda$ are in $L^2(G)$, all elements of $\mathscr{S}_0$ belong to $\mathbb{D}$ and they possess a Gâteaux's gradient which satisfies (2.5). We shall shorten into ∂_λ the notation of the derivative in the direction $\widetilde{\theta}_\lambda$.

Theorem 3.9 *One can find J and C such that, uniformly in v, for any ε, and any $F \in \mathscr{S}_0$,*

$$\int F^2 \log\left(\frac{|F|}{\|F\|_2}\right) d\mu^*_{\varepsilon,J}(\cdot \mid v) \le C\varepsilon^2 \int \sum_{\lambda \notin \Gamma_J} (\partial_\lambda F)^2 \, d\mu^*_{\varepsilon,J}(\cdot \mid v) .$$

Proof. We choose a resolution J', $J' > J$, such that F depends only on the coordinates indexed by $\Gamma_{J'}$. We observe that the estimate we want will be established if we prove its analog for twice conditionned measures $\mu^*_{\varepsilon,J}(dw \mid v, z)$, where z fixes the projection of w on the high resolution space $W^*_{J'}$, provided that the constant does not depend on J' nor z. So we have to deal with a Boltzmann measure on a finite dimensional space $\mathbb{R}^D$, where $D = (\Gamma_{J'} \setminus \Gamma_J)$, whose energy is easy to compute using the coordinates x_λ in the basis $\widetilde{\theta}_\lambda$, $\lambda \in D$:

$$U(x) = \frac{1}{2} \sum_{\lambda \in D} 2^{2sj} (x_\lambda)^2 + V\left(v + z + \sum_{\lambda \in D} x_\lambda \widetilde{\theta}_\lambda\right) .$$

We have $U(x) = U_1(x) + U_2(x)$ where,

$$U_1(x) = \frac{1}{4} \sum_{\lambda \in D} 2^{2sj} x_\lambda^2$$

$$U_2(x) = \frac{1}{4} \sum_{\lambda \in D} 2^{2sj} x_\lambda^2 + V\left(v + z + \sum_{\lambda \in D} x_\lambda \widetilde{\theta}_\lambda\right) .$$

Obviously, the lower bound on the Hessian $U_1'' \ge 2^{2sj_0 - 1} I$ is valid. We want to show that, for J large enough and any J', U_2 is convex: that will finish the proof by application of the the Bakry and Emery theorem quoted before, since the inequality needed to apply it will be strenghtened by the term U_2. While doing that, it is equivalent but more convenient to work with the coordinates y_λ in the basis θ_λ, i.e. to study the convexity of

$$\mathscr{U}(y) = \frac{1}{4} \sum_{\lambda \in D} y_\lambda^2 + V\left(v + z + \sum_{\lambda \in D} y_\lambda \theta_\lambda\right) .$$

We can write: $\mathscr{U}'' = \frac{1}{2} I + T$ with

$$T_{\lambda\mu} = \int_G \mathscr{V}''\left(u(\sigma) + z(\sigma) + \sum_{\rho \in D} y_\rho \theta_\rho(\sigma)\right) \theta_\lambda(\sigma) \theta_\mu(\sigma) \, d\sigma ,$$

so to obtain the convexity we may prove that for $\lambda \notin \Gamma_J$ $\sum_{\mu \notin \Gamma_J} |T_{\lambda\mu}| \le \frac{1}{4}$. The function $K_\mu = \mathscr{V}''\big(v + z + \sum_{\rho \in D} y_\rho \theta_\rho\big)\, \theta_\mu$ is bounded in $L^2(G)$ independently of everything since $\mathscr{V}''$ is bounded and θ_μ is the image by the bounded operator $A^{-1/2}$ of the unitary vector ψ_μ. So $A^{-1/2} K_\mu$ stay in a ball of $\mathscr{H}$ of radius r. We remark that $T_{\lambda\mu}$, $\lambda \in \Lambda$ are the wavelet coefficients $\left(A^{-1/2} K_\mu, \psi_\lambda\right)_{L^2}$ of $A^{-1/2} K_\mu$ thus, according to Theorem 2.1,

$$\sum_{\lambda \in \Lambda} 2^{2sj} (T_{\lambda\mu})^2 \leq C_2^2 r^2 .$$

We thus have by Schwarz inequality

$$\sum_{\lambda \notin \Gamma_J} |T_{\lambda\mu}| \leq \left(\sum_{\lambda \notin \Gamma_J} 2^{2sj} (T_{\lambda\mu})^2 \right)^{1/2} \left(\sum_{\mu \notin \Gamma_J} 2^{-2sj} \right)^{1/2}$$

$$\sum_{\mu \notin \Gamma_J} |T_{\lambda\mu}| \leq r C_2 \mathrm{v}(G) \sum_{j > J} 2^{j(N-2s)} = C_3\, 2^{J(N-2s)}.$$

□

By integrating the inequality given by the preceding theorem with respect to $\mu_{\varepsilon,J}$, we find:

$$\int F^2 \log |F| \, d\mu_\varepsilon \leq C \varepsilon^2 \int \sum_{\lambda \notin \Gamma_J} (\partial_\lambda F)^2 \, d\mu_\varepsilon + \int_{W_J} (F_J^*(v))^2 \log(F_J^*(v)) \, d\mu_{\varepsilon,J} \,, \qquad (3.10)$$

where we have set

$$F_J^*(v) = \left(\int F^2(w) \, d\mu^*_{\varepsilon,J}(dw \mid v) \right)^{1/2} \qquad (3.11)$$

Lemma 3.10

1 *For F in $\mathscr{S}_0$, F_J^* is a bounded $\mathscr{C}^1$ function on W_J and we have for $\lambda \in \Gamma_J$:*

$$\partial_\lambda F_J^*(v) = (F_J^*)^{-1} \left(\int F \, \partial_\lambda F \, d\mu^*_{\varepsilon,J}(\cdot \mid v) - \varepsilon^{-2} \mathrm{cov} \left(\partial_\lambda V, F^2 \mid v \right) \right) \quad (3.12)$$

where cov $(\cdot, \cdot \mid v)$ *stands for the covariance with respect to $\mu^*_{\varepsilon,J}(\cdot \mid v)$.*

2 *For any positive η, we can find a J and ε_0, so that for $\varepsilon \leq \varepsilon_0$ and $\psi \in H^1(\mu_{\varepsilon,J})$,*

$$\int \psi^2 \log \left(\frac{|\psi|}{\|\psi\|_2} \right) d\mu_{\varepsilon,J} \leq \exp((2m_S + \eta)\varepsilon^{-2}) \int \sum_{\lambda \in \Gamma_J} \partial_\lambda \psi^2 \, d\mu_{\varepsilon,J} \,. \quad (3.13)$$

Proof. The first assertion just comes from the derivation under the integral sign, (see [Z]). For the second, we remark that we can apply the Theorem (3.5) to the measure $\nu_{\varepsilon,J}$. The corresponding Hajek's constant is just m_J which differs from m_S by less than $(\eta/6)$ for J large enough. Besides, Lemmas 3.8 and 3.3 permit us, by comparison, to obtain a logarithmic Sobolev inequality for $\mu_{\varepsilon,J}$ with the constant multiplied again by $\exp((\eta/3)\varepsilon^{-2})$ for J chosen large. Thus we get a log-Sobolev constant less than $\exp(\varepsilon^{-2}(2m + \eta))$ for ε small enough. □

Proof of Theorem 3.4 We have the uniform bound $\mathrm{var}(\partial_\lambda V \mid v) \leq M'^2$, since $\mathscr{V}'$ is bounded, so $\left|\mathrm{cov}\left(\partial_\lambda V, F^2 \mid v\right)\right| \leq M'\left(\mathrm{var}(F^2 \mid v)\right)^{1/2} \leq$ $M'\left(2\,\mathrm{var}(F \mid v)\right)^{1/2}\|F\|_2 = M'\left(2\,\mathrm{var}(F \mid v)\right)^{1/2} F_J^*(v)$. Moreover, we can apply the Poincaré inequality in the space $W^*_{\varepsilon,J}$ as a byproduct of the log-Sobolev inequality stated in Theorem 3.9 and we get:

$$\left|\mathrm{cov}\left(\partial_\lambda V, F^2 \mid v\right)\right| \leq F_J^*(v)\sqrt{2C}M'\varepsilon\left(\int \sum_{\lambda \notin \Gamma_J} (\partial_\lambda F)^2\right)^{1/2}$$

We now apply the Schwarz inequality to the first term in (3.12) and the inequality just obtained:

$$\forall \lambda \in \Gamma_J \quad |\partial_\lambda F_J^*(v)| \leq \left(\int (\partial_\lambda F)^2\, d\mu^*_{\varepsilon,J}(\cdot|v)\right)^{1/2} + \sqrt{2C}M'\varepsilon^{-1}\left(\int \sum_{\lambda' \notin \Gamma_J} (\partial_\lambda F)^2\right)^{1/2} \tag{3.14}$$

Taking into account (3.10) and (3.13) for $\psi = F_J^*$, we may write:

$$\int F^2 \log\left(\frac{|F|}{\|F\|_2}\right) d\mu_\varepsilon \leq C\varepsilon^2 \int \Bigg[\sum_{\lambda \notin \Gamma_J} (\partial_\lambda F)^2\, d\mu_\varepsilon$$

$$+2\exp((2m_S + \eta)\varepsilon^{-2}) \int \sum_{\lambda \in \Gamma_J} (\partial_\lambda F)^2 + (\#\Gamma_J) 2CM'^2\varepsilon^{-2} \sum_{\lambda' \notin \Gamma_J} (\partial_\lambda F)^2\Bigg]\, d\mu_\varepsilon \tag{3.15}$$

As this inequality implies for ε small that

$$\int F^2 \log\left(\frac{|F|}{\|F\|_2}\right) d\mu_\varepsilon \leq \exp\left((2m_S + 2\eta)\varepsilon^{-2}\right) \int \sum_{\lambda \in \Lambda} (\partial_\lambda F)^2\, d\mu_\varepsilon, \tag{3.16}$$

we have proved the upper bound in Theorem 3.4. Indeed, due to the Riesz basis property of $\widetilde{\Theta}$, it is easy to show that for a constant k, $\sum_{\lambda \in \Lambda} (\partial_\lambda F)^2 \leq k\|\nabla F\|^2_{L^2(G)}$. □

4 Ergodicity at fixed temperature

Let us recall a result by Miclo on simulated annealing in a finite dimensional setting, under a form which will be also used in the next section [Mi1]. We denote by β_t the standard brownian motion with values in $\mathbb{R}^d$ (with identity covariance matrix), by $\varepsilon(t)$ a positive $\mathscr{C}^1$ function on $\mathbb{R}_+$, by σ a symetric positive definite matrix on $\mathbb{R}^d$, by v a function on $\mathbb{R} \times \mathbb{R}^d$, and we look at the stochastic differential equation:

$$dx_t = \varepsilon(t)\sigma d\beta_t - \sigma^2 \nabla v(t, x_t)\, dt\,. \tag{4.1}$$

We have to impose to v some suitable conditions of regularity and growth at infinity . We choose a very strong hypothesis which is appropriate for the present article: v is the sum of two functions v_1, v_2 in $\mathscr{C}^1(\mathbb{R} \times \mathbb{R}^d)$ such that :

$$\begin{cases} \text{for every fixed } t, \quad v_1(t,\cdot) \text{ is a positive quadratic form} \\ v_2(t,\cdot) \text{ has spatial derivatives } \partial_\alpha v_2(t,\cdot) \text{ of any order } \alpha \\ \text{for any } T > 0, \quad \sup_{0 \le t \le T} \sup_{x \in \mathbb{R}^d} |\partial_\alpha v_2(t,x)| < \infty \end{cases} \tag{4.2}$$

We denote by ν_s the Boltzmann probability on $\mathbb{R}^d$ at temperature $\varepsilon^2(s)/2$, associated with the energy v together with its density function (Eq.(3.4)). This probability is the reversible probability of the time homogeneous process

$$dx_t = \varepsilon\sigma d\beta_t - \sigma^2 \nabla v(x_t)\, dt \tag{4.3}$$

with $\varepsilon = \varepsilon(s)$ $v(x) = v(s,x)$ (see, for example, [R1]). The probability law $\mathscr{L}x_t$ is absolutely continuous with respect to Lebesgue's measure with a density differentiable any number of times with respect to space or time variables and we set: $\mathscr{L}(x_t) = f_t\, \nu_t$.

Let us denote by $I(P \mid Q)$ the relative entropy of a probability P with respect to another one, Q (see for example [DS]). We set $I(t) = I(\mathscr{L}(x_t) \mid \nu_t)$. The hypothesis (4.2) allows us to use [KS] Corollary 3.9 examples (3.14) to get:

$$\sup_{T^{-1} \le t \le T} \sup_{x \in \mathbb{R}^d} (f_t \nu_t(x)) < \infty\,, \qquad T > 0\,, \tag{4.4}$$

so by the same proof as in Proposition 3 of [Mi2] we have:

Proposition 4.1 *Suppose that the initial law $\mathscr{L}(x_0)$ has finite moments of any order: $\mathbb{E}(|x_0|^p) < \infty$. Then $I(t)$ is absolutely continuous with respect to time t, and*

$$\frac{dI(t)}{dt} = -2\varepsilon^2(t) \int_{\mathbb{R}^d} |\sigma \nabla (f_t^{1/2})|^2\, d\nu_t + 2 \int \frac{d}{dt} \left[\varepsilon^{-2}(t) v(t)\right] (1 - f_t)\, d\nu_t\ ,$$

almost surely.

Preliminaries on Langevin equation

Up to the end of this section ε will be constant and we will study the long time stabilization of the process governed by

$$dX_{\varepsilon,t} = \varepsilon\, \alpha(.,dt) - AX_{\varepsilon,t}\, dt - \mathscr{V}'(X_{\varepsilon,t})dt\ . \tag{4.5}$$

One gives a precise sense to (4.5) under an integrated form using the general setting and results of [Da] and [Fu]. We denote by $(\Omega, \mathscr{F}_t, P, B_t)$ a cylindrical brownian motion, based on $L^2(G)$. For further use, we will establish an existence result for an equation of a little more general form than (4.5), namely :

$$dX_t = \varepsilon\, dB_t - AX_t\, dt - \mathscr{G}_t(X_t)dt \quad \text{with} \quad \mathbb{E}\|X_0\| < \infty \tag{4.6}$$

where $t \mapsto \mathscr{G}_t$ is a measurable map with values in the space of lipschitzian transformations of $L^2(G)$ (we drop temporarly the reference to ε). We assume that for any $T \geq 0$, $\sup_{0\leq t\leq T} (\|\mathscr{G}_t\|_{\text{LIP}}) < \infty$. We set $\mathscr{T}_t = \exp(-tA) \quad t \geq 0$.
Since A^{-1} is a nuclear operator, Theorem 5.1 in [Da] gives:

Theorem 4.2 *Let X_0 be a $L^2(G)$-valued, $\mathscr{F}_0$-measurable random variable. Then there exists a unique adapted process X which is almost surely in $\mathscr{C}(\mathbb{R}_+, L^2(G))$ and which satifies*

$$X_t = \mathscr{T}_t X_0 + \varepsilon \int_0^t \mathscr{T}_{t-s}\, dB_s - \int_0^t \mathscr{T}_{t-s}\mathscr{G}_s(X_s)\, ds. \tag{4.7}$$

If in addition $\mathbb{E}\big[\|X_0\|_{L^2(G)}\big]$ exists,

$$\sup_{0\leq t\leq T} \mathbb{E}\big[\|X_t\|_{L^2(G)}\big] < \infty, \qquad T > 0\,.$$

Approximation by finite-dimensional processes

Let us recall that E_h is a continuous (non-orthogonal) projection of $L^2(G)$ on its finite dimensional subspace W_h. We set $B_{h,t} = E_h B_t$, $A_h = E_h A E_h$ and $\mathscr{T}_{h,t} = \exp(-tA_h)$. Let $\mathscr{G}_{h,s}$ be a family of lipschitzian transformation of W, with range W_h such that for any $T > 0$:

$$\begin{cases} \mathscr{G}_{h,s} \text{ converges pointwise to } \mathscr{G}_s \text{ when } h \to \infty \\ \sup_{h\geq j_0} \sup_{0\leq t\leq T} \|\mathscr{G}_{h,t}\|_{L^2(G)} < \infty \\ \sup_{h\geq j_0} \sup_{0\leq t\leq T} \|\mathscr{G}_{h,t}\|_{\text{LIP}} < \infty\,. \end{cases} \tag{4.8}$$

We also suppose given a sequence of $\mathscr{F}_0$-measurable random variables $X_{h,0}$ such that $X_{h,0} \in W_h$ and $\lim_{h\to\infty} \mathbb{E}\big(\|X_{h,0} - X_0\|_{L^2(G)}\big) = 0$. Then one can define W_h-valued processes $U_{h,t}$ and $X_{h,t}$ by:

$$\begin{aligned} U_{h,0} &= X_{h,0} \\ U_{h,t} &= \mathscr{T}_{h,t} U_{h,0} + \varepsilon \int_0^t \mathscr{T}_{h,(t-s)} dB_{h,s} \\ X_{h,t} &= \mathscr{T}_{h,t} X_{h,0} + \varepsilon \int_0^t \mathscr{T}_{h,(t-s)}\, dB_{h,s} - \int_0^t \mathscr{T}_{h,(t-s)} \mathscr{G}_{s,h}(X_{h,s})\, ds\,. \end{aligned}$$

The differential equation of $X_{h,t}$ is:

$$dX_{h,t} = \varepsilon dB_{h,t} - A_h X_{h,t}\, dt - \mathscr{G}_{s,h}(X_{h,t})\, dt \tag{4.9}$$

The following result appears to be sufficient for our purposes. The situation is close to the one considered by Funaki in [Fu] and probably stronger results might be proved.

Theorem 4.3 *For any fixed T, there exists a sequence $h_n \to \infty$, such that*

$$\mathbb{E}\left[\int_0^T \|X_t - X_{h_n,t}\|_{L^2(G)}\,dt\right] \to 0 .$$

This theorem will follow from several consequent lemmas. It is convenient to remark that, if we restrict the domain of A, we get a positive self-adjoint operator in $\mathscr{H}$ (we do not introduce a special notation for it) and $A_h = E_h A E_h$ is a positive self-adjoint operator on $\mathscr{H}$.

Lemma 4.4 *The operator A is the limit of A_h when $h \to \infty$ in the sense of the strong resolvent convergence of self-adjoint operators on $\mathscr{H}$:*

1 *For g in $\mathscr{H}$ and $c \geq 0$, $\quad (A_h + c\,Id)^{-1}(g) \to (A + c\,Id)^{-1}(g)$ in the topology of $\mathscr{H}$ topology.*
2 *The bounded self-adjoint operators on $\mathscr{H}$, $\quad \mathscr{T}_{h,t}$ converge strongly to $\mathscr{T}_t$.*
3 *There exists a constant k such that, uniformly in h,*

$$\|\mathscr{T}_{h,t}\,g\|_{\mathscr{H}} \leq \frac{k}{\sqrt{t}}\|g\|_{L^2}, \qquad g \in W_h .$$

4 *There exists a constant k' such that for $t \geq 1$ uniformly in h,*

$$\|\mathscr{T}_{h,t}\,g\|_{L^2} \leq k' \exp(-k't)\|g\|_{L^2}, \qquad g \in W_h .$$

Proof. We observe that the operators $(A_h + c\,Id)^{-1}$ are uniformly bounded in $\mathscr{H}$ and that $\bigcup_{j\in\Lambda} W_j$ is dense in $\mathscr{H}$. Thus it is sufficient to take g lying in some space W_j and to take $h > j$. Now, in order to work in a familiar environment (Galerkin method) we replace the first assertion by an equivalent one in L^2, using the isometric map

$$\mathscr{H} \xrightarrow{A^{1/2}} L^2(G) .$$

At the operators level this isometry transforms A, A on $\mathscr{H}$, into its counterpart, the original A on L^2, and E_h with the orthogonal projection F_h on the space $V_h = \mathrm{span}(\psi_\lambda,\ \lambda \in \Gamma_h)$ and A_h into $B_h = F_h A F_h$. So we replace g by $\varphi = A^{1/2}g \in V_j$. Since φ is in $L^2(G)$, the solution f of $(A + c\,Id)f = \varphi$ is an element of the Sobolev space $\mathscr{H} = H_0^s$. We consider on $\mathscr{H}$ the equivalent scalar product:

$$\langle u, v\rangle' = \big((A + c\,Id)u, v\big)_{L^2} .$$

Let f_h be the projection with respect to the $\langle\cdot,\cdot\rangle'$ scalar product of f on V_h. Let us check that $(B_h + c\,Id)f_h = \varphi$ for $h > j$. Since both members are in V_h we have only to check that they have the same $(\cdot,\cdot)_{L^2}$ scalar product with any vector

$v \in V_h$. As F_h is symmetric with respect to the scalar product of L^2, we can write

$$\left(v, (B_h + c\, Id)f_h\right)_{L^2} = (v, (A + c\, Id)f_h)\ ,$$

and by projection

$$= \langle v, f_h\rangle' = \langle v, f\rangle' = \left(v, (A + cI)f\right)_{L^2} = \left(v, \varphi\right)_{L^2}.$$

Now we write f as a combination of wavelets: $f = \sum_{\lambda} a_\lambda \psi_\lambda$. According to Theorem 2.1,

$$\|f\|_{\mathcal{H}} = \|A^{1/2}f\|_{H_0^s} \leq C_2\left(\sum_{\lambda} 4^{sj} a_\lambda^2\right)^{1/2} \leq C_3\|f\|_{\mathcal{H}}\ .$$

Since f_h is the closest element of V_h from f, for another constant C_2',

$$\|f - f_h\|' \leq C_2'\left(\sum_{\lambda \notin \Gamma_h} 4^{sj} a_\lambda^2\right)^{1/2}\ .$$

So f_h tends to f in $\mathcal{H}$ and this prove a stronger statement than the first wanted one, since $(A_h + cId)^{-1}g$ is the image of f_h by the bounded operator $A^{-1/2}$.

The second statement is a consequence of the first one and of Stone-Weierstraß theorem (the proof is similar to that of Theorem 8.20, p.286 in [RS]). To prove the third assertion, we remark that thanks to the spectral decomposition of the self-adjoint operator A_h in $\mathcal{H}$

$$\|\mathcal{T}_{h,t}f\|_{\mathcal{H}} \leq t^{-1}\|(A_h + \frac{1}{t}I)^{-1}(g)\|_{\mathcal{H}}$$

since $e^{-xt} \leq (xt + 1)^{-1}$ for $x \geq 0$. Like in the proof of the first assertion, we solve the equation

$$(B_h + c\, Id)f_h = \varphi, \quad c = \frac{1}{t} \quad \varphi = A^{1/2}g$$

by the variational method. The variational equation

$$\langle v, f\rangle\,' = (v, \varphi) = (v, A^{1/2}g) = \langle v, A^{-1/2}g\rangle_{\mathcal{H}}$$

leads to the inequality (we take $f = v$ and we use $\|f\|' \geq \|f\|_{\mathcal{H}}$)

$$\|f\|' \leq \|A^{-1/2}g\|_{\mathcal{H}} = \|g\|_{L^2}$$

thus by projection, uniformly in h: $\|f_h\|' \leq \|g\|_{L^2}$. It is clear from the definition that $\|\cdot\|_{L^2} \leq \sqrt{t}\,\|\cdot\|'$. Consequently,

$$\|\mathcal{T}_{h,t}f\|_{\mathcal{H}} \leq t^{-1}\|(A_h + \frac{1}{t}I)^{-1}(g)\|_{\mathcal{H}} = t^{-1}\|f_h\|_{L^2} \leq \frac{1}{\sqrt{t}}\|f_h\|' \leq \frac{1}{\sqrt{t}}\|g\|_{L^2}\ .$$

To complete the proof, we start from the semi-group property $\mathscr{T}_{h,t} = \mathscr{T}_{h,t-1}\mathscr{T}_{h,1}$, and we use the inequality just proven for $t = 1$ in conjunction with $\|\mathscr{T}_{h,s}\varphi\|_{\mathscr{H}} \leq e^{-\gamma s}\|\varphi\|_{\mathscr{H}}$ where γ is the lower bound of the spectrum of the self-adjoint operator A in $\mathscr{H}$ (the restriction of A_h to W_h is bounded from below by the same constant). □

Lemma 4.5 *Let $D_{h,t} = \mathbb{E}\left(\|U_{h,t} - U_t\|_{L^2}\right)$. For any $t > 0$, $D_{h,t} \to 0$ and for any $T > 0$ $\int_0^T D_{h,t}\,dt \to 0$ when $h \to \infty$.*

Proof. Since $\widetilde{\Theta}$ is a Riesz basis of $L^2(G)$, the L^2-norm of a vector φ is equivalent to the new norm $\|\varphi\|'' = \left(\sum_\lambda 4^{-sj}\langle\theta_\lambda,\varphi\rangle^2\right)^{1/2}$, and we will estimate $D''_{h,t} = \mathbb{E}\left(\|U_{h,t} - U_t\|''\right)$. By Lemma 4.4, we easily get the convergence of $\mathscr{T}_{h,t}U_{h,0}$ to $\mathscr{T}_t U_0$ in the required sense, so we will now study

$$D''_{2(h,t)} = \mathbb{E}\left(\left|\int_0^t \mathscr{T}_{h,(t-s)}\,dB_{h,s} - \int_0^t \mathscr{T}_{(t-s)}\,dB_s\right|''^2\right).$$

Let Q_r be an adapted family of operators. The adjoints * with respect to $\langle\cdot,\cdot\rangle$ and * with respect to the L^2 scalar product $(\cdot,\cdot)$ are related as follows: $Q_r^* A = AQ_r^\star$. So we get:

$$\mathbb{E}\left[\left\|\int_0^t Q_r\,dB_r\right\|''^2\right] = \sum_\lambda 4^{-sj}\mathbb{E}\left[\left(A\theta_\lambda, \int_0^t Q_r\,dB_r\right)^2\right]$$

$$= \sum_\lambda 4^{-sj}\mathbb{E}\left[\left(\int_0^t (Q_r^* A\theta_\lambda\,, dB_r)\right)^2\right] = \sum_\lambda 4^{-sj}\int_0^t \left(\|Q_r^* A\theta_\lambda\|_{L^2}\right)^2 dr\,.$$

$$\mathbb{E}\left[\left\|\int_0^t Q_r\,dB_r\right\|''^2\right] = \sum_\lambda 4^{-sj}\int_0^t \langle AQ_r^\star\theta_\lambda, Q_r^\star\theta_\lambda\rangle_{\mathscr{H}}\,dr. \quad (4.10)$$

We shall take advantage of the fact that the operators T_u, E_j or $T_{j,u}$ are symmetric with respect to the scalar product $\langle\cdot,\cdot\rangle$ of $\mathscr{H}$. For $Q_r = \mathscr{T}_r - \mathscr{T}_{h,r}E_h$, $Q_r^\star = \mathscr{T}_r - E_h\mathscr{T}_{h,r}$ we obtain

$$D''_{2(h,t)} = \sum_\lambda 4^{-sj}\int_0^t a_h(\lambda, t-u)\,du = \sum_\lambda 4^{-sj}\int_0^t a_h(\lambda, r)\,dr\,, \quad (4.11)$$

where the term $a_h(\lambda, r) = \langle AQ_r^\star\theta_\lambda, Q_r^\star\theta_\lambda,\rangle_{\mathscr{H}}$ expands as follows:

$a_h(\lambda, r) = \langle A(\mathscr{T}_r - E_h\mathscr{T}_{h,r})\theta_\lambda, (\mathscr{T}_r - E_h\mathscr{T}_{h,r})\theta_\lambda\rangle = b(\lambda, r) - 2c_h(\lambda, r) + d_h(\lambda, r)$,
$b(\lambda, r) = \langle\mathscr{T}_r\theta_\lambda A\mathscr{T}_r\theta_\lambda\rangle$, $\quad c_h(\lambda, r) = \langle E_h\mathscr{T}_{h,r}\theta_\lambda A\mathscr{T}_r\theta_\lambda\rangle$,
$d_h(\lambda, r) = \langle E_h\mathscr{T}_{h,r}\theta_\lambda, AE_h\mathscr{T}_{h,r}\theta_\lambda\rangle$.

It is useful to remark that we can also write:
$d_h(\lambda, r) = \langle\mathscr{T}_{h,r}\theta_\lambda, A_h\mathscr{T}_{h,r}\theta_\lambda\rangle = \langle\theta_\lambda, A_h\exp(-2rA_h)\theta_\lambda\rangle$, since the projection E_h is self-adjoint in $\mathscr{H}$. Under this form we integrate explicitly

$$\int_0^t d_h(\lambda, r)\,dr = \langle \theta_\lambda, \frac{1}{2}(1 - \exp(-2tA_h))\theta_\lambda \rangle \le \frac{1}{2}\,.$$

Thanks to Lemma 4.4, we see that this term converge to

$$\langle \theta_\lambda, \frac{1}{2}(1 - \exp(-2tA))\theta_\lambda \rangle = \int_0^t b(\lambda, r)\,dr\,.$$

As it is less than $\frac{1}{2}$, and since $\sum_{\lambda \in \Lambda} 4^{-sj}$ is finite, we have $\sum_j 4^{-sj} \int_0^t d_h(\lambda, r)\,dr \longrightarrow \sum_j 4^{-sj} \int_0^t b(\lambda, r)\,dr$. Moreover, a similar convergence takes place if we integrate one more time with respect to t on $(0, T)$.

To complete the proof we will show that $\sum_\lambda 4^{-sj} \int_0^t c_h(\lambda, r)\,dr$ converges to the same limit. The Lemma 4.4 gives the pointwise convergence of c_h to b for fixed λ and r. The term $c_{\lambda,r}$ is uniformly bounded since it can be written as $c_h(\lambda, r) = \langle E_h \mathcal{T}_{h,r}\theta_\lambda, \mathcal{T}_r(A\theta_\lambda) \rangle$ and the semigroups are contracting in $\mathcal{H}$. Therefore, by the dominated convergence on $[0, t]$ we have:

$$\int_0^t c_h(\lambda, r)\,dr \longrightarrow \int_0^t c(\lambda, r)\,dr\,.$$

Let us do some manipulations again:

$$|c_h(\lambda, r)| = |\langle A^{1/2}E_h \mathcal{T}_{h,r}\theta_\lambda, A^{1/2}\mathcal{T}_r\theta_\lambda \rangle| \le \|A^{1/2}E_h \mathcal{T}_{h,r}\theta_\lambda\|\ \|A^{1/2}\mathcal{T}_r\theta_\lambda\|$$
$$\|A^{1/2}E_h \mathcal{T}_{h,r}\theta_\lambda\|^2 = \langle A^{1/2}E_h \mathcal{T}_{h,r}\theta_\lambda, A^{1/2}E_h \mathcal{T}_{h,r}\theta_\lambda \rangle = \langle AE_h \mathcal{T}_{h,r}\theta_\lambda, E_h \mathcal{T}_{h,r}\theta_\lambda \rangle$$
$$= \langle A_h \mathcal{T}_{h,r}\theta_\lambda, \mathcal{T}_{h,r}\theta_\lambda \rangle = \langle A_h \mathcal{T}_{h,2r}\theta_\lambda, \theta_\lambda \rangle$$

By the Schwarz inequality

$$\left|\int_0^t c_h(\lambda, r)\,dr\right| \le \left(\int_0^t \langle A_h \mathcal{T}_{h,2r}\theta_\lambda, \theta_\lambda \rangle\,dr\right)^{1/2} \left(\int_0^t \langle A \mathcal{T}_{2r}\theta_\lambda, \theta_\lambda \rangle\,dr\right)^{1/2}$$
$$= \frac{1}{4}\langle \big(1 - \exp(-2tA_h)\big)\theta_\lambda, \theta_\lambda \rangle \langle \big(1 - \exp(-2tA)\big)\theta_\lambda, \theta_\lambda \rangle \le \frac{1}{4}\,.$$

and by the dominated convergence on Λ the proof is completed. □

Taking into account the two preceeding lemmas, direct estimations of perturbation in the equation of the processe X_t lead to Theorem 4.3.

Now we will see how to approximate the equation (4.5) by Langevin equations in finite dimension like (4.3). The covariance matrix σ is easily found and the estimation (4.12) below is again a consequence of Theorem 2.1. Let us identify the space W_h with $\mathbb{R}^d$, $d = (\#\Gamma_h)$ with the help of the basis $\widetilde{\theta}_\lambda, \lambda \in \Gamma_\lambda$. We put:

$$v(x) = \frac{1}{2}\left|\sum_{\lambda \in \Gamma_h} x_\lambda \widetilde{\theta}_\lambda\right|^2_{\mathcal{H}} + V\left(\sum_{\lambda \in \Gamma_h} x_\lambda \widetilde{\theta}_\lambda\right)$$
$$\varphi_\lambda = 2^{-sj} A^{1/2} \psi_\lambda \qquad \text{and} \qquad \sigma^2_{\lambda\mu} = (\varphi_\lambda, \varphi_\mu)$$

Let σ_h be the symmetric positive matrix defined by: $(\sigma_h^2)_{\lambda\mu} = \sigma_{\lambda\mu}^2, \lambda, \mu \in \Gamma_h$. Let $\mathcal{G}_{s,h} = \mathcal{G}_h$ be defined for $w \in W$ as

$$\mathcal{G}_h(w) = \sum_{\lambda \in \Gamma_h} \left[\sum_{\mu \in \Gamma_h} \sigma_{\lambda\mu}^2 \left(\widetilde{\theta}_\mu, \mathscr{V}'(w_h) \right) \right] \widetilde{\theta}_\lambda \,, \qquad w_h = E_h w \,.$$

Lemma 4.6

1 *The process x_h of coordinates of X_h is identical in law to the process defined by the equation (4.3) with $\sigma = \sigma_h$.*

2 *There exists a number k such that independently of the dimension,*

$$\frac{1}{k} Id_h \leq \sigma_h \leq k\, Id_h \,. \tag{4.12}$$

3 *The drift $\mathcal{G}_h$ satifies the hypothesis (4.8) and converges to $\mathscr{V}'$ when $j \to \infty$.*

Because our hypothesis on $\mathscr{V}$ is very strong, the law of the solution $X_{\varepsilon,t}$ can be compared to the law of the corresponding gaussian process $U_{\varepsilon,t} = \mathcal{T}_t X_{\varepsilon,0} + \varepsilon \int_0^t \mathcal{T}_{t-s}\, dB_s$ via the Girsanov theorem. A standard application of Fernique and Feldman-Hajek theorem to these processes yields:

Lemma 4.7 *Let X_0 be a deterministic starting point, not depending on ε in $L^2(G)$. For any $t > 0$, the law $\mathscr{L}(X_{\varepsilon,t})$ is absolutely continuous with respect to μ_ε at time t, and for any finite p there exists a constant C such that its density $F_{\varepsilon,t}$ satifies:*

$$\|F_{\varepsilon,t}\|_{L^p(\mu_\varepsilon)} \leq C \exp(C\varepsilon^{-2}) \qquad \varepsilon > 0 \,. \tag{4.13}$$

Moreover, for a sufficiently small constant α, $\mathbb{E}\left[\exp\left(\alpha\varepsilon^{-2}\|X_{\varepsilon,t}\|_{L^2(G)}^2\right)\right]$ is finite.

We are now ready to prove our ergodicity result.

Theorem 4.8 *If X_0 is a fixed point in $L^2(G)$, then for any $\varepsilon > 0$ there exist constants C and $\gamma(\varepsilon)$ such that*

$$\begin{cases} \text{for } t \geq 1 \qquad I(\mathscr{L}X_{\varepsilon,t} \mid \mu_\varepsilon) \leq C \exp(C\varepsilon^{-2} - \gamma_\varepsilon t) \\ \qquad\qquad\qquad \lim\limits_{\varepsilon \to 0} \varepsilon^2 \log(\gamma_\varepsilon) = -2\, m_S \end{cases}$$

where m_S is the Hajek's constant associated with the functional S.

Proof. We take 1 as the new origin of the time and we change the notations accordingly. Now $X_{0,\varepsilon}$ is a variable such that $\mathbb{E}\left[\exp \alpha(\|X_{\varepsilon,0}\|^2)\right]$ is finite and: $\mathscr{L}(X_{\varepsilon,0}) = F_{\varepsilon,0}\, \mu_\varepsilon$ with $F_{\varepsilon,0}$ in some L^p, $p > 1$. We make a finite-dimensional approximation like in Lemma 4.6. We choose $X_{h,\varepsilon,0} = E_h X_{\varepsilon,0}$. The process $x_{\varepsilon,h}$ defined as the coordinates of $X_{\varepsilon,h}$ is of the form (4.3). Its invariant measure is $\nu_{\varepsilon,h}$, already considered in Lemma 3.8 and it corresponds to the Boltzmann measure at temperature $\varepsilon^2/2$. Taking into account Proposition 4.1 we get:

$$\frac{d}{dt} I(\mathscr{L}X_{\varepsilon,h,t} \mid \nu_{\varepsilon,h}) = -2\varepsilon^2 \int |\sigma_h \nabla (f_{h,t})^{1/2}|^2 \, d\nu_{\varepsilon,h}$$

so applying the logarithmic Sobolev inequality to the function $f_{h,t}^{1/2}$ and in view of the boundeness of σ_h (see (4.12)), we find C_1 not depending on h, ε, t, such that:

$$\frac{d}{dt} I(\mathscr{L}X_{\varepsilon,h,t} \mid \nu_{\varepsilon,h}) \leq -C_1\varepsilon^2 \left(c(\nu_{\varepsilon,h})\right)^{-1} I(\mathscr{L}X_{\varepsilon,h,t} \mid \nu_{\varepsilon,h})$$
$$I(\mathscr{L}X_{\varepsilon,h,t} \mid \nu_{\varepsilon,h}) \leq I(\mathscr{L}X_{\varepsilon,h,0} \mid \nu_{\varepsilon,0}) \exp\left(-C_1\varepsilon^2 c^{-1}(\nu_{\varepsilon,h})t\right) . \quad (4.14)$$

Let $\eta > 0$ be given. We employ the log-Sobolev inequality under the form (3.16)

$$\int F^2 \log\left(\frac{|F|}{\|F\|_2}\right) d\mu_\varepsilon \leq \exp\left((2m_S + 2\eta)\varepsilon^{-2}\right) \int \sum_{\lambda \in \Lambda} (\partial_\lambda F)^2 \, d\mu_\varepsilon$$

which implies a standard log-Sobolev inequality with the same constant for the projection $\mu_{\varepsilon,h}$ on $W_h \equiv \mathbb{R}^d$ for any h, thus Lemmas 3.3 and 3.8 give for ε smaller than an $\varepsilon(\eta)$ and h large:

$$\int_{\mathbb{R}^d} g^2 \log\left(\frac{|g|}{\|g\|_2}\right) d\nu_{h,\varepsilon} \leq \exp\left((2m_S + 4\eta)\varepsilon^{-2}\right) \int \sum_{k=0}^{d} (\partial_k g)^2 \, d\nu_{\varepsilon,h}$$
$$c(\nu_{\varepsilon,h}) \leq \exp\left((2m_S + 4\eta)\varepsilon^{-2}\right) \quad (4.15)$$

On the other hand, by Lemma 3.8

$$I(\mathscr{L}X_{\varepsilon,h,0} \mid \nu_{\varepsilon,h}) = \int \log(f_{h,0})\, d\left(\mathscr{L}X_{\varepsilon,h,0}\right) \leq \eta\varepsilon^{-2}$$
$$+ \int \log\left(\frac{\partial \mathscr{L}X_{\varepsilon,h,0}}{\partial \mu_{\varepsilon,h}}\right) d\left(\mathscr{L}X_{\varepsilon,h,0}\right) \leq \eta\varepsilon^{-2} + I\left(\mathscr{L}X_{\varepsilon,h,0} \mid \mu_{\varepsilon,h}\right)$$

and in the same way

$$\left| I(\mathscr{L}X_{\varepsilon,h,0} \mid \nu_{\varepsilon,h}) - I\left(\mathscr{L}X_{\varepsilon,h,0} \mid \mu_{\varepsilon,h}\right) \right| \leq \eta\varepsilon^{-2}$$

The Radon-Nikodym derivatives $\dfrac{\partial(\mathscr{L}X_{\varepsilon,h,0})}{\partial \mu_{\varepsilon,h}} \circ X_{\varepsilon,h,0}$ define a martingale in h which, according to Lemma 4.7, converges in some L^p, thus we also have :

$$\lim_{h\to\infty} I\left(\mathscr{L}X_{\varepsilon,h,0} \mid \mu_{\varepsilon,h}\right) = I\left(\mathscr{L}X_{\varepsilon,0} \mid \mu_\varepsilon\right) \quad (4.16)$$

This relation, together with (4.15) gives, since η is arbitrary

$$\lim_{h\to\infty} I\left(\mathscr{L}X_{\varepsilon,h,0} \mid \nu_{\varepsilon,h}\right) = I\left(\mathscr{L}X_{\varepsilon,0} \mid \mu_\varepsilon\right) . \quad (4.17)$$

Now we use the lower semi-continuity dependence of the entropy $I(P \mid Q)$ on the two measures P and Q with respect to the weak convergence of measures on $L^2(G)$ (see [DS], 3.2.12). The weak convergence of $\nu_{\varepsilon,h}$ to μ_ε is easy to get using Lemma 3.8. On the other hand, it follows from Theorem 4.3 that one can find a sequence h_n and a negligible set N such that when $n \to \infty$, $t \notin N$ $\|X_t - X_{h_n,t}\|_{L^2(G)} \to 0$. Therefore $X_{h_n,t}$ converges in law in $L^2(G)$ to X_t and

$$\text{for } t \notin N \qquad I(\mathscr{L}X_{\varepsilon,t} \mid \mu_\varepsilon) \leq \varliminf_{h \to \infty} I(\mathscr{L}X_{\varepsilon,h,t} \mid \nu_{\varepsilon,h}) .$$

Letting h tend to infinity in (4.14) and taking into account (4.15), we get for any η, if ε is small enough and t outside N:

$$I(\mathscr{L}X_{\varepsilon,t} \mid \mu_\varepsilon) \leq I(\mathscr{L}X_{\varepsilon,0} \mid \mu_{\varepsilon,0}) \exp\left(-C_1\varepsilon^2 e^{-(2m_S+4\eta)\varepsilon^{-2}} t\right) .$$

Due to the semi-continuity of the entropy, the same inequality is valid for every t since the path continuity of $X_{\varepsilon,t}$ implies the weak continuity of $\mathscr{L}X_{\varepsilon,t}$ with respect to time. We observe in addition that for any ε, we have a similar inequality (see the first assertion in 3.8):

$$I(\mathscr{L}X_{\varepsilon,t} \mid \mu_\varepsilon) \leq I(\mathscr{L}X_{\varepsilon,0} \mid \mu_{\varepsilon,0}) \exp(-C_2\varepsilon^2 e^{-C_2\varepsilon^{-2}} t) .$$

So we got the existence of γ_ε as well as its asymptotic behaviour. To complete the proof, we use Lemma 4.7:

$$I(\mathscr{L}X_{\varepsilon,0} \mid \mu_{\varepsilon,0}) \leq C_2\|F_{\varepsilon,t}\|_p \leq CC_2 \exp(C\varepsilon^{-2}) .$$

□

5 Towards the simulated annealing

The functional S is lower semi-continuous on W or $L^2(G)$ and it appears as the large deviations functional of the measures μ_ε when $\varepsilon \to 0$. So these measures are more and more concentrated on, say, nearly minimal sets $\{w \in W \mid \exists w' \|w - w'\|_\infty \leq \delta \text{ and } S(w') = \inf(S)\}$ where δ is an arbitrary positive number . Having proved for fixed ε the convergence of $\mathscr{L}(X_{\varepsilon,t})$ to μ_ε, it is natural to conjecture that if we let ε decrease slowly in time, the process X_t solution of (1.1) will finally belong to any nearly minimal set with a large probability (simulated annealing), in other words X_t converges in probability in W to $\mathscr{M} = \{S = \inf(S)\}$. We shall only be able to construct for any J a process Y_t^J with values in $L^2(G)$ whose finite-dimensional projection $E_J Y^J$ converges to the compact set $E_J\mathscr{M}$. Nevertheless this solve in some sense the problem of computing the ground state of S (see our final remark).

The idea is to let diminish in time the influence of the component of the process in the direction W_J^* defined by the high-resolution wavelets. Let us set $w_J = E_J(w)$ and $w_J^* = E_J^* w$ with $E_J^* = Id - E_J$. We define Y_t^J as a solution of the following equation:

$$dY_t^J = \varepsilon(t)dB_t - \varepsilon^2(t)AY_t^J - \mathscr{B}_t^J(Y_t^J)\,dt \tag{5.1}$$

where: $$\mathscr{B}_t^J(w) = \varepsilon(t)\mathscr{V}'\left(w_J + \varepsilon(t)w_J^*\right) + (1 - \varepsilon^{-2}(t))Aw_J + (1 - \varepsilon(t))AE_JA^{-1}\mathscr{V}'\left(w_J + \varepsilon(t)w_J^*\right) \tag{5.2}$$

which comes from the more formal equation

$$dY_t^J = \varepsilon(t)\alpha(\cdot, dt) - \nabla S^J_{\varepsilon(t)}(Y_t^J)\,dt \qquad S^J_\varepsilon(w) = S(w_J + \varepsilon w_J^*) \tag{5.3}$$

Associated with S^J_ε there is a Boltzmann measure μ^J_ε at temperature $\varepsilon^2/2$. Put: $r^J_\mu(t) = \begin{cases} 1 & \text{if } \mu \in \Gamma_J \\ \varepsilon(t) & \text{if } \mu \in \Lambda \setminus \Gamma_J \end{cases}$ and define $\rho^J(t)$ as a self-adjoint operator in $\mathscr{H}$ which is diagonal in the basis Θ and admits the eigenvalues r^J_μ. Then μ^J_ε is just the image of μ_ε under $\rho^J(t)^{-1}$. Let us identify the space W_h with $\mathbb{R}^d$, where $d = \#h$, via the basis $\widetilde{\Theta}$. We consider the energy function v_h on W_h defined by

$$v^J_{h,t}(w) = S\left(E_J w + \varepsilon(t)[w - E_J(w)]\right), \qquad w \in W_h .$$

Let a process y^J_h on $\mathbb{R}^d$ be defined by the equation (4.1) with a matrix σ_h still as in Lemma 4.6, and consider the corresponding process Y^J_h in $W_h \subset L^2$ starting from $E_h(Y_0)$, where Y_0 is fixed, $Y_0 \in L^2(G)$. We may perform, like in [CHS], a time change in order to transform Y^J into an homogeneous diffusion process (with ε constant) so the following result may be viewed as a consequence of Theorem 4.3.

Lemma 5.1 *For any $T \geq 0$, $\int_0^T \|Y^J_h(t) - Y^J(t)\|_{L^2}\,dt \longrightarrow 0$, when $h \to \infty$.*

To simplify the estimates, we will only look at the case of temperature given by $\frac{1}{2}\varepsilon^2(t) = \dfrac{c}{\log(t)}$ for $t \geq 2$, which are still decreasing and of the class $\mathscr{C}^1$ on $\mathbb{R}_+$.

Lemma 5.2 *For any $\alpha > 0$, there exists a constant C and t_0 such that for arbitrary h and t, $t \geq t_0$, $\mathbb{E}\left[\|Y^J_{h,t}\|^2_{L^2(G)}\right] \leq Ct^\alpha$.*

Proof. For $h \geq J$ the process Y^J_h is a solution of

$$dY^J_h(t) = \varepsilon(t)dB_{h,t} - \left[E_hAE_J + \varepsilon^2(t)E_hA(E_h - E_J)\right] Y^J_h(t)\,dt - \mathscr{B}^{J,2}_{h,t} Y^J_h(t)\,dt$$

$$\mathscr{B}^{J,2}_{h,t}(w) = \sum_{\lambda \in \Gamma_h} \left[\sum_{\mu \in \Gamma_h} \sigma^2_{\lambda\mu} r^J_\mu(t) \left(\widetilde{\theta}_\mu , \mathscr{V}'\left(w_J + \varepsilon(t)w_J^*\right)\right)\right] \widetilde{\theta}_\lambda$$

At the first step, we will study the centered Gaussian process U^J_h that we get when $\mathscr{B}^{J,2} = 0$ and $U^J_{h,0} = 0$. We remark that the operator $\rho^J(t)$ introduced above together with its inverse are bounded in L^2, as well as in $\mathscr{H}$, by constants of less than logarithmic growth when $t \to \infty$, so it is equivalent to prove the estimate we want for $Z^J_h(t) = \rho^J(t)U^J_h(t)$ instead of $U^J_h(t)$. The gain is that Z^J_h is governed by a self-adjoint drift operator

$$D^J_h(t) = \rho^J(t)A_h\rho(t) + \frac{\varepsilon'(t)}{\varepsilon(t)}(E_h - E_J) \tag{5.4}$$

$$dZ^J_h(t) = \varepsilon(t)\rho(t)dB_{h,t} - D^J_h(t)Z^J_h(t)\,dt$$

It can be expressed as a stochastic integral. We make use of the norm $\|\cdot\|''$ and in view of formula (4.10) we can write:

$$\mathbb{E}\left[\|Z^J_{h,t}\|''\right]^2 = \sum_{\lambda\in\Gamma_h} 4^{-sj}\int_0^t \varepsilon^2(s)\langle \mathcal{S}^J_{h,st}\theta_\lambda\,,\rho^J_s A_h \rho^J_s \mathcal{S}^J_{h,st}\theta_\lambda\rangle\, ds$$

where the operators $\mathcal{S}^J_{h,st}$ are defined by:

$$\mathcal{S}^J_{h,st} = \exp\left(-\int_s^t D^J_h(u)\,du\right). \tag{5.5}$$

By (5.4):

$$\text{(5.6)}\qquad \mathbb{E}\left[\|U^J_{h,t}\|''\right]^2 \le \sum_{\lambda\in\Gamma_h} \varepsilon^2(t)4^{-sj}\int_0^t \langle \mathcal{S}^J_{h,st}\theta_\lambda, D^J_h(s)\mathcal{S}^J_{h,st}\theta_\lambda\rangle\, ds + \int_0^t |\varepsilon'(s)|\varepsilon(s)\langle \mathcal{S}^J_{h,st}\theta_\lambda, \mathcal{S}^J_{h,st}\theta_\lambda\rangle\, ds$$

Suppose that $0<\alpha<1$. By (5.4), since A is bounded from below

$$D^J_h(s) \ge \left(\gamma\varepsilon^2(u) + \frac{\varepsilon'(u)}{\varepsilon(u)}\right)E_h$$

and since $\gamma\varepsilon^2(u)+\frac{\varepsilon'(u)}{\varepsilon(u)}$ is greater than $(1-\alpha)u^{-\alpha}$ for large u

$$E_h\mathcal{S}^J_{h,st} \le C_1\exp\left(-t^{1-\alpha}+s^{1-\alpha}\right)E_h \tag{5.7}$$

For t large enough we have $|\varepsilon'(s)|\varepsilon(s)\le 2(1-\alpha)s^{-\alpha}$, so that

$$\begin{aligned}\int_0^t \langle \mathcal{S}^J_{h,st}\theta_\lambda\,, \mathcal{S}^J_{h,st}\theta_\lambda\rangle\, ds &\le C_1^2\exp(-2t^{1-\alpha})\int_0^t 2(1-\alpha)s^{-\alpha}\exp(2s^{1-\alpha})\,ds\\ &= C_1^2\left(1-\exp(-2t^{1-\alpha})\right)\le C_1^2\,.\end{aligned}$$

We can bound the other terms in (5.6) as follows:

$$\begin{aligned}2\int_0^t \langle \mathcal{S}^J_{h,st}\theta_\lambda, D^J_h(s)\mathcal{S}^J_{h,st}\theta_\lambda\rangle\, ds &= -\int_0^t \frac{d}{ds}\langle \mathcal{S}^J_{h,st}\theta_\lambda, \mathcal{S}^J_{h,st}\theta_\lambda\rangle\, ds\\ &= 1-\|\mathcal{S}^J_{h,0t}\theta_\lambda\|^2 \le 1\end{aligned}$$

The summation with respect to λ gives $\sup_t \mathbb{E}\left[\|Z^J_{h,t}\|''^2\right]<\infty$. The second step consists of finding a bound U^J_h when $U^J_h(0)$ does not vanish. We have to add a deterministic term $\mathcal{R}^J_{h,st}E_hX_0$ where $\mathcal{R}$ is defined by

$$\mathcal{R}^J_{h,ss} = Id \qquad \frac{d}{dt}\mathcal{R}^J_{h,st} = -A\left[E_J+\varepsilon^2(t)(E_h-E_J)\right]\mathcal{R}^J_{h,st}\,.$$

We remark that:

$$\frac{d}{dt}\left[\|\mathscr{R}^J_{h,st}w\|^2_{L^2(G)}\right] = -2\|E_J\mathscr{R}^J_{h,st}w\|^2_{\mathscr{H}} - 2\varepsilon^2(t)\|(E_h - E_J)\mathscr{R}^J_{h,st}w\|^2_{\mathscr{H}}$$
$$\leq -2\varepsilon^2(t)\|\mathscr{R}^J_{h,st}w\|^2_{\mathscr{H}}$$
$$\leq -2\gamma\varepsilon^2(t)\|\mathscr{R}^J_{h,st}w\|^2_{L^2}$$

which gives as in the equation (5.7) above for any $0 < \alpha < 1$

$$\|\mathscr{R}^J_{h,st}\|_{L^2\to L^2} \leq C_2\exp\left(-t^{1-\alpha} + s^{1-\alpha}\right) \tag{5.8}$$

and the second step is accomplished.

To complete the proof we go back to the general process and we form $\xi^J_h = X^J_h - U^J_h$ which satisfies:

$$\xi^J_h(t) = \int_0^t \mathscr{R}^J_{h,st}\mathscr{V}'\left(U^J_h(s) + \xi^J_h(s)\right) ds$$

$$\|\xi^J_h(t)\| \leq M\int_0^t \|\mathscr{R}^J_{h,st}\|\, ds \leq MC_2\exp(-t^{1-\alpha})\int_0^t \exp(s^{1-\alpha})\, ds$$
$$\leq MC_2\, t^\alpha \exp-(t^{1-\alpha})\int_0^t s^{-\alpha}\exp(s^{1-\alpha})\, ds \leq MC_2\, t^\alpha\,.$$

□

Theorem 5.3 *If $c > m_S$, then for any J, the distance in $L^2(G)$ between $E_JY^J(t)$ and the projection $E_J\mathscr{M}$ of the set of global minima of S vanishes in probability as $t\to\infty$.*

Proof. We will show that the relative entropy $I\left(\mathscr{L}(E_JY^J_t)\mid \mu_{J,\varepsilon(t)}\right)$ converges to 0 where $\mu_{J,\varepsilon} = E_J\mu_\varepsilon$. We remark that $E_J(\mu_{\varepsilon(t)}) = E_J(\mu^J_{\varepsilon(t)})$, and since the entropy diminishes under mapping, it will be sufficient to bound from above $I^J(t) = I(\mathscr{L}Y^J(t)\mid \mu^J_{\varepsilon(t)})$.

It is apparent that the arguments based upon the limit of finite dimensional entropy in the proof of Theorem 4.8 are still valid, so it is sufficient to bound $I^J_h(t) = I(\mathscr{L}Y^J_{h,t}\mid \nu^J_{h,s})$ uniformly in h where $\nu^J_{h,s}$ is the Boltzmann measure associated with the energy $v^J_{h,t}$ via Eq.(3.4). Our tool is the formula in Proposition 4.1, and we will first bound its last term:

$$R(t) = 2\int \frac{d}{dt}\left(\varepsilon^{-2}(t)v^J_h(t)\right)(f^J_{h,t} - 1)\, d\nu^J_{h,t}\,.$$

We have for $w \in W_h \equiv \mathbb{R}^d$

$$v^J_h(t,w) = \frac{1}{2}\left(\|E_Jw\|^2_{\mathscr{H}} + \varepsilon^2(t)\|(w - E_J(w))\|^2_{\mathscr{H}}\right) + V\left[E_J(w) + \varepsilon(t)E^*_J(w)\right]$$

$$\frac{d}{dt}[\varepsilon^{-2}(t)v^J_{h,t}] = \frac{\varepsilon'(t)}{\varepsilon^2(t)}\left(\mathscr{V}'[E_J(w) + \varepsilon(t)E^*_J(w)], E_J(w)\right)_{L^2(G)}$$
$$-\frac{\varepsilon'(t)}{\varepsilon^3(t)}\left[\|E_J(w)\|^2_{\mathscr{H}} + 2V^J_h(w)\right]$$

We remark that the $\mathscr{H}$ norm of $E_J w$ is bounded up to a constant by its L^2 norm. Since the dimension of W_J is finite, we bound ε and V and we find a C_1 such that for $t \geq 2$

$$\left|\frac{d}{dt}\left[\varepsilon^{-2}(t)\nu^J_{h,t}(w)\right]\right| \leq C_1 \frac{|\varepsilon'(t)|}{\varepsilon^3(t)}\left(\|w\|_{L^2(G)}+1\right) \leq \frac{C_1 c}{4t}\left(\|w\|_{L^2(G)}+1\right)$$

and by Lemma 5.2 for any β, such that $0 < \beta < 1$, we have a bound independent of h

$$\int\left|\frac{d}{dt}\left[\varepsilon^{-2}(t)\nu^J_{h,t}\right]\right| f_{h,t}\, d\nu^J_{h,t} = \mathbb{E}\left[\left|\frac{d}{dt}\left[\varepsilon^{-2}(t)\nu^J_{h,t}(Y^J_{h,t})\right]\right|\right] \leq C_2 t^{-\beta}$$

It is easy to complete this argument with the fact that any moment of the L^2 norm with respect to $\nu^J_{h,s}$ is bounded uniformly in h and s, thus $R(t) \leq C_3 t^{-\beta}$.

To find a logarithmic Sobolev inequality for $\nu^J_{h,t}$ we just have to transform (4.15) under the action of the restriction of ρ^J to W_h and we get for t greater than some $t(\eta)$ and any $h \geq J$:

$$\begin{aligned}\int_{\mathbb{R}^d} g^2 \log\left(\frac{|g|}{\|g\|_2}\right) d\nu^J_{h,t} &\leq \exp\left[(2m_S+4\eta)\varepsilon^{-2}(t)\right]\int \sum_{\lambda\in\Gamma_J}(\partial_\lambda g)^2 \\ &\quad + \varepsilon^{-2}(t)\sum_{\lambda\in\Gamma_h\setminus\Gamma_J}(\partial_\lambda g)^2\, d\nu^J_{t,h} \\ &\leq \exp\left[(2m_S+5)\eta\varepsilon^{-2}(t)\right]\int\sum_{\lambda\in\Gamma_h}(\partial_\lambda g)^2\, d\nu^J_{t,h}\end{aligned}$$

By the bound (4.12)

$$-2\varepsilon^2(t)\int_{W_h}|\sigma_h\nabla(f^{1/2}_{h,t})|^2\, d\nu^J_{h,t} \leq -\exp\left[(2m_S+6\eta)\varepsilon^{-2}(t)\right] I(\mathscr{L}Y^J_{h,t} \mid \nu^J_{h,t})$$

Finally, Proposition 4.1 gives that for any $\eta > 0$ and for t greater than some t_0:

$$\begin{aligned}\frac{d}{dt}(I^J_h(t)) &\leq -\exp\left[(2m_S+6\eta)\varepsilon^{-2}(t)\right] I^J_h(t) + C_3 t^{-\beta} \\ &\leq t^{-\frac{m_S+3\eta}{c}} I^J_h(t) + C_3 t^{-\beta} \qquad (5.9)\end{aligned}$$

Since $c > m_S$, we can find η such that $\frac{m_S+3\eta}{c} < 1$ and then β such that $\frac{m_S+3\eta}{c} < \beta < 1$. The differential inequality (5.9) gives then easily $\lim_{t\to\infty} I_h(t) = 0$ provided $I_h(t)$ is finite. And this can be established with the help of a Girsanov formula and of Gaussian estimates (see Lemma 4.7). □

Remark 5.4 Obviously, in order to perform numerical computations, we have to consider simulated annealing in finite dimension, that is to work with a limited resolution. In fact, the computation time grows quickly with the resolution. In view of the present results we may suggest the following procedure: we choose a J, according to the proof of Theorem 3.9, such that $S(w)$ is a convex fonction of w_J^*. Then if we know the projection $E_J w_0$ of some global minima w_0, it is easy to calculate the other components of w_0 with a gradient algorithm. But to know $E_J w_0$ with a better accuracy, it seems appropriate to put a relatively large noise intensity on some extra coordinates, in other words to use the process $Y_{h,t}^J$ considered above, for large h .

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