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Probability Theory and Related Fields

Brownian Bridges on Riemannian Manifolds

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Summary. We study properties of Brownian bridges on a complete Riemannian manifold M. Let $Q^t_{x,y}$ be the law of Brownian bridge from x to y with lifetime t. $Q^t_{x,y}$ is a probability measure on the space $\Omega_{x,y}$ of continuous paths ω with $\omega(0)=x$ and $\omega(1)=y$. We prove that $Q^t_{x,y}$ possesses the large deviation property with the rate function

$$J_{x,y}(\omega)=\frac{1}{2}\left[\int_0^1 |\dot{\omega}(s)|^2 ds-\rho(x,y)^2\right].$$

We show that if M and its metric are analytic then for *any* x, y on M there exists a probability measure $\mu_{x,y}$ which is supported by a subset of the space of minimizing geodesics joining x and y such that $Q^t_{x,y}\to\mu_{x,y}$ weakly in $\Omega_{x,y}$ as $t\to 0$. We also give a complete characterization of the exact support of $\mu_{x,y}$.

§ 1. Introduction

Let M be a complete Riemannian manifold of dimension m. The minimal heat kernel on M is denoted by $p(t, x, y)$. Let Ω_x denote the space of continuous paths $\omega: [0,1]\to M$ such that $\omega(0)=x$ and let $\Omega_{x,y}$ denote the space of paths such that $\omega(0)=x$ and $\omega(1)=y$. Ω_x and $\Omega_{x,y}$ are metric spaces under uniform convergence. The set of minimizing geodesics from x to y with uniform speed $\rho(x,y)$ (ρ is the Riemannian distance on M) is denoted by $\Gamma_{x,y}$. It is clear that $\Gamma_{x,y}\subset\Omega_{x,y}$ and since we assume M is complete, $\Gamma_{x,y}$ is never empty.

Let ${}^tX^{x,y}=\{{}^tX^{x,y}_s, 0\leqq s\leqq t\}$ be the Brownian bridge process from x to y with lifetime t. We set $X^{x,y;t}=\{X^{x,y;t}_s={}^tX^{x,y}_{st}, 0\leqq s\leqq 1\}$. We regard $X^{x,y;t}$ as map from the underlying probability space to the path space $\Omega_{x,y}$. Let $Q^t_{x,y}$ be the law of $X^{x,y;t}$ in $\Omega_{x,y}$; namely $Q^t_{x,y}=P\circ(X^{x,y;t})^{-1}$. $Q^t_{x,y}$ is closely related to the asymptotic behavior of the heat kernel $p(t,x,y)$. The purpose of this paper is to study the behavior of $Q^t_{x,y}$ *as* $t\downarrow 0$ for generally positioned x, y.

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To start with, we prove (Lemma 2.4) that the set of measures $\{Q^t_{x,y}; t > 0\}$ is sequentially compact on $\Omega_{x,y}$ as $t \downarrow 0$. This means that any sequence $t_n \downarrow 0$ has a subsequence t_{n_k} such that $Q^{t_{n_k}}_{x,y}$ converges weakly to a probability measure μ as $k \to \infty$.

The second result we will prove (Theorem 2.2) is that $Q^t_{x,y}$ possesses the large deviation property with the rate function

$$J_{x,y}(\omega) = \frac{1}{2}\left[\int_0^1 |\dot{\omega}(s)|^2\, ds - \rho(x, y)^2\right].$$

It follows that any limiting measure μ must be supported by the set of paths ω with the property $J_{x,y}(\omega) = 0$, i.e., by the set $\Gamma_{x,y}$ of minimizing geodesics joining x, y.

Now the obvious question is: does $Q^t_{x,y}$ have a unique limiting measure? or does the Brownian bridge converge (in law) to a limiting Browniann bridge? Such limiting Brownian bridge will simply consist in picking a minimizing geodesic according to a probability measure μ and then travelling along this geodesic with constant speed $\rho(x, y)$. We prove (Theorem 4.2) that the limiting measure is unique for any x, y if M and its metric are analytic. We also prove (Theorem 3.4) the uniqueness for a special case where the analyticity is not assumed.

In general the exact support of a limiting measure μ can be strictly smaller than $\Gamma_{x,y}$. We give a complete characterization of supp μ in the analytic case (Theorem 4.1). In this case, it turns out that with every $\gamma \in \Gamma_{x,y}$ we can associate a function $D(\gamma; t)$ of the form $t^{-\alpha}(\log(1/t))^{\beta}$ (α is a rational number and $m/2 \leqq \alpha \leqq m - 1/2$, and β is a non-negative integer). The contribution of γ to the heat kernel can be intuitively taken as $D(\gamma; t)\exp\{-\rho(x, y)^2/2t\}$. The order of $D(\gamma; t)$ going to infinity as $t \to 0$ is a measure of the degeneracy of the action functional

$$E(\omega) = \frac{1}{2}\int_0^1 |\dot{\omega}(s)|^2\, ds$$

near the path γ. Our result (Theorem 4.1) amounts to saying that the support of μ is exactly equal to the set of γ's in $\Gamma_{x,y}$ with the highest degeneracy.

We will use frequently various estimates on the heat kernel in the collection of papers [2]. This collection is an significant extension of the original work of Molchanov [5].

§2. Large Deviation of Brownian Bridge

Let $X^x = \{X^x_s, s \geqq 0\}$ be the Riemannian Brownian motion starting at point x. The Brownian bridge from x to y with lifetime t is obtained by conditioning X^x to hit y at time t. We make the time change $s \mapsto st$ and denote the law of the resulting process $X^{x,y;t}$ in the path space $\Omega_{x,y}$ by $Q^t_{x,y}$. Let $X^{x,t} = \{X^{x;t}_s = X^x_{st}, 0 \leqq s \leqq 1\}$ be the Brownian motion with the same time change and let Q^t_x be its law on the path space Ω_x. Then we have

$$\left.\frac{dQ^t_{x,y}}{dQ^t_x}\right|_{\mathscr{F}_s} = \frac{p(t(1-s),\ \omega(s), y)}{p(t, x, y)}, \qquad 0 \leqq s < 1\,. \tag{2.1}$$

Here $\{\mathscr{F}_s, 0 \leqq s \leqq 1\}$ is the standard filtration of σ-fields on Ω_x (or on $\Omega_{x,y}$ and $p(s, z, y)$ is the minimal heat kernel on M. (2.1) can be taken as a formal definition of the Brownian bridge $X^{x,y;t}$. The Brownian bridge is a nonhomogeneous diffusion process on M whose infinitesimal generator is

$$L_s f(z) = \frac{t}{2} \Delta f(z) + t \nabla_z \log p(t(1-s), z, y) \cdot \nabla f(z) . \tag{2.2}$$

(Δ is the Laplace-Beltrami operator and ∇_z is the gradient in the z variables.)

Remark 2.1. The transition density function of $X^{x,y;t}$ is

$$\frac{p(t(s_2 - s_1), z_1, z_2) p(t(1 - s_2), z_2, y)}{p(t(1 - s_1), z_1, y)} .$$

By the symmetry of $p(s, z, y)$ in z, y variables, we find that the processes $s \mapsto X_s^{x,y;t}$ and $s \mapsto X_{1-s}^{y,x;t}$ have the identical transition density function. Therefore they have the same law $Q_{x,y}^t$.

In this section, we prove the following large deviation property for the set of probability measures $\{Q_{x,y}^t; t > 0\}$.

Theorem 2.2. *For any open set* $G \subset \Omega_{x,y}$,

$$\liminf_{t \to 0} t \log Q_{x,y}^t(G) \geqq - \inf_{\omega \in G} J_{x,y}(\omega) \tag{2.3}$$

For any closed set $F \subset \Omega_{x,y}$,

$$\limsup_{t \to 0} t \log Q_{x,y}^t(F) \leqq - \inf_{\omega \in F} J_{x,y}(\omega) . \tag{2.4}$$

where

$$J_{x,y}(\omega) = \frac{1}{2} \left[\int_0^1 |\dot{\omega}(s)|^2 ds - \rho(x, y)^2 \right]$$

if $|\dot{\omega}(s)| \in L^2[0, 1]$; *otherwise* $J_{x,y}(\omega) = \infty$. *In other words,* $\{Q_{x,y}^t, t > 0\}$ *obeys the large deviation principle with rate function* $J_{x,y}$.

Remark 2.3. Riemannian Brownian motion Q_x^t possesses the large deviation property with rate function $I(\omega) = (1/2) \int_0^1 |\dot{\omega}(s)|^2 ds$ (see [1], p. 149 and [4], p. 155). A rough calculation shows that Theorem 2.2 is a consequence of the large deviation principle of Q_x^t and the well-known asymptotic relation

$$\lim_{t \to 0} t \log p(t, x, y) = -\frac{1}{2} \rho(x, y)^2 . \tag{2.5}$$

However, there are a few technical difficulties to overcome.

Let us start the proof of Theorem 2.2 with a preliminary result.

Lemma 2.4. *For any* $N > 0$, *there exists a compact subset* $C_N \subset \Omega_{x,y}$ *such that*

$$\limsup_{t \to 0} t \log Q_{x,y}^t (C_N^c) \leqq -N .$$

(C_N^c *denotes the complement of* C_N).

Proof: First, we show that

$$\limsup_{t\to 0} t\log Q^t_{x,y}[\rho(\omega, x) \geqq K] \leqq -\frac{1}{2}K^2 + 2\rho(x, y)^2 . \tag{2.6}$$

(Notation: $\rho(\omega, x) = \sup_{0\leqq s\leqq 1} \rho(\omega(s), x)$.) Indeed, by Remark 2.1 we have

$$Q^t_{x,y}[\rho(\omega, x) \geqq K] \leqq Q^t_{x,y}\left[\sup_{0\leqq s\leqq 1/2} \rho(\omega(s), x) \geqq K\right]$$

$$+ Q^t_{y,x}\left[\sup_{0\leqq s\leqq 1/2} \rho(\omega(s), y) \geqq K - \rho(x, y)\right]. \tag{2.7}$$

The two terms on the right-hand side can be treated in the same way. By (2.1), we write

$$Q^t_{x,y}\left[\sup_{0\leqq s\leqq 1/2} \rho(\omega(s), x) \geqq K\right] = Q^t_x\left[\frac{p(t/2, \omega(1/2), y)}{p(t, x, y)};\ \sup_{0\leqq s\leqq 1/2} \rho(\omega(s), x) \geqq K\right]$$

$$\leqq \frac{c\,t^{-N_0}}{p(t, x, y)}\, Q^{t/2}_x[\rho(\omega, x) \geqq K] .$$

Here we have used the following global estimate of the heat kernel ([2], p. 143): if M is complete, then for fixed $y\in M$, there are constants $c > 0$, $N_0 > 0$ such that for *all* $z\in M$,

$$p(t, z, y) \leqq ct^{-N_0} \tag{2.8}$$

Now using (2.5) and the large deviation principle for Q^t_x (Remark 2.3), we obtain

$$\limsup_{t\to 0} t\log Q^t_{x,y}\left[\sup_{0\leqq s\leqq 1/2} \rho(\omega(s), x) \geqq K\right] \leqq \frac{1}{2}\rho(x, y)^2 - \inf_{\substack{\omega\in\Omega_x \\ \rho(\omega, x)\geqq K}} \int_0^1 |\dot\omega(s)|^2\, ds$$

$$= \frac{1}{2}\rho(x, y)^2 - K^2 .$$

It follows from (2.7) and the above inequality that

$$\limsup_{t\to 0} t\log Q^t_{x,y}[\rho(\omega, x) \geqq K]$$

$$\leqq -\min\left\{K^2 - \frac{1}{2}\rho(x, y)^2, (K - \rho(x, y))^2 \frac{1}{2}\rho(x, y)^2\right\}$$

$$\leqq -\frac{1}{2}K^2 + 2\rho(x, y)^2 .$$

(2.6) is proved.

We continue our proof of Lemma 2.4. Take any $\alpha \in (0, 1/2)$ and let K, n be large positive numbers to be determined later. We have

$$Q_{x,y}^t\left[\sup_{\substack{0 \leqq s_1 < s_2 \leqq 2/3 \\ s_2 - s_1 \leqq 1/n}} \frac{\rho(\omega(s_1), \omega(s_2))}{|s_2 - s_1|^\alpha} \geqq K\right] \leqq n \sup_{0 \leqq s \leqq 2/3}$$

$$Q_{x,y}^t\left[\sup_{s \leqq s_1 < s_2 \leqq s + 2/n} \frac{\rho(\omega(s_1), \omega(s_2))}{|s_2 - s_1|^\alpha} \geqq K\right] \leqq n\{I_1 + I_1 + I_3\}\,. \tag{2.9}$$

where

$$I_1 = Q_{x,y}^t[\rho(\omega, x) \geqq K]$$

$$I_2 = \sup_{0 \leqq s \leqq 2/3} Q_{x,t}^t\left[\sup_{s \leqq s_1 \leqq s + 2/n} \rho(\omega(s), \omega(s_1)) \geqq \delta\right]$$

$$I_3 = \sup_{0 \leqq s \leqq 2/3} Q_{x,y}^t\left[\sup_{s \leqq s_1 \leqq s_2 \leqq s + 2/n} \frac{\rho(\omega(s_1), \omega(s_2))}{|s_2 - s_1|^\alpha} \geqq K;\ A(s, n, \delta, K)\right].$$

Here to simplify notation, we have let

$$A(s, n, \delta, K) = \left\{\sup_{s \leqq s_1 \leqq s + 2/n} \rho(\omega(s), \omega(s_1)) < \delta, \rho(\omega(s), x) < K\right\}.$$

We choose δ to be smaller than the injectivity radii at point z for all z such that $\rho(x, z) \leqq K$.

We have from (2.6),

$$\limsup_{t \to 0} t \log I_1 \leqq -\frac{1}{2}K^2 + 2\rho(x, y)^2\,.$$

By (2.1), (2.8) and the Markov property, we have

$$I_2 \leqq \sup_{0 \leqq s \leqq 2/3} Q_x^t\left[\frac{p(t(1 - s - 2/n), \omega(s + 2/n), y)}{p(t, x, y)};\ \sup_{s \leqq s_1 \leqq s + 2/n} \rho(\omega(s), \omega(s_1)) \geqq \delta\right]$$

$$\leqq \frac{c\,t^{-N_0}}{p(t,\ x,\ y)} \sup_{z \in B_K(x)} Q_z^{2t/n}[\rho(\omega, z) \geqq \delta]\,.$$

($B_K(x) = \{z \in M : \rho(z, x) < K\}$.) We have, as $t \to 0$,

$$Q_z^t[\rho(\omega, z) \geqq \delta] \sim 2Q_z^t[\rho(\omega(1), z) \geqq \delta] \sim ct^{-(m-2)/2} e^{-\delta^2/2t} \tag{2.10}$$

uniformly on the set $\{z : \rho(z, x) \leqq K\}$ ([2], Proposition 5.8 on p. 185 and Proposition 5.6 on p. 183). It follows that

$$\limsup_{t \to 0} t \log I_2 \leqq \frac{1}{2}\rho(x, y)^2 - \frac{n\delta^2}{4}\,.$$

We now estimate I_3. To simplify notation let

$$B(l, \delta, K) = \left\{\omega\colon \sup_{0 \leqq s_1 < s_2 \leqq l} \frac{\rho(\omega(s_1) - \omega(s_2))}{|s_2 - s_1|^\alpha} \geqq K;\ \sup_{0 \leqq s \leqq l} \rho(\omega(s), \omega(0)) < \delta\right\}.$$

We have by (2.1), (2.8) and the Markov property

$$I_3 = \sup_{0 \leqq s \leqq 2/3} Q_x^4 \left[\frac{p(t(1-s-2/n), \omega(s+2/n), y)}{p(t, x, y)}; B(2/n, \delta, K) \circ \theta_s, \rho(\omega(s), x) \leqq K \right]$$

$$\leqq \frac{c t^{-N_0}}{p(t, x, y)} \sup_{\substack{0 s \leqq 2/3 \\ z \in B_K(x)}} Q_z^{2t/n} [B(1, \delta, (2/n)^\alpha K)] \,. \tag{2.11}$$

Let $X^t = \{X_u^t, 0 \leqq u \leqq 1\}$ be the process whose law is Q_z^t. Let

$$\Omega^t = \{\omega: \rho(\omega, z) < \delta\} \,.$$

We need only to consider the process X^t on Ω^t. Since δ is less than the injectivity radius at $X_0^t = z$, we can choose local coordinates centered at z so that X^t is the solution of the stochastic differential equation:

$$dX_u^t = \sqrt{t}\sigma(z; X_u^t)\, dB_u + t b(z; X_u^t)\, ds \,, \qquad X_0^t = z \,.$$

We may assume that the there is a constant $C = C(K, \delta)$ such that

$$\sup_{\substack{w: \rho(w, z) \leqq \delta \\ z \in B_K(x)}} \max\{ \|\sigma(z; w)\|, |b(z; w)| \} \leqq C \,. \tag{2.12}$$

Let

$$(M_s^1, \ldots, M_s^m) = \int_0^s \sigma(z; X_u^t) dB_u \,.$$

Each M^i is a standard Brownian motion up to a time change. Thus there are m Brownian motions W^i such that $M_u^i = W_{\tau_u^i}^i$. It is clear from (2.12) that there is a constant $c_1 > 0$ such that $\tau_{s_2}^i - \tau_{s_1}^i \leqq c_1 |s_2 - s_1|$ on Ω^t for all i. We now have from the stochastic differential equation of X^t

$$\rho(X_{s_1}^t, X_{s_2}^t) \leqq c_2 |X_{s_2}^t - X_{s_1}^t| \leqq c_2 \sqrt{t} \sum_{i=1}^m |W_{\tau_{s_2}^i} - W_{\tau_{s_1}^i}^i| + c_2' t |s_2 - s_1| \,.$$

Therefore on Ω^t we have

$$\sup_{0 \leqq s_1 < s_2 \leqq 1} \frac{\rho(X_{s_1}^t, X_{s_2}^t)}{|s_2 - s_1|^\alpha} \leqq c_3 \sqrt{t} \sup_{0 \leqq s_1 < s_2 \leqq 1} \sum_{i=1}^m \frac{|W_{s_2}^i - W_{s_1}^i|}{|s_2 - s_1|^\alpha} + c_3 t |s_2 - s_1|^{1-\alpha} \,.$$

It follows that

$$Q_z^{2t/n}[B(1, \delta, (2/n)^\alpha K)] \leqq c_4 P \left[\sup_{0 \leqq s_1 < s_2 \leqq 1} \frac{|W_{s_2} - W_{s_1}|}{|s_2 - s_1|^\alpha} \geqq \frac{c_5}{\sqrt{t}} n^{1/2-\alpha} K \right]$$

$$\leqq c_6 e^{-c\tau n^{1-2\alpha} K^2 / t} \,.$$

(W stands for a one-dimensional Brownian motion.) In the last step we have used Fernique's theorem on the tail probability of a Gaussian system ([3], p. 159–p. 162). From the above inequality and (2.11), we have

$$\limsup_{t \to 0} t \log I_3 \leqq -c_7 K^2 n^{1-2\alpha} + \frac{\rho(x, y)^2}{2} \,.$$

Putting the estimates for I_1, I_2, and I_3 in (2.9) and using Remark 2.3, we have

$$\limsup_{t\to 0} t \log Q^t_{x,y}\left[\sup_{\substack{0<s_2-s_1\leqq 1/n\\ s_1,s_2\in[0,1]}} \frac{\rho(\omega(s_1),\omega(s_2))}{|s_2-s_1|^\alpha} \geqq K\right]$$

$$\leqq -\min\left\{\frac{1}{2}K^2 - 2\rho(x,y)^2, \frac{1}{4}n\delta^2 - \frac{1}{2}\rho(x,y)^2, c_7K^2n^{1-2\alpha} - \frac{\rho(x,y)^2}{2}\right\}.$$

Choose K so that $K^2/2 - 2\rho(x,y)^2 \geqq N$. Fix δ as required in the proof. Then choose n so large that $n\delta^2/4 - \rho(x,y)^2/2 \geqq N$ and $c_7K^2n^{1-2\alpha} - \rho(x,y)^2/2 \geqq N$. Now the right-hand side of the above inequality is less than $-N$. Therefore the compact set

$$C_N = \left\{\sup_{\substack{0<s_2-s_1\leqq 1/n\\ s_1,s_2\in[0,1]}} \frac{\rho(\omega(s_1),\omega(s_2))}{|s_2-s_1|^\alpha} \leqq K, \omega(0) = x\right\}$$

satisfies the requirement of the lemma. □

In the following for two paths ω_1, ω_2 we set

$$\rho(\omega_1,\omega_2) = \sup_{0\leqq s\leqq 1} \rho(\omega_1(s),\omega_2(s)).$$

We now turn to the

Proof of Theorem 2.2. The proof is naturally divided into two parts.

i) *Lower bound.* Let $G \subset \Omega_{x,y}$ be open. It is enough to show that for any $\omega^* \in G$ such that $J(\omega^*) < \infty$, we have

$$\liminf_{t\to 0} t \log Q^t_{x,y}(G) \geqq -J_{x,y}(\omega^*). \tag{2.13}$$

Let

$$O^\varepsilon_\delta = \left\{\omega\in\Omega_{x,y}: \sup_{0\leqq s\leqq 1-\varepsilon} \rho(\omega(s),\omega^*(s)) < \delta\right\}$$

$$F^\varepsilon_\delta = \left\{\omega\in\Omega_{x,y}: \sup_{1-\varepsilon\leqq s\leqq 1} \rho(\omega(s),\omega^*(s)) \geqq \delta\right\}.$$

Since G is open and $\omega^*\in G$, there exists $\delta > 0$ such that

$$\{\omega: \rho(\omega,\omega^*) < \delta\} \subset G.$$

This implies $O^\varepsilon_\delta \subset G \cup F^\varepsilon_\delta$. Now we can write

$$Q^t_{x,y}(G) \geqq Q^t_{x,y}(O^\varepsilon_\delta) - Q^t_{x,y}(F^\varepsilon_\delta) \tag{2.14}$$

The two terms on the right hand side will be estimated separately. Let $\omega^*_\varepsilon \in \Omega_x$ be defined by

$$\omega^*_\varepsilon(s) = \omega^*((1-\varepsilon)s).$$

Let

$$G^\varepsilon_\delta = \{\omega\in\Omega_x: \rho(\omega,\omega^*_\varepsilon) < \delta\}.$$

Noticing that

$$Q^t_{x,y}[\omega(1-\varepsilon)\in dz] = \frac{p(t(1-\varepsilon), x, z)p(t\varepsilon, z, y)}{p(t, x, y)} dz ,$$

we have by the Markov property,

$$Q^t_{x,y}(O^\varepsilon_\delta) = \int_M Q^{t(1-\varepsilon)}_{x,z}[G^\varepsilon_\delta]\, Q^t_{x,y}[\omega(1-\varepsilon)\in dz]$$

$$\geqq \frac{\max_{z\in B_\lambda(y)} p(t\varepsilon, z, y)}{p(t, x, y)} \int_{B_\lambda(y)} Q^{t(1-\varepsilon)}_{x,z}[G^\varepsilon_\delta]\, p(t(1-\varepsilon), x, z)\, dz$$

$$\geqq \frac{c_1(t\varepsilon)^{-m/2} e^{-\lambda^2/2\varepsilon t}}{p(t, x, y)} Q^{t(1-\varepsilon)}_x [G^\varepsilon_\delta \cap \{\omega(1)\in B_\lambda(y)\}] . \tag{2.15}$$

Here we have used the asymptotic expression

$$p(t, z, y) \sim \left(\frac{1}{2\pi t}\right)^{m/2} H(z, y)\, e^{-\rho(z,y)^2/2t}$$

uniformly and $H(z, y) \geqq c_0$ for all $z\in B_\lambda(y)$ with sufficiently small λ (see [2], p. 173).

Let

$$C_{\delta,\lambda} = G^\varepsilon_\delta \cap \{\omega(1)\in B_\lambda(y)\} .$$

$C_{\delta,\lambda}$ is open. Clearly $\omega\varepsilon^* \in O_\delta$. We also have

$$\rho(\omega^*_\varepsilon(1), y) = \rho(\omega^*(1-\varepsilon), y) \leqq \int_{1-\varepsilon}^1 |\dot\omega^*(s)|\, ds \leqq \sqrt{\varepsilon \int_{1-\varepsilon}^1 |\dot\omega^*(s)|^2\, ds} .$$

Choose

$$\lambda = \lambda(\varepsilon) = 2\sqrt{\varepsilon \int_{1-\varepsilon}^1 |\dot\omega^*(s)|^2\, ds} .$$

We see that $\rho(\omega^*(1), y) < \lambda(\varepsilon)$; hence $\omega^*_\varepsilon \in C_{\delta,\lambda(\varepsilon)}$. Now by the large deviation principle for Q^t_x (Remark 2.3), we have from (2.15)

$$\liminf_{t\to 0} t \log Q^t_{x,y}(O^\varepsilon_\delta) \geqq -\left[\frac{I(\omega^*_\varepsilon)}{1-\varepsilon} + \frac{\lambda(\varepsilon)^2}{2\varepsilon} - \frac{\rho(x, y)^2}{2}\right].$$

Now

$$I(\omega^*_\varepsilon) = (1-\varepsilon) \int_0^{1-\varepsilon} |\dot\omega^*(s)|^2\, ds .$$

From the assumption $J(\omega^*) < \infty$, we have

$$I(\omega^*_\varepsilon) \to I(\omega^*) \quad \text{and} \quad \frac{\lambda(\varepsilon)^2}{\varepsilon} \to 0 .$$

It follows that

$$\lim_{\varepsilon\to 0} \liminf_{t\to 0} t \log Q^t_{x,y}(O^\varepsilon_\delta) \geqq -J_{x,y}(\omega^*) . \tag{2.16}$$

We now have to show that the second term $Q^t_{x,y}(F^\varepsilon_\delta)$ in (2.14) is negligible, namely,

$$\limsup_{\varepsilon\to 0}\limsup_{t\to 0} t\log Q^t_{x,y}(F^\varepsilon_\delta) = -\infty\,. \tag{2.17}$$

Set

$$H^\varepsilon_\delta = \left\{\omega\in\Omega_{y,x}\colon \sup_{0\leqq s\leqq\varepsilon} \rho(y,\omega(s)) \geqq \delta/2\right\}.$$

For ε sufficiently small, we have $\int_{1-\varepsilon}^{1} |\dot\omega^*(s)|ds < \delta/2$. Using this fact, by time-reversal (Remark 2.1) we have for small ε

$$\begin{aligned} Q^t_{x,y}(F^\varepsilon_\delta) &= Q^t_{y,x}(H^\varepsilon_\delta) \\ &= Q^t_y\left[\frac{\rho(t(1-\varepsilon),\omega(\varepsilon),x)}{p(t,y,x)};\, H^\varepsilon_\delta\right] \\ &\leqq \frac{c_1 t^{-N}}{p(t,x,y)}\, Q^t_y(H^\varepsilon_\delta)\,. \end{aligned}$$

Without loss of generality, we may assume that δ is less than the injectivity radius at y. We then have (see (2.10))

$$Q^t_y(H^\varepsilon_\delta) \sim 2Q^{t\varepsilon}_y[\rho(\omega(1),y) > \delta/2] \sim c(t\varepsilon)^{-(m-2)/2}e^{-\delta^2/8t\varepsilon}\,.$$

It follows that

$$\limsup_{t\to 0} t\log Q^t_{x,y}(F^\varepsilon_\delta) \leqq \frac{\rho(x,y)^2}{2} - \frac{\delta^2}{2\varepsilon}\,.$$

(2.17) follows immediately by letting $\varepsilon\to 0$.

Combining (2.14), (2.16) and (2.17) we obtain (2.13). The lower bound (2.3) is proved.

ii) Upper bound. Let us first prove the upper bound for a compact set $C\in\Omega_{x,y}$. Let

$$C^\varepsilon = \{\omega\in\Omega_{x,y}\colon \exists\omega^\circ\in C \quad \text{such that} \quad \omega(s)=\omega^\circ(s) \quad \text{for} \quad 0\leqq s\leqq 1-\varepsilon\}$$

and

$$C^\varepsilon_* = \{\omega\in\Omega_x\colon \exists\omega^\circ\in C \quad \text{such that}\ \omega(s)=\omega^\circ((1-\varepsilon)s) \quad \text{for} \quad 0\leqq s\leqq 1\}\,.$$

C^ε is closed both in Ω_x and $\Omega_{x,y}$ and $C\subset C^\varepsilon$. By the Markov property, we have

$$\begin{aligned} Q^t_{x,y}(C) &\leqq Q^t_{x,y}(C^\varepsilon) \\ &\leqq \int_M Q^{t(1-\varepsilon)}_{x,z}(C^\varepsilon_*)[\omega(1-\varepsilon)\in dz] \\ &\leqq \frac{\max_{z\in M} p(t\varepsilon,\ z,\ y)}{p(t,\ x,\ y)} \int_M Q^{t(1-\varepsilon)}_{x,z}(C^\varepsilon_*)\,p(t(1-\varepsilon),x,z)dz\,. \end{aligned}$$

It follows that

$$Q^t_{x,y}(C) \leqq \frac{c(t\varepsilon)^{-N}}{p(t,x,y)}\, Q^{t(1-\varepsilon)}_x(C^\varepsilon_*)\,.$$

Clearly C_*^ε is closed in Ω_x. We have by the large deviation principle for Q_x^t,

$$\limsup_{t\to 0} t \log Q_{x,y}^t(C) \leqq \frac{\rho(x,y)^2}{2} - \frac{1}{1-\varepsilon} \inf_{\omega\in C_*^\varepsilon} I(\omega) . \tag{2.18}$$

Let

$$l = \limsup_{\varepsilon\to 0} \inf_{\omega\in C_*^\varepsilon} I(\omega) .$$

We claim that $l \geqq \inf_{\omega\in C} I(\omega)$. To prove this claim we note that there exists a sequence $\varepsilon_n \downarrow 0$ and $\omega_n \in C_*^{\varepsilon_n}$ such that $l = \lim_{n\to\infty} I(\omega_n)$. Let $\omega_n^\circ \in C$ be such that $\omega_n^\circ((1-\varepsilon_n)s) = \omega_n(s)$, $0 \leqq s \leqq 1$. Since C is compact, so is $\{\omega_n^\circ, n \geqq 1\}$. Hence we can assume that $\omega_n \to \omega^*$ for an $\omega^* \in C$. Because I is lower semicontinuous on Ω_x, we have

$$l = \lim_{n\to\infty} I(\omega_n) \geqq I(\omega^*) \geqq \inf_{\omega\in C} I(\omega) .$$

This proves our claim. The upper bound for C now follows from (2.18) by letting $\varepsilon \to 0$.

Finally, we prove the upper bound for a closed set F. Let C_N be as in Lemma 2.2. We have

$$Q_{x,y}^t(F) \leqq Q_{x,y}^t(F \cap C_N) + Q_{x,y}^t(C_N^c) .$$

Since $F \cap C_N$ is compact, by the upper bound for compact sets, we have

$$\limsup_{t\to 0} t \log Q_{x,y}^t(F) \leqq -\min\left\{ \inf_{\omega\in F\cap C_N} J(\omega), N \right\} .$$

Letting $N \to \infty$, we obtain the upper bound (2.4). Theorem 2.2 is proved. □

§3. Weak Convergence of $Q_{x,y}^t$

Lemma 2.4 shows that the measures $\{Q_{x,y}^t; t > 0\}$ is sequentially compact as $t \to 0$. Let μ be a limiting measure. Recall that $\Gamma_{x,y}$ is the set of minimizing geodesics joining x, y. A little geometry shows that

$$\Gamma_{x,y} = \{\omega \in \Omega_{x,y} : J_{x,y}(\omega) = 0\} .$$

Lemma 3.1. *Let μ be a limiting measure of $\{Q_{x,y}^t;\ t > 0\}$ as $t \to 0$. Then μ is concentrated on the set $\Gamma_{x,y}$; i.e., $\mu(\Gamma_{x,y}) = 1$.*

Proof. Fix $\varepsilon > 0$. Let

$$G = \{\omega : \rho(\omega, \Gamma_{x,y}) > \varepsilon\} .$$

We have to show $\mu(G) = 0$. Let

$$\varepsilon_G = \inf_{\omega\in \overline{G}} J_{x,y}(\omega) .$$

We claim that $\varepsilon_G > 0$. Suppose $\varepsilon_G = 0$. Then there exists a sequence $\omega_n \in \bar{G}$ such that $J_{x,y}(\omega_n) \to 0$. Since $\{J_{x,y}(\omega_n), n \geqq 1\}$ is bounded, by extracting a subsequence if necessary, we can assume that $\omega_n \to \omega$ for some $\omega \in \bar{G}$. But $J_{x,y}$ is lower semicontinuous we have therefore $J_{x,y}(\omega) = 0$, or $\omega \in \Gamma_{x,y}$, a contradiction. It follows that $\varepsilon_G > 0$.

Now by the upper bound, we have

$$\mu(G) \leqq \liminf_{n \to \infty} Q^{t_n}_{x,y}(\bar{G}) = 0 .$$

The lemma is proved. □

In this section we prove the uniqueness of the limiting measure for a special case. In the next section we prove the uniqueness for analytic manifolds.

Let $S = \Gamma_{x,y}(1/2)$ be the middle cross-section of $\Gamma_{x,y}$. Consider the probability measure μ^t on M defined by

$$\mu^t(A) = Q^t_{x,y}\left[\omega\left(\frac{1}{2}\right) \in A\right].$$

If $Q^t_{x,y} \to \mu$ weakly in $\Omega_{x,y}$ as $t \to 0$ through a sequence, then clearly through the same sequence we have $\mu^t \to \mu^0$ weakly on M where $\mu^0(A) = \mu[\omega(1/2) \in A]$. From Lemma 3.1 we know that $\mu^0(S) = 1$. Now we prove

Lemma 3.2. *$Q^t_{x,y} \to \mu$ weakly on $\Omega_{x,y}$ if and only $\mu^t \to \mu^0$ weakly on M.*

Proof. We have just proved "only if" part. For the "if" part, suppose $\mu^t \to \mu^0$ weakly on M. Define μ on $\Omega_{x,y}$ by $\mu(O) = \mu^0(O(1/2))$ [$O(1/2)$ is the middle cross-section of O]. We want to show $Q^t_{x,t} \to \mu$ weakly on $\Omega_{x,y}$.

Let

$$O_\varepsilon = \{\omega : \rho(\omega, \Gamma_{x,y}) < \varepsilon\} .$$

Since $\Gamma_{x,y}(1/2) = S$ intersects the cut-locus of neither x nor y, and the cut-locus of a point is closed and S is compact, there is an $\varepsilon_0 > 0$ such that $O_{\varepsilon_0}(1/2)$ intersects the cut-locus of neither x nor y. For each $z \in O_{\varepsilon_0}(1/2)$ we define the path

$$\lambda^z(s) = \begin{cases} \gamma_{x,z}(2s) , & 0 \leqq s \leqq 1/2 \\ \gamma_{z,y}(2s-1) , & 1/2 \leqq s \leqq 1 \end{cases}$$

where $\gamma_{x,z}$ is the unique minimizing geodesic with uniform speed $\rho(x,z)$ joining x, z. The map $z \mapsto \lambda^z$ from $O_{\varepsilon_0}(1/2)$ to $\Omega_{x,y}$ is continuous. For $\omega \in O_{\varepsilon_0}$, we set $\omega^* = \lambda^{\omega(1/2)}$. We claim:

$$\forall \varepsilon > 0 , \qquad Q^t_{x,y}[\omega \in O_{\varepsilon_0}, \rho(\omega, \omega^*) \geqq \varepsilon] \to 0 \quad \text{as} \quad t \to 0 . \tag{3.1}$$

Indeed, we have

$$Q^t_{x,t}[\omega \in O_{\varepsilon_0}, \rho(\omega, \omega^*) \geqq \varepsilon] \leqq Q^t_{x,y}[O^c_\delta] + Q^t_{x,y}[\rho(\omega, \omega^*) \geqq \varepsilon, \omega \in O_\delta] . \tag{3.2}$$

By a simple geometric argument we know there exists a $\delta > 0$ such that if $\omega_1 \in \Gamma_{x,y}$ and $\rho(\omega_1(1/2), z) \leqq \delta$ then $\rho(\omega_1, \lambda^z) < \varepsilon/2$. We may assume that $2\delta \leqq \varepsilon \wedge \varepsilon_0$. Now if $\omega \in O_\delta$, then there exists $\omega_1 \in \Gamma_{x,y}$ such that $\rho(\omega, \omega_1) < \delta$. This implies by the above observation $\rho(\omega_1, \omega^*) < \varepsilon/2$ and therefore $\rho(\omega, \omega^*) < \delta + \varepsilon/2 \leqq \varepsilon$. This means that the set in the last term of (3.2) is empty. The first term on the right-hand side of (3.2) tends to zero by Lemma 3.1. This proves (3.1).

Now let $F:\Omega_{x,y}\to R$ be bounded, continuous and positive. We have from (3.1) and Lemma 3.1

$$\int_{\Omega_{x,y}} F(\omega)\,Q^t_{x,y}(d\omega) = \int_{O_{\varepsilon_0}} F(\omega^*)Q^t_{x,y}(d\omega) + o(1)$$

$$= \int_{O_{\varepsilon_0}(1/2)} F(\lambda^z)\mu^t(dz) + o(1)$$

$$\to \int_S F(\lambda^z)\mu^0(dz)$$

$$= \int_{\Omega_{x,y}} F(\omega)\,\mu(d\omega)\,.$$

This means $Q^t_{x,y}\to\mu$ weakly. The lemma is proved. □

Lemma 3.3. *Let x, y be arbitrary two points on M. Then for any neighborhood O of the compact set $\Gamma_{x,y}(1/2)$ and any $F\subset M$, we have as $t\to 0$,*

$$\mu^t(F)\sim\frac{1}{p(t,x,y)}\int_{F\cap O} p\left(\frac{t}{2},x,z\right)p\left(\frac{t}{2},z,y\right)dz\,.$$

In particular

$$p(t,x,y)\sim\int_O p\left(\frac{t}{2},x,z\right)p\left(\frac{t}{2},z,y\right)dz\,.$$

Proof. It is enough to show that $\mu^t(O^c)\to 0$ as $t\to 0$. But $\mu^t(O^c) = Q^t_{x,y}(\Omega^c_O)$, where

$$\Omega_O = \left\{\omega\in\Omega_{x,y}:\omega\left(\frac{1}{2}\right)\in O\right\}.$$

The lemma follows immediately from Lemma 3.1. □

As mentioned earlier, the set $S=\Gamma_{x,y}(1/2)$ intersects the cut-locus of neither x nor y. An ε-neighborhood of S with a sufficiently small ε will have the same property. Let O be such a neighborhood. We have *uniformly* for $z\in O$:

$$p\left(\frac{t}{2},x,z\right)\sim\left(\frac{1}{\pi t}\right)^{m/2} H(x,z)\,e^{-\rho(x,z)^2/t}$$

where $H(x,z)=\det[d\exp_x(\dot\gamma_{x,z}(0))]^{-1/2}$. Similar assertion holds for $p(t/2,z,y)$ ([2], p. 173). Let

$$E(z) = 2[\rho(x,z)^2+\rho(z,y)^2]-\rho(x,y)^2\,.$$

It follows from Lemma 3.3 that for any $F\subset M$

$$p(t,x,y)\sim e^{-\rho(x,y)^2/2t}\left(\frac{1}{\pi t}\right)^m\int_O H(x,z)H(z,y)\,e^{-E(z)/2t}\,dz \tag{3.3}$$

and

$$\mu^t(F)\sim\frac{e^{-\rho(x,y)^2/2t}}{p(t,x,y)}\left(\frac{1}{\pi t}\right)^m\int_{F\cap O} H(x,z)H(z,y)e^{-E(z)/2t}\,dz\,. \tag{3.4}$$

Theorem 3.4. *Suppose that $\Gamma_{x,y}$ is a smooth manifold of dimension k and each geodesic in $\Gamma_{x,y}$ have exactly multiplicity k (this means that the dimension of the space of Jacobi fields which vanish at both x and y is equal to k). Then $Q^t_{x,y}$ converges weakly to a probability measure μ as $t \to 0$.*

Proof. Under our hypothesis, $S = \Gamma_{x,y}(1/2)$ is a smooth compact submanifold of M of dimension k. Let $\pi: N \to S$ be the normal bundle of S in M. Then

$$S_\varepsilon = \{z = \exp_s b : b \in N_s,\ \|b\| < \varepsilon\} .$$

S_ε is a neighborhood S_ε of S in M. Denote $z = \exp_s b$ by $z = (s, b)$. Let σ be the induced volume element on S and db the standard volume element on the normal bundle fibre. Then inside S_ε the volume element of M can be written as $dz = \beta(s, b)\sigma(ds)\, db$, where $\beta(s, b)$ is a smooth function such that $\beta(s, b) \to 1$ as $\|b\| \to 0$.

By Lemma 3.2, it suffices to show that $\mu^t \to \mu^0$ weakly on M. Let $f: M \to R$ be continuous, bounded and positive. We have by (3.4)

$$\int_M f(z)\mu^t(dz) \sim \frac{e^{-\rho(x,y)^2/2t}}{p(t,x,y)}\left(\frac{1}{\pi t}\right)^m \int_{S_\varepsilon} f(z)H(x,z)\,H(z,y)e^{-E(z)/2t}\,dz$$
$$= \frac{e^{-\rho(x,y)^2/2t}}{p(t,x,y)}\left(\frac{1}{\pi t}\right)^m \int_S \Omega(s;t)\ \sigma(ds) \tag{3.5}$$

where

$$\Omega(s;t) = \int_{B_\varepsilon(0)} f(s,b)H(x,(s,b))\,H((s,b),y)\beta(s,b)e^{-E(s,b)/2t}\,db .$$

Here $B_\varepsilon(0)$ is the ball in the fibre space N_s of radius ε and centered at the origin. The assumption that each geodesic has multiplicity k implies that for fixed $s \in S$, the function $E(s, b)$ has a unique nondegenerate isolated critical point on $B_\varepsilon(0)$ at $b = 0$ (see [2], p. 117). Hence by Laplace's method, we have the asymptotic relation

$$\Omega(s;t) \sim (4\pi t)^{(m-k)/2}\ H(x,s)\,H(s,y)f(s)\det[\mathrm{Hess}\, E|_{N_s}]^{-1/2} .$$

($\mathrm{Hess}\, E|_{N_s}$ is the restriction of the hessian of E on the fibre N_s.) It follows from (3.3) and (3.5) that

$$\int_M f(z)\mu^t(dz) \to \int_M f(z)\mu^0(dz)$$

with μ^0 defined by

$$\mu^0(A) = C \int_{A\cap S} H(x,s)H(s,y)\ \det[\mathrm{Hess}\, E|_{N_s}]^{-1/2}\,\sigma(ds)$$

(C is the normalizing constant so that $\mu(S) = 1$). □

Remark 3.5. Theorem 3.3 remains true if $\Gamma_{x,y}$ is a disjoint union of smooth manifolds such that each geodesic has multiplicity equal to the dimension of the connected component it is in. In this case μ is concentrated on the components of $\Gamma_{x,y}$ which have the maximum dimension.

Let us look at a two special cases.

Example 3.6. Let N and S be the north and south poles of m-dimensional sphere S^m in R^{m+1}. Then we see that $\Gamma_{N,S}$ is S^{m-1} and μ is the uniform distribution on S^{m-1}.

Example 3.7. Let $\Gamma_{x,y} = \{\gamma_1, \ldots, \gamma_l\}$ and along each of γ_i the endpoints x, y are not conjugate. Then we have

$$\mu(\{\gamma_i\}) = \frac{\det\left[\underset{x}{d\exp}(\dot{\gamma}_i(0))\right]^{-1/2}}{\sum_{j=1}^{l} \det\left[\underset{x}{d\exp}(\dot{\gamma}_j(0))\right]^{-1/2}} .$$

This is because if x, y are not conjugate along a minimizing geodesic γ joining them then ([2], p. 125)

$$H(x, \gamma(1/2)) H(\gamma(1/2), y) \det[\operatorname{Hess} E(\gamma(1/2))]^{-1/2} = 2^{-m} \det[d\exp_x(\dot{\gamma}(0))] .$$

§4. Analytic Manifolds

In general, the geometry of $\Gamma_{x,y}$ can be quite complicated. It is usually a subvariety of M with various singularities. We consider in this section analytic manifold M with analytic metric. We first give a characterization of exact support of limiting measures of $Q^t_{x,y}$. Then we show that the limiting measure is unique.

Consider the function $E(z) = 2[\rho(x, z)^2 + \rho(z, y)^2] - \rho(x, y)^2$. E is analytic, nonnegative and vanishes exactly on $S = \Gamma_{x,y}(1/2)$. We will use the analyticity of E via the following fact: for any open set $O \subset M$ we have

$$\left(\frac{1}{\pi t}\right)^m \int_O e^{-E(z)/2t} dz \sim C(O) t^{-\alpha} \left(\log \frac{1}{t}\right)^\beta , \tag{4.1}$$

where $\alpha \in [m/2, m - 1/2]$ is rational and β is nonnegative integral. Furthermore, if $z \in M$ and O is a neighborhood of z then there exists $\varepsilon = \varepsilon(z) > 0$ such that α and β in 4.1 depends only on z but not on O provided $\operatorname{diam}(O) \leqq \varepsilon(z)$ ([2], p. 172). This fact gives rise to two functions α and β on M. Let $D: M \to [m/2, m - 1/2] \times N$ be the map $D(z) = (\alpha(z), \beta(z))$. Also define

$$D(z; t) = t^{-\alpha(z)} \left(\log \frac{1}{t}\right)^{\beta(z)} .$$

It is clear that the growth rate of $D(z; t)$ as $t \to 0$ is a measurement of the degeneracy of E at point z.

We now give the set $[m/2, m - 1/2] \times N$ the lexicographic order; i.e., $(\alpha_1, \beta_1) \leqq (\alpha_2, \beta_2)$ if $\alpha_1 \leqq \alpha_2$ or if $\alpha_1 = \alpha_2$ and $\beta_1 \leqq \beta_2$. With this ordering we verify easily that D is upper semicontinuous. Since S is compact, D must attain its maximum value on S. Let (α_0, β_0) be this maximum value. Set $S^0 = \{z \in S: D(z) = (\alpha_0, \beta_0)\}$ and $\Gamma^0_{x,y} = \{\gamma \in \Gamma_{x,y}: \gamma(1/2) \in S^0\}$.

We have the following result.

Theorem 4.1. *For any limiting measure μ, the support of μ is exactly equal to $\Gamma^0_{x,y}$.*

Proof. We first of all note that by (3.3), (4.1) and the continuity of $H(x,z)H(z,y)$ in z, we can show (see the proof of Theorem 4.2 below) that there is a positive constant c such that

$$p(t,x,y) \sim c\, t^{-\alpha_0}\left(\log\frac{1}{t}\right)^{\beta_0} e^{-\rho(x,y)^2/2t}\,. \tag{4.2}$$

Let

$$F = \{z\in M\colon \rho(z,S^0) \geqq \varepsilon\}\,.$$

Then by (3.3) and (4.2)

$$\mu^t(F) \sim \text{const.}\; t^{\alpha_0-\alpha_1}\left(\log\frac{1}{t}\right)^{-(\beta_0-\beta_1)}$$

where $(\alpha_1, \beta_1) = \max_{z\in F\cap S} D(z)$. Obviously we have $(\alpha_1, \beta_1) < (\alpha_0, \beta_0)$. Therefore $\mu^t(F)\to 0$. It follows that the support of measure μ^0 is contained in S_0.

To show that the support of μ contains S_0, let $z_0\in S_0$ and let G be any neighborhood of z_0 in M. Since $z_0\in S_0$, by the definition of S_0, we have from (4.1)

$$\left(\frac{1}{\pi t}\right)^m \int_G e^{-E(z)/2t}\,dz \sim c_1 t^{-\alpha_0}\left(\log\frac{1}{t}\right)^{\beta_0}$$

with a positive c_1. It follows from (3.4) and (4.2) that

$$\mu(G) = \frac{c_1}{c} > 0\,.$$

We thus have proved $\operatorname{supp}\mu^0 \subset S^0$. Therefore $\operatorname{supp}\mu^0 = S^0$, which implies the desired theorem. □

If $\gamma\in\Gamma_{x,y}$, then $D(\gamma(1/2);\, t)$ is a measurement of the degeneracy of the minimizing geodesic γ. Thus we paraphrase the above theorem by saying that the limiting measure is concentrated on the most degenerate minimizing geodesics.

We now turn to the main theorem in this section.

Theorem 4.2. *If M and its metric are analytic, then for any fixed $x, y\in M$, the set of measures $\{Q^t_{x,y},\, t>0\}$ converges weakly as $t\to 0$.*

Proof. Recall that O_{ε_0} is a neighborhood of $\Gamma_{x,y}$ whose middle cross-section $O_{\varepsilon_0}(1/2)$ intersects neither the cut-locus of x nor that of y (see the proof of Lemma 3.2). For fixed $\varepsilon>0$, let $\mathscr{F}^\varepsilon = \{\Delta_1, \ldots, \Delta_n\}$ be a finite collection of disjoint open sets of M such that $\Delta_i \subset O_{\varepsilon_0}(1/2)$, $\operatorname{diam}(\Delta_i) < \varepsilon$ for $i = 1, \ldots, n$; $\bigcup_{i=1}^n \bar\Delta_i \supset O_{\varepsilon_0/2}(1/2)$. Let $f\colon M\to R$ be continuous, bounded and positive. Let $f^*(z) = f(z)H(x,z)H(z,y)$, $z\in O_{\varepsilon_0}(1/2)$. Finally let

$$\delta(\varepsilon) = \sup_{\substack{\rho(z_1,z_2)\leqq\varepsilon\\ z_1,z_2\in O_{\varepsilon_0}(1/2)}} |f^*(z_2) - f^*(z_1)|\,.$$

We have as $t \to 0$,

$$\int_M f(z)\mu^t(z) \sim \sum_{\Delta \in \mathscr{F}^\varepsilon} \int_\Delta f(z)\mu^t(dz)$$
$$\sim \frac{e^{-\rho(x,y)^2/2t}}{p(t,x,y)} \sum_{\Delta \in \mathscr{F}^\varepsilon} \left(\frac{1}{\pi t}\right)^m \int_\Delta f^*(z) e^{-E(z)/2t} dz$$
$$\sim \frac{e^{-\rho(x,y)^2/2t}}{p(t,x,y)} \sum_{\Delta \in \mathscr{F}^\varepsilon} f^*(z_\Delta)\left(\frac{1}{\pi t}\right)^m \int_\Delta e^{-E(z)/2t} dz + O(\delta(\varepsilon)) .$$

Here z_Δ is any point in Δ. Note that the term $O(\delta(\varepsilon))$ is independent of t. It is clear that

$$t^{\alpha_0}\left(\log\frac{1}{t}\right)^{-\beta_0}\left(\frac{1}{\pi t}\right)^m \int_\Delta e^{-E(z)/2t} dz \to C(\Delta)\begin{cases} = 0 & \text{if } \Delta \cap S^0 = \emptyset \\ > 0 & \text{if } \Delta \cap S^0 \neq \emptyset \end{cases}$$

Therefore, letting $t \to 0$ in (4.3), we obtain that for any limiting measure μ^0 of μ^t,

$$\int_M f(z)\mu^t(dz) \to \int_M f(z)\mu^0(dz) = \sum_{\substack{\Delta \in \mathscr{F}^\varepsilon \\ \Delta \cap S^0 \neq \emptyset}} C(\Delta) f^*(z_\Delta) + O(\delta(\varepsilon))$$

Now let $\varepsilon \to 0$. Since $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$, the limit on the right-hand side is independent of the sequence along which $t \to 0$. It follows that μ^t converges weakly to a unique measure. Our theorem now follows from Lemma 3.1. □

References

1. Azencott, R.: Grandes déviations et applications. In: Hennequin, P. (ed.) Ecole d'Eté de Probabilités de Saint Flour, VIII. (Lect. Notes Math., vol. 774) Berlin Heidelberg New York: Springer 1980
2. Azencott, R.: Géodesiques et Diffusions en Temps Petit. Astéristique, no. 84–85. Soc. Math. France
3. Kuo, H-H.: Gaussian measures in Banach spaces. (Lect. Notes Math., vol. 463.) Berlin Heidelberg New York: Springer
4. Freidlin, M.I., Ventcell, A.D.: Random perturbation of dynamic systems. Berlin, Heidelberg New York: Springer
5. Molchanov, S.A.: Diffusion processes and Riemannian geometry. Russ. Math. Surv. **30**, 1–63 (1975)

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