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Probability Theory and Related Fields

Martingale Measures and Stochastic Calculus

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Summary. In this paper, martingale measures, introduced by J.B. Walsh, are investigated. We prove, with techniques of stochastic calculus, that each continuous orthogonal martingale measure is the time-changed image martingale measure of a white noise.

We also exhibit a representation theorem for certain vector martingale measures as stochastic integrals of orthogonal martingale measures. Thus we can study the following martingale problem:

$$f(X_t) - f(X_0) - \int_0^t \int_E Lf(s, X_s, x) q_s(dx) ds \text{ is a } P\text{-martingale },$$

where L is a second order differential operator and q a predictable random measure-valued process. We prove that this problem is bound to a stochastic differential equation with a term integral with respect to a martingale measure.

Introduction

The purpose of this paper is to study in details some properties of a class of martingale measures (continuous and orthogonal). This concept has been introduced by Walsh [14].

In section 1, we will review the principal results he obtained: existence of a random measure, called intensity, similar to the quadratic variation process for a martingale; construction of a stochastic integral. One fundamental example of these martingale measures is the white noise: it can be characterized by the deterministic nature of its intensity. Other examples will be described in the second section of the paper.

The third part contains our main results. We give essentially two representation theorems. Firstly, we prove that each continuous martingale measure is the time-changed image measure of white noise. We use for the proof the fact that the intensity of a continuous martingale measure has the form $q_t(dx)dk_t$, where q_t is a predictable family of random measures and k_t is a nondecreasing continuous process. By a generalization of a theorem of Skorohod, q_t can be interpreted as the image measure of a deterministic measure λ. If k_t is deterministic, the martingale measure with intensity $q_t(dx)dk_t$ is the image measure of a white noise with intensity $\lambda(dx)dk_t$. If k_t is not deterministic, we need a time change to obtain the representation. This method is thus similar to the one used to represent point processes as image measures of Poisson processes [2]. Our representation theorem is applied in [10], [11] to give a sense to a stochastic differential equation in a space of vector measures for a certain class of measure-valued branching processes.

The second representation theorem describes vector martingale measures in terms of stochastic integrals with respect to orthogonal martingale measures, and generalizes the well known representation of continuous martingales as stochastic integrals of a Brownian motion. A fundamental application is to represent n continuous martingales $(m^i)_{i=1}^n$ with quadratic variation processes

$$\langle m^i, m^j\rangle_t = \int_0^t \int_E a_{ij}(s,x) q_s(dx) dk_s$$

(a being a quadratic matrix), as stochastic integrals with respect to n continuous martingale measures with intensity $q_t(dx)dk_t$:

$$m_t^i = \sum_{k=1}^n \int_0^t \int_E \sigma_{ik}(s,x) M^k(ds,dx)\,, \quad \text{where} \quad a(s,x) = \sigma\sigma^*(s,x).$$

In the last section, we study the following martingale problem: let q be a $\mathscr{F}_t$-predictable random measure-valued process on a space $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$, we define a solution of $(\mathscr{P})$ as a couple $(X, \tilde{P})$ on an extension such that

$$\forall f \in C_b^2(\mathbb{R}^d)\,,\ f(X_t) - f(X_0) - \int_0^t \int_E Lf(s, X_s, x) q_s(dx) ds$$

is a $\tilde{P}$-martingale, with L a second order differential operator depending on the parameter x, $(Lf = 1/2 \Sigma a_{ij} \partial_{ij} f + \Sigma b_i \partial_i f)$, and q_s a predictable random measure on a Lusin space E.

This type of problems appears for example in many particles systems, when there is an interaction in the term of diffusion [9], or in control problems, when relaxed controls are introduced [3]. When the measure-valued process q is deterministic, by interpreting the measure $q_s(dx)ds$ as the intensity of a white noise W (unique in law), we prove that a solution of $(\mathscr{P})$ is exactly the solution of a stochastic differential equation whose martingale term is a stochastic integral with respect to W. When q is random, the problem is not so well-posed: for a given process q, there is an infinity of martingale measures with intensity $q_s(dx)ds$. But the representation theorem above described allows to give relations between a solution of $(\mathscr{P})$ and such a martingale measure.

1. Definition and Basic Properties of Martingale Measures

Let us define, as Walsh [14], the notions of L^2-valued σ-finite measure and martingale measure. In the following, $(\Omega, \mathscr{F}, P)$ will denote a probability space and $(E, \mathscr{E})$ a Lusin space. We consider a set function $U(A, \omega)$ defined on $\mathscr{A} \times \Omega$, where

$\mathscr{A}$ is a subring of $\mathscr{E}$ which satisfies:

$$\|U(A)\|_2^2 = E[U(A)^2] < \infty \quad \forall A \in \mathscr{A}$$

$$A \cap B = \varnothing \Rightarrow U(A) + U(B) = U(A \cup B) \quad \text{a.s.} \quad \forall A \quad \text{and} \quad B \text{ in } \mathscr{A}\,.$$

We will say that the map U is σ-finite when there exists an increasing sequence (E_n) of E such that:

1) $\bigcup_n E_n = E$
2) $\forall n,\ \mathscr{E}_n = \mathscr{E}_{|E_n} \subseteq \mathscr{A}$
3) $\sup\{\|U(A)\|_2,\ A \in \mathscr{E}_n\} < \infty$

The set function U will be said countably additive if for each n, for each sequence (A_j) of $\mathscr{E}_n$ decreasing to $\varnothing$, $\|U(A_j)\|_2$ tends to zero. Then it is easy to extend U by $U(A) = \lim_n U(A \cap E_n)$ on every set of $\mathscr{E}$ such that the limit exists in $L^2(\Omega, \mathscr{F}, P)$.

A set function which satisfies all these properties is called a σ-finite L^2-valued measure.

Definition I-1. Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geqq 0}, P)$ be a filtered probability space satisfying the "usual conditions" [1].
$\{M_t(A),\ t \geqq 0,\ A \in \mathscr{A}\}$ is a $\mathscr{F}_t$-martingale measure if and only if:

1) $M_0(A) = 0 \quad \forall A \in \mathscr{A}$
2) $\{M_t(A),\ t \geqq 0\}$ is a $\mathscr{F}_t$-martingale, $\forall A \in \mathscr{A}$
3) $\forall t > 0$, $M_t(.)$ is a L^2-valued σ-finite measure
4) $A \cap B = \varnothing \Rightarrow M(A)$ and $M(B)$ are orthogonal martingales, $\forall A$ and B in $\mathscr{A}$.

Remarks. (1) For each $T > 0$, the localizing family $(E_n)_n$ of M_t can be chosen independently of t, $0 \leqq t \leqq T$ (cf. [12]).

(2) Walsh studies a more general class of martingale measures, which does not satisfy the condition (4). In his terminology, the martingale measures defined in I-1 are called orthogonal martingale measures.

Definition I-2. If M is a martingale measure and if, moreover, for all A of $\mathscr{A}$, the map $t \to M_t(A)$ is continuous, we will say that M is continous.

Definition I-3. If M and N are two $\mathscr{F}_t$-martingale measures on E and E' which satisfy: for $A \in \mathscr{E}_n$ and $B \in \mathscr{E}'_m$, $\forall n$, $\forall m$, $\{M_t(A)N_t(B),\ t \geqq 0\}$ is a $\mathscr{F}_t$-martingale, then M and N are called orthogonal martingale measures.

It is clear that we can associate with each set A of $\mathscr{A}$ the increasing process $\langle M(A)\rangle$ of the martingale $\{M_t(A),\ t \geqq 0\}$. The process can be regularized in a positive measure on $\mathbb{R}_+ \times E$, in the following sense:

Theorem I-4. Walsh [14]. *If M is a $\mathscr{F}_t$-martingale measure, there exists a random σ-finite positive measure $\nu(ds, dx)$ on $\mathbb{R}_+ \times E$, $\mathscr{F}_t$-predictable, such that for each A of $\mathscr{A}$ the process $(\nu(0, t] \times A)_t$ is predictable, and satisfies:*

$$\forall A \in \mathscr{A}\,, \quad \forall t > 0\,, \quad \nu((0, t] \times A) = \langle M(A)\rangle_t \quad P \text{ a.s.}\,.$$

If M is continuous, ν is continuous, i.e. $\nu(\{t\} \times E_n) = 0 \; \forall t > 0, \forall n \in \mathbb{N}$. The measure ν is called the intensity of M.

Remark I-5. 1) By point (4) of definition I-1,

$$\forall A, B \in \mathscr{A}, \quad \forall t > 0, \quad \langle M(A), M(B) \rangle_t = \langle M(A \cap B) \rangle_t = \nu((0,t] \times A \cap B) \quad P \text{ a.s.}$$

The measure ν characterizes thus completely all quadratic variations of the martingale measure M.

2) In the following, measures on $\mathbb{R}_+ \times E$ are positive and σ-finite.

We can construct a stochastic integral with respect to M, by the method which is used in the construction of Itô's integral (Walsh [14]). Let us consider:

$\mathscr{S} = \{h(\omega,s,x) = \sum_{i=1}^{n} h_i(\omega) 1_{]u_i,v_i]}(s) 1_{B_i}(x),\ B_i \in \mathscr{A},\ h_i$ a $\mathscr{F}_{u_i}$ measurable bounded function$\}$ and

$$L^2_\nu = \left\{ f(\omega, s, x) \ \mathscr{P} \otimes \mathscr{E} \text{ measurable}, \ E\left(\int_{\mathbb{R}_+ \times E} f^2(\omega, s, x) \nu(\omega, ds, dx) \right) < \infty \right\}$$

where $\mathscr{P}$ is the predictable σ-field.

If h is a function of $\mathscr{S}$, it is easy to verify that we can define a martingale measure with intensity $h^2(s,x)\nu(ds,dx)$ by:

$$h.M_t(A) = \sum_{i=1}^{n} h_i(M_{v_i \wedge t}(A \cap B_i) - M_{u_i \wedge t}(AnB_i)) \quad \forall A \in \mathscr{A}.$$

Since $\mathscr{S}$ is dense in L^2_ν, the linear mapping $h \to \{h.M_t(A), t \geqq 0, A \in \mathscr{A}\}$ can be extended to L^2_ν as usual. If $f \in L^2_\nu$, $f.M$ is called the stochastic integral of f with respect to M.

We then find usual properties of the stochastic integral.

Proposition I-6. 1) *Let f belong in L^2_ν, then $f.M$ is a martingale measure with respect to M. Moreover, if M is continuous, $f.M$ is continuous.*

2) *If f and g belong to L^2_ν, A and B to $\mathscr{A}$,* then

$$\langle f.M(A), g.M(B) \rangle_t = \int_{(0,t]} \int_{A \cap B} f(s,x) g(s,x) \nu(ds,dx).$$

This property characterizes continuous martingale measures, in the following sense.

Corollary I-7. *Let M be a martingale measure on E and $\nu(ds,dx)$ a random continuous positive measure on $\mathbb{R}_+ \times E$. Then M is a continuous martingale measure with intensity ν if and only if:*

$$E\left(\exp\left\{ \int_0^t \int_E f(s,x) M(ds,dx) - 1/2 \int_{(0,t]xE} f^2(s,x) \nu(ds,dx) \right\} \right) = 1 \qquad (*)$$

$\forall f \in L^2_\nu$, $\forall y > 0$.

Remark. By extension, $\int_0^t \int_E f(s,x) M(ds,dx)$ will be denoted by $M_t(f)$.

Proof. The condition is clearly necessary.
Conversely, let us consider f in L^2_ν and the following functions F:

$$F(\omega,u,x) = \theta f(\omega,u,x)\, 1_{]s,t]}(u) 1_{G_s}(\omega)\,,$$

where $G_s \in \mathscr{F}_s$, $0 \le s < t$, $\theta \in \mathbb{R}$.

The condition $(*)$ implies that:

$$E\left[\exp\left\{\theta 1_{G_s}(M_t(f) - M_s(f)) - 1_{G_s}\frac{\theta^2}{2}\int_s^t\int_E f^2(u,x)\nu(du,dx)\right\}\right] = 1, \quad \text{i.e.}$$

$$E\left[1_{G_s}\exp\left\{\theta(M_t(f) - M_s(f)) - \frac{\theta^2}{2}\int_s^t\int_E f^2(u,x)\nu(du,dx)\right\}\right] = P(G_s)\,.$$

Then, *for* $f \in L^2_\nu$, $M_t(f)$ is a continuous martingale with quadratic variation $\int_0^t\int_E f^2(u,x)\nu(du,dx)$, according to the result of Jacod and Memin [8] about the characterization of continuous martingales.

II. Examples of Martingale Measures

(1) Let us suppose that E is a finite space $\{a_1, a_2 \ldots a_n\}$. A martingale measure is uniquely determined by the n orthogonal square integrable martingales $(M_t(\{a_i\}))_{i=1}^n$. Conversely, let $m_t^1, \ldots, m_t^n$ be n orthogonal martingales with increasing processes $(C_t^i)_{i=1}^n$; then the mapping $M_t(A) = \sum_{i=1}^n m_t^i\, \delta_{\{a_i\}}(A)$ defines a martingale measure on E with intensity $\sum_{i=1}^n dC_t^i \delta_{\{a_i\}}(dx)$.

(2) More generally:

Proposition II-1. *Let E be a Lusin space and $(u_s)_{s \ge 0}$ an E-valued predictable process. Let us consider moreover a square integrable martingale m_t with quadratic variation process C_t. Let $M_t(A) = \int_0^t 1_A(u_s)dm_s$, for A in $\mathscr{E}$, then $\{M_t(A),\ A \in \mathscr{A},\ t \ge 0\}$ is a martingale measure with intensity equal to $\delta_{u_s}(dx)dC_s$. If m is continuous, M is continuous.*

Conversely, all martingale measures with intensity $\delta_{u_s}(dx)dC_s$ are of this form, with $m_t = M_t(E)$.

Proof. We get immediately the first assertion.

Conversely, let us study the difference $M_t(A) - M_t(f 1_E)$, $A \in \mathscr{E}$, where $f(\omega,s) = 1_A(u_s(\omega))$

Let us remark that

$$\begin{aligned} M_t(f1_E) &= \int_0^t \int_E 1_A(u_s) M(ds, dx) \\ &= \int_0^t 1_A(u_s) dm_s\,, \quad \text{if } m_s = M_s(E)\,, \end{aligned}$$

because f is not depending on x.
$M_t(A) - M_t(f1_E)$ is a martingale with increasing process:

$$\int_0^t \int_E (1_A(x) - f(s))^2 \delta_{u_s}(dx) dC_s = \int_0^t (1_A(u_s) - f(s))^2 dC_s = 0\,. \quad \text{(Prop I-6).}$$

Then, $M_t(A) = M_t(f1_E) = \int_0^t 1_A(u_s) dm_s \quad P$ a.s. .

(3) *White noises.* As the Brownian motion in the theory of continuous martingales, there exist fundamental martingale measures: white noises. Let us consider a centered Gaussian measure W on $(\mathbb{R}_+ \times E, \mathscr{B}(\mathbb{R}_+) \otimes \mathscr{E}, \mu)$, where μ is a positive σ-finite measure on $\mathbb{R}_+ \times E$, defined by:

$$\forall h \in L^2_\mu\,, \quad E(\exp W(h)) = \exp\left(1/2 \int_{\mathbb{R}_+ \times E} h^2(y)\mu(dy)\right). \tag{C}$$

A construction of such a measure is given by Neveu [12].

The process $B_t(A) = W(0,t] \times A)$, defined for the state $A \in \mathscr{A}$ which satisfy $\mu((0,t] \times A) < \infty$, $\forall t > 0$, is then a Gaussian process with independent increments and intensity μ, with cadlag trajectories. It is easy to show that $\{B_t(A), t \geqq 0, A \in \mathscr{A}\}$ is a martingale measure with a deterministic intensity, with respect to its natural filtration. When μ is continuous, its continuity is proven according to Corollary I-7 and the characterization (C).

Definition II-2. When the measure μ is continuous, the family $\{B_t(A), t \geqq 0, A \in \mathscr{A}\}$ is called white noise with intensity μ.

White noises are completely determined by the deterministic nature of their intensity:

Proposition II-3. *Let $\{M_t(A),\ t \geqq 0,\ A \in \mathscr{A}\}$ be a $\mathscr{F}_t$-martingale measure with a deterministic continuous intensity ν. Then, M is a white noise (with respect to its natural filtration).*

Proof. This result is immediate according to Corollary II-7 and to the characterization (C) of the centered Gaussian measures.

(4) *Image martingale measures*

Proposition and Definition II-4. $(E, \mathscr{E})$ and $(U, \mathscr{U})$ are two Lusin spaces. Let N be a martingale measure with intensity $\nu(ds, dx)$ on $\Omega \times \mathbb{R}_+ \times U$ and $\phi(\omega, s, u)$ a $\mathscr{P} \otimes \mathscr{U}$-measurable E-valued process.

Let $M_t(\omega, B) = \int_0^t \int_U 1_B(\phi(\omega, s, u)) N(\omega, ds, du)$.

$\{M_t(B), t \geqq 0, B \in \mathscr{E}\}$ defines a martingale measure with intensity μ, where μ is given by:

$$\mu((0, t]xB) = \int_{(0,t]} \int_U 1_B(\phi(s, u))\nu(ds, du) .$$

M is called image martingale measure of N under ϕ. Let us remark that if N is continuous, M is also continuous.

III. Representation of Martingale Measures

1. Intensity Decomposition. Construction of Martingale Measures

We will prove at first that the form $q_t(dx)dk_t$ for a martingale measure intensity is not a restrictive assumption.

Lemma III-1. *Let $\nu(dt, dx)$ be a random predictable σ-finite measure. ν can be decomposed as follows; $\nu(dt, dx) = q_t(dx)dk_t$ where k_t is a random predictable increasing process and $(q_t(dx))_{t \geqq 0}$ is a predictable family of random σ-finite measures.*

Proof. We will use the notations of section I.

If ν is a finite measure, the lemma is well known. Otherwise, there exists a $P \otimes \mathscr{E}$-measurable function $W: \Omega \times \mathbb{R}_+ \times E \to (0, \infty)$ such that

$$\nu'(dt, dx) = \nu(dt, dx) \cdot W(t, x)$$

is finite. Then we can decompose

$$\nu'(dt, dx) = q'_t(dx)dk_t ;$$

the result follows by setting

$$q_t(dx) = W(t, x)^{-1} \cdot q'_t(dx) .$$

Remark. This decomposition is not unique, and it is always possible to assume that the process k_t is increasing, for example by replacing k_t by $t + k_t$. In the following, we will use this decomposition of the intensity in which the time coordinate plays a special role, and we will denote the intensities of martingale measures in the form $q_t(dx)dk_t$, with an increasing processes $(k_t)_{t \geqq 0}$.

An important result is that it is always possible to give a representation of the random measures $(q_t(dx)_{t \geqq 0}$ as image measures of deterministic measures (cf. A.V. Skorohod [13], N. El Karoui and J.P. Lepeltier [2], B. Grigelionis [6]):

Theorem III-2. *Let $(q_t(dx)_{t \geqq 0}$ be a predictable family of random σ-finite measures, defined on a Lusin space $(E, \mathscr{E})$.*

Let us also consider a Lusin space $(U, \mathscr{U})$ and a deterministic diffuse σ-finite measure λ on U which satisfies:

$$q_t(E) \leqq \lambda(U) \quad \forall t \in \mathbb{R}_+ , \quad \forall \omega \in \Omega .$$

Then there exists a predictable process $\varphi(t, u)$, with values in $E \cup \{\delta\}$, (δ is the cemetery point), such that:

$$q_t(A) = \int_U 1_A(\varphi(t, u))\lambda(du) \qquad \forall A \in \mathscr{E}, \quad \forall \omega \in \Omega \tag{$**$}$$

and a predictable kernel from E to U, $Q(t, x, du)$ which satisfies:

$$\int_U 1_B(u) f(\varphi(t, u))\lambda(du) = \int_E f(x) Q(t, x, B) q_t(dx) \tag{$***$}$$

$\forall \omega \in \Omega$, $\forall f$ measurable positive, $\forall B \in \mathscr{U}$.

The kernel $Q(t, x, .)$ is the conditional law of u with respect to the σ-field generated by φ.

According to this theorem, the existence of a continuous martingale measure with intensity $q_t(dx)dk_t$, follows immediately from the existence of a white noise, as our construction will show it. When k_t is deterministic, the martingale measure is given as image measure of a white noise, and the general case follows by using a time-change.

Theorem III-3. *Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geqq 0}, P)$ be a filtered space and ν a random positive continuous σ-finite measure, satisfying:*

$$\nu(dt, dx) = q_t(dx)dk_t\,, \quad (k_t) \text{ continuous and increasing}$$
$$(q_t) \text{ predictable.}$$

There exists on an extenison $\hat{\Omega} = (\Omega \times \tilde{\Omega}, \ \mathscr{F} \otimes \tilde{\mathscr{F}}, \ (\mathscr{F}_t \otimes \tilde{\mathscr{F}}_t)_{t \geqq 0}, \ P \otimes \tilde{P})$ a continuous martingale measure N with intensity ν, obtained as time-changed image measure of a white noise.

Moreover, N is orthogonal to each continuous $(\mathscr{F}_t, P)$ martingale measure M.

Proof. i) Let us assume that k_t is deterministic.

We can build on an auxiliary space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathscr{F}}_t, \tilde{P})$ a white noise B with intensity $\lambda(du)dk_t$ where λ satisfies the assumptions of theorem III-2. On the extension $(\hat{\Omega}, \hat{\mathscr{F}}, (\hat{\mathscr{F}}_t)_{t \geqq 0}, \hat{P}) = (\Omega \times \tilde{\Omega}, \mathscr{F} \otimes \tilde{\mathscr{F}}, (\mathscr{F}_t \otimes \tilde{\mathscr{F}}_t)_{t \geqq 0}, P \otimes \tilde{P})$, B is a continuous martingale measure with a deterministic intensity and then a $(\hat{\mathscr{F}}_t)$-white noise (Proposition II-3). Let $\varphi(t, u)$ be the predictable process satisfying $(**)$. It is clear that φ is $\hat{\mathscr{P}} \otimes \mathscr{U}$ measurable, $\hat{\mathscr{P}}$ being the predictable σ-field on the extension $\hat{\Omega}$.

By the example II-4 and $(**)$, the family

$$N_t(\omega, \omega', A) = \int_0^t \int_U 1_A(\varphi(\omega, s, u)) B(\omega', ds, du)\,, \quad A \in \mathscr{E}\,,$$

is a continuous martingale measure with intensity

$$\int_{(0,t]} \int_E 1_A(\varphi(\omega, s, u))\lambda(du)dk_s = \nu((0, t] \times A)\,.$$

Moreover, B and each $(\mathscr{F}_t, P)$-martingale measure M are orthogonal (by construction, M is again in a $\hat{\mathscr{F}}_t$-martingale measure). We verify that for each predictable step function h, the martingale measures $\int\limits_0^t \int\limits_U h(\varphi(s,u))B(ds,du)$ and M are orthogonal, and that this property is more generally satisfied for h in $L^2(dP \otimes q_t(dx)dk_t)$. That implies immediately the orthogonality for M and N.

ii) If k_t is not deterministic, let us consider $\tau_t = \inf\{s > 0, k_s \geqq t\}$. τ_t is the then the increasing inverse of k_t. We can consider the σ-finite random measure $\gamma(dt, dx) = q_{\tau_t}(dx)dt$, where q_τ is predictable (for the filtration $\mathscr{F}_{\tau_t}$).

According to i), we construct a white noise B with intensity $\lambda(du)dt$, φ a predictable process (for $\mathscr{F}_{\tau_t}$), such that

$$N_t(A) = \int\limits_0^t \int\limits_U 1_A(\varphi(\omega, s, u))B(ds, du) \quad \text{defines for} \quad t \geqq 0 \,,\ A \in \mathscr{E} \,,$$

a $\mathscr{F}_{\tau_t}$-martingale measure, with intensity $\gamma(dt, dx)$.

Let us now consider the $\mathscr{F}_t$-martingale measure $\{M_t(A),\ A \in \mathscr{A},\ t \geqq 0\}$ defined by $M_t(A) = N_{k_t}(A)$. The intensity of M is then $q_t(dx)dk_t$, since:

$$\langle M(A)\rangle_t = \int\limits_0^{k_t} \int\limits_E 1_A(x)q_{\tau_s}(dx)ds = \int\limits_0^t \int\limits_E 1_A(x)q_u(dx)dk_u \,.$$

2. Extension and Representation of Martingale Measures as Image Measures of a White Noise

The main result of this part is to obtain a converse to theorem III-3, that is to describe all martingale measures as time changed image measures of white noises. To obtain this property, it is necessary to use an extension result, (this idea is due to Funaki [5]), and the following theorem is thus fundamental.

Theorem III-4. *Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geqq 0}, P)$ be a filtered space, E and $\tilde{E}$ two Lusin spaces and M a continuous martingale measure with intensity $q_t(dx)dk_t$ on $\mathbb{R}_+ \times E$, where k_t is a continuous increasing process and $(q_t(dx))_{t \geqq 0}$ is a $\mathscr{F}_t$-predictable family of random measures.*

Let $r_t(x, d\tilde{x})$ be a predictable probability transition kernel from E to $\tilde{E}$ and define the predictable σ-finite measure $p_t(dx, d\tilde{x})$ on $\mathbb{R}_+ \times E \times \tilde{E}$ as follows: $p_t(dx, d\tilde{x}) = q_t(dx)r_t(x, d\tilde{x})$. Then there exists on an extension $(\Omega \times \tilde{\Omega},\ \mathscr{F} \otimes \tilde{\mathscr{F}},\ P \otimes \tilde{P})$ a continuous martingale measure $\tilde{M}_t(dx, d\tilde{x})$ with intensity $dk_t p_t(dx, d\tilde{x})$ and whose projection on $\mathbb{R}_+ \times E$ is M, i.e. $\tilde{M}_t(A \times \tilde{E},\ (\omega, \tilde{\omega})) = M_t(A, \omega)$, $\forall A \in \mathscr{A}$, $(\omega, \tilde{\omega}) \in \Omega \times \tilde{\Omega}$, $\forall t \geqq 0$.

Proof. Let N be the continuous martingale measure on $E \times \tilde{E}$, built on an auxiliary space $(\tilde{\Omega}, \tilde{\mathscr{F}}, (\tilde{\mathscr{F}}_t)_{t \geqq 0}, \tilde{P})$ with intensity $dk_t p_t(dx, d\tilde{x})$ such that N and each $\mathscr{F}_t$-martingale measure are orthogonal (Theorem III-3).

Let us consider the mapping:

$$\tilde{M}_t(C) = \int\limits_0^t \int\limits_E r_s(x, C)M(ds, dx) + \int\limits_0^t \int\limits_{E \times \tilde{E}} (1_C(x, \tilde{x}) - r_s(x, C))N(ds, dx, d\tilde{x})$$

$\forall C \in \mathscr{E} \otimes \tilde{\mathscr{E}}$, where $r_s(x, C) = \int\limits_{\tilde{E}} 1_C(x, \tilde{x})r_s(x, d\tilde{x})$.

The two terms on the right of the above equality are orthogonal continuous martingale measures. $\{\tilde{M}_t(C),\ C \in \mathscr{E} \otimes \tilde{\mathscr{E}},\ t \geqq 0\}$ is then a continuous martingale measure with intensity given by:

$$\int_{(0,t]} dk_s \left[\int_E r_s^2(x, C) q_s(dx) + \int_{E \times \tilde{E}} p_s(dx,\ d\tilde{x})\, (1_C(x, \tilde{x}) - r_s(x, C))^2 \right]$$

$$= \int_{(0,t]} dk_s \left[\int_E r_s^2(x, C) q_s(dx) + \int_{E \times \tilde{E}} q_s(dx) r_s(x, d\tilde{x})\, (1_C(x, \tilde{x}) + \right.$$

$$\left. + r_s^2(x, C) - 2 r_s(x, C) 1_C(x, \tilde{x}) \right]$$

$$= \int_{(0,t]} dk_s \int_E r_s(x, C) q_s(dx) \quad (r_s(x,.) \text{ is a probability})$$

$$= \int_0^t dk_s p_s(C)\,.$$

(b) Let us assume that C is in $\mathscr{E}$.

$$1_C(x) - \int_{\tilde{E}} r_s(x, d\tilde{x}) 1_C(x) = 0 \quad \text{and then} \quad \tilde{M}_t(C) = M_t(C)\,.$$

We can apply this result to continuous square integrable martingales, by interpreting them as degenerated martingale measures.

Corollary III-5. *Let n_t be a continuous square integrable martingale with increasing process $\langle n \rangle_t(\omega) = \int_0^t \int_E \sigma^2(\omega s, x) q_s(\omega, dx) dk_s, (k_t)$ being increasing and continuous, $(q_t(dx))_{t \geqq 0}$ being a predictable family of random measures and $\sigma(s, x)$ a function of $L^2(q_s(dx)dk_s)$. We assume moreover that $n_0 = 0$.*

There exists on an extension a continuous martingale measure N with intensity $\sigma^2(s, x) q_s(dx) dk_s$ such that:

$$n_t = N_t(E)\,.$$

Remark. This result will be generalized in the next section to vector continuous square integrable martingales (theorem III-7).

Proof. The proof is nearly natural.

Let us define $S = \left\{ s > 0,\ \int_E \sigma^2(s, x) q_s(dx) \neq 0 \right\}$.

The random measures $r_s(dx)$ equal to $1_S(s) \dfrac{\sigma^2(s, x) q_s(dx)}{\int_E \sigma^2(s, x) q_s(dx)} + 1_{S^c}(s) \delta_{\{\alpha\}}(x),\ \alpha \in E,$

constitute a predictable family of probabilities and we can apply theorem III-4 with the kernel $r_s(dx)$ and the nondecreasing process $\int_E \sigma^2(s,x)q_s(dx)dk_s$. Therefore, there exists a martingale measure N whose projection on $\mathbb{R}_+$ is n, with intensity

$$r_s(dx)\int_E \sigma^2(s,x)q_s(dx)dk_s = \sigma^2(s,x)q_s(dx)dk_s\,.$$

Using theorem III-4, we can now state one of our main results: each martingale measure is representable as time-changed image martingale measure of a white noise. An application of this result is given in Méléard, Roelly-Coppoletta [10]; [11]: it allows to give a sense to a stochastic differential equation in the space of vector measures with values in $L^2(\Omega)$ for a certain class of measure-valued branching processes.

Theorem III-6. *Let M be a continuous martingale measure on $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, P)$ with intensity $q_t(dx)dk_t$. Let λ be the diffuse σ-finite measure and φ be the predictable process given in Theorem III-2.*

(1) *If (k_t) is deterministic, there exist an extension $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{\mathscr{F}}_t, \hat{P})$ of $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$ and a white noise $B_t(\hat{\omega}, du)$ with intensity $\lambda(du)dk_t$ such that:*

$$\forall f\in L^2(q_s(dx)dk_s),\ M_t(f) = \int_0^t \int_U f(\varphi(s,u))B(ds,du)$$

(2) *In the general case, M is a time-changed image martingale measure of a white noise.*

Proof. (1) We use the predictable kernel $Q_t(x,du)$ defined in Theorem III-2 by $(***)$.

We consider the measure $p_t(dx,du) = Q_t(x,du)q_t(dx)$.

It satisfies:

$$\forall B\in\mathscr{E}\,,\quad A\in\mathscr{U}\,,\quad \int_U 1_B(\varphi(t,u))1_A(u)\,\lambda(du) = \int_{E\times U} 1_B(x)1_A(u)p_t(dx,du)$$

According to theorem III-4, we build on $E\times U$ a continuous martingale measure $\hat{M}$ with intensity $p_t(dx,du)dk_t$ and whose projection onto E is M. The martingale measure $N(dt,du) = \int_E \hat{M}(dt,dx,du)$ has thus the intensity:

$$\int_E Q_t(x,du)q_t(dx)dk_t = dk_t 1_{\{\varphi(t,u)\neq\delta\}}\lambda(du)\quad (\delta\text{ cemetry point}).$$

N_t is not a white noise, because its intensity is not deterministic. We build then on an auxiliary space a white noise $W_t(du)$ with intensity $\lambda(du)dk_t$ and we consider the martingale measure $B_t(du) = N_t(du) + 1_{\{\delta\}}(\varphi(t,u))\,W_t(du)$. Then, B is a continuous martingale measure with deterministic intensity and is therefore a white noise (Proposition II-3).

Let f be in $L^2(q_s(dx)dk_s)$, then $f \circ \varphi$ belongs to $L^2(dk_t \lambda(du))$ and

$$\begin{aligned}\int_0^t \int_U f(\varphi(s,u)) B(ds,du) &= \int_0^t \int_U f(\varphi(s,u)) N(ds,du) \\ &\quad + \int_0^t \int_U f(\varphi(s,u)) 1_{\{\delta\}}(\varphi(s,u)) W(ds,du) \\ &= \int_0^t \int_U f(\varphi(s,u)) N(ds,du) \\ &= \int_0^t \int_U f(\varphi(s,u)) \int_E \hat{M}\,(ds,dx,du) \\ &= \int_0^t \int_E \int_U f(\varphi(s,u)) \hat{M}\,(ds,dx,\mathrm{du})\end{aligned}$$

We want to compare this quantity to

$$\int_0^t \int_E f(x) M(ds,dx) = \int_0^t \int_E \int_U f(x) \hat{M}\,(ds,dx,du)\,.$$

$$\begin{aligned}&E\left[\left(\int_0^t \int_E \int_U f(\varphi(s,u)) \hat{M}\,(ds,dx,du) - \int_0^t \int_E \int_U f(x) \hat{M}\,(ds,dx,du)\right)^2\right] \\ &= E\left[\int_0^t \int_E \int_U (f(\varphi(s,u)) - f(x))^2 Q_s(x,du) q_s(dx) dk_s\right] \\ &= E\left[\int_0^t \int_U (f(\varphi(s,u)) - f(\varphi(s,u)))^2 \lambda(du) dk_s\right] = 0\,.\end{aligned}$$

Thus

$$\begin{aligned}\int_0^t \int_E \int_U f(\varphi(s,u) \hat{M}\,(ds,dx,du) &= \int_0^t \int_E \int_U f(x) \hat{M}\,(ds,dx,du) \\ &= \int_0^t \int_E f(x)\; M\,(ds,dx) \quad P \quad \text{a.s.}\end{aligned}$$

(2) The proof of the generalization is similar to the proof of theorem III-3 (ii).

3. *Representation of Vector Martingale Measures*

The first theorem of this section gives a representation of vector martingale measures in terms of orthogonal martingale measures, which generalizes the representation theorem for continuous martingales in terms of Brownian motions.

Theorem III-7. *Let $(M^i)_{i=1}^n$ be n continuous martingale measures on a Lusin space E, with intensities*:

$$\langle M^i(\varphi), M^j(\psi)\rangle_t = \int_0^t \int_E \varphi(x)\psi(x) a_{ij}(s,x) q_s(dx) dk_s$$

where:

$$a_{ij}(s, x) = \sum_{k=1}^{n} \sigma_{ik}(s, x)\sigma_{jk}(s, x) ,$$

$\forall i,k \in \{1, \ldots, n\}$, *$\sigma_{ik}(s, x) \in L^2(q_s(dx)dk_s)$, (k_t) is a continuous increasing process, $(q_t(dx))$ is a predictable process of random finite measures.*

There exists on an extension n continuous orthogonal martingale measures $(\hat{M}_s^i(dx))_{i=1}^n$ with intensity $q_s(x)dk_s$ which satisfy:

$$M_t^i(\varphi) = \sum_{k=1}^{n} \int_0^t \int_E \varphi(x)\sigma_{ik}(s, x)\hat{M}^k(ds, dx) \quad \forall i \in \{1, \ldots, n\} .$$

Proof. This theorem is proven with the same method as in [7]. Let us describe quickly the principal steps of the proof:

We can suppose that $\sigma(s, x) = a^{1/2}(s, x)$ is the symmetric square root of $a(s, x)$ and define:

$$\tilde{\sigma}(s, x) = \lim_{\varepsilon \downarrow 0} a^{1/2}(s, x)(a(s, x) + \varepsilon I)^{-1} , \qquad \forall (s, x) \in \mathbb{R}_+ \times E .$$

We have: $\tilde{\sigma}(s, x)\sigma(s, x) = \sigma(s, x)\tilde{\sigma}(s, x) = E_R(s, x)$, where $E_R(s, x)$ is the orthogonal projection onto range $a(s, x)(\mathbb{R}^d)$ and denote $E_N(s, x) = I - E_R(s, x)$.

We define then, for i in $\{1, \ldots, n\}$, the continuous martingale measure

$$\hat{M}_s^i(f) = \sum_{k=1}^{n} \int_0^t \tilde{\sigma}_{ik}(s, x)f(x)M^k(ds, dx) + \sum_{k=1}^{n} \int_0^t \int_E E_N(s, x)f(x)\tilde{M}^k(ds, dx)$$

where $(\tilde{M}^k)_{k=1}^n$ are n continuous orthogonal martingale measures with intensity $q_s(dx)dk_s$ built on an auxiliary space. It is therefore easy to verify that

$$\langle \hat{M}^i(f), \hat{M}^j(g)\rangle_t = \delta_{ij} \int_0^t \int_E f(x)g(x)q_s(dx)dk_s , \qquad \forall f, g \in L^2(q_s(dx)dk_s)$$

and that

$$\sum_{k=1}^{n} \int_0^t \int_E f(x)\sigma_{ik}(s, x)\hat{M}^k(ds, dx) = M_t^i(f) .$$

(The calculations are carried out in the book of Ikeda and Watanabe [7] p. 90.)

Corollary III-8. *If we use the notations and the result of theorem III-6, and if the process (k_t) is deterministic, we can represent the martingale measures $(M^i)_{i=1}^n$ with n orthogonal white noises $(B^i)_{i=1}^n$* by:

$$M_t^i(f) = \sum_{k=1}^{n} \int_0^t \int_U f(\varphi(s, u))\sigma_{ik}(s, \varphi(s, u)B^k(ds, du) .$$

A very interesting problem is to obtain a similar representation theorem for vector square integrable martingales $(m_t^i)_{i=1}^n$ whose quadratic variation process has the special form: $\langle m^i, m^j \rangle_t = \int_0^t \int_E a_{ij}(s, x) q_s(dx) dk_s$ (where a is a quadratic matrix). The aim is to represent them in terms of orthogonal martingale measures with intensity $q_s(dx)dk_s$. It will be used in particular to describe solutions of martingale problems; this will be the subject of the last section of the paper. To obtain this result, we need an extension property, which generalizes to vector martingales the extension property obtained in corollary III-5 for the dimension one.

Proposition III-9. *Let $(m_t^i)_{i=1}^n$ be n continuous square integrable martingales such that $m_0^i = 0$. We assume that the quadratic variation process corresponding to m_i and m_j is $\langle m^i, m^j \rangle_t = \int_0^t \int_E a_{ij}(s, x) q_s(dx) dk_s$, where: $a(s, x) = \sigma(s, x)\sigma^*(s, x)$ is a $\mathscr{P} \otimes \mathscr{E}$ measurable matrix such that $a_{ij}(s, x) \in L^2(q_s(dx)dk_s)$ $\forall i, j \in \{1, \ldots, n\}$, $(k_t)_{t \geqq 0}$ is a continuous increasing process, $(q_t(dx))_{t \geqq 0}$ is a predictable finite measure-valued process.*

Then on an extension, there exist n continuous martingale measures $(M_s^i(dx))_{i=1}^n$ such that $\forall B, C \in \mathscr{E}$,

$$\langle M^i(B), M^j(C) \rangle_t = \int_0^t \int_E 1_B(x) 1_C(x) a_{ij}(s, x) q_s(dx) dk_s$$

and $M_t^i(E) = m_t^i \quad \forall t \geqq 0$.

Proof. (a) We suppose first that the symmetic matrix $\Delta(s) = ((\int a_{ij}(s, x) q_s(dx)))_{\substack{1 \leqq i \leqq n \\ 1 \leqq i \leqq n}}$ is invertible. Let us denote by $\delta(s)$ its inverse. For f in $L^2(q_s(dx)dk_s)$, we will denote $Q(s, f)$ the symmetric matrix $((\int a_{ij}(s, x) f(x) q_s(dx)))_{\substack{1 \leqq i \leqq n \\ 1 \leqq j \leqq n}}$; $Q(s, 1) = \Delta(s)$.

It is easy to build on a larger space n martingale measures $(\hat{N}^i)_{i=1}^n$ which satisfy, $\forall f, g \in L^2(q_s(dx)dk_s)$:

$$\langle \hat{N}^i(f), \hat{N}^j(g) \rangle_t = \int_0^t \int_E f(s, x) g(s, x) a_{ij}(s, x) q_s(dx) dk_s \,,$$

In fact, we can define on $\mathbb{R}_+ \times E \times \{1, \ldots, n\}$ a martingale measure N with intensity $\sum_{i=1}^n q_s(dx) dk_s \delta_{\{i\}}(dj)$ (see Theorem III-3) and construct the martingale measures $(\hat{N}_s^i(dx))_{i=1}^n$ as follows:

$$\forall A \in \mathscr{E}, \hat{N}_s^i(A) = \sum_{k=1}^n \int_0^t \int_A \sigma_{ik}(s, x) N(ds, dx, \{k\}) \,.$$

We may take therefore:

$$i \in \{1, \ldots, n\} \,, \quad t \geqq 0 \,, \quad f \in L^2(q_s(dx)dk_s) \,,$$

$$M_t^i(f) = \sum_{k=1}^n \int_0^t (Q(s, f)\delta(s))_{ik} \, dm_s^k + \sum_{k=1}^n \int_0^t \int_E (f(x) I - Q(s, f)\delta(s))_{ik} \hat{N}^k(ds, dx)$$

(I identity matrix of $\mathscr{M}_n(\mathbb{R})$).

It is immediate to verify that $M_t^i(E) = m_t^i$, since $Q(s, E) = \Delta(s)$. Let us calculate the intensity of $(M^i)_{i=1}^n$: For every f and g in $L^2(q_s(dx)dk_s)$, we set

$$\begin{aligned}\langle M^i(f), M^j(g)\rangle_t &= \sum_{k,l=1}^{n} \int_0^t (Q(s,f)\delta(s))_{ik}(Q(s,g)\delta(s))_{jl} \int_E a_{kl}(s,x)q_s(dx)dk_s \\ &+ \sum_{k,l=1}^{n} \int_0^t \int_E (f(x)I - Q(s,f)\delta(s))_{ik}(g(x)I - Q(s,g)\delta(s))_{jl} a_{kl}(s,x)q_s(dx)dk_s \\ &= \int_0^t [Q(s,f)\delta(s)\Delta(s)(Q(s,g)\delta(s))^*]_{ij}dk_s \\ &+ \int_0^t \int_E [(f(x)I - Q(s,f)\delta(s))a(s,x)(g(x)I - Q(s,g)\delta(s))^*]_{ij} q_s(dx)dk_s\end{aligned}$$

$Q(s,.)$ and $\delta(s)$ are symmetric matrices for every s in $\mathbb{R}_+$. Thus,

$$\begin{aligned}Q(s,f)\delta(s)\Delta(s)(Q(s,g)\delta(s))^* &= Q(s,f)\delta(s)\Delta(s)\delta(s)Q(s,g) \\ &= Q(s,f)\delta(s)Q(s,g)\end{aligned}$$

and,

$$\begin{aligned}&\int_E (f(x)I - Q(s,f)\delta(s))a(s,x)(g(x)I - Q(s,g)\delta(s))^* q_s(dx) \\ &= \int_E (f(x)I - Q(s,f)\delta(s))a(s,x)(g(x)I - \delta(s)Q(s,g))q_s(dx) \\ &= \int_E [f(x)g(x)a(s,x) - f(x)a(s,x)\delta(s)Q(s,g) - Q(s,f)\delta(s)g(x)a(s,x) \\ &\quad + Q(s,f)\delta(s)a(s,x)\delta(s)Q(s,g)]q_s(dx) \\ &= \int_E f(x)g(x)a(s,x)q_s(dx) - Q(s,f)\delta(s)Q(s,g)\end{aligned}$$

So, $\langle M^i(f), M^j(g)\rangle_t = \int_0^t \int_E f(x)g(x)a_{ij}(s,x)q_s(dx)dk_s$.

(b) When $\Delta(s)$ is not invertible, we use a method similar to that one of Ikeda and Watanabe [7]: We introduce the symmetric matrix $\tilde{\delta}(s)$ which satisfies: $\forall s \in \mathbb{R}_+$, $\tilde{\delta}(s)\Delta(s) = \Delta(s)\tilde{\delta}(s) = E_R(s)$, where $E_R(s)$ is the orthogonal projection onto range $\Delta(s)\mathbb{R}^d$. We have:

$$I - E_R(s) = E_N(s)\,, \quad \text{with} \quad E_N(s)\Delta(s) = 0\,, \quad \tilde{\delta}(s)\Delta(s)\tilde{\delta}(s) = 0\,.$$

Let us consider now:

$$M_t^i(f) = \sum_{k=1}^{n} \int_0^t (Q(s,f)\tilde{\delta}(s))_{ik}dm_s^k + \sum_{k=1}^{n} \int_0^t \int_E (f(x)I - Q(s,f)\tilde{\delta}(s))_{ik}\tilde{N}^k(ds,dx)\,.$$

We get

$$M_t^i(E) = \sum_{k=1}^{n} \int_0^t (\Delta(s)\tilde{\delta}(s))_{ik} dm_s^k + \sum_{k=1}^{n} \int_0^t \int_E (I - \Delta(s)\tilde{\delta}(s))_{ik} \hat{N}^k(ds, dx)$$

$$= \sum_{k=1}^{n} \int_0^t (E_R(s))_{ik} dm_s^k + \sum_{k=1}^{n} \int_0^t \int_E (E_N(s))_{ik} \hat{N}^k(ds, dx)$$

$$= m_t^i - \sum_{k=1}^{n} \int_0^t (E_N(s))_{ik} dm_s^k + \sum_{k=1}^{n} \int_0^t \int_E (E_N(s))_{ik} \hat{N}^k(ds, dx)\,.$$

The two right-hand terms have the intensity

$$\sum_{k,l=1}^{n} \int_0^t (E_N(s))_{ik} (E_N(s))_{jl} \int_E a_{kl}(s, x) q_s(dx) dk_s$$

$$= \sum_{l=1}^{n} \int_0^t (E_N(s)\Delta(s))_{il} (E_N(s))_{jl} dk_s = 0$$

and thus vanish.

We verify easily, with an analogous calculation, that the quadratic variation $\langle M^i(f), M^j(g) \rangle_t$ is $\int_0^t \int_E f(x) g(x) a_{ij}(s, x) q_s(dx) dk_s$. □

Let us give now the main theorem, which is obtained immediately by application of theorem III-7 and proposition III-9:

Theorem III-10. *Let $(m_t^i)_{i=1}^n$ be n continuous square integrable martingales, with (matrix valued) quadratic variation process*

$$\langle m^i, m^j \rangle_t = \int_0^t \int_E a_{ij}(s, x) q_s(dx) dk_s\,.$$

There exist on an extension n continuous orthogonal martingale measures $(\hat{M}_s^i(dx))_{i=1}^n$ with intensity *$q_s(dx)dk_s$ which satisfy:*

$$m_t^i = \sum_{k=1}^{n} \int_0^t \int_E \sigma_{ik}(s, x) \hat{M}^k(ds, dx) \qquad \forall i \in \{1, \dots, n\}\,.$$

IV. Study of a Martingale Problem

One of the motivations for the investigation of the martingale measures theory was the study of solutions of problems as below:
given a filtered space $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$ and q a finite measure-valued $\mathscr{F}_t$-predictable random process on this space, we call solution of the martingale problem $(\mathscr{P})$ a couple $(X, \tilde{P})$ defined on an extension of $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$, X being a continuous process, satisfying:

$$(\mathscr{P}):\ \forall f \in C_b^2(\mathbb{R}^d),\quad f(X_t) - f(X_0) - \int_0^t \int_E Lf(s, X_s, x) q_s(dx) ds$$

is a $\tilde{P}$ martingale, where L is an elliptic operator, defined on $\mathbb{R}_+ \times \mathbb{R}^d \times E$ (E a Lusin space) by:

$$Lf(s, y, x) = \left(\frac{1}{2} \sum_{i,j=1}^{d} a_{ij} \frac{\partial^2 f}{\partial y_i \partial y_j} + \sum_{i=1}^{d} b_i \frac{\partial f}{\partial y_i} \right)(s, y, x)$$

with bounded coefficients and $a(s, y, x) = \sigma(s, y, x)\sigma^*(s, y, x)$, $\forall (s, y, x) \in \mathbb{R}_+ \times \mathbb{R}^d \times E$.

We meet here some difficulties provided firstly by the integrated form of the generator in the martingale problem and secondly by the randomness of the measure q. In the first step, we assume that q is deterministic and we give an appropriate representation of the solutions of $(\mathscr{P})$ with respect to white noises, which allows to obtain their existence and uniqueness with Lipschitz continuous coefficients. Secondly, we generalize this method to a random process q, by giving a representation of the solutions of $(\mathscr{P})$ with respect to martingale measures and we study the problem of the uniqueness of these solutions.

1. A Particular Case: q is Deterministic

To clarify the situation, let us assume first that $d = 1$. We can always represent the solutions of $(\mathscr{P})$ as follows:

$$X_t = X_0 + \int_0^t \sqrt{\sigma^2(s, X_s, x)q_s(dx)}\, dB_s + \int_0^t \int_E b(s, X_s, x)q_s(dx)ds$$

where B is a brownian motion defined on an extension of $(\Omega, \mathscr{F}, P)$. But we are not able to assure the existence and the uniqueness of solutions of this equation, not even under Lipschitz continuity hypothesis on σ and b. This representation is then not well adapted to the problem. The other idea is to interpret the deterministic measure $q_s(dx)ds$ as the intensity of a white noise $W_s(dx)$ defined on an extension of $(\Omega, \mathscr{F}, P)$ (Theorem III-10). A solution of $(\mathscr{P})$ is then equivalent to a solution of:

$$X_t = X_0 + \int_0^t \int_E \sigma(s, X_s, x) W(ds, dx) + \int_0^t \int_E b(s, X_s, x)q_s(dx)ds\,,$$

whose existence and uniqueness are immediate, under Lipschitz continuity conditions on σ and b.

More generally, we obtain in the d-dimensional case:

Proposition IV-1 *Let us assume that q is deterministic.*

1) *The existence of a solution of $(\mathscr{P})$ is equivalent to the existence of a solution of the stochastic differential equation $\forall i \in \{1, \ldots, d\}$,*

$$dX_t^i = \sum_{k=1}^{d} \int_E \sigma_{ik}(t, X_t, x) W^k(dt, dx) + \int_E b_i(t, X_t, x)q_t(dx)\, dt\,, \tag{E}$$

where $(W^k)_{k=1}^d$ are k orthogonal white noises defined on an extension of $(\Omega, \mathscr{F}, P)$.

2) *If σ and b are Lipschitz continuous in the second variable, uniformly in t and x, the stochastic differential equaton* (E) *has a unique pathwise solution.*

Proof. The representation is a particular case of theorem III-10. The law of a white noise being exactly defined by its intensity, the proposition is shown by applying similar methods developed in the book of Ikeda and Watanabe (p. 151). So we omit the proof.

2. General Case

We generalize the above point of view. When q is random, theorem III-10 allows us to give a tie between a solution of $(\mathscr{P})$ and some martingale measures:

Theorem IV-2. *Let $(X_t)_{t \geqq 0}$ be a d-dimensional continuous adapted process, solution of $(\mathscr{P})$. Then, on an extension, there exist d continuous orthogonal martingale measures $(M^k)_{k=1}^d$ with intensity $q_s(dx)ds$ such that $(X_t)_{t \geqq 0}$ satisfies $\forall i \in \{i, \ldots, d\}$,*

$$dX_t^i = \sum_{k=1}^{d} \int_E \sigma_{ik}(t, X_t, x) M^k(dt, dx) + \int_E b_i(t, X_t, x) q_t(dx) dt . \tag{E'}$$

The proof of Theorem IV-2 is routine (see proposition IV-1).

The problem $(\mathscr{P})$ is a martingale problem with a random coefficient q and thus, it is not well-posed. In fact, just to know the random variable q on a space gives an infinity of solutions, since there is an infinity of martingale measures with intensity $q_s(dx)ds$. Anyhow, if a d-dimensional process $(X_t)_{t \geqq 0}$ satisfies (E′), Itô's formula implies that X satisfies $(\mathscr{P})$.

For given orthogonal martingale measures $(M^k)_{k=1}^d$, and if σ and b are uniform Lipschitz continuous, (E′) has a unique pathwise solution; the proof is obtained by imitating the usual arguments for stochastic differential equations (Ikeda-Watanabe p. 165).

Remark IV-5. We assume here that the drift coefficient is the linear function of q: $b(s, y, q) = \int_E b(s, y, x) q_s(dx) ds$. But we could generalize all the results to a function b defined on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathscr{P}(E)$, $\mathscr{P}(E)$ being the space of probabilities on E. It can be useful for example to model particle systems with a non linear interaction in the drift coefficient [9].

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