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Probability Theory and Related Fields

Self-avoiding Paths on the Pre-Sierpinski Gasket

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Summary. We study a statistical mechanics of self-avoiding paths on the pre-Sierpinski gasket. We first show the existence of the thermodynamic limit of the (appropriately scaled) free energy. Then we show that there are two domains in the weight parameters (i.e. two phases) between which the scaling differs; i.e. there is a certain kind of phase transition in our model, and we find the critical exponents of the free energy at the phase transition point. We also show the convergence of the distribution of the scaled length of the paths at thermodynamic limit.

0. Introduction

Let us define the pre-Sierpinski Gasket as follows. Let $O = (0, 0)$, $a_0 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $b_0 = (1, 0)$, and let F_0 be a graph which consists of the vertices and the edges of the equilateral triangle $\Delta O a_0 b_0$. Let us define a sequence of graphs (Fig. 1) inductively by

$$F_{n+1} = F_n \cup (F_n + 2^n a_0) \cup (F_n + 2^n b_0), \qquad n = 0, 1, 2, \ldots,$$

where, $A + a = \{x + a | x \in A\}$, and $kA = \{kx | x \in A\}$. Let $F = \bigcup_{n=0}^{\infty} F_n$. F is the pre-Sierpinski Gasket. Let G_0 be the set of the vertices in F, and $a_n = 2^n a_0$, $b_n = 2^n b_0$.

We define the set of self-avoiding paths W_0 on G_0 to be the set of mappings $w: \mathbb{Z}_+ \to G_0$ such that there exists $L(w) \in \mathbb{Z}_+ \cup \{\infty\}$ for which $w(i) = w(L(w))$, $i \geqq L(w)$, $w(i_1) \neq w(i_2)$, $0 \leqq i_1 < i_2 \leqq L(w)$, and $|w(i) - w(i+1)| = 1$, and $w(i) w(i+1) \subset F$, $0 \leqq i \leqq L(w) - 1$. We call $L(w)$ the length of the path w.

In this paper, we study the special subsets $W^{(n)}$, $n \geqq 1$, in W_0, the set of self avoiding paths from O to a_n that do not pass through b_n, i.e.,

$$W^{(n)} = \{w \in W_0;\ w(0) = O,\ w(L(w)) = a_n,\ w(i) \neq b_n,\ i \geqq 0\}.$$

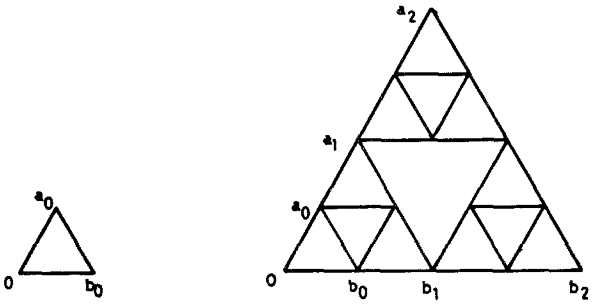

Fig. 1

We define probability measures $\mu_n(\beta)$, $\beta \in R$, $n \geqq 1$, in $W^{(n)}$ by

$$\mu_n(\beta)(A) = Z_n(\beta)^{-1} \sum_{w \in A} e^{-L(w)\beta}, \qquad A \subset W^{(n)},$$

where $Z_n(\beta) = \sum_{w \in W^{(n)}} e^{-L(w)\beta}$.

Our results are the following.

(0.1) **Theorem.** *There is a* $\beta_c > 0$ *such that*

(*i*) *if* $\beta < \beta_c$, *then* $f_0(\beta) = \lim_{n \to \infty} 3^{-n} \log Z_n(\beta)$ *exists and is positive,*

(*ii*) *if* $\beta > \beta_c$, *then* $f_1(\beta) = \lim_{n \to \infty} 2^{-n} \log Z_n(\beta)$ *exists and is negative,*

and

(*iii*) $Z_n(\beta_c) \to \dfrac{\sqrt{5} - 1}{2}$ *as* $n \to \infty$.

In terms of statistical mechanics, we may call f_0 and f_1 the free energy, and β_c the critical point. We can interpret our result in the manner that there are effectively 3^{-n} (times a constant) degrees of freedom for $\beta < \beta_c$ and 2^{-n} (times a constant) degrees of freedom for $\beta > \beta_c$. Such a kind of phase transition does not seem to be conventional in standard statistical mechanics.

(0.2) **Theorem.** (*i*) $f_0 : (-\infty, \beta_c) \to (0, \infty)$ *is continuous and strictly decreasing. Moreover,* $f_0(\beta) \sim (\beta_c - \beta)^a$ *as* $\beta \uparrow \beta_c$. *Here*

$$a = \left(\log \frac{7 - \sqrt{5}}{2} \right)^{-1} \log 3 \simeq 1.26579 .$$

(*ii*) $f_1 : (\beta_c, \infty) \to (-\infty, 0)$ *is continuous and strictly decreasing. Moreover,* $f_1(\beta) \sim -(\beta - \beta_c)^b$ *as* $\beta \downarrow \beta_c$. *Here*

$$b = \left(\log \frac{7 - \sqrt{5}}{2} \right)^{-1} \log 2 \simeq 0.79862 .$$

(0.3) **Theorem.** *(i) If $\beta < \beta_c$, then the probability law of $3^{-n}L(w)$ under $\mu_n(\beta)(dw)$ converges in law to $\delta_{l(\beta)}$ for some $l(\beta) > 0$.*
(ii) If $\beta > \beta_c$, then the probability law of $2^{-n}L(w)$ under $\mu_n(\beta)(dw)$ converges in law to $\delta_{l'(\beta)}$ for some $l'(\beta) > 0$.
(iii) The probability law of $\left(\frac{7-\sqrt{5}}{2}\right)^{-n} L(w)$ under $\mu_n(\beta_c)(dw)$ converges in law to a certain probability measure ν in $(0, \infty)$. Moreover, the Laplace transform $g(\xi) = \int_0^\infty \exp(\xi x)\nu(dx)$ is an entire function in ξ and satisfies

$$g\left(\left(\frac{7-\sqrt{5}}{2}\right)\xi\right) = \frac{3-\sqrt{5}}{2}\cdot g(\xi)^3 + \frac{\sqrt{5}-1}{2}\cdot g(\xi)^2, \qquad \xi \in \mathbb{C}, \quad \text{and} \quad g'(0) > 0\,.$$

The value of the critical point β_c can be evaluated numerically and we have $\beta_c = 0.827691\ldots$ ($\exp(-\beta_c) = 0.437057\ldots$).

We remark that the value of critical exponent b in Theorem (0.2)(ii) has been known to R. Rammal, G. Toulouse, J. Vannimenus [8] and D.J. Klein, W.A. Seiz [6] as the value of an exponent for end-to-end separation of self-avoiding paths. The exponent a in Theorem (0.2)(i) is not considered in their works. We also remark that the recursion relation in Proposition (1.1) below is also given in these works, and in D. Ben-Avraham, S. Havlin [2].

1. Recursion Formula

Let T_0 denote the set of subgraphs in F which are the translation of F_0. For $w \in W_0$, let

$S_1(w) = \{i \in \{0, \ldots, L(w)-1\}$; the triangle in T_0 containing $X(i)$ and $X(i+1)$ does not contain $X(i-1)$ if $i \geqq 1$, and does not contain $X(i+2)$ if $i \leqq L(w)-2\}$

and

$S_2(w) = \{i \in \{0, \ldots, L(w)-2\}$; a triangle in T_0 contains $X(i)$, $X(i+1)$, and $X(i+2)\}$.

Let $s_1(w)$ and $s_2(w)$ be the numbers of the elements of $S_1(w)$ and $S_2(w)$, respectively. Then we have $s_1(w) + 2s_2(w) = L(w)$.

For a subset W of W_0, let $\Phi(W)$ be the generating function for W defined by

$$\Phi(W)(x, y) = \sum_{w \in W} x^{s_1(w)} y^{s_2(w)}, \quad \text{for } (x, y) \in [0, \infty)^2\,.$$

Let $\tilde{W}^{(n)} = \{w \in W_0;\ w(0) = O,\ w(L(w)) = a_n,\ w(i) = b_n$ for some

$$i \leqq L(w)\,,\ w(i) \in F_n \quad \text{for all} \quad i = 0, \ldots, L(w)\}\,,$$

and let $\Phi_n(x, y) = \Phi(W^{(n)})(x, y)$ and $\Theta_n(x, y) = \Phi(\tilde{W}^{(n)})(x, y)$. Then we have the following.

(1.1) **Proposition.** (1) $\Phi_1(x, y) = (x+y)^2 + x^2(x+2y)$ *and* $\Theta_1(x, y) = xy(x+2y)$,

(2) *For* $n \geqq 1$,

$$\Phi_{n+1}(x, y) = \Phi_1(\Phi_n(x, y), \Theta_n(x, y)) , \tag{1.2}$$

and

$$\Theta_{n+1}(x, y) = \Theta_1(\Phi_n(x, y), \Theta_n(x, y)) . \tag{1.3}$$

Proof. The assertion (1) is a simple exercise (see Fig. 2). To prove the assertion (2), let $G_n = 2^n G_0$ and T_n be the set of subgraphs in F which are the translation of F_n, $n \geqq 1$. For each $w \in W_0$ and $n, k \geqq 0$, let

$$T_0^{(n)}(w) = \min\{i \geqq 0 \; ; \; w(i) \in G_n\} ,$$

and

$$T_k^{(n)}(w) = \min\{i \geqq T_{k-1}^{(n)} \; ; \; w(i) \in G_n\} \wedge L(w) .$$

For $w \in W^{(n+1)} \cup \tilde{W}^{(n+1)}$, let $\pi^{(n)}(w)(k) = 2^{-n} w(T_k^{(n)})$, $k \geqq 0$. Then it is easy to see that $\pi^{(n)}(w) \in W^{(1)}$, if $w \in W^{(n+1)}$, and $\pi^{(n)}(w) \in \tilde{W}^{(1)}$ if $w \in \tilde{W}^{(n+1)}$.

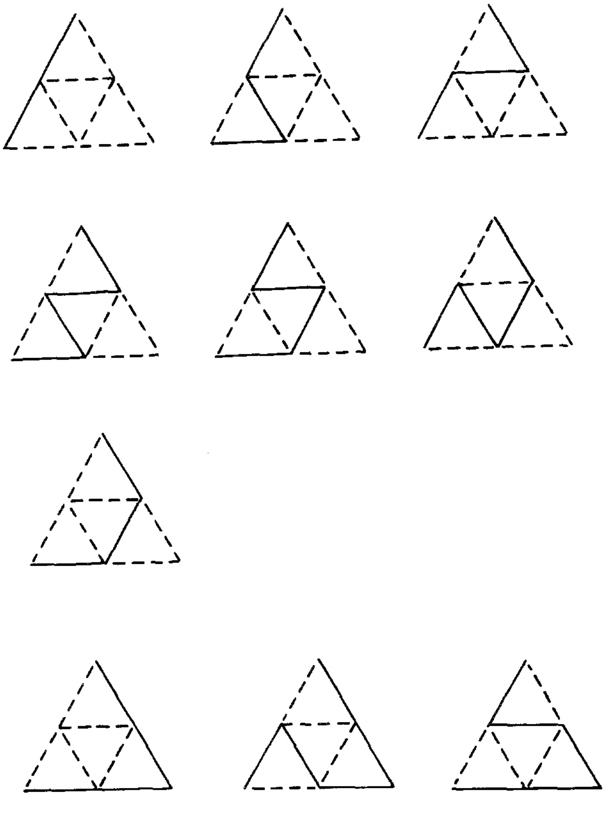

Fig. 2

Let $w_0 \in W^{(1)}$. Suppose that $\pi^{(n)}w = w_0$, $w \in w^{(n+1)}$. Then we see that $w(i)$'s, $T_k^{(n)}(w) \leqq i \leqq T_{k+1}^{(n)}(w)$ are in a triangle of T_n if $k \in S_1(w_0)$, and that $w(i)$'s, $T_k^{(n)}(w) \leqq i \leqq T_{k+2}^{(n)}(w)$, are in a triangle of T_n if $k \in S_2(w_0)$. Moreover, these triangles are different from each other. So to count the number of $w \in W^{(n+1)}$ with $\pi^{(n)}w = w_0$, we can think of the subgraphs inside these triangles independently. Therefore the self-similarity of the Sierpinski Gasket implies that

$$\Phi(\{w \in W^{(n+1)}; \pi^{(n)}w = w_0\})(x, y) = \Phi_n(x, y)^{s_1(w_0)}\Theta_n(x, y)^{s_2(w_0)} .$$

This shows that $\Phi_{n+1}(x, y) = \Phi_1(\Phi_n(x, y), \Theta_n(x, y))$. Similarly, we can show that $\Theta_{n+1}(x, y) = \Theta_1(\Phi_n(x, y), \Theta_n(x, y))$.

This completes the proof.

2. Basic Results

Let $G^0(x, y) = (x, y)$ and $G(x, y) = G^1(x, y) = (\Phi_1(x, y), \Theta_1(x, y))$, and define $G^n(x, y)$ by $G^n(x, y) = G^{n-1}(\Phi_1(x, y), \Theta_1(x, y))$. Then $G^n(x, y) = (\Phi_n(x, y), \Theta_n(x, y))$, $n \geqq 1$. Note that $G^n(x, y)$ are polynomials in x and y. Let $R_0(x, y) = y/x$ and

$$R_n(x, y) = \Theta_n(x, y)/\Phi_n(x, y) , \quad n \geqq 1 , \quad x, y > 0 .$$

Then it is easy to see the following.

$$\Phi_{n+1}(x, y) = \Phi_n(x, y)^3\{(1 + 2R_n(x, y)) + \Phi_n(x, y)^{-1}(1 + R_n(x, y))^2\} , \tag{2.1}$$

$$\Phi_{n+1}(x, y) = \Phi_n(x, y)^2\{(1 + R_n(x, y))^2 + \Phi_n(x, y)(1 + 2R_n(x, y))\} , \tag{2.2}$$

and

$$\begin{aligned} R_{n+1}(x, y) &= R_n(x, y)\left\{1 + \Phi_n(x, y)^{-1}\left(1 + \frac{R_n(x, y)^2}{1 + 2R_n(x, y)}\right)\right\}^{-1} \\ &\leqq (1 + \Phi_n(x, y)^{-1})^{-1} \cdot R_n(x, y) , \\ &\leqq \Phi_n(x, y) \cdot R_n(x, y) , \end{aligned} \tag{2.3}$$

for $x, y > 0$.
Therefore from (2.3), we see that

$$R_{n+1}(x, y) < R_n(x, y) , \quad x, y > 0 . \tag{2.4}$$

Also, we have

$$\log \Phi_{n+1}(x, y) = 3 \log \Phi_n(x, y) + \log\{(1 + 2R_n(x, y)) + \Phi_n(x, y)^{-1}(1 + R_n(x, y))^2\} . \tag{2.5}$$

$$\log \Phi_{n+1}(x, y) = 2 \log \Phi_n(x, y) + \log\{(1 + R_n(x, y))^2 + \Phi_n(x, y)(1 + 2R_n(x, y))\} . \tag{2.6}$$

(2.7) **Lemma.** *For any* $a, b > 0$, *let* $g(x) = g(x; a, b) = ax^3 + bx^2$, $x > 0$. *Also, let* g^n *be the* n*-fold composition of* g. *Then* $g^n(x) \downarrow 0$ *as* $n \to \infty$

for any $x \in \left(0, \frac{(4a+b^2)^{1/2}-b}{2a}\right)$ *and* $g^n(x) \uparrow \infty$ *as* $n \to \infty$ *for any* $x \in \left(\frac{(4a+b^2)^{1/2}-b}{2a}, \infty\right)$.

Proof. Suppose that $x \in \left(0, \frac{(4a+b^2)^{1/2}-b}{2a}\right)$. Then it is easy to see that $0 < g(x) < x$. Since $g\colon (0, \infty) \to (0, \infty)$ is increasing, we see that $g^n(x) \downarrow$ as $n \uparrow$. Let $y = \lim_{n \to \infty} g^n(x)$. Then we see that $0 \leqq y < x$ and $g(y) = y$. This implies that $y = 0$. The case $x \in \left(\frac{(4a+b^2)^{1/2}-b}{2a}, \infty\right)$ is similar. Q.E.D.

(2.8) **Proposition.** *Let* $x, y > 0$.

(1) If $\sup_n \Phi_n(x, y) > \frac{\sqrt{5}-1}{2}$, *then* $\lim_{n\to\infty} 3^{-n} \log \Phi_n(x, y) > 0$.

(2) *Suppose that* $\sup_n \Phi_n(x, y) \leqq \frac{\sqrt{5}-1}{2}$. *Then* $\lim_{n\to\infty} \Phi_n(x, y) = \frac{\sqrt{5}-1}{2}$ *or* $\overline{\lim_{n\to\infty}}\, 2^{-n} \log \Phi_n(x, y) < 0$. *Moreover,* $\lim_{n\to\infty} R_n(x, y) = 0$.

Proof. (1) Suppose that $\sup_n \Phi_n(x, y) > \frac{\sqrt{5}-1}{2}$. Since $\Phi_{n+1}(x, y) \geqq \Phi_n(x, y)^3 + \Phi_n(x, y)^2 = g(\Phi_n(x, y); 1, 1)$ by (2.1), we see from Lemma (2.7) that $\lim_{n\to\infty} \Phi_n(x, y) = \infty$. By (2.5), we have $\log \Phi_{n+m}(x, y) \geqq 3^m \log \Phi_n(x, y)$. These imply that $\lim_{n\to\infty} 3^{-n} \log \Phi_n(x, y) > 0$.

(2) Suppose that $\sup_n \Phi_n(x, y) \leqq \frac{\sqrt{5}-1}{2}$. Then by (2.3) we have $R_{n+1}(x, y) \leqq R_n(x, y)(1 + \Phi_n(x, y)^{-1})^{-1} \leqq \left(\frac{\sqrt{5}+3}{2}\right)^{-1} R_n(x, y)$. This shows that

$$R_n(x, y) \leqq \left(\frac{\sqrt{5}+3}{2}\right)^{-n} \left(\frac{y}{x}\right). \tag{2.9}$$

Therefore $\lim_{n\to\infty} R_n(x, y) = 0$. Note that

$$\Phi_{n+1}(x, y) = g(\Phi_n(x, y); 1 + 2R_n(x, y), (1 + R_n(x, y))^2).$$

Thus by Lemma (2.7), if $R_n(x, y) < \varepsilon$ for any $n \geqq N$ and

$$\Phi_N(x, y) < \frac{\{4(1+2\varepsilon) + (1+\varepsilon)^4\}^{1/2} - (1+\varepsilon)^2}{2(1+2\varepsilon)}, \quad \text{then} \quad \lim_{n\to\infty} \Phi_n(x, y) = 0.$$

Therefore if $\underline{\lim}_{n\to\infty} \Phi_n(x, y) < \frac{\sqrt{5}-1}{2}$, then $\lim_{n\to\infty} \Phi_n(x, y) = 0$. So we see that $\lim_{n\to\infty} \Phi_n(x, y) = 0$ or $\frac{\sqrt{5}-1}{2}$.

Now let $\lim_{n\to\infty} \Phi_n(x, y) = 0$. By (2.6) we have

$$\overline{\lim_{n\to\infty}} \left\{ \log \Phi_{n+1}(x, y) - \frac{3}{2} \log \Phi_n(x, y) \right\} = -\infty .$$

Therefore we see that

$$\left(\frac{3}{2}\right)^{-n} \log \Phi_{n+N}(x, y) \leqq \log \Phi_N(x, y)$$

for sufficiently large N. This implies that

$$\overline{\lim_{n\to\infty}} \left(\frac{3}{2}\right)^{-n} \log \Phi_N(x, y) < 0.$$

Then, again by (2.6), we see that

$$\overline{\lim_{n\to\infty}} \left\{ \log \Phi_{n+1}(x, y) - \left(2 - \left(\frac{2}{3}\right)^n\right) \cdot \log \Phi_n(x, y) \right\} < 0.$$

Therefore we see that

$$\left\{ \prod_{k=1}^{n} \left(1 - \frac{1}{2}\left(\frac{2}{3}\right)^{k+N-1}\right) \right\}^{-1} \cdot 2^{-n} \log \Phi_{n+N}(x, y) \leqq \log \Phi_N(x, y)$$

for sufficiently large N. This implies that

$$\overline{\lim_{n\to\infty}} 2^{-n} \log \Phi_n(x, y) < 0. \qquad \text{Q.E.D.}$$

Let $D = \left\{ (x, y) \in (0, \infty)^2; \sup_n \Phi_n(x, y) \leqq \frac{\sqrt{5}-1}{2} \right\}$. Then we have the following.

(2.10) **Proposition.** (1) *D is a closed set in $(0, \infty)^2$.*

(2) *If $(x, y) \in D$, $0 < x' \leqq x$ and $0 < y' \leqq y$, then $(x', y') \in D$.*

(3) *If $(x, y) \in D$, $0 < x' < x$ and $0 < y' < y$, then $\lim_{n\to\infty} \Phi_n(x', y') = 0$.*

In particular, if $(x, y) \in D \backslash \partial D$, then $\lim_{n\to\infty} \Phi_n(x, y) = 0$.

(4) *If $(x, y) \in \partial D \cap (0, \infty)^2$, $x' > x$ and $y' > y$, then $(x', y') \in (0, \infty)^2 \backslash D$.*

(5) *If $(x, y) \in \partial D \cap (0, \infty)^2$, then $\lim_{n\to\infty} \Phi_n(x, y) = \frac{\sqrt{5}-1}{2}$. Moreover,*

$$\Phi_n(x, y) \geqq \frac{\{4(1 + 2R_n(x, y)) + (1 + R_n(x, y))^4\}^{1/2} - (1 + R_n(x, y))^2}{2(1 + 2R_n(x, y))} . \tag{2.11}$$

Proof. (1) and (2) is obvious, since $\Phi_n(x, y)$ is continuous and increasing in x and y.

(3) Let $\lambda = \max\{x'/x, y'/y\} < 1$. Then it is easy to see by induction that $\Phi_n(x', y') \leqq \lambda^{2^n}\Phi_n(x, y) \leqq \lambda^{2^n} \cdot \frac{\sqrt{5}-1}{2}$. Thus we have our assertion.

(4) If $(x', y') \in D$, then by (2) we see that $(x, y) \in (0, x') \times (0, y') \subset D$. This contradicts the assumption that $(x, y) \in \partial D$.

(5) Let $(x_t, y_t) = (tx, ty)$, $t > 0$. Then by (4), we see that $(x_t, y_t) \in (0, \infty)^2 \backslash D$ if $t > 1$.

Let $a_n(t) = 1 + 2R_n(x_t, y_t)$ and $b_n(t) = (1 + R_n(x_t, y_t))^2$. Then $a_n(t)$ and $b_n(t)$ are continuous in t. By (2.1) and (2.4), we see that $\Phi_{n+1}(x_t, y_t) \leqq g(\Phi_n(x_t, y_t); a_m(t), b_m(t))$ for $n \geqq m$ and $t \in (0, \infty)$.

Suppose that $\Phi_m(x, y) < \frac{(4a_m(1) + b_m(1)^2)^{1/2} - b_m(1)}{2a_m(1)}$. Then there is a $t > 1$ such that $\Phi_m(x_t, y_t) < \frac{(4a_m(t) + b_m(t)^2)^{1/2} - b_m(t)}{2a_m(t)}$. Then by Lemma (2.7), we have

$$\overline{\lim_{n\to\infty}}\, \Phi_n(x_t, y_t) \leqq \lim_{n\to\infty} g^n(\Phi_m(x_t, y_t); a_m(t), b_m(t)) = 0 .$$

But this contradicts Proposition (2.8) and the fact that $(x_t, y_t) \in (0, \infty)^2 \backslash D$. This proves (2.11). Since $R_n(x, y) \to 0$ as $n \to \infty$, by Proposition (2.8)(2), (2.11) implies that

$$\lim_{n\to\infty} \Phi_n(x, y) = \frac{\sqrt{5}-1}{2} .$$ Q.E.D.

(2.12) **Proposition.** (1) $R_n(x, y) \leqq \left(\frac{\sqrt{5}+3}{2}\right)^{-n}\left(\frac{y}{x}\right)$ *if* $(x, y) \in D$.

(2) $\overline{\lim_{n\to\infty}}\, 2^{-n} \log R_n(x, y) < 0$ *if* $(x, y) \in D \backslash \partial D$.

(3) $R_n(x, y)^{-1} = \left(\frac{y}{x}\right)^{-1} \cdot \prod_{k=0}^{n-1}\left\{1 + \Phi_k(x, y)^{-1}\left(1 + \frac{R_k(x, y)^2}{1 + 2R_k(x, y)}\right)\right\}$

for $(x, y) \in (0, \infty)^2$.

(4) $R(x, y) = \lim_{n\to\infty} R_n(x, y)$ *exists and is continuous in* $(x, y) \in (0, \infty)^2$. *Moreover,* $R(x, y) > 0$ *if* $(x, y) \in (0, \infty)^2 \backslash D$, *and* $R(x, y) = 0$ *if* $(x, y) \in D$.

(5) $0 \leqq \log \frac{R_n(x, y)}{R(x, y)} \leqq \left(1 + \frac{R_0(x, y)^2}{1 + 2R_0(x, y)}\right) \cdot \sum_{k=n}^{\infty} \Phi_n(x, y)^{-1}$

for any $(x, y) \in (0, \infty)^2 \backslash D$ *and* $n \geqq 0$.

Proof. (1) is already proved in (2.9). (2) follows from Propositions (2.8) and (2.10) and the fact that $R_n(x, y) \leqq \left(\frac{y}{x}\right) \cdot \prod_{k=0}^{n-1} \Phi_k(x, y)$, $(x, y) \in D$. (3) is obvious from (2.3).

Now let us prove (4). (1) implies that $\lim_{n\to\infty} R_n(x, y) = 0$ if $(x, y) \in D$. So $R(x, y) = 0$ and is continuous in D. Let $(x, y) \in (0, \infty)^2 \backslash D$. Then there is a $(x', y') \in (0, \infty)^2 \backslash D$ with $x' < x$ and $y' < y$. It is easy to see that

$$\varliminf_{n \to \infty} 3^{-n} \log \inf\{\Phi_n(x'', y''); (x'', y'') \in [x', \infty) \times [y', \infty)\}$$

$$= \varliminf_{n \to \infty} 3^{-n} \log \Phi_n(x', y') > 0 .$$

Thus by (3), we see that $R_n(x, y)^{-1}$ converges uniformly on compacts in $[x', \infty) \times [y', \infty)$. Therefore we see that $R(x, y) > 0$ and is continuous in $(0, \infty)^2 \backslash D$.

The final point which we have to show is the continuity of R at $(x, y) \in \partial D \cap (0, \infty)^2$. However, by (2.4) we see that if $(x_n, y_n) \to (x, y)$ as $n \to \infty$, then

$$0 \leqq \varlimsup_{n \to \infty} R(x_n, y_n) \leqq \varlimsup_{n \to \infty} R_m(x_n, y_n) = R_m(x, y) \to 0 \text{ as } m \to \infty .$$

Thus R is continuous at (x, y).

(5) is obvious from (3) and (4). Q.E.D.

(2.13) **Proposition.** *If* $(x, y) \in (0, \infty)^2 \backslash D$, *then* $\lim_{n \to \infty} 3^{-n} \log \Phi_n(x, y)$ *exists. If* $(x, y) \in D \backslash \partial D$, *then* $\lim_{n \to \infty} 2^{-n} \log \Phi_n(x, y)$ *exists.*

Proof. From (2.5) and (2.6), we have

$$3^{-n} \log \Phi_n(x, y) = \log x + \sum_{k=0}^{n-1} 3^{-k-1} \log\{(1 + 2R_k(x, y)) + \Phi_k(x, y)^{-1}(1 + R_k(x, y))^2\} , \tag{2.14}$$

and

$$2^{-n} \log \Phi_n(x, y) = \log x + \sum_{k=0}^{n-1} 2^{-k-1} \log\{(1 + R_k(x, y))^2 + \Phi_k(x, y)(1 + 2R_k(x, y))\} . \tag{2.15}$$

By (2.4) and Proposition (2.8) we see that $\lim_{n \to \infty} 3^{-n} \log \Phi_n(x, y)$ exists, and also by Proposition (2.8) we see that $\lim_{n \to \infty} 2^{-n} \log \Phi_n(x, y)$ exists.

Let us summarize the results in this section.

(2.16) **Theorem.** (I) $R(x, y) = \lim_{n \to \infty} R_n(x, y)$ *exists for* $(x, y) \in (0, \infty)^2$ *and is continuous in* (x, y).

(II) *Let* $D = \left\{(x, y) \in (0, \infty)^2 ; \sup_n \Phi_n(x, y) \leqq \frac{\sqrt{5} - 1}{2}\right\}$. Then D is a closed set in $(0, \infty)^2$. Moreover,

(1) *If* $(x, y) \in (0, \infty)^2 \backslash D$, *then* $R(x, y) > 0$, *and* $\lim_{n\to\infty} 3^{-n} \log \Phi_n(x, y)$ *exists and is positive.*

(2) *If* $(x, y) \in \partial D \cap (0, \infty)^2$, *then* $R(x, y) = 0$ *and* $\lim_{n\to\infty} \Phi_n(x, y) = \dfrac{\sqrt{5}-1}{2}$.

(3) *If* $(x, y) \in D \backslash \partial D$, *then* $R(x, y) = 0$, *and* $\lim_{n\to\infty} 2^{-n} \log \Phi_n(x, y)$ *exists and is negative.*

3. Preliminary Results

(3.1) **Lemma.** *Let* A *and* A_n, $n = 1, 2, \ldots$, *be* 2×2 *matrices. We assume that there is an invertible* 2×2 *matrix* P *and* $PAP^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, $\lambda_1 > \lambda_2 > 0$. *We assume moreover that there is a* $\varphi : N \to (0, \infty)$ *such that* $\|A_n - A\| \leqq \varphi(n)$, $n = 1, 2, \ldots$ *and* $\sum_{n=1}^{\infty} \varphi(n) < \infty$. *Then we have*

$$
\begin{aligned}
&\|\lambda_1^{-(n+m)}(A_{n+m}A_{n+m-1} \cdots A_1) - \lambda_1^{-m} \cdot QA_mA_{m-1} \cdots A_1\| \\
&\leqq \|P\|^3 \|P^{-1}\|^3 \left\{ \left(\frac{\lambda_2}{\lambda_1}\right)^n + \lambda_1^{-1} \left(\sum_{k=m+1}^{\infty} \varphi(k) \right) \cdot \exp\left(\lambda_1^{-1} \|P\| \|P^{-1}\| \sum_{k=1}^{\infty} \varphi(k) \right) \right\}
\end{aligned} \tag{3.2}
$$

for any $n, m \geqq 0$. *Here* $Q = P^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} P$. *In particular,* $\lim_{n\to\infty} \lambda_1^{-n} \cdot A_n \ldots A_1$ *exists. Moreover, if the elements of* A_n *are positive for any* $n \geqq 1$, *and if* $(1, 0)Q \neq 0$, *then*

$$
\lim_{n\to\infty} \lambda_1^{-n}(1, 0)A_n \ldots A_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} > 0 \quad \text{and} \quad \lim_{n\to\infty} \lambda_1^{-n}(1, 0)A_n \ldots A_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} > 0 .
$$

Proof. First observe that

$$
\begin{aligned}
\lambda_1^{-n} \cdot \|A_n \ldots A_1\| &\leqq \|P\| \|P^{-1}\| \prod_{k=1}^{n} \{\lambda_1^{-1}(\|PAP^{-1}\| + \|P(A_k - A)P^{-1}\|)\} \\
&\leqq \|P\| \|P^{-1}\| \cdot \exp\left(\lambda_1^{-1} \|P\| \|P^{-1}\| \sum_{k=1}^{n} \varphi(k) \right)
\end{aligned} \tag{3.3}
$$

for any $n \geqq 1$. Also, observe that

$$
\begin{aligned}
&\lambda_1^{-n} \cdot \|(A_{m+n} \ldots A_{m+1}) - A^n\| \\
&\leqq \lambda_1^{-n} \sum_{k=1}^{n} \|P^{-1}(PAP^{-1})^{n-k}P(A_{k+m} - A)P^{-1}(PA_{k+m-1}P^{-1}) \ldots (PA_{m+1}P^{-1})P\| \\
&\leqq \lambda_1^{-1} \|P\|^2 \|P^{-1}\|^2 \sum_{k=1}^{n} \varphi(k+m) \cdot \exp\left(\lambda_1^{-1} \|P\| \|P^{-1}\| \sum_{l=1}^{k} \varphi(l+m) \right)
\end{aligned}
$$

for any $n, m \geqq 0$. This implies that

$$\begin{aligned}&\|\lambda_1^{-n}\cdot(A_{n+m}\ldots A_{m+1})-Q\| \\ &\leqq \lambda_1^{-1}\|P\|^2\|P^{-1}\|^2\left(\sum_{k=m+1}^{\infty}\varphi(k)\right)\cdot\exp\left(\lambda_1^{-1}\|P\|\|P^{-1}\|\sum_{k=m+1}^{\infty}\varphi(k)\right) \\ &\quad+\|P\|\|P^{-1}\|\left(\frac{\lambda_2}{\lambda_1}\right)^n.\end{aligned} \tag{3.4}$$

Our assertion (3.2) is an easy consequence of (3.3) and (3.4). By using (3.2), we see easily that $\lim_{m\to\infty}\overline{\lim_{n\to\infty}}\|\lambda_1^{-n}(A_n\ldots A_1)-\lambda_1^{-m}Q(A_m\ldots A_1)\|=0$. This proves the existence of $\lim_{n\to\infty}\lambda_1^{-n}\cdot A_n\ldots A_1$.

Now assume that the elementes of A_n's are positive. By (3.4), we see that $\lim_{m\to\infty}\lim_{n\to\infty}\|\lambda_1^{-n}(A_{n+m}\ldots A_{m+1})-Q\|=0$. Therefore, if $(1,0)Q\neq 0$, then $\lim_{n\to\infty}\lambda_1^{-n}\cdot(1,0)(A_{n+m}\ldots A_{m+1})\neq 0$ for sufficiently large m. Therefore we see that $\lim_{n\to\infty}\lambda_1^{-n}\cdot(1,0)(A_{n+m}\ldots A_{m+1})\begin{pmatrix}1\\0\end{pmatrix}>0$ or $\lim_{n\to\infty}\lambda_1^{-n}\cdot(1,0)(A_{n+m}\ldots A_{m+1})\begin{pmatrix}0\\1\end{pmatrix}>0$. Since the elements of $A_m\ldots A_1\begin{pmatrix}1\\0\end{pmatrix}$ and $A_m\ldots A_1\begin{pmatrix}0\\1\end{pmatrix}$ are positive, we obtain the last assertion. Q.E.D.

(3.5) **Proposition.** (1) *If* $(x, y)\in(0,\infty)^2\backslash D$, *then* $\lim_{n\to\infty}3^{-n}\cdot\frac{\partial}{\partial x}(\log\Phi_n(x,y))$ *and* $\lim_{n\to\infty}3^{-n}\cdot\frac{\partial}{\partial y}(\log\Phi_n(x,y))$ *exist and are positive.*

(2) *If* $(x, y)\in\partial D\cap(0,\infty)^2$, *then* $\lim_{n\to\infty}\left(\frac{7-\sqrt{5}}{2}\right)^{-n}\cdot\frac{\partial}{\partial x}(\log\Phi_n(x,y))$ *and* $\lim_{n\to\infty}\left(\frac{7-\sqrt{5}}{2}\right)^{-n}\cdot\frac{\partial}{\partial y}(\log\Phi_n(x,y))$ *exist and are positive.*

(3) *If* $(x, y)\in D\backslash\partial D$, *then* $\lim_{n\to\infty}2^{-n}\cdot\frac{\partial}{\partial x}(\log\Phi_n(x,y))$ *and* $\lim_{n\to\infty}2^{-n}\cdot\frac{\partial}{\partial x}(\log\Phi(x,y))$ exist and are positive.

Proof. Let $B(x, y)=\left(\frac{\partial}{\partial x}{}^tG(x,y), \frac{\partial}{\partial y}{}^tG(x,y)\right)$, $x, y\in[0,\infty)^2$, where ${}^tG$ denotes the transposition of G: ${}^tG(x,y)=\begin{pmatrix}\Phi(x,y)\\ \Theta(x,y)\end{pmatrix}$. Then we have

$$B(x,y)=\begin{pmatrix}2(x+y)+3x^2+4xy & 2(x+y)+2x^2\\ 2xy+2y^2 & x^2+4xy\end{pmatrix}. \tag{3.6}$$

Note that

$$\left(\frac{\partial}{\partial x}{}^tG^n(x,\ y),\ \frac{\partial}{\partial y}{}^tG^n(x, y)\right) = B(G^{n-1}(x,\ y)) \ldots B(G(x,\ y))B(x,\ y)\,. \quad (3.7)$$

(1) Now we assume that $(x, y) \in (0, \infty)^2 \backslash D$. Let

$$A_n = \begin{pmatrix} 3 + 4R_n(x, y) + 2\Phi_n(x, y)^{-1}(1 + R_n(x, y)) & 2 + 2\Phi_n(x, y)^{-1}(1 + R_n(x, y)) \\ 2R_n(x, y)(1 + R_n(x, y)) & 1 + 4R_n(x, y) \end{pmatrix}$$

and

$$A = \begin{pmatrix} 3 + 4R(x, y) & 2 \\ 2R(x,\ y)(1 + R(x,\ y)) & 1 + 4R(x,\ y) \end{pmatrix}.$$

Then we have $B(G^n(x,\ y)) = \Phi_n(x,\ y)^2 A_n$. By Proposition (2.12)(5) and Theorem (2.16)(II)(1), we see that $\sum_{n=1}^{\infty} \Phi_n(x, y)^{-1} < \infty$ and $\sum_{n=1}^{\infty} |R_n(x,\ y) - R(x,\ y)| < \infty$. Therefore we have

$$\sum_{n=0}^{\infty} \|A - A_n\| < \infty\ . \quad (3.8)$$

The eigenvalues of A are $1 + 2R(x, y)$ and $3(1 + 2R(x,\ y))$. Moreover, $Q = \dfrac{1}{1 + 2R(x,\ y)}\begin{pmatrix} 1 + R(x,\ y) & 1 \\ R(x,\ y)(1 + R(x,\ y)) & R(x,\ y) \end{pmatrix}$ for this matrix A. Thus $(1, 0)\,Q \neq 0$. Therefore by Lemma (3.1), we see that

$\lim_{n\to\infty} (3(1 + 2R(x,\ y))^{-n} A_{n-1} \ldots A_0$ exists and

$\lim_{n\to\infty} (3(1 + 2R(x,\ y))^{-n}(1, 0) A_{n-1} \ldots A_1\begin{pmatrix}1\\0\end{pmatrix} > 0$ and

$\lim_{n\to\infty} (3(1 + 2R(x, y))^{-n}(1, 0) A_{n-1} \ldots A_1\begin{pmatrix}0\\1\end{pmatrix} > 0.$

By (2.1), we have

$$\Phi_n(x, y) = x \cdot \prod_{k=0}^{n-1} \Phi_k(x, y)^2 \{(1 + 2R_k(x,\ y)) + \Phi_k(x,\ y)^{-1}(1 + R_k(x,\ y))^2\}\,. \quad (3.9)$$

Therefore again by Theorem (2.16), we see that

$$\lim_{n\to\infty} \Phi_n(x, y)^{-1}((1 + 2R(x, y))^n \prod_{k=0}^{n-1} \Phi_k(x,\ y)^2) \text{ exists and is positive.}$$

Noting that

$$3^{-n} \cdot \frac{\partial}{\partial x} \log \Phi_n(x, y) = \Phi_n(x,\ y)^{-1}\left\{(1 + 2R(x,\ y))^n \prod_{k=0}^{n-1} \Phi_k(x, y)^2\right\}$$

$$\times\ (3(1 + 2R(x,\ y)))^{-n}(1, 0) A_{n-1} \ldots A_0\begin{pmatrix}1\\0\end{pmatrix},$$

and

$$3^{-n}\cdot\frac{\partial}{\partial y}\log\Phi_n(x,y)=\Phi_n(x,y)^{-1}\left\{(1+2R(x,y))^n\prod_{k=0}^{n-1}\Phi_k(x,y)^2\right\}$$
$$\times(3(1+2R(x,y)))^{-n}(1,0)A_{n-1}\dots A_0\begin{pmatrix}0\\1\end{pmatrix},$$

we have our assertion (1).

Since the proof for the assertions (2) and (3) are similar, we only give a sketch for them.

(2) Let $(x,y)\in\partial D\cap(0,\infty)^2$. Then we see that

$$B(G^n(x,y))=\begin{pmatrix}2\Phi_n(x,y)+3\Phi_n(x,y)^2+R_n(x,y)(2\Phi_n(x,y)+4\Phi_n(x,y)^2) & 2\Phi_n(x,y)(\Phi_n(x,y)+1+R_n(x,y))\\ 2\Phi_n(x,y)^2R_n(x,y)(1+R_n(x,y)) & \Phi_n(x,y)^2(1+4R_n(x,y))\end{pmatrix}. \tag{3.10}$$

Let $A=\begin{pmatrix}(7-\sqrt{5})/2 & 2\\ 0 & (3-\sqrt{5})/2\end{pmatrix}$. Then by Proposition (2.10)(5) and Proposition (2.12)(1), we see that

$$\sum_{n=0}^{\infty}\|B(G^n(x,y))-A\|<\infty\ . \tag{3.11}$$

The eigenvalues of A are obviously $\frac{7-\sqrt{5}}{2}$ and $\frac{3-\sqrt{5}}{2}$, and $Q=\begin{pmatrix}1&1\\0&0\end{pmatrix}$ for this matrix A. Thus we have our assertion (2) similarly.

(3) Let $(x,y)\in D\backslash\partial D$. Let

$$A_n=\begin{pmatrix}2(1+R_n(x,y))+\Phi_n(x,y)(3+4R_n(x,y)) & 2(1+R_n(x,y))+2\Phi_n(x,y)\\ 2\Phi_n(x,y)R_n(x,y)(1+R_n(x,y)) & \Phi_n(x,y)(1+4R_n(x,y))\end{pmatrix},$$

and

$$A=\begin{pmatrix}2&2\\0&0\end{pmatrix}.$$

Then $B(G^n(x,y))=\Phi_n(x,y)A_n$ and by Proposition (2.12)(2) and Theorem (2.16)(II)(3), we see that

$$\sum_{n=0}^{\infty}\|A_n-A\|<\infty\ . \tag{3.12}$$

The eigenvalues of A are 2 and 0, and $Q=\begin{pmatrix}1&1\\0&0\end{pmatrix}$. By (2.2), we have

$$\Phi_n(x,y)=x\cdot\prod_{k=0}^{n-1}\Phi_k(x,y)\{(1+R_k(x,y))^2+\Phi_k(x,y)(1+2R_k(x,y))\}\ , \tag{3.13}$$

and so $\lim_{n\to\infty} \Phi_n(x, y)^{-1}\left(\prod_{k=0}^{n-1} \Phi_k(x, y)\right)$ exists and is positive. Thus we have our assertion (3) similarly. Q.E.D.

4. The Convergence of Laplace Transformations

For any $(x, y) \in (0, \infty)^2$, we define a probability measure $\nu_n^{(x,y)}$ in $\mathbb{N} \times \mathbb{N}$ by

$$\nu_n^{(x,y)}(\{(k, l)\}) = \Phi_n(x, y)^{-1}\left(\frac{1}{k!l!}\left(\frac{\partial}{\partial x}\right)^k\left(\frac{\partial}{\partial y}\right)^l \Phi_n(x, y)|_{x=y=0}\right)x^k y^l, \quad k, l \in N \,. \tag{4.1}$$

Let $g(s, t; \nu_n^{(x,y)})$ be the moment generating function of the probability measure $\nu_n^{(x,y)}$, i.e., $g(s, t; \nu_n^{(x,y)}) = \int_{\mathbb{N}\times\mathbb{N}} \exp(ks + lt)\nu_n^{(x,y)}(dk \otimes dl)$, $s, t \in \mathbb{C}$. Then it is easy to see that

$$g(s, t; \nu_n^{(x,y)}) = \Phi_n(x, y)^{-1}\Phi_n(xe^s, ye^t)\,.$$

(4.2) **Proposition.** (1) *Let* $(x, y) \in (0, \infty)^2 \backslash D$. *Then there are* $a > 0$ *and* $b > 0$ *such that* $g(3^{-n}s, 3^{-n}t; \nu_n^{(x,y)}) \to \exp(as + bt)$ *as* $n \to \infty$ *for any* $s, t \in \mathbb{C}$.

Proof. Let $F_n(x, y) = (\log \Phi_n(x, y), R_n(x, y))$, $x, y \in (0, \infty)^2$. Also let
$H(q, r) = \left(2q + \log((1 + r)^2 + e^q(1 + 2r)),\ \ r\left(1 + e^{-q}\left(1 + \frac{r^2}{1 + 2r}\right)\right)^{-1}\right)$, $q \in \mathbb{R}$ and $r > 0$. Then we have

$$F_{n+1}(x, y) = H(F_n(x, y))\,, \quad n \geqq 0,\ x, y \in (0, \infty)^2\,, \tag{4.3}$$

and

$$F_0(x, y) = \left(\log x, \frac{y}{x}\right). \tag{4.4}$$

Let $B(q, r) = \left(\frac{\partial}{\partial q}{}^tH(q, r), \frac{\partial}{\partial r}{}^tH(q, r)\right)$, $q \in \mathbb{R}$ and $r > 0$, where ${}^tH$ denotes the transposition of H.

First we prove our assertion (1). Take an element (x', y') in $(0, \infty)^2 \backslash D$ with $x' < x$ and $y' < y$. Let $\delta = \min\left\{\log\left(\frac{x}{x'}\right), \log\left(\frac{y}{y'}\right)\right\}$. Note that

$$B(q, r) = \begin{pmatrix} 3 - e^{-q}(1 + r)^2(1 + 2r + e^{-q}(1 + r)^2)^{-1} & 2(1 + e^{-q}(1 + r))(1 + 2r + e^{-q}(1 + r)^2)^{-1} \\ re^{-q}\left(1 + \frac{r^2}{1 + 2r}\right)\left(1 + e^{-q}\left(1 + \frac{r^2}{1 + 2r}\right)\right)^{-2} & \left(1 + e^{-q}\left(1 + \frac{r^2}{1 + 2r}\right)\right)^{-1} - e^{-q}\cdot\frac{2r^2(1 + r)}{(1 + 2r)^2}\left(1 + e^{-q}\left(1 + \frac{r^2}{1 + 2r}\right)\right)^{-2} \end{pmatrix}. \tag{4.5}$$

Let $A(r) = \begin{pmatrix} 3 & 2(1+2r)^{-1} \\ 0 & 1 \end{pmatrix}$. Then,

$$D \equiv A(R(xe^s, ye^t)) - B(F_n(xe^s, ye^t))$$
$$= \begin{pmatrix} \Phi_n^{-1}(1+R_n)^2 S_n - \Phi_n^{-1} R_n (1+R_n)^2 (1+2R_n) S_n^2 & 4(R_n - R)(1+2R)^{-1} S_n + 2\Phi_n^{-1}(1+R_n)(R_n - 2R)(1+2R)^{-1} S_n \\ & \Phi_n^{-1}((1+R_n)^2 S_n + 2R_n^2(1+R_n) S_n^2) \end{pmatrix},$$

where, $\Phi_n = \Phi_n(xe^s, ye^t)$, $R_n = R_n(xe^s, ye^t)$, $R = R(xe^s, ye^t)$, and $S_n = (1 + 2R_n + \Phi_n^{-1}(1+R_n)^2)^{-1}$. Therefore, by Theorem (2.16), we see that there is a constant $C' < \infty$ such that

$$\|D\| \leqq \sum_{i=1}^{2} \sum_{j=1}^{2} |D_{ij}| \leqq C'(\Phi_n(xe^s, ye^t)^{-1} + |R_n(xe^s, ye^t) - R(xe^s, ye^t)|),$$

for any $s, t \in [-\delta, 1]$ and $n \geqq 0$. Since $\Phi_n(x, y)$ is increasing in x and y, we see by Proposition (2.12)(5) that there is a constant $C < \infty$ such that

$$\|A(R(xe^s, ye^t)) - B(F_n(xe^s, ye^t))\| \leqq C \sum_{k=n}^{\infty} \Phi_k(x', y')^{-1} \tag{4.6}$$

for any $s, t \in [-\delta, 1]$ and $n \geqq 0$. Note that eigenvalues of $A(r)$ are 3 and 1, and that

$$\left(\frac{\partial}{\partial s} {}^tF_{n+1}(xe^s, ye^t), \frac{\partial}{\partial t} {}^tF_{n+1}(xe^s, ye^t)\right)$$
$$= B(F_n(xe^s, ye^t)) \ldots B(F_0(xe^s, ye^t)) \begin{pmatrix} 1 & 0 \\ -\exp(t-s)y/x & \exp(t-s)y/x \end{pmatrix}. \tag{4.7}$$

Therefore by Lemma (3.1), we see that there is a $C' < \infty$ such that

$$\left\| 3^{-(n+m)} \left(\frac{\partial}{\partial s} {}^tF_{n+m}(xe^s, ye^t), \frac{\partial}{\partial t} {}^tF_{n+m}(xe^s, ye^t)\right) - 3^{-m} Q(R(xe^s, ye^t)) \left(\frac{\partial}{\partial s} {}^tF_m(xe^s, ye^t), \frac{\partial}{\partial t} {}^tF_m(xe^s, ye^t)\right) \right\| \leqq C' \sum_{k=m+1}^{\infty} \sum_{l=k}^{\infty} \Phi_l(x', y')^{-1} \tag{4.8}$$

for any $s, t \in [-\delta, 1]$ and $n, m \geqq 0$. Here, $Q(r) = \begin{pmatrix} 1 & (1+2r)^{-1} \\ 0 & 0 \end{pmatrix}$. From (4.8), we can conclude that

$$\sup_n \sup\{|\log \Phi_n(xe^s, ye^t) - \log \Phi_n(x, y)|;\ s, t \in [-c \cdot 3^{-n}, c \cdot 3^{-n}]\} < \infty \tag{4.9}$$

for any $c > 0$.
Also, noting $(0, 1)Q(r) = 0$, we see that

$$\overline{\lim_{n \to \infty}}\, 3^{-n} \log(\sup\{|R_n(xe^s, ye^t) - R_n(x, y)|\ ;\ s, t \in [-c \cdot 3^{-n}, c \cdot 3^{-n}]\}) < 0\,. \tag{4.10}$$

Therefore we see that

$$\sup_n \sup\{|\Phi_n(x, y)^{-1}\Phi_n(xe^{3^{-n}s}, ye^{3^{-n}t})|\ ;\ s, t\in[-c, c]\} < \infty\ , \tag{4.11}$$

and

$$\sup_n \sup\{|\Phi_n(x, y)^{-1}\Theta_n(xe^{3^{-n}s}\ , \ ye^{3^{-n}t})|\ ;\ s, t\in[-c, c]\} < \infty \tag{4.12}$$

for any $c > 0$.
$\Phi_n(x, y)^{-1}\Phi_n(xe^{3^{-n}s}, ye^{3^{-n}t})$ and $\Phi_n(x, y)^{-1}\Theta_n(xe^{3^{-n}s}, ye^{3^{-n}t})$ are entire functions in $\mathbb{C}^2$ and the coefficients of their Taylor's expansion at (0, 0) are nonnegative. Thus $\{\Phi_n(x, y)^{-1}\Phi_n(xe^{3^{-n}s}, ye^{3^{-n}t})\}_{n=1}^{\infty}$ and $\{\Phi_n(x, y)^{-1}\Theta_n(xe^{3^{-n}s}, ye^{3^{-n}t})\}_{n=1}^{\infty}$ are normal families. Therefore (4.10) implies that $R_n(xe^{3^{-n}s}, ye^{3^{-n}t}) \to R(x, y)$, $n\to\infty$, uniformly on some neighborhood of (0, 0) in $\mathbb{C}^2$.

Let $\log(\Phi_n(x, y)^{-1}\Phi_n(xe^{3^{-n}s}, ye^{3^{-n}t})) = \sum_{k,l=0}^{\infty} a_{k,l}^{(n)}s^k t^l$ and $R_n(xe^{3^{-n}s}, ye^{3^{-n}t}) = \sum_{k,l=0}^{\infty} b_{k,l}^{(n)}s^k t^l$. Then we already saw that $b_{0,0}^{(n)} \to R(x, y)$ and $b_{k,l}^{(n)} \to 0$ unless $(k, l) = (0, 0)$. Also, we see that $a_{0,0}^{(n)} = 0$. By Proposition (3.5), we see that there are $a_{1,0} > 0$ and $a_{0,1} > 0$ such that $a_{1,0}^{(n)} \to a_{1,0}$ and $a_{0,1}^{(n)} \to a_{0,1}$.

By (2.5), we have

$$\begin{aligned}\sum_{k,l=0}^{\infty} a_{k,l}^{(n+1)}s^k t^l &= \sum_{k,l=0}^{\infty} 3^{-(k+l-1)}a_{k,l}^{(n)}s^k t^l + \log\Bigg\{\left(1 + 2\sum_{k,l=0}^{\infty} 3^{-k-l}b_{k,l}^{(n)}s^k t^l\right)\\ &\quad + \Phi_n(x, y)^{-1}\exp\left(-\sum_{k,l=0}^{\infty} 3^{-k-l}a_{k,l}^{(n)}s^k t^l\right)\cdot\left(1 + \sum_{k,l=0}^{\infty} 3^{-k-l}b_{k,l}^{(n)}s^k t^l\right)^2\Bigg\}\\ &\quad - \log(\Phi_{n+1}(x, y)\cdot\Phi_n(x, y)^{-3})\,.\end{aligned} \tag{4.13}$$

Expanding the right hand side of (4.13) by s and t, and comparing the $s^k t^l$ terms of both sides, we find that there are polynomials $P_{k,l,q}(b_{k',l'})$, $k, l \geqq 0$, $k + l \geqq 2$, $q \geqq 1$, in $b_{k',l'}$, $0 \leqq k' \leqq k$, $0 \leqq l' \leqq l$, and $P_{k,l,m,q}(a_{k',l'}, b_{k',l'})$, $k, l \geqq 0$, $m, q \geqq 1$, $k + l \geqq 2$, in $a_{k',l'}$ and $b_{k',l'}$, $0 \leqq k' \leqq k$, $0 \leqq l' \leqq l$ such that

$$\begin{aligned}a_{k,l}^{(n+1)} &= 3^{-(k+l-1)}a_{k,l}^{(n)} + \sum_{q=1}^{k+l}\Bigg\{P_{k,l,q}(b_{k',l'}^{(n)})\\ &\quad + \sum_{m=1}^{k+l}\Phi_n(x, y)^{-m}\cdot P_{k,l,m,q}(a_{k',l'}^{(n)}, b_{k',l'}^{(n)})\Bigg\}\\ &\quad \times\{1 + 2R_n(x, y) + \Phi_n(x, y)^{-1}(1 + R_n(x, y))^2\}^{-q}\,,\end{aligned}$$

for $k, l \geqq 0$ with $k + l \geqq 2$.
Since $\{a_{k,l}^{(n)}\}_{n=1}^{\infty}$ is bounded for any $k, l \geqq 0$, (4.14) implies that $a_{k,l}^{(n)} \to 0$ as $n\to\infty$, for $k + l \geqq 2$. This proves that $\Phi_n(xe^{3^{-n}s}, ye^{3^{-n}t})/\Phi_n(x, y)$ converges to $\exp(a_{1,0}s + a_{0,1}t)$ uniformly on compacts in $(s, t)\in\mathbb{C}^2$. This completes the proof of the assertion (1).

Since the proof of the assertion (2) is similar, we give a sketch of the proof only. Take an element (x', y') in $D\backslash\partial D$ with $x' > x$ and $y' > y$. Let

$\delta = \min\left\{\log\left(\frac{x'}{x}\right), \log\left(\frac{y'}{y}\right)\right\}$. Then we have in this case

$$B(q, r) = \begin{pmatrix} 2 + e^q(1+2r)((1+r)^2 + e^q(1+2r))^{-1} & 2((1+r) + e^q)((1+r)^2 + e^q(1+2r))^{-1} \\ re^q\left(1 + \frac{r^2}{1+2r}\right)\left(e^q + \left(1 + \frac{r^2}{1+2r}\right)\right)^{-2} & e^q\left(e^q + \left(1 + \frac{r^2}{1+2r}\right)\right)^{-1} - e^q \cdot \frac{2r^2(1+r)}{(1+2r)^2}\left(e^q + \left(1 + \frac{r^2}{1+2r}\right)\right)^{-2} \end{pmatrix}. \tag{4.15}$$

Let $A(r) = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}$. Then, since $\Phi_n(x, y)$ is increasing in x and y, by Proposition (2.12)(1) and Theorem (2.16) we see that there is a constant $C < \infty$ such that

$$\|A(R(xe^s, ye^t)) - B(F_n(xe^s, ye^t))\| \leqq C\left\{\Phi_n(x', y') + \left(\frac{\sqrt{5}-1}{2}\right)^n\right\} \tag{4.16}$$

for any $s, t \in [-\delta, \delta]$ and $n \geqq 0$. The eigenvalues of $A(r)$ are 2 and 0. Therefore, similarly to the proof of the assertion (1), we see that

$$\{\Phi_n(x, y)^{-1}\Phi_n(xe^{2^{-n}s}, ye^{2^{-n}t})\}_{n=1}^{\infty} \quad \text{and} \quad \{\Phi_n(x, y)^{-1}\Theta_n(xe^{2^{-n}s}, ye^{2^{-n}t})\}_{n=1}^{\infty}$$

are normal families, and moreover, $\Phi_n(x, y)^{-1}\Theta_n(xe^{2^{-n}s}, ye^{2^{-n}t}) \to 0$, $n \to \infty$, uniformly on compacts in $(s, t) \in \mathbb{C}^2$. By using (2.6), we obtain the assertion (2) similarly. Q.E.D.

(4.17) **Proposition.** *Let* $(x, y) \in \partial D \cap (0, \infty)^2$. *Then there is an entire function* $\varphi: \mathbb{C}^2 \to \mathbb{C}$ *such that*

$$\Phi_n\left(x \cdot \exp\left(\left(\frac{7-\sqrt{5}}{2}\right)^{-n} s\right),\ y \cdot \exp\left(\left(\frac{7-\sqrt{5}}{2}\right)^{-n} t\right)\right) \to \varphi(s, t), \quad n \to \infty, \tag{4.18}$$

uniformly in $\{(s, t) \in \mathbb{C}^2; |s| \leqq R, |t| \leqq R\}$ *for all* $R > 0$, *and*

$$\varphi\left(\left(\frac{7-\sqrt{5}}{2}\right)s, \left(\frac{7-\sqrt{5}}{2}\right)t\right) = \varphi(s, t)^3 + \varphi(s, t)^2 \tag{4.19}$$

for any $(s, t) \in \mathbb{C}^2$.

Moreover, $\frac{\partial}{\partial s}\varphi(0, 0) > 0$ *and* $\frac{\partial}{\partial t}\varphi(0, 0) > 0$.

Proof. We prove this assertion in two steps.

Step 1. Let $\lambda = \frac{7-\sqrt{5}}{2}$. We will show that there is a $\delta > 0$ such that

$$\sup_n \Phi_n((1 + \lambda^{-n}\delta)x, \quad (1 + \lambda^{-n}\delta)y) \leqq 1. \tag{4.20}$$

Let $u=\left(\frac{\sqrt{5}-1}{2},0\right)$. Then $G(u)=u$ and $\left(\frac{\partial}{\partial x}{}^tG(u), \frac{\partial}{\partial y}{}^tG(u)\right) = \begin{pmatrix} (7-\sqrt{5})/2 & 2 \\ 0 & (3-\sqrt{5})/2 \end{pmatrix}$. By (2.11) and Proposition (2.12)(1), we see that there is a constant $C<\infty$ such that

$$|u-G^n(x, y)| \leqq C\left(\frac{\sqrt{5}-1}{2}\right)^n, \quad n \geqq 0 . \tag{4.21}$$

Let $\bar{G}(v)=G(u+v)-u$, $v \in \mathbb{R}^2$, and $v_n=\bar{G}^n((x, y)-u)=G^n(x, y)-u$, $n \geqq 0$. Here $\bar{G}^n$ and G^n are the n-fold composition of $\bar{G}$ and G, respectively, and $G^0(x, y)=(x, y)$, and $\bar{G}^0((x, y)-u)=(x, y)-u$. Then we see by (4.21) that

$$|v_n| \leqq C\left(\frac{\sqrt{5}-1}{2}\right)^n, \quad n \geqq 0 . \tag{4.22}$$

By the mean-value theorem, we have

$$\begin{aligned} |\bar{G}(v_n+z)-v_{n+1}| &= |G(u+v_n+z)-G(u+v_n)| \\ &\leqq \lambda(1+C'(|v_n|+|z|))|z| \end{aligned} \tag{4.23}$$

for any $n \geqq 1$ and $z \in \mathbb{R}^2$ with $|z| \leqq 1$.

Take a positive number $a>0$ such that $a \cdot \prod_{k=0}^{\infty}(1+C'|v_k|) \cdot \prod_{k=0}^{\infty}(1+C'\lambda^{-k}) \leqq 1$. Then by (4.23) and induction, we see that

$$\begin{aligned} |\bar{G}^k(v_0+z)-v_k| &\leqq \lambda^k|z| \cdot \prod_{l=0}^{k-1}(1+C'|v_l|) \cdot \prod_{l=0}^{k-1}(1+C'\lambda^{-(n-l)}) \\ &\leqq \lambda^k|z|/a , \end{aligned} \tag{4.24}$$

for $k \in \mathbb{Z}$ and $z \in \mathbb{R}^2$ with $0 \leqq k \leqq n$ and $|z| \leqq a \cdot \lambda^{-n}$. In fact, we have

$$\begin{aligned} |\bar{G}^{k+1}(v_0+z)-v_{k+1}| &\leqq \lambda \cdot (1+C'|v_k|)(1+C'|\bar{G}^k(v_0+z)-v_k|)|\bar{G}^k(v_0+z)-v_k| \\ &\leqq \lambda \cdot (1+C'|v_k|)(1+C'\lambda^{-(n-k)})|\bar{G}^k(v_0+z)-v_k| . \end{aligned}$$

Since

$$\begin{aligned} G^n((x, y)+z) &= (\Phi_n((x, y)+z), \quad \Theta_n((x, y)+z)) \\ &= u+v_n+(\bar{G}^n(v_0+z)-v_n) , \end{aligned}$$

and $|u|<1$, (4.22) and (4.24) imply our assertion.

Step 2. By (4.20), we see that there is a $\delta>0$ such that

$$\sup_n \Phi_n(xe^{\lambda^{-n\delta}}, ye^{\lambda^{-n\delta}}) \leqq 1 . \tag{4.25}$$

Then we see that there exists a positive constant $C_0<1$ such that $\Phi_k(xe^{\lambda^{-n\delta}}, ye^{\lambda^{-n\delta}})<C_0$, $0 \leqq k \leqq n-1$. Thus by (2.3), we have $R_n(xe^{\lambda^{-n\delta}}, ye^{\lambda^{-n\delta}}) \leqq C_0^n\left(\frac{y}{x}\right)$. This implies that $\lim_{n\to\infty}\Theta_n(xe^{\lambda^{-n\delta}}, ye^{\lambda^{-n\delta}})=0$.

Therefore we see that

$$\lim_{n\to\infty} \sup\{|\Theta_n(xe^{\lambda^{-n}s}, ye^{\lambda^{-n}t})| \; ; \; s, t\in\mathbb{C}\, , |s|, |t| \leqq \delta\} = 0 \, . \tag{4.26}$$

Also, (4.25) implies that $\{\Phi_n(xe^{\lambda^{-n}s}, ye^{\lambda^{-n}t})\}_{n=0}^{\infty}$ is a normal family of holomorphic functions in $\{(s, t)\in\mathbb{C}^2; |s|, |t| < \delta\}$.

Let $\Phi_n(xe^{\lambda^{-n}s}, ye^{\lambda^{-n}t}) = \sum_{k,l=0}^{\infty} a_{k,l}^{(n)} s^k t^l$ and $\Theta_n(xe^{\lambda^{-n}s}, ye^{\lambda^{-n}t}) = \sum_{k,l=0}^{\infty} b_{k,l}^{(n)} s^k t^l$.

Then $b_{k,l}^{(n)} \to 0$, $n\to\infty$, for all k and l. It is obvious that $a_{0,0}^{(n)} \to \dfrac{\sqrt{5}-1}{2}$, $n\to\infty$. By Proposition (3.5)(3), we see that there are $a_{1,0} > 0$ and $a_{0,1} > 0$ such that $a_{1,0}^{(n)} \to a_{1,0}$ and $a_{0,1}^{n} \to a_{0,1}$, $n\to\infty$. By (1.2), we have

$$\sum_{k,l=0}^{\infty} a_{k,l}^{(n+1)} s^k t^l = \left(\sum_{k,l=0}^{\infty} \lambda^{-(k+l)}(a_{k,l}^{(n)} + b_{k,l}^{(n)}) s^k t^l\right)^2 + \left(\sum_{k,l=0}^{\infty} \lambda^{-(k+l)} a_{k,l}^{(n)} s^k t^l\right)^2 \left(\sum_{k,l=0}^{\infty} \lambda^{-(k+l)}(a_{k,l}^{(n)} + 2b_{k,l}^{(n)})\, s^k t^l\right). \tag{4.27}$$

Therefore, we see that there are polynomials $P_{k,l}(a_{k',l'}, b_{k'',l''})$, $k, l \geqq 0$, $k + l \geqq 2$, in $a_{k',l'}$'s and $b_{k'',l''}$'s, $0 \leqq k', k'' \leqq k$, $0 \leqq l', l'' \leqq l$, $0 < k' + l' \leqq k + l$, such that

$$a_{k,l}^{(n+1)} = \lambda^{-(k+l)}(2(a_{0,0}^{(n)} + b_{0,0}^{(n)}) + (3a_{0,0}^{(n)2} + 4a_{0,0}^{(n)} b_{0,0}^{(n)}))a_{k,l}^{(n)} + P_{k,l}(a_{k',l'}^{(n)}, b_{k'',l''}^{(n)}) \tag{4.28}$$

for any $n \geqq 0$ and $k, l \geqq 0$ with $k + l \geqq 2$.

Note that $\lambda^{-(k+l)}(2(a_{0,0}^{(n)} + b_{0,0}^{(n)}) + (3a_{0,0}^{(n)2} + 4a_{0,0}^{(n)} b_{0,0}^{(n)})) \to \lambda^{-(k+l-1)}$ as $n\to\infty$. So by induction we see that there are $a_{k,l}$'s such that $a_{k,l}^{(n)} \to a_{k,l}$ as $n\to\infty$ and

$$a_{k,l} = \lambda^{-(k+l-1)} a_{k,l} + P_{k,l}(a_{k',l'}, 0) \tag{4.29}$$

for $k, l \geqq 0$ with $k + l \geqq 2$. Therefore we see that there is a holomorphic function $\varphi:\Omega\to\mathbb{C}$, $\Omega = \{(s, t)\in\mathbb{C}^2; |s|, |t| < \delta\}$, such that

$$\Phi_n(xe^{\lambda^{-n}s}, ye^{\lambda^{-n}t}) \to \varphi(s, t)\, , \tag{4.30}$$

and

$$\Theta_n(xe^{\lambda^{-n}s}, ye^{\lambda^{-n}t}) \to 0\, , \tag{4.31}$$

as $n\to\infty$, uniformly in $\{(s, t)\in\mathbb{C}^2; |s|, |t| \leqq \delta/2\}$, and

$$\varphi(s, t) = \varphi(\lambda^{-1}s, \lambda^{-1}t)^3 + \varphi(\lambda^{-1}s, \lambda^{-1}t)^2 \tag{4.32}$$

for any $(s, t)\in\mathbb{C}^2$ with $|s|, |t| \leqq \delta/2$.

For any $R > 0$, take an $m\in\mathbb{N}$ such that $\lambda^{-m}R < \delta/2$. Then we see that

$$\Phi_{n+m}(xe^{\lambda^{-(n+m)}s}, ye^{\lambda^{-(n+m)}t}) = \Phi_m(\Phi_n(xe^{\lambda^{-n}\lambda^{-m}s}, ye^{\lambda^{-n}\lambda^{-m}t}),$$
$$\Theta_n(xe^{\lambda^{-n}\lambda^{-m}s}, ye^{\lambda^{-n}\lambda^{-m}t})) \to \Phi_m(\varphi(\lambda^{-m}s, \lambda^{-m}t), 0)\, , \quad n\to\infty\, , \tag{4.33}$$

uniformly in $\{(s, t)\in\mathbb{C}^2; |s|, |t| \leqq R\}$. This also shows that φ can be extended to an entire function in $\mathbb{C}^2$ which satisfies (4.32) for all $(s, t)\in\mathbb{C}^2$. This completes the proof.

Setting $g(s, t) = \left(\frac{\sqrt{5}-1}{2}\right)^{-1} \varphi(s, t)$, we have the following assertion.

(4.34) **Corollary.** *Let* $(x, y) \in \partial D \cap (0, \infty)^2$. *Then there is an entire function* $g: \mathbb{C}^2 \to \mathbb{C}$ *such that*

$$g\left(\left(\frac{7-\sqrt{5}}{2}\right)^{-n} s, \left(\frac{7-\sqrt{5}}{2}\right)^{-n} t\,;\, v_n^{(x,y)}\right) \to g(s, t), \quad n \to \infty, \tag{4.35}$$

uniformly in $\{(s, t) \in \mathbb{C}^2; |s|, |t| \leqq R\}$ for all $R > 0$, and

$$g\left(\left(\frac{7-\sqrt{5}}{2}\right)s, \left(\frac{7-\sqrt{5}}{2}\right)t\right) = \frac{3-\sqrt{5}}{2} \cdot g(s, t)^3 + \frac{\sqrt{5}-1}{2} \cdot g(s, t) \tag{4.36}$$

for any $(s, t) \in \mathbb{C}^2$.

Moreover, $\frac{\partial}{\partial s} g(0, 0) > 0$ and $\frac{\partial}{\partial t} g(0, 0) > 0$.

5. Critical Exponents

Let $\lambda = \frac{\sqrt{5}-1}{2}$ and $\tilde{\lambda} = \frac{3-\sqrt{5}}{2}$, and let

$$f_0(x, y) = \lim_{n \to \infty} 3^{-n} \log \Phi_n(x, y), \ (x, y) \in (0, \infty)^2 \backslash D\,, \tag{5.1}$$

and

$$f_1(x, y) = \lim_{n \to \infty} 2^{-n} \log \Phi_n(x, y), \ (x, y) \in D\,. \tag{5.2}$$

(5.3) **Proposition.** (1) $f_0(\Phi_n(x, y), \ \Theta_n(x, y)) = 3^n f_0(x, y)$ *for any* $n \geqq 1$ *and* $(x, y) \in (0, \infty)^2 \backslash D$.

(2) $f_1(\Phi_n(x, y), \Theta_n(x, y)) = 2^n f_1(x, y)$ *for any* $n \geqq 1$ *and* $(x, y) \in D$.

(3) $f_0(x, y) = \log x + \sum_{k=0}^{\infty} 3^{-k-1} \log\{(1 + 2R_k(x, y)) + \Phi_k(x, y)^{-1}(1 + R_k(x, y))^2\}$

for any $(x, y) \in (0, \infty)^2 \backslash D$. *In particular,*

$$\log x \leqq f_0(x, y) \leqq \log x + \log\left\{\left(1 + \left(\frac{1}{x}\right)\right)\left(1 + \frac{y}{x}\right)\right\}, \ (x, y) \in (0, \infty)^2 \backslash D\,.$$

(4) $f_1(x, y) = \log x + \sum_{k=0}^{\infty} 2^{-k-1} \log\{(1 + R_k(x, y))^2 + \Phi_k(x, y)(1 + 2R_k(x, y))\}$

for any $(x, y) \in D$. *In particular,*

$$\log x \leqq f_1(x, y) \leqq \log x + \log\left\{\frac{\sqrt{5}+1}{2}\left(1 + \left(\frac{y}{x}\right)\right)^2\right\}, \quad (x, y) \in D\,.$$

Proof. (1) and (2) are obtained easily from the definitions, (5.1), and (5.2). The equality in (3) is obtained by the iterated use of (2.5).

To prove the inequality in (3), let us first show that $\Phi_{n+1}(x, y) > \Phi_n(x, y)$, for $(x, y) \in D^c$ and $n \geqq 0$. Assume, for some n and some $(x, y) \in D^c$, that $\Phi_{n+1}(x, y) \leqq \Phi_n(x, y)$. Then from (2.4) and Lemma (2.7), we have

$$\Phi_n(x, y) \leqq \frac{\{4(1 + 2R_n(x, y)) + (1 + R_n(x, y))^4\}^{1/2} - (1 + R_n(x, y))^2}{2(1 + R_n(x, y))}$$
$$\leqq \frac{\{4(1 + 2R_{n+1}(x, y)) + (1 + R_{n+1}(x, y))^4\}^{1/2} - (1 + R_{n+1}(x, y))^2}{2(1 + R_{n+1}(x, y))} .$$

Again by Lemma (2.7), this implies that $\Phi_{n+2}(x, y) \leqq \Phi_{n+1}(x, y)$, which indicates that Φ_n is decreasing with respect to n, which contradicts the statement of Proposition (2.13). Using the monotonicity of $R_k(x, y)$ and $\Phi_k(x, y)$, we have

$$1 + 2R_k(x, y) + \Phi_k(x, y)^{-1}(1 + R_k(x, y))^2 \leqq (1 + \Phi_0(x, y)^{-1})(1 + R_0(x, y))^2$$
$$\leqq \left(1 + \frac{1}{x}\right)\left(1 + \frac{y}{x}\right)^2 .$$

From this, we obtain the inequality. (4) is obtained similarly. Q.E.D.

(5.4) **Lemma.** *Let $g(x) = x^3 + x^2$, $x \geqq 0$, and let $\lambda_n = g^{-n}(1)$, where g^{-n} is the n-fold composition of g^{-1}, the inverse map of g. Then $\lambda \leqq \lambda_n \leqq \lambda + 2^{-n}$, $n \geqq 0$. In particular, $C_0 = \prod_{n=0}^{\infty} ((1 + \lambda_n^{-1})^{-1} \cdot \tilde{\lambda}^{-1}) < \infty$.*

Proof. Since $g\colon [0, \infty) \to [0, \infty)$ is a continuous strictly increasing function and $g(\lambda) = \lambda$, we see that $\lambda_n \geqq \lambda$. Note that $g'(x) = 3x^2 + 2x \geqq 3\lambda^2 + 2\lambda > 2$, $x \geqq \lambda$. Thus we see that

$$\lambda_n - \lambda = g(\lambda_{n+1}) - g(\lambda) \geqq 2(\lambda_{n+1} - \lambda) , \quad n \geqq 0 .$$

Therefore we have $\lambda_n - \lambda \leqq 2^{-n}(1 - \lambda)$, $n \geqq 0$. Since $\tilde{\lambda} = (1 + \lambda^{-1})^{-1}$, this implies that $C_0 < \infty$. Q.E.D.

Let $N(x, y) = \min\{n \geqq 0; \Phi_n(x, y) \geqq 1\}$, $(x, y) \in (0, \infty)^2 \backslash D$. Then we have the following.

(5.5) **Proposition.** *For any $(x, y) \in (0, \infty)^2 \backslash D$ with $x < 1$ and $n \leqq N(x, y) - 1$, $\Phi_n(x, y) \leqq \lambda + 2^{-N(x, y) + n + 1}$ and $R_n(x, y) \leqq \tilde{\lambda}^n \cdot C_0\left(\frac{y}{x}\right)$.*

Proof. Since $\Phi_{N(x, y) - 1}(x, y) < 1$ and $\Phi_{n+1}(x, y) \geqq g(\Phi_n(x, y))$ by (2.1), we have $\Phi_{N(x, y) - 1 - m}(x, y) \leqq \lambda_m$, $m \leqq N(x, y) - 1$. Also by (2.3), we have $R_n(x, y) \leqq \prod_{k=0}^{n-1} (1 + \Phi_k(x, y)^{-1})^{-1} \times \left(\frac{y}{x}\right)$. These and Lemma (5.4) imply our assertion. Q.E.D.

(5.6) **Proposition.** *For any $(x, y) \in (0, \infty)^2 \backslash D$ with $x < 1$,*

$$- N(x, y) \log 3 + (\log\log 12 - 2\log 3) \leqq \log f_0(x, y)$$
$$\leqq - N(x, y)\log 3 + \log 3 + \log\log\left\{2\left(1 + \tilde{\lambda}^{N(x, y) - 1} C_0\left(\frac{y}{x}\right)\right)\right\} .$$

Proof. Let $m = N(x, y) + 2$. Then by (2.1), we have $\Phi_m(x, y) \geqq 12$. Thus by Proposition (5.3), we have $\log 12 \leqq f_0(\Phi_m(x, y), \Theta_m(x, y))$ and so $\log\log 12 \leqq \log f_0(x, y) + (N(x, y) + 2)\log 3$.

Now let $m = N(x, y) - 1$. Then $\Phi_m(x, y) < 1$. Thus by Propositions (5.3) and (5.5), we have

$$\log f_0(x, y) + (N(x, y) - 1)\cdot\log 3 \leqq \log\log\left\{2\left(1 + \tilde{\lambda}^{N(x,y)-1} C_0\left(\frac{y}{x}\right)\right)\right\}.$$

These prove our assertion.

(5.7) **Proposition.** *Let* $(x, y) \in \partial D \cap (0, \infty)^2$. *Then we have*

$$\sup\{|N(x\cdot e^u, y\cdot e^v) + \log(u + v)/\log\alpha|\,;\quad (u, v) \in [0, 1]^2 \setminus \{(0, 0)\}\} < \infty .$$

Here $\alpha = \dfrac{7 - \sqrt{5}}{2}$.

Proof. Since $\varphi(0, 0) = \dfrac{\sqrt{5} - 1}{2} < \dfrac{3}{4}$, $\dfrac{\partial}{\partial s}\varphi(0, 0) > 0$ and $\dfrac{\partial}{\partial t}\varphi(0, 0) > 0$, we see by using (4.19) that there is an $\varepsilon \in (0, 1)$ such that $\varphi(\varepsilon^{-1}, 0) > 2$, $\varphi(0, \varepsilon^{-1}) > 2$ and $\varphi(\varepsilon, \varepsilon) < \frac{3}{4}$. By Proposition (4.17), we see that $\Phi_n(xe^{\alpha^{-n}s}, ye^{\alpha^{-n}t}) \to \varphi(s, t)$, $n \to \infty$, uniformly in $(s, t) \in [0, \varepsilon^{-1}]^2$. Then, there is an $N \in \mathbb{N}$ such that

$$\Phi_n(x\cdot e^{\alpha^{-n}\varepsilon^{-1}}, y) \geqq 2\,, \tag{5.8}$$

$$\Phi_n(x, y\cdot e^{\alpha^{-n}\varepsilon^{-1}}) \geqq 2\,, \tag{5.9}$$

and

$$\Phi_n(xe^{\alpha^{-n}\varepsilon}, ye^{\alpha^{-n}\varepsilon}) \leqq \frac{3}{4} \tag{5.10}$$

for any $n \geqq N$.

Let $(u, v) \in [0, \alpha^{-N}\varepsilon]^2 \setminus \{(0, 0)\}$ and let $m = -\log(\max\{u, v\})/\log\alpha$. Then we see that

$$\Phi_n(x\cdot e^u, y\cdot e^v) \geqq \min\{\Phi_n(x\cdot e^{\alpha^{-n}\varepsilon^{-1}}, y), \Phi_n(x, y\cdot e^{\alpha^{-n}\varepsilon^{-1}})\} \geqq 2 \tag{5.11}$$

if $m \leqq n + \log\varepsilon/\log\alpha$, and

$$\Phi_n(x\cdot e^u, y\cdot e^v) \leqq \Phi_n(x\cdot e^{\alpha^{-n}\varepsilon}, y\cdot e^{\alpha^{-n}\varepsilon}) \leqq \frac{3}{4} \tag{5.12}$$

if $m \geqq n - \log\varepsilon/\log\alpha$.

Therefore we have

$$|N(x\cdot e^u, y\cdot e^v) - m| \leqq |\log\varepsilon/\log\alpha|\,. \tag{5.13}$$

Since $\frac{1}{2}(u + v) \leqq \max\{u, v\} \leqq u + v$, this proves our assertion.

(5.14) **Corollary.** *Let* $(x, y) \in \partial D \cap (0, \infty)^2$. *Then we have*

$$\sup\left\{\left|\log f_0(x\cdot e^u, y\cdot e^v) - \frac{\log 3}{\log\alpha}\log(u + v)\right|;\ (u, v) \in [0, 1]^2 \setminus \{(0, 0)\}\right\} < \infty .$$

(5.15) **Proposition.** *Let* $(x, y) \in \partial D \cap (0, \infty)^2$. *Then we have*

$$\sup\left\{\left|\log R(x \cdot e^u, y \cdot e^v) - \frac{\log \beta}{\log \alpha} \log(u+v)\right|; (u, v) \in [0, 1]^2 \backslash \{(0, 0)\}\right\} < \infty .$$

Here $\beta = \dfrac{\sqrt{5}+3}{2}$.

Proof. Since $\Phi_{N(x', y')}(x', y') \geqq 1$, we have by (2.2) $\Phi_{N(x', y')+n}(x', y') \geqq 2^n$, $n \geqq 0$. Therefore from Proposition (2.12), we have

$$R(xe^u, ye^v)^{-1} R_{N(xe^u, ye^v)}(xe^u, ye^v) \leqq \exp[2\{1 + (x^{-1}ye^{v-u})\}^2/\{1 + 2(x^{-1}ye^{v-u})\}] . \tag{5.16}$$

Similarly to the proof of (2.11), we have

$$\Phi_n(xe^u, ye^v) \geqq \frac{\{4(1+2R_n(xe^u, ye^v)) + (1+R_n(xe^u, ye^v)^4\}^{1/2} - (1+R_n(xe^u, ye^v))^2}{2(1+2R_n(xe^u, ye^v))} . \tag{5.17}$$

By Proposition (5.5), we have $\Phi_n(xe^u, ye^v) \leqq \lambda + 2^{-N(xe^u, ye^v)+n+1}$ and $R_n(xe^u, ye^v) \leqq \tilde{\lambda}^n C_0\left(\dfrac{y}{x}\right)e^{v-u}$, $n \leqq N(xe^u, ye^v) - 1$. Therefore, noting that $\beta = \tilde{\lambda}^{-1} = \lambda^{-1} + 1$, we have

$$\sup\left\{\sum_{k=0}^{N(xe^u, ye^v)-1} |\Phi_n(xe^u, ye^v)^{-1} - (\beta - 1)|; u, v \in [0, 1]\right\} < \infty . \tag{5.18}$$

Now observe that

$$\left|\log R_m(xe^u, ye^v) - \left(\log\left(\frac{y}{x}\right) + v - u\right) + m \log \beta\right|$$
$$= \left|\sum_{k=0}^{m-1} \log\left(1 + \Phi_k(xe^u, ye^v)^{-1}\left(1 + \frac{R_k(xe^u, ye^v)^2}{1 + 2R_k(xe^u, ye^v)}\right)\right) - m \cdot \log \beta\right| .$$

Note also that $|\log(1+a) - \log(1+b)| \leqq |a-b|$, $a, b \geqq 0$. Since we have

$$\left|\left\{1 + \Phi_n(xe^u, ye^v)^{-1}\left(1 + \frac{R_n(xe^u, ye^v)^2}{1 + 2R_n(xe^u, ye^v)}\right)\right\} - \beta\right|$$
$$\leqq |\Phi_n(xe^u, ye^v)^{-1} - (\beta - 1)| \cdot (1 + R_n(xe^u, ye^v)^2) + \frac{1}{2}(\beta - 1) \cdot R_n(xe^u, ye^v)$$
$$\leqq |\Phi_n(xe^u, ye^v)^{-1} - (\beta - 1)|(1 + (x^{-1}ye^{v-u})^2) + 2(\beta - 1)\tilde{\lambda}^n C_0\left(\frac{y}{x}\right)$$

for $n \leqq N(xe^u, ye^v) - 1$, we can conclude that

$$\sup\{|\log R_{N(xe^u, ye^v)}(xe^u, ye^v) + N(xe^u, ye^v) \log \beta|; \quad (u, v) \in [0, 1]^2 \backslash \{(0, 0)\}\} < \infty .$$

Combining this with Proposition (5.7), we obtain our assertion. Q.E.D.

Now let $\tilde{N}(u, v) = \tilde{N}_{xy}(u, v) = \min\left\{n \geqq 0;\quad \Phi_n(xe^{-u}, ye^{-v}) \leqq \frac{x}{2}\right\}$, $(x, y) \in \partial D \cap (0, \infty)^2$, $(u, v) \in [0, \log 2)^2 \backslash \{(0, 0)\}$. Then similarly to Proposition (5.5), we obtain the following.

(5.19) **Proposition.** *Let* $\alpha = \frac{7 - \sqrt{5}}{2}$. *Then*

$$\sup\{|\tilde{N}(u, v) + \log(u + v)/\log\alpha|;\quad (u, v) \in [0, \varepsilon]^2 \backslash \{(0, 0)\}\} < \infty .$$

By Proposition (5.3), *combined with Propositions* (2.12) *and* (5.19), *we have*

(5.20) **Proposition.** *There are constants* C_1 *and* C_2 *such that*

$$-\tilde{N}(u, v)\log 2 + C_1 \leqq \log\{-f_1(xe^{-u}, ye^{-v})\} \leqq -\tilde{N}(u, v)\log 2 + C_2$$

for any $(u, v) \in [0, \varepsilon]^2 \backslash \{(0, 0)\}$.

(5.21) **Corollary.** *Let* $(x, y) \in \partial D \cap (0, \infty)^2$. *Then we have*

$$\sup\left\{\left|\log\{-f_1(xe^{-u}, ye^{-v})\} - \frac{\log 2}{\log\alpha}\log(u + v)\right|;\quad (u, v) \in [0, 1]^2 \backslash \{(0, 0)\}\right\} < \infty .$$

6. Proofs of Theorems in Introduction and Remarks

It is obvious that

$$Z_n(\beta) = \Phi_n(e^{-\beta}, e^{-2\beta}), \quad \beta \in \mathbb{R}, \tag{6.1}$$

and

$$\int_{W^{(n)}} \exp(sL(w))\mu_n(\beta)(dw) = \Phi_n(e^{-\beta}e^{s}, e^{-2\beta}e^{2s}), \quad s \in \mathbb{C}. \tag{6.2}$$

Then Theorem (0.1) follows from Proposition (2.10) and Theorem (2.16). Theorem (0.2) follows from Corollaries (5.14) and (5.21). Also, Theorem (0.3) follows from Propositions (4.2), (4.17) and Corollary (4.34). Thus we proved all theorems in Introduction.

We can obtain a rough shape of the domain D by numerical calculations by using the following.

(6.3) **Lemma.** *Let* $(x, y) \in (0, \infty)^2$.

(1) *If there exists* $n \geqq 0$ *such that* $\Phi_n(x, y) > \frac{\sqrt{5} - 1}{2}$, *then* $(x, y) \in (0, \infty)^2 \backslash D$.

(2) *If there exists* $n \geqq 0$ *such that* $\Phi_{n+1}(x, y) \leqq \Phi_n(x, y)$, *then* $(x, y) \in D \backslash \partial D$.

Proof. (1) is a direct consequence of the definition of D. As in the proof of Proposition (2.8), we have $\Phi_{n+1}(x, y) = g(\Phi_n(x, y); 1 + 2R_n(x, y), (1 + R_n(x, y))^2)$. From (2.4) and Lemma (2.7) we obtain (2). Q.E.D.

In Fig. 3 we give a shape of D numerically obtained using the above criterion. Next, we let $\tilde{R}_n(\beta) = R_n(e^{-\beta}, e^{-2\beta})$. We list up some properties of $\tilde{R}_n(\beta)$.

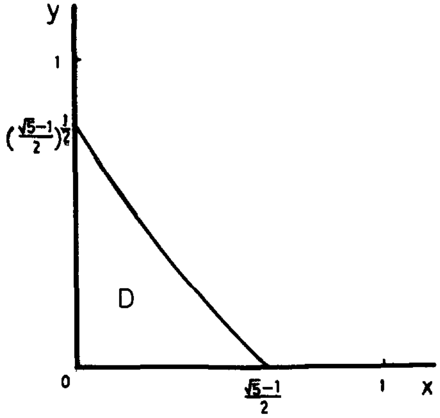

Fig. 3

(6.4) **Proposition.** (1) $\tilde{R}(\beta) = \lim_{n\to\infty} \tilde{R}_n(\beta)$, $\beta \in \mathbb{R}$, *exists and is continuous in* β.

(2) $\tilde{R}(\beta) = 0$, *if* $\beta \geqq \beta_c$, *and* $\tilde{R}(\beta) > 0$, *if* $\beta < \beta_c$.

(3) $\tilde{R}(\beta)$ *is decreasing with respect to* β.

(4) $\sup_{\beta_c > \beta \geqq \beta_c - 1} |\log \tilde{R}(\beta) - \frac{\log \tilde{\lambda}}{\log \lambda} \log(\beta_c - \beta)| < \infty$, *where* $\lambda = \frac{7-\sqrt{5}}{2}$, and $\tilde{\lambda} = \frac{3+\sqrt{5}}{2}$.

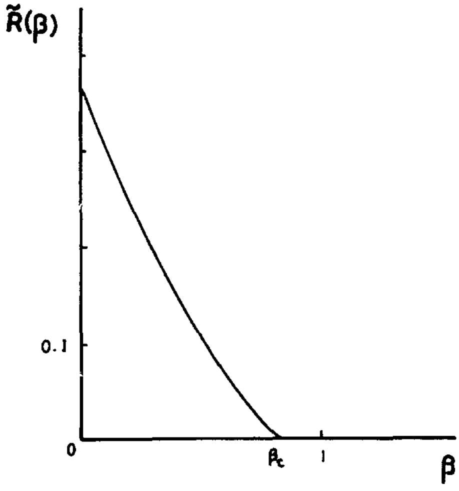

Fig. 4

Proof. The assertions (1) and (2) follow from Proposition (2.12) and Theorem (2.16). By (2.3), we have

$$R_{n+1}(x, y)^{-1} = R_n(x, y)^{-1} + \Phi_n(x, y)^{-1} \cdot (R_n(x, y)^{-1} + (R_n(x, y)^{-1} + 2)^{-1}) .$$

Since the function $f\colon [0, \infty) \to [0, \infty)$ given by $f(t) = t + (t+2)^{-1}$ is increasing and $\Phi_n(e^{-\beta}, e^{-2\beta})$ is decreasing in β, we see by induction that $R_n(e^{-\beta}, e^{-2\beta})^{-1}$ is increasing in β. This implies the assertion (3). (4) comes from Proposition (5.15). Q.E.D.

Proposition (6.2)(4) implies, in terms of statistical mechanics, that the critical exponent of $\tilde{R}(\beta)$ is $\rho \equiv \frac{\log \tilde{\lambda}}{\log \lambda} = 1.108877\ldots$. In conventional notation, $\tilde{R}(\beta) \sim (\beta_c - \beta)^{\rho}$, $\beta \uparrow \beta_c$. We give a figure of $\tilde{R}(\beta)$, obtained numerically, in Fig. 4.

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