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Probability Theory and Related Fields

A Mean Field Limit of the Contact Process with Large Range

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Summary. A mean field limit of the contact process is obtained as the range M approaches ∞. Fluctuations about the deterministic limit are identified as a Generalized Ornstein Uhlenbeck process.

1. Introduction

The interacting particle system which we are considering is the contact process with large range. This process is a subset of $\frac{1}{M}Z$, where Z is the integer lattice, and M is a large number. Given an initial configuration of occupied sites (particles), the system evolves according to the following rules: (i) particles die at rate one, (ii) a living particle at site x attempts to give birth to another particle at rate λ, and sends the new particle to a site chosen uniformly from sites located in $[x-1, x+1]$, (iii) if the chosen site is already occupied, the birth is suppressed, or, alternatively, the two particles coalesce into one. We will denote the set of occupied sites at time t by ${}^M\xi_t^\mu$, where μ indicates the distribution of the initial configuration. As M increases, the number of sites from which a single particle can choose to place an offspring increases—consequently, we call this the contact process with *large range*. A more detailed construction of this process and related processes will be provided later.

Define: $u_M(t, x) = P(x \in {}^M\xi_t^\mu)$, *where we will now always take the distribution of the initial configuration* μ *to be the product measure* $P(x \in {}^M\xi_0^\mu) = u_M(0, x)$. It should be remarked that, while u_M is presently defined only for $x \in \frac{1}{M}Z$, when the context of a statement requires a definition over the entire real line, we will take $u_M(t, r) = u_M(t, x)$ where x is the rightmost point in the lattice $\frac{1}{M}Z$ which is less than r. The mean field limit is stated in Theorem 2 and the Corollary. Many authors have established deterministic limits for interacting particle systems. In [DFL], for

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example, a reaction-diffusion limit is obtained for a system with rapid exclusion and slow Ising spin flips. There the nontrivial interaction is the Ising interaction, and the rapid exclusion is the device by which desired asymptotic independences are established. In this work, the interaction is of a completely different kind: the interference due to the restriction of at most one particle per site, as is the vehicle by which independence is established: the divergence of the range of interaction.

Theorem 1. *Consider the contact process with λ and T fixed. If $u_M(0, x) \to u(0, x)$ uniformly on compact sets, where $u(0, x)$ is continuous with bounded derivative, then*

$$u_M(t, x) \to u(t, x) \quad \text{for all} \quad t \in [0, T] \quad \text{and} \quad x \in R \tag{1.1}$$

where the function $u(t, x)$ satisfies:

$$\frac{\partial u(t, x)}{\partial t} = -u(t, x) + \lambda(1 - u(t, x)) \int_{x-1}^{x+1} u(t, y) \frac{dy}{2} . \tag{1.2}$$

Theorem 1 is a statement about the convergence of occupation probabilities as $M \to \infty$. The following results embody the desired limit in the sense that random elements of the spaces $D([0, T], R)$ and $D([0, T], S'(R))$ converge to deterministic limits. Here $S(R)$ is the space of Schwartz functions and $S'(R)$ is its topological dual. In what follows, we will write ${}^M\xi_t^\mu(x) = 1$ when $x \in {}^M\xi_t^\mu$, and ${}^M\xi_t^\mu(x) = 0$ otherwise.

Theorem 2. *For a given continuous function ϕ such that $\sup_x |x^2 \phi(x)| < \infty$, consider $X_t^M(\phi) = \frac{1}{M} \sum_{x \in \frac{1}{M}Z} \phi(x) {}^M\xi_t^\mu(x)$ as an element of $D([0, T], R)$. Then $X^M(\phi) \Rightarrow X(\phi)$, where,*

$$X(\phi) = \int_{-\infty}^{+\infty} \phi(x) u(\cdot, x) dx .$$

Theorem 2 essentially states that as the range becomes large, the randomness associated with the values of ${}^M\xi_t^\mu(x)$ which appear in $X_t^M(\phi)$ disappears as in the weak law of large numbers. This is due to the fact (which will be proven), that as $M \to \infty$ the occupation events of two different sites become uncorrelated. Results given in [Mil] relating tightness in $D([0, T], R)$ to tightness in $D([0, T], S'(R))$ imply the following

Corollary to Theorem 2. *Consider $X_t^M(\phi) = \frac{1}{M} \sum_{x \in \frac{1}{M}Z} \phi(x) {}^M\xi_t^\mu(x)$ where $\phi \in S(R)$. Viewing X^M as an element of $D([0, T], S'(R))$, $X^M \Rightarrow X$, where,*

$$X(\phi) = \int_{-\infty}^{+\infty} \phi(x) u(\cdot, x) dx .$$

The next result is a central limit theorem complementing the weak law results of Theorem 2 and its Corollary. As in the previous result, we restrict ourselves to the space $S(R)$. The following theorem identifies the fluctuations about the deterministic limit. The statement of the theorem involves a Generalized Ornstein Uhlenbeck (GOU) process – a random element N_t of $C([0, T], S'(R))$ with law P

characterized by two operators $A_t(\phi)$ and $B_t(\phi)$ which satisfy appropriate conditions (see Appendix) including the condition that:

$$f(N_t(\phi)) - \int_0^t N_u(A_u\phi)f'(N_u\phi)du - \int_0^t \frac{1}{2}\|B_u\phi\|^2 f''(N_u\phi)du$$

is a P-martingale for all $f \in C_0^\infty(R)$, the space of infinitely differentiable functions of compact support.

Theorem 3. *Suppose that* $u(0,x) \in S(R)$. *Consider*

$$Y_t^M(\phi) = \frac{1}{\sqrt{M}} \sum_{x \in \frac{1}{M}Z} \phi(x)({}^M\xi_t^\mu(x) - u_M(t,x))$$

as an element of $D([0,T], S'(R))$, *and denote its law by* P_M. *Let* P *be the law of the* GOU *process prescribed by:*

$$A_t(\phi) = \lambda\phi * \beta - \phi - \lambda\phi X_t(\beta_{-x}) - \lambda X_t(\phi\cdot(\beta)_{-x}) \tag{1.3}$$

and

$$\|B_t(\phi)\|^2 = X_t\{\lambda\phi^2 * \beta + \phi^2 - \lambda X_t(\phi^2\cdot(\beta)_{-x})\} \tag{1.4}$$

where

$$\beta(x) \equiv \tfrac{1}{2} 1_{\{|x| \leq 1\}}$$

$$f_{-x}(y) \equiv f(y-x)$$

$$f * \beta(x) \equiv \int_{-\infty}^{+\infty} f(y)\beta(y-x)dy\,,$$

and $X_t(\phi)$ *is the deterministic process defined in Theorem* 2.
Then $P_M \Rightarrow P$.

All of the results above are stated for the contact process in which offspring are placed uniformly over the sites within one unit of the parent particle. In fact, the proofs of Theorems 1 and 2 generalize to any offspring distribution which is piecewise continuous with bounded derivative, and we believe that Theorem 3 is also valid in this situation.

The paper is organized in the following manner. In Sect. 2 we will describe the contact process and couplings to other related processes which will be used in the proofs of the results. Section 3 will contain the proofs of Theorems 1 and 2, and the proof of Theorem 3 is given in Sect. 4. The Appendix contains a brief discussion of tightness criteria and Generalized Ornstein-Uhlenbeck processes.

2. Some Important Lemmas

We begin this section with a detailed construction of the contact process (CP). In the course of this construction, several related branching random walks (BRWs) will also be described. These related processes are used extensively in the proofs of the Theorems.

The first task involved in constructing the desired processes is to prescribe a means of identifying particles. Particles living at time $t = 0$ are considered to be the first generation, and are identified by distinct integers which we denote by i_1, the subscript indicating the generation. A particle in the n^{th} generation is denoted by a sequence of positive integers $(i_1, \ldots, i_n)$, where $i_n = 3$, for example, implies that the designated particle is chronologically the third offspring of the particle designated by $(i_1, \ldots, i_{n-1})$.

We will now construct a branching random walk on R with birth rate λ and death rate 1. To each possible particle x, identified by some n-vector, we assign a Poisson process $T(x)$ at rate λ, and an exponential random variable $D(x)$ with parameter 1. The T processes indicate when particles give birth to new particles, and the D random variables indicate when living particles die. If a particle located at $x \in R$ gives birth to another paticle, the offspring is placed at a location y selected uniformly from the interval $[x-1, x+1]$. We designate this process starting with a single particle at x by: Z_t^x.

Minor modifications of the procedure specified in the preceding paragraph construct the branching random walk and the contact process on the lattice $\frac{1}{M}Z$. We will begin with the BRW. The same processes $T(x)$ and $D(x)$ are used to designate births and deaths of particles for the branching random walk on $\frac{1}{M}Z$. Therefore, the two BRWs are coupled in the sense that births and deaths occur at the same time for particles identified by the same n-vectors. However, the offspring distribution is modified for the BRW on $\frac{1}{M}Z$. Given that a birth occurs from a particle at site x, the offspring is placed at a site y chosen uniformly from the sites in $[x-1, x+1] \cap \frac{1}{M}Z$. We designate this BRW starting from a single particle at x: ${}^M Z_t^x$.

The offspring locations of the two processes Z_t^x and ${}^M Z_t^x$ can also be coupled in the following way. If a particle in Z_t^x gives birth to a particle displaced a distance $\delta (0 < \delta \leqq 1)$, the corresponding particle in ${}^M Z_t^x$ places the offspring at a distance $\frac{1}{M}\{[M\delta] + 1\}$ from its location, where $[x]$ denotes the integer part of x. In words, for an $x \in \frac{1}{M}Z$, any offspring placed in the interval $\left(x - \frac{1}{M}, x\right]$ in Z_t^x is placed at x in ${}^M Z_t^x$.

The contact process on $\frac{1}{M}Z$ is exactly like the BRW with the restriction that there can exist only a single particle per site. Therefore, when a particle at a particular site encounters a birth-time and attempts to place a new particle on a site which is already occupied by a particle, no new particle is formed. While we do not place another particle at the site which is already occupied, we note that such an attempt was made at the particular time in question and we shall call such an occurrence a "hit." The CP starting from a fixed set A is designated by ${}^M\xi_t^A$, and when the CP is started from a random initial configuration with distribution μ, we will denote the process by ${}^M\xi_t^\mu$.

One feature of the CP which is of significant utility is the following duality relation:

$$P({}^M\xi_t^A \cap B \neq \varnothing) = P({}^M\tilde{\xi}_t^B \cap A \neq \varnothing), \tag{2.1}$$

where ${}^M\tilde{\xi}_t^B$ is equal in distribution to the original CP starting at B: ${}^M\xi_t^B$. Because of this last fact, the CP is said to be "self-dual." For a complete description of contact process duality see [Du] or [Li]. We shall denote the dual process starting with a single occupied site x by ${}^M\tilde{\xi}_t^x$. Because of the original construction of the CP and self-duality, we can view the dual process ${}^M\tilde{\xi}_t^x$, starting at x, as being coupled to a BRW ${}^MZ_t^x$, starting at x, in the sense that birth and death times, as well as offspring locations (or attempted offspring locations) are the same.

The following lemmas are essential to the proofs of the theorems stated in the previous section. Lemma 2.1 establishes a bound on the probability of k or more "hits" occurring between two CP duals starting at different points x and y in $\frac{1}{M}Z$, and Lemma 2.2 uses Lemma 2.1 to place bounds on certain covariance terms. Let

$$\begin{aligned} I_M[T,x,y] = \{(s,z)\colon\ 0 \leqq s \leqq T,\ {}^M\tilde{\xi}_s^y(z) \quad &\text{and} \\ [{}^M\tilde{\xi}_{s^-}^x(z) = 0 \quad \text{and} \quad {}^M\tilde{\xi}_{s^-}^y(z) = 1 \quad &\text{or} \\ {}^M\tilde{\xi}_{s^-}^x(z) = 1 \quad \text{and} \quad {}^M\tilde{\xi}_{s^-}^y(z) = 0]\}\,. \end{aligned}$$

In words, each element of $I_M[T,x,y]$ prescribes the occurrence and space-time location of a hit.

Lemma 2.1. *Let $B_k = \{|I_M[T,x,y]| \geqq k\}$. Then there are constants d_k such that for all x and y in $\frac{1}{M}Z$, with $x \neq y$,*

$$P(B_k) \leqq \frac{d_k}{M^k}$$

Proof. Recalling the construction of the contact process, it is clear that we can define a BRW by ignoring all death times D_n. The associated branching process is known as the Yule process ([AN]). We shall call this BRW starting at a point x: ${}^M\eta_t^x$. It is clear that we can extend our definition of $I_M[T,x,y]$ to ${}^M\eta_t^x$ and ${}^M\eta_t^y$, and that this new $I_M[T,x,y]$ dominates the old one almost surely. It is adequte, therefore, to bound $P(B_k)$ for the η processes.

Let θ be the (random) number of births which occur in ${}^M\eta_t^x \cup {}^M\eta_t^y$ by time $t = T$. Then $N = \theta + 2$ is the total number of particles in ${}^M\eta_t^x \cup {}^M\eta_t^y$ at T.

If we condition of the set $\{\theta = m\}$, then the probability of any single birth event placing a particle on a site already occupied is bounded by $\frac{m+2}{2M}$. This can be seen by considering the extreme case, where all of the $m+2$ particles alive at $t = T$ are within a unit distance of the parent particle. Consequently,

$$P(B_k) \leqq E\left\{\sum_{m=0}^{\infty} 1_{\{\theta=m\}}\left(\frac{m+2}{2M}\right)^k m^k\right\},$$

where the term m^k is an upper bound for the $\binom{m}{k}$ ways of selecting the k hits from the m births. By the finiteness of the moments of the Yule process, the proof is complete.

Lemma 2.2. *Define* $\gamma_M^{x_1} \equiv {}^M\xi_t^\mu(x_1) - u_M(t, x_1)$. *There exist constants* C_2, C_3, *and* C_4 *such that for all* $t \in [0, T]$:

$$(i)\ E\{\gamma_M^{x_1}\gamma_M^{x_2}\} \leqq \frac{C_2}{M}$$

$$(ii)\ E\{\gamma_M^{x_1}\gamma_M^{x_2}\gamma_M^{x_3}\} \leqq \frac{C_3}{M^2}$$

$$(iii)\ E\{\gamma_M^{x_1}\gamma_M^{x_2}\gamma_M^{x_3}\gamma_M^{x_4}\} \leqq \frac{C_4}{M^3}\,.$$

Remark. As will be seen below, the method of proof will clearly yield:

$$E\left\{\prod_1^n \gamma_M^{x_i}\right\} \leqq \frac{C_n}{M^{n-1}}\,.$$

Proof. We begin the proof of (i) by observing:

$$\begin{aligned} E\{\gamma_M^{x_1}\gamma_M^{x_2}\} &= E\{{}^M\xi_t^\mu(x_1)\,{}^M\xi_t^\mu(x_2)\} - P(x_1 \in {}^M\xi_t^\mu)P(x_2 \in {}^M\xi_t^\mu) \\ &= P(x_1 \in {}^M\xi_t^\mu, x_2 \in {}^M\xi_t^\mu) - P(x_1 \in {}^M\xi_t^\mu)P(x_2 \in {}^M\xi_t^\mu) \\ &= P(x_1 \notin {}^M\xi_t^\mu, x_2 \notin {}^M\xi_t^\mu) - P(x_1 \notin {}^M\xi_t^\mu)P(x_2 \notin {}^M\xi_t^\mu)\,. \end{aligned}$$

The last equality is obtained by taking complements. We will also use the following consequence of contact process duality stated in (2.1). The probability that a site x is not occupied at time t is equal to the probability that the dual process starting at x does not hit any initially occupied site:

$$P(x \notin {}^M\xi_t^\mu) = P({}^M\xi_t^\mu \cap \{x\} = \varnothing) = P({}^M\tilde{\xi}_t^x \cap {}^M\xi_0^\mu = \varnothing)$$

Now, if $z \in {}^M\tilde{\xi}_t^\mu$, then the probability that $z \notin {}^M\xi_0^\mu$ is equal to $1 - u_M(0, z)$. Because the initial configuration is a product measure, the probability that all of the sites in the dual process are disjoint from the initial set ${}^M\xi_0^\mu$ is just the product of the respective probabilities, and we have:

$$P(x \notin {}^M\xi_t^\mu) = P({}^M\tilde{\xi}_t^x \cap {}^M\xi_0^\mu = \varnothing) = E\left\{\prod_{x \in {}^M\tilde{\xi}_t^x} (1 - u_M(0, z))\right\}, \tag{2.2}$$

where the product over the empty set is defined to be one.

We will now consider independent versions of the two dual processes: ${}^M\tilde{\xi}_t^{x_1}$ and ${}^M\tilde{\xi}_t^{x_2}$, which we denote by $\chi_t^{x_1}$ and $\chi_t^{x_2}$, and we define the process $\zeta_t^{x_2}$ to be the $\chi_t^{x_2}$ process in which particles are killed upon hitting or being hit by particles in the $\chi_t^{x_1}$ process. Note that $\zeta_t^{x_2}$ depends upon x_1 and the probability space on which $\chi_t^{x_1}$ is defined, but we omit this in the notation. This procedure is another way of constructing the contact process, and $\chi_t^{x_1} \cup \zeta_t^{x_2}$ is identical in distribution to ${}^M\tilde{\xi}_t^{x_1} \cup {}^M\tilde{\xi}_t^{x_2}$.

For convenience we define the following two random variables:

$$f^x \equiv \prod_{z \in \chi_t^x} (1 - u_M(0, z))$$
$$g^x \equiv \prod_{z \in \zeta_t^x} (1 - u_M(0, z))$$

We can now write the following:

$$P(x_1 \notin {}^M\xi_t^\mu) = E\{f^{x_1}\}$$
$$P(x_2 \notin {}^M\xi_t^\mu) = E\{f^{x_2}\}$$
$$P(x_1 \notin {}^M\xi_t^\mu \cap x_2 \notin {}^M\xi_t^\mu) = E\{f^{x_1} g_{x_1}^{x_2}\}$$

where f^{x_1} and f^{x_2} are independent. Therefore,

$$\begin{aligned} E\{\gamma_M^{x_1}\gamma_M^{x_2}\} &= E\{f^{x_1} g_{x_1}^{x_2} - f^{x_1} f^{x_2}\} \\ &= E\{(f^{x_1} g_{x_1}^{x_2} - f^{x_1} f^{x_2}) 1_{\{|I_M[T, x, y]| \geqq 1\}}\} \\ &\leqq P(|I_M[T, x, y]| \geqq 1) \leqq \frac{d_1}{M}, \end{aligned}$$

where the second equality follows from the fact that $f^{x_2} = g_{x_1}^{x_2}$ on the set $\{|I_M[T, x, y]| = 0\}$, and the last inequality follows from Lemma 2.1. Note that, from the proof of Lemma 2.1, the bounds on the hit probabilities hold for the η processes, and therefore, the use of the bound for the independent χ processes is valid. This completes the proof of (i).

To prove (ii), we write:

$$\begin{aligned} E\{\gamma_M^{x_1}\gamma_M^{x_2}\gamma_M^{x_3}\} &= P(x_1, x_2, x_3 \in {}^M\xi_t^\mu) - P(x_1, x_2 \in {}^M\xi_t^\mu) P(x_3 \in {}^M\xi_t^\mu) \\ &\quad - P(x_1, x_3 \in {}^M\xi_t^\mu) P(x_2 \in {}^M\xi_t^\mu) - P(x_2, x_3 \in {}^M\xi_t^\mu) P(x_1 \in {}^M\xi_t^\mu) \\ &\quad + 2P(x_1 \in {}^M\xi_t^\mu) P(x_2 \in {}^M\xi_t^\mu) P(x_3 \in {}^M\xi_t^\mu) \\ &= -P(x_1, x_2, x_3 \notin {}^M\xi_t^\mu) + P(x_1, x_2 \notin {}^M\xi_t^\mu) P(x_3 \notin {}^M\xi_t^\mu) \\ &\quad + P(x_1, x_3 \notin {}^M\xi_t^\mu) P(x_2 \notin {}^M\xi_t^\mu) + P(x_2, x_3 \notin {}^M\xi_t^\mu) P(x_1 \notin {}^M\xi_t^\mu) \\ &\quad - 2P(x_1 \notin {}^M\xi_t^\mu) P(x_2 \notin {}^M\xi_t^\mu) P(x_3 \notin {}^M\xi_t^\mu), \end{aligned}$$

where the last equality followed by taking complements.

As in the proof of part (i), we define independent versions of the contact process: $\chi_t^{x_1}$, $\chi_t^{x_2}$, and $\chi_t^{x_3}$ starting from the points appearing in the superscripts. We now let $\zeta_t^{(x_i, x_j)}$ be the $\chi_t^{x_i}$ process in which particles are killed upon hitting or begin hit by particles in the $\chi_t^{x_j}$ process. Additionally, we let $\iota_t^{x_i}$ to be the $\chi_t^{x_i}$ process in which particles are killed upon hitting or being hit by particles in either of the $\chi_t^{x_j}$ processes, where $j \neq i$.

For convenience we make the following definitions:

$$f^x \equiv \prod_{z \in \chi_t^x} (1 - u_M(0, z))$$

$$g_y^x \equiv \prod_{z \in \zeta_t^{(x, y)}} (1 - u_M(0, z))$$

$$h^x \equiv \prod_{z \in \iota_t^x} (1 - u_M(0, z)) .$$

We are now in a position to use duality to write:

$$E\{\gamma_M^{x_1}\gamma_M^{x_2}\gamma_M^{x_4}\} = E\{f^{x_1}g_{x_1}^{x_2}h^{x_3} - f^{x_1}g_{x_1}^{x_2}f^{x_3}$$
$$- f^{x_1}g_{x_1}^{x_3}f^{x_2} - f^{x_2}g_{x_2}^{x_3}f^{x_1} - 2f^{x_2}f^{x_2}f^{x_3}\} .$$

We now observe that on the sets $\{|I_M[T, x, y]| = 0\}$ and $\{|I_M[T, x, y]| = 1\}$ the integrand above vanishes. Additionally, the integrand is bounded, so that (ii) follows from Lemma 2.1. Part (iii) follows from similar reasoning.

The following Lemma will also be required in the proof of Theorem 1.

Lemma 2.3. *If* $z_1, \ldots, z_m$ *and* $u_1, \ldots, u_m$ *are complex numbers with modulus less than or equal to* 1, *then*:

$$|z_1 \ldots z_m - u_1 \ldots u_m| \leqq \sum_{i=1}^{m} |z_i - u_i| .$$

Proof. The proof follows by induction using the following relation.

$$z_1 \ldots z_m - u_1 \ldots u_m = (z_1 - u_1)(z_2 \ldots z_m) + u_1(z_2 \ldots z_m - u_2 \ldots u_m) .$$

3. Proofs of Theorems 1 and 2

Proof of Theorem 1. The proof will be done in two steps. The first step is showing that $\lim_{M \to \infty} u_M(t, x)$ actually exists, and the second step is to show that the limit $u(t, x)$ satisfies Eq. 1.2.

Define $A = (|^M Z_t^x| = |^M \tilde{\xi}_t^x|)$, the set on which the number of particles alive in the two processes is the same. Due to the coupled construction of these two processes, A is the set on which no hits occur in ${}^M\tilde{\xi}_t^x$. By Eq. (2.2):

$$1 - u_M(t, x) = E\left\{ \prod_{z \in {}^M\tilde{\xi}_t^x} (1 - u_M(0, z)) \right\}$$
$$= E\left\{ {}^M\prod({}^M\tilde{\xi}_t^x)1_{(A)} \right\} + E\left\{ {}^M\prod({}^M\tilde{\xi}_t^x)1_{(A^c)} \right\}$$

where,

$${}^M\prod(\eta) = \prod_{z \in \eta} (1 - u_M(0, z))$$

$$\prod(\eta) = \prod_{z \in \eta} (1 - u(0, z)) .$$

The last definition will be used later. By the definition of A:

$$1 - u_M(t, x) = E\left\{{}^M\prod({}^M Z_t^x)1_{(A)}\right\} + E\left\{{}^M\prod({}^M\tilde{\xi}_t^x)1_{(A^c)}\right\}$$

$$= E\left\{{}^M\prod({}^M Z_t^x)\right\} + E\left\{({}^M\prod({}^M\tilde{\xi}_t^x) - {}^M\prod({}^M Z_t^x))1_{(A^c)}\right\}.$$

Now, the boundedness of the integrand in the second term on the right hand side and Lemma 2.1 yield a bound on this term of the form $\frac{C}{M}$, which vanishes in the limit. We need only show that:

$$\lim_{M\to\infty} E\left\{{}^M\prod({}^M Z_t^x))\right\} = E\left\{\prod(Z_t^x)\right\} \equiv 1 - u(t, x). \tag{3.1}$$

We will actually prove a stronger version of Eq. 3.1:

Lemma 3.1. $\lim_{M\to\infty} \sup_{0\leqq t\leqq T} E\{|{}^M\prod({}^M Z_t^x) - \prod(Z_t^x)|\} = 0.$

Proof. Recall the definitions of the two processes ${}^M\eta_t^x$ and η_t^x, which were introduced in the proof of Lemma 2.1. $|{}^M\eta_T^x|$ and $|\eta_T^x|$ are the total number of births that occur in ${}^M Z_t^x$ and Z_t^x up to $t = T$ respectively. By Chebysheff's inequality and coupling:

$$P(|{}^M\eta_T^x| > N) = P(|\eta_T^x| > N) \leqq \frac{E(|\eta_T^x|^2)}{N^2} \tag{3.2}$$

For the remainder of this proof we will let $K = E(|\eta_T^x|^2)$. We shall denote the locations of the particles in ${}^M\eta_T^x$ and η_T^x by z_i^M and z_i. Since we know that the integrand in the statement of the Lemma is bounded by 1, we can use (3.2) and Lemma 2.3 to write:

$$E\left\{\left|{}^M\prod({}^M Z_t^x) - \prod(Z_t^x)\right|\right\} \leqq \frac{K}{N^2} + E\left\{\sum_{i=1}^{|\eta_T^x|} |u_M(0, z_i^M) - u(0, z_i)| 1_{\{|\eta_T^x| \leqq N\}}\right\}$$

$$\leqq \frac{K}{N^2} + E\left\{\sum_{i=1}^{|\eta_T^x|} |u_M(0, z_i^M) - u(0, z_i^M)| + \sum_{i=1}^{|\eta_T^x|} |u(0, z_i^M) - u(0, z_i)|\right\}.$$

for any $t\in[0, T]$.

For any $\varepsilon > 0$, select N such that $\frac{K}{N^2} < \frac{\varepsilon}{3}$. Now select M_1 so that for every $M > M_1$: $|u_M(0, y) - u(0, y)| < \frac{\varepsilon}{3N}$ whenever $y\in[x - N, x + N]$. The motivation for this last statement is that on the set $(|\eta_T^x| \leqq N)$, the locations of the particles are confined to the interval $[x - N, x + N]$. Observe that it was at this point that we used the uniform convergence of the initial conditions on compact sets. Finally, we make the observation that on the set $\{|\eta_T^x| \leqq N\}$, the maximum number of jumps that can occur between the initial position x and the location of any particle at time t is N. Consequently, on this set the largest difference between the location of any two coupled particles in the two processes ${}^M\eta_t^x$ and η_t^x is $\frac{N}{M}$ and we have:

$$|u(0, z_i^M) - u(0, z_i) \leqq \frac{N}{M} \sup_{y \in [x-N, x+N]} |u'(0, y)| .$$

Therefore, selecting M_2 so that for any $M > M_2$:

$$\frac{N}{M} \sup_{y \in [x-N, x+N]} |u'(0, x)| < \frac{\varepsilon}{3} ,$$

yields that for $M > \max(M_1, M_2)$:

$$\sup_{0 \leqq t \leqq T} E\left\{\left| {}^M\prod ({}^M Z_t^x) - \prod (Z_t^x)\right|\right\} < \varepsilon ,$$

which completes the proof of the Lemma.

We will now proceed to show that the limit function $u(t, x)$ satisfies Eqn. 1.2. It is clear from the definition that $u_M(t, x)$ is presently defined on $x \in \frac{1}{M} Z$. For the remainder of this paper the functions $u_M(t, x)$ will be defined over R in the following way. Letting $[r]_M$ be the lattice point $x \in \frac{1}{M} Z$ so that $x \leqq r < x + \frac{1}{M}$ we define:

$$u_M(t, r) = u_M(t, x) \quad \text{when} \quad [r]_M = x .$$

A simple generator calculation yields:

$$\begin{aligned} \frac{\partial u_M}{\partial t}(t, x) &= -u_M(t, x) + \sum_{j=-M}^{M} \frac{\lambda}{2M} P\left(\left([x]_M \notin {}^M\xi_t^\mu\right) \cap \left([x]_M + \frac{j}{M} \in {}^M\xi_t^\mu\right)\right) \\ &= -u_M(t, x) + (1 - u_M(t, x)) \frac{\lambda}{2M} \sum_{j=-M}^{M} \left\{u_M\left(t, x + \frac{j}{M}\right) + O\left(\frac{1}{M}\right)\right\} \end{aligned} \tag{3.3}$$

where the second equality follows from Lemma 2.2. We will now use the fact (see (3.1)) that $u_M(t, x) \to u(t, x)$, pointwise in x for any given t, to establish:

$$\lim_{M \to \infty} \frac{1}{M} \sum_{j=-M}^{M} u_M\left(t, x + \frac{j}{M}\right) = \int_{x-1}^{x+1} u(y, t)dy \tag{3.4}$$

$$\lim_{M \to \infty} \frac{\partial u_M(t, x)}{\partial t} = \frac{\partial u(t, x)}{\partial t} \tag{3.5}$$

To establish (3.4), we note that the left hand side is equal to:

$$\int_{x-1}^{x+1} u_M(t, y)dy + O\left(\frac{1}{M}\right) .$$

Now, by the boundedness of the integrand and the pointwise convergence of u_M to u, the dominated convergence theorem establishes the result.

Proceeding to (3.5), we can write:

$$u_M(t, x) = u_M(0, x) + \int_0^t \frac{\partial u_M(t, x)}{\partial t}(s, x)ds \, .$$

Since we know that for any x: $u_M(0, x) \to u(0, x)$, as $M \to \infty$, we need only show that for any given x, $\frac{\partial u_M(t, x)}{\partial t}$ converges uniformly on $t \in [0, T]$. Specifically, we will show that:

For any $\varepsilon > 0$, there exists a constant $L(\varepsilon, x)$ so that:

$$\left| \frac{\partial u_M(t, x)}{\partial t} - \frac{\partial u_N(t, x)}{\partial t} \right| \leqq \varepsilon$$

$$\text{for all } t \in [0, T] \text{ and for all } M, N > L(\varepsilon, x) \, . \tag{3.6}$$

Defining:

$$\Sigma_M(t, x) \equiv \frac{1}{2M} \sum_{j=-M}^{M} u_M\left(t, [x]_M + \frac{j}{M} \right),$$

(3.3) immediately yields:

$$\begin{aligned} \frac{\partial u_M(t, x)}{\partial t} - \frac{\partial u_N(t, x)}{\partial t} &= (u_N(t, x) - u_M(t, x)) + \lambda(1 - u_M(t, x))\Sigma_M(t, x) \\ &\quad - \lambda(1 - u_N(t, x))\Sigma_N(t, x) \\ &= (u_N - u_M) + \lambda(u_N - u_M)\Sigma_M + \lambda(1 - u_N)[\Sigma_M - \Sigma_N] \end{aligned}$$

Because $|u_N(t, x)| \leqq 1$ for all N, x, and t, an application of the triangle inequality yields the fact that (3.6) will follow upon establishing:

(i) For any $\varepsilon > 0$, there exists an $L_1(\varepsilon, x)$ so that for any $M, N > L_1(\varepsilon, x)$ and for all $t \in [0, T]$:

$$|u_M(t, x) - u_N(t, x)| < \varepsilon \, .$$

and

(ii) For any $\varepsilon > 0$, there exists an $L_2(\varepsilon, x)$ so that for any $M, N > L_2(\varepsilon, x)$ and for all $t \in [0, T]$:

$$|\Sigma_M - \Sigma_N| < \varepsilon \, .$$

Condition (i) follows directly from the triangle inequality and (3.1). To establish (ii) we observe that:

$$\begin{aligned} \Sigma_M - \Sigma_N = \Sigma_M &- \int_{x-1}^{x+1} u_M(t, y)\frac{dy}{2} + \int_{x-1}^{x+1} u_N(t, y)\frac{dy}{2} - \Sigma_N \\ &+ \int_{x-1}^{x+1} \{u_M(t, y) - u(t, y)\}\frac{dy}{2} + \int_{x-1}^{x+1} \{u(t, y) - u_N(t, y)\}\frac{dy}{2} \, . \end{aligned}$$

The first two pairs of terms on the RHS vanish in the limit, and an application of the dominated convergence theorem implies that the last two terms also vanish, establishing the result.

Proof of Theorem 2. The proof will proceed in two main steps. Given a continuous function ϕ such that $\sup_x |x^2\phi(x)| < \infty$, we will first establish that the laws of the $X^M(\phi)$ are tight in $D([0, T], R)$, and we will then prove that the finite dimensional distributions (fdds) of the processes $X_\cdot^M(\phi)$ converge to the fdds of $X_\cdot(\phi)$.

We will begin by showing that:

$$\sup_M E\left\{ \sup_{0 \leqq t \leqq T} X_s^M(\phi)^2 \right\} < \infty \tag{3.7}$$

For any $\eta > 0$, there exists a $\delta > 0$ and an M_1 so that

$$\sup_{M > M_1} P\left(\sup_{\substack{0 \leqq s, t \leqq T \\ |t-s| < \delta}} |X_t^M(\phi) - X_s^M(\phi)| > \eta \right) < \eta \, . \tag{3.8}$$

These are stronger conditions than (*i*) and (*ii*) in Lemma A.1 in the Appendix, and they imply, therefore, that the laws of the $X_\cdot^M(\phi)$ are tight in $D([0, T], R)$.

Condition (3.7) is established directly. For any $t \in [0, T]$:

$$\begin{aligned} X_t^M(\phi)^2 &= \frac{1}{M^2} \sum_{x \in \frac{1}{M}Z} \sum_{y \in \frac{1}{M}Z} \phi(x)\phi(y)^M\xi_t^\mu(x)^M\xi_t^\mu(y) \\ &\leqq \left(\frac{1}{M} \sum_{x \in \frac{1}{M}Z} |\phi(x)| \right)^2 \\ &= \left(\frac{1}{M} \sum_{|x| \leqq 1} |\phi(x)| + \frac{1}{M} \sum_{|x| > 1} \frac{|x^2\phi(x)|}{|x|^2} \right)^2 \\ &\leqq (3\|\phi\|_{0,0} + C\|\phi\|_{2,0})^2 < \infty \, . \end{aligned}$$

where $\|\phi\|_{\alpha,\beta} = \sup_x \left| x^\alpha \frac{\partial^\beta}{\partial^\beta x} \phi(x) \right|$.

To verify (3.8) we begin by selecting a positive integer N so that:

$$\frac{1}{M} \sum_{|x| > N} \frac{\|\phi\|_{2,0}}{x^2} < \frac{\eta}{2}$$

where $\eta > 0$ is already given. Note that, for any $x \in [-N, N]$, $|\phi(x)| \leqq \|\phi\|_{0,0}$. The idea is the following. If all of the sites in $[-N, N]^c$ were to change state at the same time, the large possible change in $X_t^M(\phi)$ would be $\frac{\eta}{2}$. Our attention is now directed to sites in the interval $[-N, N]$. We select a positive number r so that: $r\|\phi\|_{0,0} < \frac{\eta}{4}$. In order for the event prescribed in (3.8) to occur, there would have to exist a time interval: $I_n = \left[\frac{n\delta}{2}, \frac{(n+1)\delta}{2}\right]$ during which at least rM births or deaths occur among the sites in $[-N, N]$. Observing that λ is an upper bound on the rate at which births or deaths occur at any given site, and letting J_n be the number of jumps (births or deaths) which occur among the sites in $[-N, N]$ in the time interval I_n, we have:

$$P(J_n > rM) \leqq P(Y_\mu > rM)$$

where Y_μ has Poisson distribution with mean $\mu = (2MN + 1)\lambda\frac{\delta}{2}$. Using Chebysheff's inequality with the Laplace transform of the Poisson distribution we have:

$$\begin{aligned} P(J_n > rM) &\leqq P(Y_\mu > rM) = P(e^{\theta Y_\mu} > e^{\theta rM}) \\ &\leqq e^{-\theta rM} e^{(e^\theta - 1)\mu} . \end{aligned}$$

Setting $\theta = 1$ and using $e - 1 \leqq 2$:

$$P(J_n > rM) \leqq e^{(2MN+1)\lambda\delta - rM} .$$

Selecting δ so that $2N\lambda\delta < \frac{r}{2}$, we have:

$$P\left(J_i > rM \text{ some } i = 1, \ldots, \left[\frac{T}{\delta}\right] + 1\right) \leqq \left(\left[\frac{T}{\delta}\right] + 1\right) e^{\lambda\delta - 1/2rM} ,$$

which decreases exponentially in M. Therefore, there exists an M_1 so that for $M > M_1$ the right hand side is less than η, establishing (3.8).

The remaining task is to verify that the fdds of $X^M_.(\phi)$ converge weakly to the fdds of $X_.(\phi)$. We will establish that for any $t \in [0, T]$:

$$\lim_{M \to \infty} E\{X_t^M(\phi)\} = \int_{-\infty}^{+\infty} \phi(x)u(t, x)dx \tag{3.9}$$

$$\lim_{M \to \infty} E\{(X_t^M(\phi) - E[X_t^M(\phi)])^2\} = 0 \tag{3.10}$$

It follows immediately from (3.9) and (3.10) that for any $t_i \in [0, T]$ and any continuous ϕ_i such that $\sup_x |x^2 \phi_i(x)| < \infty$, for $i = 1, \ldots, n$, given any $\zeta > 0$:

$$\lim_{M \to \infty} P\left(\left|X_{t_i}^M(\phi_i) - \int_{-\infty}^{+\infty} \phi_i(x)u(t_i, x)dx\right| < \zeta \quad \forall i = 1, \ldots, n\right) = 1 ,$$

which implies the desired weak convergence of the fdds.

We establish (3.9) by using Fubini's Theorem:

$$\begin{aligned}\lim_{M\to\infty} E\{X_t^M(\phi)\} &= \lim_{M\to\infty} E\left\{\frac{1}{M}\sum_{x\in\frac{1}{M}Z} \phi(x)^M\xi_t^\mu(x)\right\} \\ &= \lim_{M\to\infty} \frac{1}{M}\sum_{x\in\frac{1}{M}Z} \phi(x)u_M(t,x) \\ &= \lim_{M\to\infty} \sum_{x\in\frac{1}{M}Z} \int_{x-1/2M}^{x+1/2M} [\phi(x)-\phi(s)]u_M(t,s)ds \\ &\quad + \lim_{M\to\infty} \sum_{x\in\frac{1}{M}Z} \int_{x-1/2M}^{x+1/2M} \phi(s)u_M(t,s)ds\,.\end{aligned}$$

The first term on the RHS vanishes as $M\to\infty$ due to the continuity of ϕ. The second term can be rewritten as:

$$\lim_{M\to\infty} \int_{-\infty}^{+\infty} \phi(s)u_M(t,s)ds\,.$$

Now, $|\phi u_M| \leqq |\phi|$ and ϕ is integrable. Also $\phi u_M \to \phi u$ pointwise in x. Therefore, the Dominated Convergence Theorem implies (3.9).

(3.10) follows in much the same way, but requires a result stated in Lemma 2.2:

$$\begin{aligned}\text{LHS of (3.10)} &= \lim_{M\to\infty} E\left\{\left(\frac{1}{M}\sum_{x\in\frac{1}{M}Z} \phi(x)^M\xi_t^\mu(x) - E\left(\frac{1}{M}\sum_{x\in\frac{1}{M}Z}\phi(x)^M\xi_t^\mu(x)\right)\right)^2\right\} \\ &= \lim_{M\to\infty} E\left\{\left(\frac{1}{M}\sum_{x\in\frac{1}{M}Z}\phi(x)({}^M\xi_t^\mu(x)-u_M(t,x))\right)^2\right\}\end{aligned}$$

(by Fubini's Theorem)

$$\begin{aligned}&= \lim_{M\to\infty} E\left\{\frac{1}{M^2}\sum_{x\in\frac{1}{M}Z}\sum_{y\in\frac{1}{M}Z}\phi(x)\phi(y)\gamma_M^x\gamma_M^y\right\} \\ &\leqq \lim_{M\to\infty}\left\{\frac{C_2}{M}\left(\frac{1}{M}\sum_{x\in\frac{1}{M}Z}|\phi(x)|\right)^2\right\} \quad \text{(by Lemma 2.2)} \\ &= 0\,,\end{aligned}$$

completing the proof of Theorem 2.

Proof of Corollary to Theorem 2. By Lemma A.3, tightness of the laws of the X^M processes in the space $D([0,T], S'(R))$ follows from tightness of the processes $X_\cdot^M(\phi)$ in $D([0,T], R)$ for each $\phi\in S(R)$. Since every $\phi\in S(R)$ is continuous with $\sup_x|x^2\phi(x)|<\infty$, this follows from the proof of Theorem 2. Additionally, (3.9) and (3.10) above indentify the limit uniquely.

4. Proof of Theorem 3

The first step is the explicit calculation of the generator of the process $G(Y_t^M(\phi))$ for $\phi\in C_0^\infty(R)$, and $G\in C^2(R)$, where, as before:

$$Y_t^M(\phi) = \frac{1}{\sqrt{M}}\sum_{x\in\frac{1}{M}Z}\phi(x)({}^M\xi_t^\mu(x)-u_M(t,x))\,.$$

The fact that C_0^∞ is dense in $S(R)$ allows us to define our generator for all $\phi \in S(R)$. If we define

$$\beta_M(x) \equiv \begin{cases} \dfrac{1}{2M} & \text{if} \quad x \in \dfrac{1}{M} Z \quad \text{and } |x| \leqq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f * \beta_M(x) \equiv \sum_{y \in \frac{1}{M} Z} f(y)\beta_M(y-x) = \frac{1}{2M} \sum_{j=-M}^{M} f\left(x + \frac{j}{M}\right)$$

$$f_{-x}(y) \equiv f(y-x) ,$$

we find after a little work that the generator is:

$$\begin{aligned} L_M G(Y_t^M(\phi)) = G'(Y_t^M(\phi)) Y_t^M \{ &\lambda\phi * \beta_M - \phi - \lambda\phi X_t^M((\beta_M)_{-x}) \\ &- \lambda E X_t^M(\phi(\beta_M)_{-x})\} + \tfrac{1}{2} G''(Y_t^M(\phi)) \\ &X_t^M\{\lambda\phi^2 * \beta_M + \phi^2 - \lambda X_t^M(\phi^2(\beta_M)_{-x})\} . \end{aligned} \tag{4.1}$$

Now we will verify that the set of measures $\{P_M\}$ on $D([0, T], S'(R))$ corresponding to the Y^M processes are tight. The tightness condition implies that any subsequence of $\{P_M\}$ has a further subsequence which converges to a probability measure on $C([0, T], S'(R))$. Then we will use the martingale characterization of the Generalized Ornstein Uhlenbeck processes given in [HS1] (see Lemma A.5 in the Appendix). To establish that all subsequential limits are indeed the same GOU process specified by the particular forms of two operators A_t and B_t given in the statement of the theorem.

Proof of tightness. It is sufficient to verify the following conditions (see [HS2]). Let P^M be the law of Y^M, and let F_t denote the filtration:

$$F_t = \sigma(Y_s(\phi): 0 \leqq s \leqq t; \phi \in S(R)) .$$

For any $\phi \in S(R)$,

$$\sup_M E\{Y_t^M(\phi)^2\} < \infty \text{ for all } t \in [0, T] . \tag{4.2}$$

There exist $\alpha_i^M(\phi, t) \in F$ $(i = 1, 2)$ such that:

$$J_t(\phi) \equiv Y_t^M(\phi) - \int_0^t \alpha_1^M(\phi, s)ds$$

$$K_t(\phi) \equiv J_t(\phi)^2 - \int_0^t \alpha_2^M(\phi, s)ds$$

are (P^M, F_t) martingales, and such that

$$\sup_M E\left\{\sup_{0 \leqq s \leqq t} |\alpha_i^M(\phi, s)|^2\right\} < \infty \tag{4.3}$$

There exists an $\eta(\phi, M) \downarrow 0$ as $M \to \infty$ such that:

$$\lim_{M\to\infty} P\left(\sup_{0 \leq s \leq t} |Y^M_{t^+}(\phi) - Y^M_{t^-}(\phi)| > \eta(\phi, M) \right) = 0 . \qquad (4.4)$$

Verifying (4.2) is easy. To establish (4.3) we first find α^M_1 and α^M_2. If we let

$$C^M_s(\phi)(x) \equiv \frac{\lambda}{2M} \sum_{j=-M}^{M} \phi\left(x + \frac{j}{M} \right) - \phi(x) - \phi(x) \frac{\lambda}{2M} \sum_{j=-M}^{M} {}^M\xi^\mu_t\left(x + \frac{j}{M} \right)$$
$$- \frac{\lambda}{2M} \sum_{j=-M}^{M} u_M\left(t, x + \frac{j}{M} \right) \phi\left(x + \frac{j}{M} \right) .$$

and

$$D^M_t(\phi)(x) = \frac{\lambda}{2M} \sum_{j=-M}^{M} \left\{ \left(1 - {}^M\xi^\mu_t\left(x + \frac{j}{M} \right) \right) \phi^2\left(x + \frac{j}{M} \right) \right\} + \phi^2(x) .$$

a little work yields

$$\alpha^M_1(\phi, t) = Y^M_s(C^M_t(\phi)) \qquad \alpha^M_2(\phi, t) = X^M_t(D^M_t(\phi)) .$$

Now that we know the α's, we must establish (4.3). For α^M_2 this task is relatively straightforward. For α^M_1 we will outline the procedure. First, viewing $\alpha^M_1(\phi, t)$ as a stochastic process, we calculate the drift: $b^M_1(\phi, t)$. We can then write:

$$\alpha^M_1 = \alpha^M_1 - \int_0^t b^M_1 ds + \int_0^t b^M_1 ds .$$

The first two terms comprise a martingale, and we can use the following consequence of Doob's inequality to bound them. Bounding the third term is laborious but straightforward.

Proposition 4.1.

$$E\left\{ \sup_{0 \leq s \leq t} |\alpha^M_1(\phi, s)|^2 \right\}^{\frac{1}{2}} \leq 4E\{\alpha^M_1(\phi, t)^2\}^{\frac{1}{2}} + 4t^{\frac{1}{2}} E\left\{ \int_0^t |b^M_1(\phi, s)|^2 ds \right\}^{\frac{1}{2}} . \qquad (4.9)$$

Bounding $\sup_M E\left\{ \sup_{0 \leq s \leq t} |\alpha^M_2(\phi, s)|^2 \right\}$: From (4.6) and (4.7) we find that

$$|\alpha^M_2(\phi, s)|^2 \leq \frac{1}{M^2}\left(\frac{\lambda}{2M} \right)^2 \sum_{x \in \frac{1}{M}Z} \sum_{y \in \frac{1}{M}Z} \sum_{j=-M}^{M} \sum_{k=-M}^{M} \phi^2\left(x + \frac{j}{M} \right) \phi^2\left(y + \frac{j}{M} \right)$$
$$+ \frac{1}{M^2} \sum_{x \in \frac{1}{M}Z} \sum_{y \in \frac{1}{M}Z} \phi^2(x)\phi^2(y)$$
$$\leq (\lambda^2 + 1)(C\|\phi\|_{0,0} + D\|\phi\|_{2,0})^2$$

independent of the value of t or M, completing the verification of condition (4.3).

To finish the proof of tightness, we must establish (4.4). However, this follows from standard Poisson process theory, as, given any M, the set on which two jumps occur at the same time is a set of measure zero. Therefore, we now know that the sequence of measures P^M are tight.

Uniqueness and identification of subsequential limits. Now that we know that set $\{P^M\}$ is tight, we will establish that if a subsequence $P^{M_k} \Rightarrow P$, then P is the law of the GOU process prescribed by the operators given in (1.3) and (1.4).

After observing that the two operators satisfy the required conditions, we use the martingale condition to establish uniqueness; namely that if $P^{M_k} \Rightarrow P$, then

$$G(Y_t(\phi)) - \int_0^t Y_s(A_s\phi)G'(Y_s(\phi))ds - \int_0^t \frac{1}{2} \| B_s\phi \|^2 G''(Y_s(\phi))ds$$

is a P martingale. We first note that

$$G(Y_t(\phi)) - \int_0^t Y_s(A_s^M\phi)G'(Y_s(\phi))ds - \int_0^t \frac{1}{2} \| B_s^M\phi \|^2 G''(Y_s(\phi))ds$$

is a P_M martingale, where A_t^M and $\| B_s^M \|$ are expressed implicitly in the expression (4.1) for the generator $L_M G(Y_t^M(\phi))$. This means that

$$\begin{aligned} G(Y_t(\phi)) &- \int_0^t Y_s(A_s\phi)G'(Y_s(\phi))ds - \int_0^t \frac{1}{2} \| B_s\phi \|^2 G''(Y_s(\phi))ds \\ &+ \int_0^t Y_s\{(A_s - A_s^M)\phi\} G'(Y_s(\phi))ds \\ &+ \int_0^t \frac{1}{2}\{ \| B_s\phi \|^2 - \| B_s^M\phi \|^2\} G''(Y_s(\phi))ds \end{aligned}$$

is a P^M martingale: The following proposition establishes the differentiability of $u(t, x)$ which is essential in the remainder of the proof.

Proposition 4.2. *If $u(0, x) \in S(R)$, then $u(t, x)$ has bounded x-derivatives of all order.*

Proof. Recall the following expression for $u(t, x)$ given in (3.1):

$$u(t, x) = 1 - E\left\{ \prod_{z \in Z_t^x} (1 - u(0, z)) \right\}.$$

Consequently,

$$u^{(n)}(x) = -E\left\{ \sum_{k=1}^{\infty} 1_{\{|Z_t^x| = k\}} \frac{d^n}{dx^n} \left[\prod_{i=1}^{k} (1 - u(0, z_i)) \right] \right\},$$

where z_i denotes the position of particle i. The above interchange is permissible, since we will find that the nth derivative of the integrand is bounded by an integrable function, for any given $n \geqq 0$. We now observe that there are k^n ways to distribute the n derivatives over the product of k terms using the product rule. Because $u(0, x) \in S(R)$, each of the k^n permutations is bounded in absolute value by a constant M_n given by:

$$M_n = \sup_{\substack{(j_1, \ldots, j_l) \\ j_1 + \ldots j_l = n}} \left\{ \sup_x |u^{(j_1)}(0, x)| \cdot \ldots \cdot \sup_x |u^{(j_l)}(0, x)| \right\}.$$

Therefore,

$$|u^{(n)}(x)| \leqq E\left\{\sum_{k=1}^{\infty} 1_{\{|Z_t^x| = k\}} k^n M_n\right\},$$

and the boundedness of the moments of the branching process yields the required interchange condition and the desired result.

Considering the form of $A_t(\phi)$ given in (1.3), Proposition 4.2 implies that $A_t: S(R) \to S(R)$. We use this fact, the continuity and boundedness of G and its derivatives, along with

Proposition 4.3. *If W_t^M is a martingale with respect to a filtration F_t and if:*

$$\lim_{M\to\infty} E\{|W_t^M - W_t|^2\} = 0 \quad \text{for all } t,$$

then W_t is a martingale with respect to F_t.

Proposition 4.3 implies that the proof of Theorem 3 will be complete upon showing:

$$\lim_{M\to\infty} E\left\{\left(\int_0^t Y_s^M((A_s - A_s^M)\phi)G'(Y_s^M(\phi))ds\right)^2\right\} = 0$$

$$\lim_{M\to\infty} E\left\{\left(\int_0^t \frac{1}{2}(\|B_s\phi\|^2 - \|B_s^M\phi\|^2)G''(Y_s^M(\phi))ds\right)^2\right\} = 0,$$

which follow from straightforward computations. The martingale condition given in Lemma A.5 is now verified, and we have identified the limit process P, completing the proof of Theorem 3.

Appendix: Topological Considerations

Schwartz Functions

The space of Schwartz functions denoted by $S(R)$ is a Fréchet space consisting of $C^\infty(R)$ functions topologized by the following seminorms:

$$\|\phi\|_{\alpha,\beta} = \sup_{x\in R}\left|x^\alpha \frac{d^\beta\phi}{dx^\beta}(x)\right|, \tag{A1}$$

where α and β run over the nonnegative integers. Consequently, $S(R)$ consists of those functions which, together with their derivatives, decrease faster than any polynomial as the argument approaches $\pm\infty$.

The space $D([0, T], R)$. is the space of functions from the interval $[0, T]$ to the real line R which are right-continuous with left-hand limits. We topologize this function space with the following metric:

$$d(x, y) = \inf\left\{\varepsilon: \exists\lambda\in\Lambda,\ \sup_{0\leqq t\leqq T}|\lambda(t) - t| \leqq \varepsilon, \text{ and } \sup_{0\leqq t\leqq T}|x(t) - y(\lambda(t))| \leqq \varepsilon\right\},$$

where Λ is the set of strictly increasing continuous mappings of $[0, T]$ onto itself.

To state tightness conditions, we first must define the following modulus of continuity:

$$w'_x(\delta) = \inf_{\{t_i\}} \sup_{0<i\leqq r} w_x[t_{i-1}, t_i) ,$$

where:

$$w_x[a, b) = \sup\{|x(s) - x(t)| : s, t \in [a, b)\} ,$$

and where the infimum extends over finite sequences $\{t_i\}$ with: $0 = t_0 < t_1 < \ldots < t_r = 1$, and $t_i - t_{i-1} > \delta$ for all $i = 1, \ldots, r$.

Lemma A.1. ([Bi], p. 125) *A sequence of measures $\{P^M\}$ on $D([0, T], R)$ is tight if and only if these two conditions hold:*

(*i*) *For each $\eta > 0$, there exists an a such that:*

$$P^M\left\{x: \sup_{0\leqq t\leqq T} |x(t)| > a\right\} \leqq \eta , \quad M \geqq 1$$

(*ii*) *For each $\varepsilon > 0$ and $\eta > 0$, there exists a $\delta: 0 < \delta < 1$, and an integer N such that:*

$$P^M\{x: w'_x(\delta) \geqq \varepsilon\} \leqq \eta , \quad M \geqq N .$$

The following is a martingale condition for tightness which we use in the proof of Theorem 3 and later in this Appendix. This result appears in a disguised form in [HS2] in the discussion preceding Theorem (1.15).

Lemma A.2. *Let X^M be elements of $D([0, T], R)$ with laws P^M. Let $F_s = \sigma(X_s : 0 \leqq s \leqq t)$. Then the $\{P^M\}$ are tight if the following conditions hold:*

(*i*) $E\left\{\sup_{0\leqq t\leqq T} (X_s^M)^2\right\} < \infty$

(*ii*) *There exist non-anticipating functions $\alpha_1^M(t)$ and $\alpha_2^M(t)$ such that for $t \in [0, T]$:*

$$J_t^M \equiv X_t^M - \int_0^t \alpha_1^M(s) ds$$

and

$$K_t^M \equiv (J_t^M)^2 - \int_0^t \alpha_2^M(s) ds$$

are (P^M, F_t) martingales, where:

$$\sup_M E\left\{\sup_{0\leqq t\leqq T} |\alpha_i^M(t)|^2\right\} < \infty \quad i = 1, 2 .$$

The proof of this result given in [HS2] uses standard martingale theory in conjunction with the Censov criterion for tightness ([Bi], Theorem 15.6).

The space $D([0, T], S'(R))$. is the space of functions from the interval $[0, T]$ to $S'(R)$ which are right-continuous with left-hand limits, where, as before, $S'(R)$ is given the strong topology. The topology of $D([0, T], S'(R))$ is generated by the following seminorms indexed to Θ, the bounded sets of $S(R)$:

$$d_\theta(x, y) = \inf\left\{\varepsilon : \exists \lambda \in \Lambda : \sup_{0\leqq t\leqq T} |\lambda(t) - t| \leqq \varepsilon, \text{ and } \sup_{0\leqq t\leqq T} \|x(t) - y(\lambda(t))\|_\theta \leqq \varepsilon\right\} ,$$

where, as before, Λ is the set of strictly increasing mappings of $[0, T]$ onto itself. To state the following tightness criterion we must define the following collection of projections: Given $\phi \in S(R)$ and $x \in D([0, T], S'(R))$, $\Pi_\phi : D([0, T], S'(R)) \to D([0, T], R)$ is given by:

$$\Pi_\phi(x) \to x(\phi) \in D([0, T], R)\,. \tag{A2}$$

Lemma A.3. ([Mi1]) *Consider a sequence of probability measures* $\{P_n\}$ *on* $D([0, T], S'(R))$. *If, for each* $\phi \in S(R)$ *the measure* $\{P_n \Pi_\phi^{-1}\}$ *are tight in* $D([0, T], R)$, *then the measures* $\{P_n\}$ *are tight in* $D([0, T], S'(R))$.

Once again, tightness in $D([0, T], S'(R))$ is essentially a question of tightness of the projection processes in $D([0, T], R)$. The following tightness criteria are used in the proof of Theorem 3.

Lemma A.4. *Let* X^M *be elements of* $D([0, T], S'(R))$ *with laws* P^M. *Let* $F_s = \sigma(X_s(\phi)\colon 0 \leqq s \leqq t,\ \phi \in S(R))$. *Suppose that, for each* $\phi \in S(R)$, *the following conditions hold*:

(*i*) $E\left\{\sup_{0 \leqq t \leqq T} (X_s^M(\phi))^2\right\} < \infty$

(*ii*) *There exist non-anticipating functions* $\alpha_1^M(\phi, t)$ *and* $\alpha_2^M(\phi, t)$ *such that for* $t \in [0, T]$:

$$J_t^M(\phi) \equiv X_t^M(\phi) - \int_0^t \alpha_1^M(\phi, s)ds$$

and

$$K_t^M(\phi) \equiv (J_t^M(\phi))^2 - \int_0^t \alpha_2^M(\phi, s)ds$$

are (P^M, F_t) *martingales, where*:

$$\sup_M E\left\{\sup_{0 \leqq t \leqq T} |\alpha_i^M(\phi, t)|^2\right\} < \infty \quad i = 1, 2\,.$$

(*iii*) *There exists an* $\eta(\phi, M) \downarrow 0$ *as* $M \to \infty$ *such that*:

$$\lim_{M \to \infty} P\left(\sup_{0 \leqq t \leqq T} |X_{t^+}^M(\phi) - X_{t^-}^M(\phi)| > \eta(\phi, M)\right) = 0\,.$$

Then the $\{P^M\}$ *are tight in* $D([0, T], S'(R))$, *and all subsequential limits are concentrated on* $C([0, T], S'(R))$.

Remark. This Lemma is stated in a similar form in [DIPP] as Theorem 4.5. It follows directly from Lemmas A.2 and A.3 above, and results in [Mi2].

The Generalized Ornstein-Uhlenbeck Process

This section is devoted to the description of certain probability measures on the space $C([0, T], S'(R))$ which appear as limits of the fluctuation field mentioned in Theorem 3. These measures are known as Generalized Ornstein-Uhlenbeck processes. The relevant result appears in [HS1] as Theorem 1.4. We will state the result with a minor modification – our transition kernel is not time-homogeneous. The resulting differences are slight, and the proof provided in [HS1] is easily extended to our case. First, a little notation is required. Let $T_t^s(\phi)$ be a strongly continuous bounded semigroup on $S(R)$ with a generator:

$$A_t(\phi) = \lim_{h \to 0} \left(\frac{T_{t+h}^t \phi - \phi}{h} \right), \tag{A3}$$

where $0 \leqq s < t \leqq T$. The semigroup property referred to is:

$$T_t^s T_u^t(\phi) = T_u^s(\phi) \quad \text{when} \quad 0 \leqq s \leqq t \leqq u \leqq T.$$

We require that A_t be bounded on $S(R)$ for any $t \in [0, T]$, and we let B_u be a positive bounded linear operator on $L^2(R)$ for any given $u \in [0, T]$.

Lemma A.5. *Let P be a probability measure on $\Omega \equiv C([0, T], R)$ such that $\forall f \in C_0^\infty(R)$ and for all stopping times τ such that:*

$$\sup_{\omega \in \Omega} \sup_{0 \leqq t \leqq T} |N_{t \wedge \tau}(A_{t \wedge \tau}\phi)| < \infty,$$

with N the element of Ω, then the following is a P-martingale:

$$f(N_{t \wedge \tau}(\phi)) - \int_0^{t \wedge \tau} N_u(A_u \phi) f'(N_u(\phi)) du - \int_0^{t \wedge \tau} \frac{1}{2} \| B_u \phi \|^2 f''(N_u(\phi)) du$$

where $\| \cdot \|$ designates the L^2 norm and the relevant filtration is:

$$F_t = \sigma(N_s(\phi): 0 \leqq s \leqq t, \phi \in S(R)).$$

Then,

$$f(N_t(\phi)) - \int_0^t N_u(A_u \phi) f'(N_u(\phi)) du - \int_0^t \frac{1}{2} \| B_u \phi \|^2 f''(N_u(\phi)) du \tag{A4}$$

is a P-martingale, and additionally, for any Borel set $\Gamma \subset R$,

$$P(N_t(\phi) \in \Gamma | F_s) = \int_\Gamma g\left(\int_s^t \| B_u(T_t^u \phi) \|^2 du, y - N_s(T_t^s \phi) \right) dy \quad P - a.s., \tag{A5}$$

where

$$g(t, y) = \frac{1}{\sqrt{2\pi t}} e^{\frac{-y^2}{2t}}.$$

In particular, P is uniquely determined by (A5), *and, therefore, by* (A4).

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