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On the continuity of Ornstein–Uhlenbeck processes in infinite dimensions*

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Summary. Existence and continuity of Ornstein–Uhlenbeck processes in Banach and Hilbert spaces are investigated under various assumptions.

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0 Introduction

The study of the existence and path behaviour of infinite dimensional Ornstein–Uhlenbeck processes is almost twenty years old now. Dawson in his seminal 1972 paper [7] initiated the subject. It is however in recent years that the problem has been intensively studied by many people in different settings (e.g. [1, 5, 17, 10, 16]).

In this paper we also study existence and continuity of Ornstein–Uhlenbeck processes in infinite dimensions. Our approach is purely Gaussian. We state our results in terms of the covariance function of the investigated process and we do not use the fact that the process satisfies certain infinite dimensional stochastic Itô equations. In this approach we follow Antoniadis and Carmona [1]. It enables us to state and prove results outside the boundaries of the existing theory of Itô equations in infinite dimensions. We extend and complement some results from [1, 5] and [17]; in [10] and [16] the diagonal l_2 case is studied in greater depth.

We use the framework and the language of Gross' Abstract Wiener Spaces (AWS) and cylindrical (weak) random variables as exposed in Kuo [19], and Badrikian and Chevet [2].

Let $B' \subsetneq H \subsetneq B$ be an abstract Wiener space and let $(S(t), t \geqq 0)$ be a measurable (and sufficiently integrable) semigroup of bounded linear operators either on H or on B. We investigate the existence and the continuity of a B-valued Gaussian process (X_t) with covariance given by

$$E[f'(X_s)g'(X_t)] = \int_0^{s \wedge t} \langle S'(s-u)f', S'(t-u)g' \rangle_H \, du, \quad \forall s, t > 0, \forall f', g' \in B' . \quad (0.1)$$

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In Sect. 2 we give some sufficient conditions for the existence and continuity of (X_t) in the case of a measurable (in general non C_0-) semigroup $S(t)$ on B. The results obtained generalize those proved by Da Prato et al. [5] when B is Hilbert, Smoleński [21] when $S(t)$ is C_0 both on B and H; the proof is based on the same "factorization method", which uses a continuous mapping from $L^p([0, T]; B)$ into $C([0, T]; B)$. Note that when $S(t)$ is an arbitrary C_0-semigroup on a Banach space B, the continuity of (X_t) is still an open problem.

The main result of Sect. 3 is the existence and continuity of (X_t) when $S(t)$ is an analytic semigroup on H (in general without extension on B). Thus, this section extends a result proved by Antoniadis and Carmona [1] for self-adjoint operators, and complements a result proved by Kotelenez [17] for analytic semigroups on B in the case of B being Hilbert. The proof is based on the comparison of the covariance under study with that of a continuous Gaussian B-valued process, as in Carmona [3]. In Sect. 1, we show that the approaches of Sects. 2 and 3 are not comparable.

In the case of a C_0-semigroup $(S(t), t \geqq 0)$ on a Hilbert space B, the process (X_t) given by (0.1) can be represented as the stochastic integral

$$X_t = \int_0^t S(t-s)\,\mathrm{d}W_s \,,$$

where (W_t) is a B-valued Wiener process and H is the reproducing kernel Hilbert space of the law of W_1. Therefore, (X_t) is a so-called "mild" solution of the Langevin equation

$$\begin{cases} \mathrm{d}X_t = AX_t\mathrm{d}t + \mathrm{d}W_t \\ X_0 = 0 \qquad\qquad \text{a.s.}\,, \end{cases} \tag{0.2}$$

where A is the infinitesimal generator of $S(t)$ (cf. e.g. Chojnowska–Michalik [4]). The B-valued process (W_t) is continuous, and thus if A is not "too bad", the continuity of (X_t) is intuitively clear. In fact (X_t) is known to behave much better than (W_t) does; Dawson [7] showed that (X_t) may be H-continuous while (W_t) is not even H-valued.

In the fourth section we investigate the interplay between the existence and the continuity of (X_t). Roughly speaking, we prove that in the case of self-adjoint strictly negative A, the continuity and the existence of (X_t) are equivalent if and only if A is bounded. Here we use the connection between Ornstein–Uhlenbeck processes and stationary Gaussian processes, for example studied in [15, 16, 10].

1 Preliminaries

Let $B' \subsetneq H \subsetneq B$ be an abstract Wiener space, and let γ be the associated Gaussian probability on B, i.e., such that H is the reproducing kernel Hilbert space (RKHS) of γ.

Remark 1.1 Let $(S(t), t \geqq 0)$ be a measurable semigroup of linear continuous operators on B, that is for every $x \in B$ the map $t \to S(t)x$ is measurable. This implies that for any $x \in B$, the map $t \in]0, +\infty[\to S(t)x \in B$ is continuous when B is endowed with the strong topology (cf. e.g. Theorem 10.2.2 in [14]), and that the map $(t, x) \in [0, +\infty[\times B \to S(t)x \in B$ is jointly measurable. Fix $T > 0$, $0 \leqq s, t \leqq T, f', g' \in B'$; then the map

$$r \in [0, T] \to \langle S(s-r)'f', S(t-r)'g' \rangle_H = \int_B f'[S(s-r)x]g'[S(t-r)x]\gamma(dx)$$

is measurable by Fubini's theorem. Fix $a \geqq 0$; under proper integrability assumptions on this semigroup, the integral $\int_0^T u^{-2a}|S(u)'f'|_H^2 du$ converges, and

$$\int_0^{s \wedge t} (s-u)^{-a}(t-u)^{-a}\langle S(s-u)'f', \quad S(t-u)'g' \rangle_H du$$

is the covariance of some cylindrical B-valued Gaussian process $(Z_t, 0 \leqq t \leqq T)$. To show that (Z_t) is a genuine B-valued process, it suffices to prove the existence of a Gaussian measure μ_T on B such that

$$\forall f' \in B', \int_B |f'(x)|^2 d\mu_T(x) = \int_0^T u^{-2a}|S(u)'f'|_H^2 du, \tag{1.1}$$

i.e., to show the existence of the B-valued random variable Z_T. Indeed, let $t \in [0, T]$, and let μ_t denote the cylindrical law of Z_t. Then for every $f' \in B'$,

$$\begin{aligned}\int_B f'(x)^2 \mu_t(dx) &= \int_0^t u^{-2a}|S(u)'f'|_H^2 du \\ &\leqq \int_0^T u^{-2a}|S(u)'f'|_H^2 du = \int_B f'(x)^2 \mu_T(dx) .\end{aligned}$$

Hence the RKHS of μ_t is included in the RKHS of μ_T, so that μ_t is also tight on B. When $a = 0$, we will denote by $(X_t, 0 \leqq t \leqq T)$ the corresponding cylindrical process $(Z_t, 0 \leqq t \leqq T)$.

Let us now suppose that $(S(t), t \geqq 0)$ is a measurable semigroup of linear continuous operators on H, which is a separable Hilbert space. Then for any $x, y \in H$, the map $t \in [0, T] \to \langle S(t)'x, y \rangle_H = \langle x, S(t)y \rangle_H$ is measurable. This in turn implies the joint measurability of the map $(t, x) \in [0, T] \times H \to S(t)x$. Therefore, under proper integrability assumptions on $|S(t)|_{L(H,H)}$, for $f', g' \in B' \subset H' \simeq H, 0 \leqq s, t \leqq T$,

$$\Gamma_{s,t}(f', g') = \int_0^{s \wedge t} \langle S(s-u)'f', S(t-u)'g' \rangle_H du \tag{1.2}$$

is again the covariance of some cylindrical B-valued Gaussian process $(X_t, 0 \leqq t \leqq T)$. This integrability requirement is trivially satisfied if the semigroup $(S(t), t \geqq 0)$ is C_0 on H, or more generally if $\int_0^T |S(u)|_{L(H,H)}^2 du < \infty$. In this case there exists a (genuine) B-valued Gaussian process $(X_t, 0 \leqq t \leqq T)$ with covariance $\Gamma_{s,t}(f', g')$ defined by (1.2). Indeed, for any $f' \in B'$,

$$\begin{aligned}\int_0^T |S(u)'f'|_H^2 du &\leqq T|f'|_H^2 \int_0^T |S(u)'|_{L(H,H)}^2 du \\ &\leqq C|f'|_H^2 ,\end{aligned}$$

so that the law μ_T of X_T is tight.

In the sequel, we will assume that the semigroup $(S(t), t \geqq 0)$ acts on H or on B. The following result shows that these assumptions cannot be compared.

Proposition 1.2 *Let $(S(t), t \geqq 0)$ be a measurable semigroup of bounded linear operators on a separable infinite dimensional Banach space E, which is not of the form $S(t) = \lambda^t \mathrm{Id}_E$ for some $\lambda > 0$. Given $T > 0$:*

(i) If E is Hilbert, there exists a separable Hilbert space B such that $B' \subsetneq E \subsetneq B$ is an abstract Wiener space, and a subset I of $]0, T]$ of positive Lebesgue measure for which $S(t)$ does not extend to a bounded linear operator of B for any $t \in I$.

(ii) There exists a separable Hilbert space H such that $E' \subsetneq H \subsetneq E$ is an abstract Wiener space, and a subset I of $]0, T]$ of positive Lebesgue measure for which $S(t)$ does not leave H invariant for any $t \in I$.

Proof. Fix $T > 0$; there exists $t \in]0, T]$ such that $|\lambda \mathrm{Id} - S(t)|_{L(E,E)} > 0$ for each $\lambda \in \mathbb{R}$. Set $F = S(t)$.

(i) Let E be Hilbert; there exists $h_0 \in E' \simeq E$ such that h_0 and $F'h_0$ are linearly independent. We may and do assume that $|h_0|_E = 1$, and that $(f_n; n \geqq 0)$ is a CONS of E such that $f_0 = h_0$, and the spans of $\{f_0, f_1\}$ and of $\{h_0, F'h_0\}$ are equal. Set

$$e_0 = f_0, \quad e_1 = \frac{1}{\sqrt{2}}(f_1 + f_2), \quad \text{and} \quad e_n = \frac{1}{\sqrt{n+n^2}}\left(f_1 - \sum_{k=2}^{n} f_k\right) + \frac{n}{\sqrt{n+n^2}} f_{n+1} \quad \text{for } n \geqq 2 .$$

Then (e_n) is a CONS of E, and $\sum_{n \geqq 0} n^2 \langle F'h_0, e_n \rangle_E^2 = +\infty$. Define $\|\cdot\|_B$ on E by

$$\|h\|_B^2 = \sum_{n=0}^{\infty} \frac{\langle h, e_n \rangle_E^2}{(n+1)^2} .$$

Then $\|\cdot\|_B$ is Gross-measurable, and $F'h_0 \notin B'$; this shows that $S(t)$ does not extend to a bounded linear operator of B. Let J denote the set of reals s in $]0, t[$ such that $S(s)$ extends to a bounded linear operator of B. Then J is measurable, and $t - s \notin J$ for any $s \in J$; therefore, the Lebesgue measure of $]0, t] \backslash J$ is at least $t/2$.

(ii) There exists $x \in E$ such that $y = F(x)$ and x are linearly independent. Let $(\zeta_n, n \geqq 1)$ be a sequence of independent standard Gaussian variables, $(c_n, n \geqq 1)$ a sequence of strictly positive numbers and $(x_n, n \geqq 1)$ a sequence of linearly independent elements of E such that $\mathrm{span}(x_n, n \geqq 1)$ is dense in E, $x_1 = x$, and the series $Y = \sum_{n \geqq 1} c_n x_n \zeta_n$ converges a.s. in E. Let K denote the RKHS of Y. If $y \notin K$, set $H = K$ and let γ denote the law of Y. If $y \in K$, let $H \subsetneq K$ be a dense Hilbert subspace of K which is the RKHS of some Gaussian measure γ, and such that $x \in H$ and $y \notin H$. Then $S(t)$ does not leave H invariant, and the proof is completed as in part (i). □

2 The method of factorization for a semigroup acting on B

In this section we suppose that $B' \subsetneq H \subsetneq B$ is an abstract Wiener space, that $(S(t), t \geqq 0)$ is a measurable semigroup of linear continuous operators on B. The method of factorization consists in deducing the continuity of a process from the L^p-integrability property of a "worsened" process. It depends on the following lemma, which is a slight generalization of Lemma 1 in [5].

Lemma 2.1 *Suppose that there exist $\beta \in [0, 1[$ and $C \in \mathbb{R}$ such that*

$$\forall u \in]0, T], \quad \| S(u) \|_{L(B,B)} \leqq C u^{-\beta} .$$

Let $\alpha \in]\beta, 1]$, let p be a real number such that $p(\alpha - \beta) > 1$. Then for $f \in L^p([0, T], B)$, the map

$$t \to R_\alpha f(t) = \int_0^t (t-s)^{\alpha-1} S(t-s) f(s) \mathrm{d}s \tag{2.1}$$

is continuous from $[0, T]$ in $(B, \| \quad \|)$.

Proof. It suffices to check that if the operator R_α is defined by $R_\alpha f(t) = \int_0^t (t-s)^{\alpha-1} S(t-s) f(s) \mathrm{d}s$, then there exists a constant C' such that $\| R_\alpha f(t) \|_B \leqq C' \| f \|_p$ for every $f \in C([0, T]; B)$. Let $q = p/(p-1)$; then

$$\begin{aligned} \| R_\alpha f(t) \|_B &\leqq C \int_0^t (t-s)^{\alpha-1} (t-s)^{-\beta} \| f(s) \|_B \mathrm{d}s \\ &\leqq C \left(\int_0^T u^{(\alpha-\beta-1)q} \mathrm{d}s \right)^{1/q} \| f \|_p \\ &\leqq C' \| f \|_p , \end{aligned}$$

where $C' = C T^{\alpha-\beta-1/p} [(\alpha - \beta - 1)q + 1]^{-1/q}$. Then the continuity of $R_\alpha f$ extends by a standard density argument from smooth functions to arbitrary functions $f \in L^p([0, T]; B)$. □

The following theorem is the main result of this section; it extends continuity properties of Ornstein–Uhlenbeck processes proved in [5] and [21]. Condition (ii) is crucial to obtain the existence of a B-valued process with a "worsened" covariance function, and by Lemma 2.1, condition (i) is used to prove the continuity of the Ornstein–Uhlenbeck process.

Theorem 2.2 *Let $(S(t), t \geqq 0)$ be a measurable semigroup of bounded linear operators on B such that there exist reals $\beta \in [0, \frac{1}{4}[$, $\alpha \in]\beta, \frac{1}{2} - \beta[$, $T > 0$ and $C > 0$ for which*

(i) $$\forall t \in]0, T], \quad \| S(t) \|_{L(B,B)} \leqq C t^{-\beta} . \tag{2.2}$$

(ii) $\Gamma^\alpha_{f',g'} = \int_0^T t^{-2\alpha} \langle S(t)' f', S(t)' g' \rangle_H \mathrm{d}t, f', g' \in B'$ *is the covariance of some Gaussian measure ν on B (i.e., $\Gamma^\alpha_{f',g'} = \int_B f'(x) g'(x) \nu(\mathrm{d}x)$).*

Then there exists a Gaussian B-valued stochastic process $(X_t, 0 \leqq t \leqq T)$ with continuous sample paths, and with covariance $\Gamma_{s,t}(f', g')$ defined for $0 \leqq s, t \leqq T$, and $f', g' \in B'$ by

$$\Gamma_{s,t}(f', g') = E[f'(X_s) g'(X_t)] = \int_0^{s \wedge t} \langle S(s-u)' f', S(t-u)' g' \rangle_H \mathrm{d}u . \tag{2.3}$$

Proof. The proof consists of four steps.
Step 1. The assumptions (i) and (ii) and the inequality $|S(u)' f'|_H \leqq c \| S(u) \|_{L(B,B)} \cdot \| f' \|_{B'}$ imply that for any $f' \in B'$, $\int_0^T u^{-2\alpha} \| S(u)' f' \|^2 \mathrm{d}u < \infty$. Thus, as shown in

Remark 1.1, the existence of the measure ν imposed in (ii) yields the existence of a B-valued Gaussian process $(Y_t, 0 < t \leqq T)$ with covariance

$$E[f'(Y_s)g'(Y_t)] = \int_0^{s \wedge t} (s-u)^{-\alpha}(t-u)^{-\alpha}\langle S(s-u)'f', S(t-u)'g'\rangle_H \, du ,$$

for $0 < s, t \leqq T, f', g' \in B'$. Set $Y_0 = 0$.

We prove that $(Y_t, 0 \leqq t \leqq T)$ is weakly continuous in probability. Indeed, fix $f' \in B'$, $0 \leqq s, t \leqq T$, and set $\delta = t - s$. Then $E[f'(Y_t - Y_s)^2] = A_1 + A_2$ where

$$A_1 = \int_0^{\delta} u^{-2\alpha}|S(u)'f'|_H^2 du ,$$

$$A_2 = \int_0^{s} |u^{-\alpha}S(u)'f' - (u+\delta)^{-\alpha}S(u+\delta)'f'|_H^2 du .$$

Since $u^{-2\alpha}|S(u)'f'|_H^2$ is integrable on $]0, T]$ because of assumption (2.2), A_1 converges to 0 as $\delta \to 0$. Fix $\varepsilon > 0$, and let $\eta > 0$ be chosen such that

$$\int_0^{3\eta} u^{-2\alpha}|S(u)'f'|_H^2 du \leqq \varepsilon .$$

Then for $0 < \delta < \eta$, $|A_2| \leqq 4\varepsilon$ if $s \leqq \eta$. Let $\eta \leqq s \leqq T$, and set

$$F(u, \delta) = |u^{-\alpha}S(u)'f' - (u+\delta)^{-\alpha}S(u+\delta)'f'|_H^2$$

for $s \leqq u \leqq T$. Then for every $u \in [\eta, T]$, $\lim_{\delta \to 0} F(u, \delta) = 0$.

Furthermore, for $M = \sup\{\|S(u)\|_{L(B,B)}; \eta \leqq u \leqq T\}$, since $(u+\delta)^{-\alpha}u^{\alpha} \leqq 1$, the function $F(u, \delta)$ is dominated by $2M^2|f'|_H^2 u^{-2\alpha}$. Therefore, writing A_2 as the sum of integrals over the intervals $[0, \eta[$ and $[\eta, s]$ yields that

$$|A_2| \leqq 4\varepsilon + \int_\eta^T F(u, \delta) du ,$$

and hence that $|A_2| \leqq 5\varepsilon$ for small δ.

Step 2. Therefore, given any $f' \in B'$, the map $t \to f'(Y_t)$ is continuous in probability on $[0, T]$, and hence has a measurable version, still denoted by $f'(Y_.)$. This in turn implies the existence of a measurable version $(t, \omega) \in [0, T] \times \Omega \to Y_t(\omega) \in B$ of the process Y (see [8, chapitre 4, Théorème 30] or [11, Théorème 4.1.5].

We check that $Y_.(\omega) \in L^p([0, T], B)$ for almost every ω and any $p \geqq 1$. Indeed, by Fubini's theorem and Slepian's lemma, we have that:

$$\begin{aligned} E\int_0^T \|Y_t\|^p dt &= \int_0^T \int_0^\infty pu^{p-1}P(\|Y_t\| > u) du\, dt \\ &\leqq \int_0^T \int_0^\infty pu^{p-1}P(\|Y_T\| > u) du\, dt \\ &\leqq TE(\|Y_T\|^p) , \end{aligned}$$

which is finite by the Fernique–Skorohod Theorem.

Step 3. Let p be such that $p(\alpha-\beta)>1$. Then by Lemma 2.1, the B-valued process $X=(\sin(\pi\alpha)/\pi)\,R_\alpha Y=\tilde{R}_\alpha Y$ such that

$$X_t=\frac{\sin(\pi\alpha)}{\pi}\int_0^t (t-s)^{\alpha-1}S(t-s)\,Y_s\mathrm{d}s,\quad 0\leqq t\leqq T, \tag{2.4}$$

is well-defined and continuous. To conclude the proof, it suffices to compute the covariance of $X_\cdot$.

Set $\mathscr{H}=L^2([0,T],H)$, let $\tilde{\mu}$ denote the law of $Y_\cdot$, and let $\mu=\tilde{\mu}\circ\tilde{R}_\alpha^{-1}$ denote the law of X. Since H is continuously embedded in B, given any $\varphi\in\mathscr{H}$, $\varphi\in L^2([0,T],B)$. Since $2(1-\alpha-\beta)>1$, Lemma 2.1 implies that the process defined by

$$(T_\alpha\varphi)_t=\int_0^t (t-s)^{-\alpha}S(t-s)\varphi(s)\mathrm{d}s=(R_{1-\alpha}\varphi)_t$$

belongs to $\mathscr{C}([0,T],B)\subset L^p([0,T],B)$. Furthermore,

$$\begin{aligned}(R_\alpha T_\alpha\varphi)_t&=\int_0^t (t-s)^{\alpha-1}S(t-s)\left[\int_0^s (s-u)^{-\alpha}S(s-u)\varphi(u)\mathrm{d}u\right]\mathrm{d}s\\&=\int_0^t\left\{\int_u^t (t-s)^{\alpha-1}(s-u)^{-\alpha}\mathrm{d}s\right\}S(t-u)\varphi(u)\mathrm{d}u\\&=\int_0^t\left\{\int_0^1 x^{-\alpha}(1-x)^{\alpha-1}\mathrm{d}x\right\}S(t-u)\varphi(u)\mathrm{d}u\\&=\frac{\pi}{\sin(\pi\alpha)}\int_0^t S(t-u)\varphi(u)\mathrm{d}u\,.\end{aligned}$$

The RKHS of the law μ of $X_\cdot$ is the image of the RKHS of $\tilde{\mu}$ by $\tilde{R}_\alpha$. Therefore, to show that the RKHS of μ is equal to $V(\mathscr{H})=(\tilde{R}_\alpha\circ T_\alpha)(\mathscr{H})$, it suffices to check that the RKHS of the law $\tilde{\mu}$ of Y is equal to $T_\alpha(\mathscr{H})$. Define the Hilbert norm $\|x\|_{V(\mathscr{H})}=\inf\{\|y\|_{\mathscr{H}};\ V(y)=x\}$. Then, if we identify $V(\mathscr{H})$ as the RKHS of the measure on B with covariance $\Gamma_{s,t}(f',g')=\int_0^{s\wedge t}\langle S(s-u)'f',S(t-u)'g'\rangle_H\mathrm{d}u$, we will obtain that $X_\cdot$ has the required covariance Γ.

Step 4. Thus, the proof reduces to identify the RKHS of some cylindrical law with required covariance:

$$\Gamma^\alpha_{s,t}(f',g')=\int_0^{s\wedge t}(s-u)^{-a}(t-u)^{-a}\langle S(s-u)'f',\quad S(t-u)'g'\rangle_H\mathrm{d}u$$

for $a=0$ or $a=\alpha$. The argument, which is well known (cf. e.g. Lemma 1 in [22]), is sketched for the sake of completeness.

Suppose that $a=0$, let λ be the cylindrical Gaussian probability on $C([0,T],B)$ with covariance Γ^0, i.e., such that for any $f'\in B'$, $t\in[0,T]$, $E_\lambda[|f'(\omega_t)|^2]=\Gamma^0_{t,t}(f',f')$. Let $V(\mathscr{H})$ be endowed with the Hilbert space structure defined above and let us identify $V(\mathscr{H})$ and $V(\mathscr{H})'$ in a usual way. Then $\delta_t\otimes f'$, as an element of $V(\mathscr{H})$, has its norm given by:

$$\|\delta_t\otimes f'\|^2_{V(\mathscr{H})}=\inf\{\|y\|^2_{\mathscr{H}};\quad V(y)=\delta_t\otimes f',\ \|y\|_{\mathscr{H}}\leqq 1\}\,.$$

For any $\varphi \in \mathscr{H}$, the action of $\delta_t \otimes f' = V(y)$ on $V(\varphi)$ is

$$(\delta_t \otimes f')(V(\varphi)) = f'[V(\varphi)_t] = f'\left(\int_0^t S(t-s)\varphi(s)\mathrm{d}s\right),$$

and hence the norm of $\delta_t \otimes f'$ in $V(\mathscr{H})$ is such that:

$$\|\delta_t \otimes f'\|^2_{V(\mathscr{H})} = \sup\left\{\left|f'\left(\int_0^t S(t-s)\varphi(s)\mathrm{d}s\right)\right|^2 ; \|\varphi\|_{\mathscr{H}} \leqq 1\right\}$$

$$= \sup\left\{\left\langle \varphi, \int_0^t S(t-s)' f' \mathrm{d}s\right\rangle^2_{\mathscr{H}} ; \|\varphi\|_{\mathscr{H}} \leqq 1\right\}$$

$$= \int_0^t |S(t-s)'f'|^2_H \mathrm{d}s = \Gamma^0_{t,t}(f', f') .$$

A similar argument for $a = \alpha$ show that the RKHS of the law $\tilde{\mu}$ of $Y_.$, with covariance Γ^α, is equal to $T_\alpha(\mathscr{H})$; this concludes the proof. □

The following results show that Theorem 2.2 extends the continuity of Ornstein–Uhlenbeck processes proved in [5] for Hilbert spaces B and for C_0-semigroups $S(t)$, and in [21] for C_0-semigroups on Banach spaces B being at the same time C_0 on H. Indeed, if B is a Hilbert space, stochastic integration provides a B-valued random variable with covariance $\Gamma^\alpha_{f',g'}$.

Theorem 2.3 *Suppose that B is a Hilbert space, and let $(W(t), t \geqq 0)$ be a B-valued Brownian motion. Let $(S(t), t \geqq 0)$ be a measurable semigroup of linear continuous operators on B such that:*

$$\exists \beta \in [0, \tfrac{1}{4}[, \exists C \in \mathbb{R}, \forall t \in]0, T], \|S(t)\|_{L(B,B)} \leqq Ct^{-\beta} . \tag{2.5}$$

Then there exists a B-valued continuous Gaussian process $(X_t, t \in [0, T])$ with covariance given by

$$E[f'(X_s)g'(X_t)] = \int_0^{s \wedge t} \langle S(s-u)'f', \quad S(t-u)'g'\rangle_H \mathrm{d}u .$$

Proof. Let H denote the RKHS of the law of $W(T)$. Since for $\alpha < \frac{1}{2} - \beta$,

$$\int_0^T (T-s)^{-2\alpha}(T-s)^{-2\beta}\mathrm{d}s < \infty ,$$

the condition (ii) of Theorem 2.2 is satisfied for the Gaussian measure $\nu = P \circ Y_T^{-1}$ on B, where

$$Y_T = \int_0^T (t-s)^{-\alpha} S(t-s)\mathrm{d}W_s . \qquad \square$$

If $(S(t), t \geqq 0)$ is also a measurable semigroup on H with "not too big" norm as $t \to 0$ (e.g., is C_0 on H), then the condition (ii) in Theorem 2.2 is fulfilled. More precisely we have the following:

Theorem 2.4 *Let $B' \subsetneq H \subsetneq B$ be an abstract Wiener space and let $(S(t), t \geqq 0)$ be a measurable semigroup of linear continuous operators on B and on H such that:* $\exists \beta \in [0, \frac{1}{4}[, \exists C \in \mathbb{R}, \forall t \in]0, T]$

$$\| S(t) \|_{L(B,B)} \leqq C t^{-\beta} ,$$

$$|S(t)|_{L(H,H)} \leqq C t^{-\beta} . \tag{2.6}$$

Then there exists a continuous B-valued Gaussian process with covariance Γ given by

$$E[f'(X_s) g'(X_t)] = \int_0^{s \wedge t} \langle S(s-u)' f', S(t-u)' g' \rangle_H du .$$

Proof. For $\alpha \in]\beta, \frac{1}{4}[$, the cylindrical Gaussian measure ν with covariance defined by

$$\forall f', g' \in B', \int_0^T (T-s)^{-2\alpha} \langle S(T-s)' f', \quad S(T-s)' g' \rangle_H ds ,$$

is tight on B. Indeed, given any $f' \in B'$, $\beta < \alpha < \frac{1}{4} < \frac{1}{2} - \beta$, so that $2(\alpha + \beta) < 1$; thus

$$\int_0^T (T-s)^{-2\alpha} |S(T-s)' f'|_H^2 ds \leqq C^2 \left(\int_0^T (T-s)^{-2(\alpha+\beta)} ds \right) |f'|_H^2$$

$$\leqq C' |f'|_H^2 ,$$

where $C' = (T^{1-2\alpha-2\beta}/1 - 2\alpha - 2\beta) C^2$. Theorem 2.2 concludes the proof. □

Let us state the following obvious questions which remain open: if $(S(t), t \geqq 0)$ is C_0 on a Banach space B and $B' \subsetneq H \subsetneq B$ is an abstract Wiener space, does there exist a B-valued process (X_t) with covariance given by (2.3), and if yes is it continuous?

We finally give an example of a measurable semigroup of linear continuous operators on B which satisfies (2.2), but is not C_0.

Example 2.5 This example is taken from [18, p. 161]. Let $E = L^2([0, +\infty[, \mathbb{R}^2)$, let $k > 2$, and let A be the operator on E defined by:

$$A \begin{pmatrix} f \\ g \end{pmatrix} (x) = \begin{pmatrix} -x^2 f(x) \\ x^k f(x) - x^2 g(x) \end{pmatrix} .$$

Then A generates the semigroup $(S(t) = e^{At}, t \geqq 0)$ on E, defined by:

$$S(t) \begin{pmatrix} f \\ g \end{pmatrix} (x) = \begin{pmatrix} e^{-tx^2} f(x) \\ x^k t e^{-tx^2} f(x) + e^{-tx^2} g(x) \end{pmatrix} .$$

Then the components of $S(t)(f, g)$ are the Fourier transforms (with respect to the variable y) of functions $V_1(t, y)$ and $V_2(t, y)$ which satisfy a partial pseudo differential equation. We estimate the norm of $S(t)$ for $0 < t \leqq T$. Fix $t \in]0, T]$; the supremum of the function $x \to x^k t e^{-tx^2}$ is achieved for x_0 such that $x_0^2 = k/2t$, and is equal to $(k/2)^{k/2}\, e^{-(k/2)} t^{1-(k/2)}$. Set $\beta = (k/2) - 1$, and let $f, g \in L^2([0, +\infty[, \mathbb{R})$ be such that $\int_0^\infty [f^2(x) + g^2(x)]\mathrm{d}x \leqq 1$. Then

$$\left\| S(t)\begin{pmatrix} f \\ g \end{pmatrix} \right\|^2 = \int_0^\infty e^{-2tx^2}[f^2(x) + x^{2k}t^2 f^2(x) + 2x^k t f(x)g(x) + g^2(x)]\mathrm{d}x$$

$$\leqq 2 + 2\int_0^\infty e^{-2tx^2} x^{2k} t^2 f^2(x)\mathrm{d}x$$

$$\leqq 2 + 2\left(\frac{k}{2}\right)^k e^{-k} t^{2-k},$$

which yields that $\| S(t) \| \leqq C_1 t^{-\beta}$ for $t \in]0, T]$.

Choosing $g = 0$ and letting $f = 0$ outside a neighborhood of x_0 on which $x^k t e^{-tx^2} \geqq \frac{1}{2}(k/2)^{k/2} e^{-(k/2)} t^{-\beta}$, and such that $\int_0^\infty [f^2(x) + g^2(x)]\mathrm{d}x = 1$, we obtain that there exists a constant $C_2 > 0$ such that for $t \in]0, T]$, $\| S(t) \| \geqq C_2 t^{-\beta}$. Therefore, if $k > 2$ the semigroup $S(t)$ is not C_0 on E. However, if $2 < k < 5/2$ one has that $0 < \beta < \frac{1}{4}$, and hence for any $\alpha \in]\beta, \frac{1}{2} - \beta[$ and $f' \in E'$,

$$\int_0^T u^{-2\alpha} \| S(u)' f' \|_{E'}^2 \mathrm{d}u \leqq \int_0^T u^{-2\alpha - 2\beta} \mathrm{d}u < \infty .$$

3 Semigroups acting on H

Let $B' \subsetneq H \subsetneq B$ be an abstract Wiener space, let $(S(t), t \geqq 0)$ be a "regular" semigroup on H. In this section we prove the existence of a B-valued continuous process $(X_t, t \geqq 0)$ with given covariance Γ defined by

$$\forall s, t \in [0, T], \forall f', g' \in B', \Gamma_{s,t}(f', g') = \int_0^{s \wedge t} \langle S(s-u)' f', S(t-u)' g' \rangle_H \mathrm{d}u . \tag{3.1}$$

We do not require anything on the action of the semigroup on B. Our results generalize Theorem II.1′ of Antoniadis and Carmona [1], and therefore cover their interesting examples 1, 2, 4, pp. 39–40. Again we impose two assumptions. The first one ensures the existence of a B-valued process with covariance Γ defined in (3.1). The second one is a "continuity" property of the semigroup, which is used to compare the covariance Γ with that of a continuous B-valued Gaussian process.

Proposition 3.1 *Let $B' \subsetneq H \subsetneq B$ be an abstract Wiener space, and let $(S(t), t \geqq 0)$ be a measurable semigroup on H, such that:*
(i) *There exists $p > 1$ such that*

$$\int_0^T | S(u) |_{L(H,H)}^{2p} \mathrm{d}u < \infty . \tag{3.2}$$

(ii) *There exists $C > 0$ and $\alpha \in]0, 1]$ for which:*

$$\forall t \in [0, T], \forall x \in H, \int_0^T | [S(t)' - \mathrm{Id}] S(u)' x |_H^2 \mathrm{d}u \leqq C t^\alpha | x |_H^2 . \tag{3.3}$$

Then there exists a B-valued continuous Gaussian process $(X_t, 0 \leqq t \leqq T)$ *with covariance* Γ *given by* (3.1).

Proof. Let $(\xi_t, 0 \leqq t \leqq T)$ denote the cylindrical B-valued Gaussian process with covariance Γ given by (3.1). The integrability condition (3.2) on the norm of the semigroup implies the tightness of the law of ξ_T. Furthermore, given any $f' \in B'$, the variance of $f'(\xi_t)$ is clearly increasing with t, so that the argument in Remark 1.1 implies that the cylindrical law of each ξ_t is tight. Hence $(\xi_t, 0 \leqq t \leqq T)$ is a genuine B-valued process.

To prove that $\xi.$ has a continuous modification, we apply Proposition 5 in [3]. Set $Z(f') = f'(\xi_T)$ for every $f' \in B'$, and let U' denote the unit ball of B'. Then the Gaussian process $(Z(f'), f' \in U')$ is weakly continuous and satisfies:

$$\forall t \in [0, T], \forall (f', g') \in U' \times U', |\Gamma_{t,t}(f' - g', f' - g')| \leqq E[|Z(f') - Z(g')|^2] \,. \tag{3.4}$$

Let $0 \leqq s \leqq t \leqq T, f' \in B'$, and set $q = p/(p-1)$; *then*:

$$\begin{aligned} E[|f'(\xi_t) - f'(\xi_s)|^2] &= \int_0^s |S(t-u)'f' - S(s-u)'f'|_H^2 du + \int_s^t |S(t-u)'f'|_H^2 du \\ &\leqq \int_0^s |[S(t-s)' - \mathrm{Id}]S(u)'f'|_H^2 du \\ &\quad + |f'|_H^2 \left(\int_0^{t-s} |S(u)'|_{L(H,H)}^{2p} du \right)^{1/p} (t-s)^{1/q} \\ &\leqq C \| f' \|_{B'}^2 (t-s)^{\beta} \,, \end{aligned}$$

where $\beta = \inf(\alpha, q^{-1}) \in]0, 1]$. Let $(Y_t, t \in [0, T])$ be a Gaussian continuous process such that $E(|Y_s - Y_t|^2) = C|t-s|^{\beta}$ for $(s, t) \in [0, T]^2$. Then:

$$\forall (s, t) \in [0, T]^2, \forall f' \in B', E[|f'(\xi_s) - f'(\xi_t)|^2] \leqq E[|Y_s - Y_t|^2] \,. \tag{3.5}$$

By Proposition 5 in [3], (3.4) and (3.5) imply the existence of a B-valued continuous stochastic process with covariance given by (3.1). □

Remark 3.2 The semigroup described in Example 2.5 also provides an example of a semigroup $(S(t), t \geqq 0)$ on $H = E$ which satisfies the assumptions of Proposition 3.1 for $2 < k < 3$. Furthermore, if $\beta = (k/2) - 1$, $\| S(t) \|_{L(E,E)} \sim Ct^{-\beta}$. Therefore, for $\frac{5}{2} < k < 3$, conditions (3.2) and (3.3) hold, while (2.6) fails. Proposition 3.1 shows that for any abstract Wiener space $B' \subsetneq H \subsetneq B$, (X_t) is a B-valued continuous process, while Proposition 1.2(i) yields the existence of such an AWS such that $(S(t), t \geqq 0)$ does not extend to B.

Furthermore, Example 2.5, Theorem 2.3 and Proposition 1.2(ii) show that there exists a semigroup $(S(t), t \geqq 0)$ on a Hilbert space B such that $\| S(t) \|_{L(B,B)} \sim Ct^{-\beta}$ for $0 < \beta < \frac{1}{4}$, $(S(t), t \geqq 0)$ does not act on the RKHS H of some measure γ on B, and yet the process (X_t) is defined and continuous on B. Condition (2.6) clearly implies (3.2).

Proposition 3.1 applies to sectorial semigroups on H. Indeed, let $(S(t) = e^{-tA}, t \geqq 0)$ be a semigroup generated by a sectorial operator A with spectrum included in $\{\mathrm{Re}(z) > \delta > 0\}$; then (see e.g. [13, p. 26] for any $\alpha \in]0, \frac{1}{2}[$

there exists constants δ, C_α and M_α such that for any $u > 0$, $S(u)x \in \mathrm{Dom}(A^\alpha)$. Thus for any $x \in H$, $t \in [0, T]$,

$$|S(t)S(u)x - S(u)x|_H \leqq C_\alpha t^\alpha |A^\alpha S(u)x|_H$$
$$\leqq C_\alpha M_\alpha t^\alpha u^{-\alpha} e^{-\delta u} |x|_H ,$$

and hence

$$\int_0^T |[S(t)' - \mathrm{Id}]S(u)'x|_H^2 du \leqq C_\alpha^2 M_\alpha^2 t^{2\alpha} |x|_H^2 \int_0^T u^{-2\alpha} du$$
$$= Ct^{2\alpha}|x|_H^2 ,$$

so that both conditions (3.2) and (3.3) are satisfied. We next give a continuity result for $X_.$ in the case of an arbitrary analytic semigroup on H. Its simple proof is based on a maximal regularity result for the solution of an Abstract Cauchy Problem ([9]; cf. also [20] or [6]), which we apply to obtain the direct comparison of the RKHS of $X_.$ and the RKHS of the Wiener process which is canonically associated with H.

Theorem 3.3 *Let $B' \subsetneq H \subsetneq B$ be an abstract Wiener space with a separable space B, and let $(S(t), t \geqq 0)$ be an analytic semigroup on H. Then there exists a B-valued continuous Gaussian process $(X_t, 0 \leqq t \leqq T)$ with covariance Γ defined by* (3.1).

Proof. An analytic semigroup is C_0 (see e.g. [13]), so that $\int_0^T |S(u)|_{L(H,H)}^2 du < \infty$. Hence there exists a B-valued process $(X_t, 0 \leqq t \leqq T)$ with covariance Γ. Let μ denote the law of $X_.$ on $L^2([0, T], B)$. The RKHS associated with the Gaussian measure μ is

$$H_\mu = \left\{ y(\cdot) | y(t) = \int_0^t S(t-u)x(u)du;\ x \in L^2([0, T], H) \right\} , \tag{3.6}$$

(cf. e.g. [22, p. 229]). To show that the inclusion $i: H_\mu \to C([0, T], B)$ is Gross-measurable, it suffices to check that there exists a Hilbert space $K \supset H_\mu$ such that the inclusion $j: K \to C([0, T], B)$ is Gross-measurable. Set

$$K = \left\{ z(\cdot) = \int_0^\cdot \varphi(s)ds \,|\, \varphi \in L^2([0, T], H) \right\} .$$

Then K is the RKHS of the law of a B-valued continuous Wiener process $(W_t, 0 \leqq t \leqq T)$ with $E(|f'(W_1)|^2) = |f'|_H^2$ for $f' \in B'$.

We will show that $H_\mu \subset K$. Let $-A$ denote the infinitesimal generator of $S(t)$; since the semigroup is analytic, for every $x \in L^2([0, T], H)$ and almost every $t \in [0, T]$ we have:

$$\int_0^t S(t-u)x(u)du \in \mathrm{Dom}(A), \quad A \int_0^\cdot S(.-u)x(u)du \in L^2([0, T], H) ,$$

where the last integral on $[0, .]$ is defined as the limit of the corresponding ones on $[0, . - \delta]$ as $\delta \to 0$, and

$$\int_0^t S(t-u)x(u)du = \int_0^t \varphi(s)ds \quad \text{for} \quad \varphi(s) = -A\int_0^s S(s-u)x(u)du + x(s) ,$$

which concludes the proof ([9, Lemma 3.1]; cf. also [20] or [6, p. 315]). □

Remark 3.4 In [17] the continuity of $X_.$ is proved in the case of B being Hilbert and $S_.$ being analytic on B (actually, a much more general result is proved for analytic semigroups on a Hilbert space). The following question is still unanswered: is the conclusion of Theorem 3.3 still true if $S(t)$ is C_0 on H; we believe it is.

Remark 3.5 The proof of Theorem 3.3 shows that its conclusion remains valid if the analyticity hypothesis made on the semigroup $(S(t), t \geqq 0)$ is replaced by the following assumptions (i)–(iii):
(i) $(S(t), t \geqq 0)$ is a measurable semigroup on H with infinitesimal generator $-A$.
(ii) $\int_0^T |S(u)|^2_{L(H,H)} du < \infty$.
(iii) For every $x \in L^2([0, T], H)$ and almost every $t \in [0, T]$, $\int_0^t S(t-u)x(u)du = \int_0^t \varphi(s)ds$, where $\varphi(\cdot) = -\psi(\cdot) + x(\cdot)$ and the map ψ defined by

$$\psi(t) = -A \int_0^t S(t-u)x(u)du = \lim_{\delta \to 0} \int_0^{t-\delta} AS(t-u)x(u)du$$

belongs to $L^2([0, T], H)$.

Remark 3.6 Let $B' \subsetneq H \subsetneq B$ be an abstract Wiener space with a separable space B, let K be a Hilbert space such that $K \subset H \simeq H' \subset K'$. In the statements of Proposition 3.1 and Theorem 3.3 one can replace the conclusion by the existence of a B-valued Gaussian process with covariance Γ^K defined by:

$$\forall s, t \in [0, T], \forall f', g' \in B', \Gamma^K_{s,t}(f', g') = \int_0^{s \wedge t} \langle S(s-u)'f', S(t-u)'g' \rangle_{K'} du \,. \tag{3.7}$$

Remark 3.7 Let $(S(t) = e^{-tA}, t \geqq 0)$ be a sectorial (hence analytic) semigroup of self-adjoint operators defined on a Hilbert space H. Suppose that the spectrum of A, say $\Sigma(A)$, is included in $[\lambda_0, +\infty[$ for some $\lambda_0 \in \mathbb{R}$. Let f: $[\lambda_0, +\infty[\to \mathbb{R}$ be such that $(S(t), t \geqq 0)$ has an analytic extension to $\mathrm{Dom}[f(A)]$, let $K \subset \mathrm{Dom}[f(A)]$ be a Hilbert space, and let B be a Banach space such that the inclusion $K \subset B$ is Gross-measurable. Then for any $T > 0$, Theorem 3.3 implies the existence of a B-valued Gaussian process $(X_t, 0 \leqq t \leqq T)$ with covariance Γ^K defined by (3.7).

Remark 3.8 The two previous remarks cover the case of Theorem II.1′ in [1], and the corresponding examples on pp. 39 and 40. Indeed, Antoniadis and Carmona's theorem can be formulated as follows: Let A be a self-adjoint operator of H with spectrum $\Sigma(A) \subset [\lambda_0, +\infty[$ for some $\lambda_0 \in \mathbb{R}$. Then $-A$ is a sectorial operator, and generates an analytic semigroup $(S(t), t \geqq 0)$ on H (see e.g. [13]). Let $a = \frac{1}{2}$, $H_a = \mathrm{Dom}(\mathrm{Id} + |A|)^a$ be endowed with the norm $\|f\|_a = |(\mathrm{Id} + |A|)^a f|_H$, let H_{-a} denote the completion of H with respect to the norm associated with the scalar product:

$$\langle f, g \rangle_{-a} = \langle (\mathrm{Id} + |A|)^{-a} f, \quad (\mathrm{Id} + |A|)^{-a} g \rangle_H \,.$$

Then $(S(t), t \geqq 0)$ is a semigroup on H_a, with infinitesimal generator $-A$ (with domain $[(\mathrm{Id} + |A|)^a]^{-1}(\mathrm{Dom}(A))$). Let H_a be the RKHS of a Gaussian measure μ_a on some Banach space B, and let K be another Hilbert space such that $H_{-a} \simeq H'_a \subset K'$. Then Remarks 3.6 and 3.7 yield the existence of a B-valued continuous Gaussian process with covariance $\int_0^{s \wedge t} \langle S(s-u)'f', S(t-u)'g' \rangle_{K'} du$.

Remark 3.9 Theorem II.1′ in [1] can also be deduced from Proposition 3.1. Indeed, $(S(t), t \geqq 0)$ is a C_0-semigroup on H_a with $a = \frac{1}{2}$. Let $E(\Lambda)$ denote the spectral resolution of the identity associated with A, and set $R = (\mathrm{Id} + |A|)^a$. Clearly, for $u, t \in [0, T]$ and $x \in \mathrm{Dom}(A)$,

$$\|[S(t) - \mathrm{Id}]S(u)x\|_a^2 = |R[S(t) - \mathrm{Id}]S(u)x|_H^2$$
$$= \int_{\Sigma(A)} (1 + |\lambda|)(e^{-\lambda t} - 1)^2 e^{-2\lambda u} d\langle E(\lambda)x, x\rangle .$$

If $\lambda_0 \leqq \lambda < 0$,

$$(e^{-\lambda t} - 1)^2 \leqq e^{2t|\lambda|} - 1 \leqq 2|\lambda| t e^{2|\lambda_0|T} .$$

If $\lambda > 0$,

$$(1 - e^{-\lambda t})^2 \leqq 1 - e^{-2\lambda t} \leqq 2|\lambda| t \leqq 2|\lambda| t e^{2|\lambda_0|T} .$$

Therefore, for $C = 2e^{2|\lambda_0|T}$,

$$\|[S(t) - \mathrm{Id}]S(u)x\|_a^2 \leqq Ct|R^2 S(u)x|_H^2 ,$$

and

$$\int_0^T \|[S(t) - \mathrm{Id}]S(u)x\|_a^2 \,\mathrm{d}u \leqq Ct \int_0^T \int_{\Sigma(A)} (1 + |\lambda|)^2 e^{-2\lambda u} d\langle E(\lambda)x, x\rangle \mathrm{d}u$$
$$\leqq CtT \int_{\Sigma(A) \cap [\lambda_0, 0]} (1 + |\lambda|)(1 + |\lambda_0|)e^{2|\lambda_0|} d\langle E(\lambda)x, x\rangle$$
$$+ Ct \int_{\Sigma(A) \cap [0, +\infty[} (1 + |\lambda|)^2 \frac{1 - e^{-2\lambda T}}{2\lambda} d\langle E(\lambda)x, x\rangle$$
$$\leqq C' t \|x\|_a^2 ,$$

for some constant $C' > 0$. This inequality extends from $\mathrm{Dom}(A)$ to H_a by a standard density argument. This completes the proof of (3.6) with $\alpha = 1$. Thus, given any Hilbert space $K \subset H_a$, Proposition 3.1 and Remark 3.6 yield the existence of a B-valued continuous Gaussian process with covariance $\int_0^{s \wedge t} \langle S(s-u)'f', S(t-u)'g'\rangle_{K'} \,\mathrm{d}u$.

4 Relationship between the existence and the continuity of Ornstein–Uhlenbeck processes

In this section we suppose that $K \simeq K'$ is a given Hilbert space and that $(S(t), t \geqq 0)$ is a C_0-semigroup of self-adjoint operators on K with infinitesimal generator $-A$. Then

$$\Gamma_{s,t}(f', g') = \int_0^{s \wedge t} \langle S(s-u)'f', S(t-u)'g'\rangle_{K'} \,\mathrm{d}u \qquad (4.1)$$

is defined for any $s, t \geqq 0$ and $f', g' \in K'$. Given $T > 0$ let H denote the RKHS of the cylindrical measure on K with covariance $\Gamma_{T,T}(f', g')$; then $H \subset K$.

Theorem 4.1 *Let K, $S(t)$, A, H and Γ be as above. Assume moreover that $\Sigma(A) \subset [\lambda_0, \infty[$ for some $\lambda_0 > 0$. Then the following are equivalent:*

(i) *A is bounded.*

(ii) *For any Gross-measurable norm $\|\ \|$ on H and $B = \bar{H}^{\|\ \|}$, there exists a B-valued almost surely continuous Gaussian process with covariance Γ defined by* (4.1).

Proof. (i) implies that $H = K$, and thus it implies (ii) by the Theorem 3.3 above. Note that if A is a self-adjoint operator on K with $\Sigma(A) \subset [\lambda_0, \infty[$ for some $\lambda_0 > 0$, and if $E(\Lambda)$ denotes the spectral decomposition of A, then for every $f' \in K'$,

$$\begin{aligned}\int_0^\infty |S(t)'f'|_{K'}^2 \, dt &= \int_0^\infty \int_{\Sigma(A)} e^{-2\lambda t} d\langle E(\lambda)f', f'\rangle \, dt \\ &= \int_{\Sigma(A)} \frac{1}{2\lambda} d\langle E(\lambda)f', f'\rangle \\ &\leqq \frac{1}{2\lambda_0} |f'|_{K'}^2 .\end{aligned}$$

Therefore, $\Gamma(f', g') = \int_0^\infty \langle S(t)'f', S(t)'g'\rangle_{K'} \, dt$ is the covariance of a cylindrical Gaussian random variable on K, say Y_0; we also assume that for any $f \in K'$, $Y_0(f)$ is independent of $f(X_.)$ where $X_.$ has covariance Γ. Then the cylindrical process defined by $Y_t = S(t) Y_0 + X_t$ is stationary. Indeed, for every $f \in K' \simeq K$, $s \leqq t$,

$$\begin{aligned}E[f(Y_s)f(Y_t)] &= E[S(s)'f(Y_0)S(t)'f(Y_0)] + E[f(X_s)f(X_t)] \\ &= \int_0^\infty \langle S(u+s)'f, S(u+t)'f\rangle \, du \\ &\quad + \int_0^s \langle S(s-u)'f, S(t-u)'f\rangle \, du \\ &= \int_{\Sigma(A)} \left[\int_0^\infty e^{-\lambda(s+t+2u)} du + \int_0^s e^{-\lambda(t-u)-\lambda(s-u)} du \right] d\langle E(\lambda)f, f\rangle \\ &= \int_{\Sigma(A)} e^{-\lambda(t-s)} \frac{1}{2\lambda} d\langle E(\lambda)f, f\rangle \\ &= \frac{1}{2} \left| A^{-\frac{1}{2}} S\left(\frac{t-s}{2}\right) f \right|_K^2 .\end{aligned}$$

The RKHS of the law of Y_0 is equal to $H_\infty = A^{-\frac{1}{2}}K$.

To prove that (ii) implies (i), let A be self-adjoint, unbounded and such that $\lambda_0 = \inf \Sigma(A) > 0$. We construct a Banach B such that the process (X_t) is B-valued, and such that

$$\forall x > 0, \forall a > 0, P\left(\sup_{0 \leqq s,t \leqq a} \|X_t - X_s\|_B \geqq x \right) = 1 , \tag{4.2}$$

which implies that (X_t) is almost surely discontinuous. Since $\sup \Sigma(A) = +\infty$, there exists an increasing sequence of numbers $(\alpha_k, k \geqq 0)$, such that $\alpha_0 = \lambda_0$, and for $k \geqq 1$:

$$\alpha_{k+1} > \alpha_k + 1, \quad k^{-1}\alpha_k > t_k \geqq 1 ,$$

where t_k will be specified later,

$$F_k = E(I(k))(K) \neq \{0\} \quad \text{for } I(2k) = [\alpha_k, \alpha_k + 1[, I(2k+1) = [\alpha_k + 1, \alpha_{k+1}[,$$

$$F_1 = E([\lambda_0, \alpha_1[)(K) \neq \{0\} .$$

Then $K = \oplus_{k \geqq 1} F_k$; for every $k \geqq 1$ let $(e_i^k, 1 \leqq i < J(k))$ be a complete orthonormal system of F_k, with $2 \leqq J(k) \leqq +\infty$. For any $k \geqq 1$, we construct by induction a complete orthonormal system of F_k, say $(\varepsilon_i^k; 1 \leqq i < J(k))$ and an increasing sequence of integers $(N^k(i), 1 \leqq i < J(k))$ such that for all $1 \leqq i < J(k)$,

$$(i+2) \wedge J(k) \leqq N^k(i) + 1 \leqq (2i+2) \wedge J(k), \tag{4.3}$$

$$\operatorname{span}(e_j^k, 1 \leqq j \leqq i) \subset \operatorname{span}(\varepsilon_j^k; 1 \leqq j \leqq N^k(i)), \tag{4.4}$$

$$A^{-\frac{1}{2}} \varepsilon_i^k \in \operatorname{span}(\varepsilon_j^k; 1 \leqq j \leqq N^k(i)). \tag{4.5}$$

Let us fix $k \geqq 1$ and drop k from the notation for simplicity.

Set $\varepsilon_1 = e_1$. If $J = 2$, set $N(1) = 1$; if $J > 2$, set $N(1) = 2$ and let $f_0 = A^{-\frac{1}{2}}\varepsilon_1 - \langle A^{-\frac{1}{2}}\varepsilon_1, \varepsilon_1\rangle \varepsilon_1$. If $f_0 = 0$ set $\varepsilon_2 = e_2$, and otherwise set $\varepsilon_2 = f_0 |f_0|_K^{-1}$.

Suppose that $N(i) < J$ and the family of orthonormal vectors $(\varepsilon_j; 1 \leqq j \leqq N(i))$ have been defined, and satisfy the induction hypothesis (4.3) up to (4.5).

If $N(i) + 2 \leqq J$, then set

$$f_i = A^{-\frac{1}{2}}\varepsilon_{i+1} - \sum_{1 \leqq j \leqq N(i)} \langle A^{-\frac{1}{2}}\varepsilon_{i+1}, \varepsilon_j\rangle \varepsilon_j, \tag{4.6}$$

$$L(i) = \inf\{n \geqq 1; e_n \notin \operatorname{span}(\varepsilon_j; 1 \leqq j \leqq N(i))\}, \tag{4.7}$$

$$g_i = e_{L(i)} - \sum_{1 \leqq j \leqq N(i)} \langle e_{L(i)}, \varepsilon_j\rangle \varepsilon_j \neq 0. \tag{4.8}$$

Suppose that $N(i) + 2 < J$. If $f_i = 0$, set $N(i+1) = N(i) + 1$, and $\varepsilon_{N(i+1)} = g_i |g_i|_K^{-1}$. If $f_i \neq 0$, let $\varepsilon_{N(i)+1} = f_i |f_i|_K^{-1}$, and let

$$h_i = e_{L(i)} - \sum_{1 \leqq j \leqq N(i)+1} \langle e_{L(i)}, \varepsilon_j\rangle \varepsilon_j.$$

If $h_i = 0$, set $N(i+1) = N(i) + 1$, and otherwise set $N(i+1) = N(i) + 2$, and $\varepsilon_{N(i+1)} = h_i |h_i|_K^{-1}$. Then the family $(\varepsilon_j; 1 \leqq j \leqq N(i+1))$ is again orthonormal and satisfies the conditions (4.3)–(4.5).

If $N(i) + 2 = J < \infty$, then for $i + 1 \leqq j < J$, set $N(j) = J - 1$, and let f_i, $L(i)$, and g_i be defined by (4.6), (4.7) and (4.8) respectively. If $f_i = 0$, set $\varepsilon_{N(i+1)} = g_i |g_i|_K^{-1}$, and otherwise set $\varepsilon_{N(i+1)} = f_i |f_i|_K^{-1}$. Then the family $(\varepsilon_j; 1 \leqq j \leqq N(i+1))$ is an orthonormal basis of F.

Finally, if $N(i) + 1 = J$, then $(\varepsilon_j; 1 \leqq j \leqq N(i))$ is an orthonormal basis of F, and the construction is over.

The preceding construction ensures that $(\varepsilon_i^k; 1 \leqq i < J(k))$ is again a complete orthonormal system of F_k. Let $\mathscr{E}$ denote the span of $(\varepsilon_j^k; 1 \leqq j < J(k), k \geqq 1)$. Define a linear operator $L: \mathscr{E} \to \mathscr{E}$ by setting

$$\begin{cases} L(\varepsilon_1^{2k}) = k^{-1}\sqrt{\alpha_k}\,\varepsilon_1^{2k} \\ L(\varepsilon_i^{2k}) = 2^{-(2k+i)}\varepsilon_i^{2k} \quad \text{for } 2 \leqq i < J(2k) \\ L(\varepsilon_i^{2k-1}) = 2^{-(2k-1+i)}\varepsilon_i^{2k-1} \quad \text{for } 1 \leqq i < J(2k-1). \end{cases}$$

Let us remind that $H_\infty = A^{-\frac{1}{2}}K$. Observe moreover that the RKHS of the law of X_T, say $\mathscr{H}$, is equal to H_∞ as a set, and that there exists a constant $C > 0$ such that

$$C|x|_{\mathscr{H}} \leqq |A^{\frac{1}{2}}x|_K \leqq \frac{1}{C}|x|_{\mathscr{H}}.$$

We show that $LA^{-\frac{1}{2}}$ is Hilbert Schmidt on K, and define a new norm $\| \ \|_B$ on $\mathscr{H}$ by setting $\|x\|_B = |Lx|_K$. Indeed, (4.5) implies that $\langle A^{-\frac{1}{2}}\varepsilon_i^k, \varepsilon_j^k\rangle = \langle \varepsilon_i^k, A^{-\frac{1}{2}}\varepsilon_j^k\rangle = 0$ except if $j \leqq 2i + 1$ and $i \leqq 2j + 1$. Thus

$$\begin{aligned}\sum &= \sum_{k \geqq 1} \sum_{1 \leqq i < J(k)} |LA^{-\frac{1}{2}}\varepsilon_i^k|_K^2 \\ &= \sum_{k \geqq 1} [k^{-2}\alpha_k(\langle A^{-\frac{1}{2}}\varepsilon_1^{2k}, \varepsilon_1^{2k}\rangle + \langle A^{-\frac{1}{2}}\varepsilon_2^{2k}, \varepsilon_1^{2k}\rangle + \langle A^{-\frac{1}{2}}\varepsilon_3^{2k}, \varepsilon_1^{2k}\rangle)^2 \\ &\quad + 2^{-4k}(\langle A^{-\frac{1}{2}}\varepsilon_1^{2k-1}, \varepsilon_1^{2k-1}\rangle + \langle A^{-\frac{1}{2}}\varepsilon_2^{2k-1}, \varepsilon_1^{2k-1}\rangle \\ &\quad + \langle A^{-\frac{1}{2}}\varepsilon_3^{2k-1}, \varepsilon_1^{2k-1}\rangle)^2] + \sum_{k \geqq 1} \sum_{2 \leqq i < J(k)} 2^{-2(k+i)} \\ &\quad \cdot \Bigg(\sum_{[i-1/2] \leqq j \leqq 2i+1} \langle A^{-\frac{1}{2}}\varepsilon_j^k, \varepsilon_i^k\rangle \Bigg)^2 \\ &\leqq \sum_{k \geqq 1} [k^{-2}\alpha_k 3\alpha_{k-1}^{-1} + 2^{-4k} 3\alpha_k^{-1}] + \sum_{k \geqq 1} \alpha_{k-1}^{-1} \sum_{2 \leqq i < J(k)} 2^{-2(k+i)}(2i+1) \\ &\leqq 3 \sum_{k \geqq 1} k^{-2} + \lambda_0^{-1} \sum_{k \geqq 1} 2^{-2k} \sum_{i \geqq 1} (2i+1)2^{-2i} < \infty \ .\end{aligned}$$

Let B denote the completion of $\mathscr{H}$ with respect to the norm $\| \ \|_B$. It follows that $H \subsetneq B$ is an abstract Wiener space. Let $(X_t, 0 \leqq t \leqq T)$ be the cylindrical Gaussian process with covariance Γ defined by (4.1). The law of X_T is tight on B, and hence $(X_t, 0 \leqq t \leqq T)$ is a genuine B-valued process. In the last step of the proof we show that:

$$\forall a > 0, \forall M > 0, P\Big(\sup_{0 \leqq s,t \leqq a} \|X_s - X_t\|_B > M \Big) = 1 \ . \tag{4.9}$$

For every $k \geqq 1$, set

$$p_k = P\Big(\sup_{(2\alpha_k)^{-1} \leqq s,t \leqq k^{-1}} \|X_s - X_t\|_B \geqq k \Big) ;$$

then

$$\begin{aligned} p_k &\geqq P\Big(\sup_{(2\alpha_k)^{-1} \leqq s,t \leqq k^{-1}} \langle \varepsilon_1^{2k}, LX_s - LX_t\rangle_K \geqq k \Big) \\ &\geqq P\Big(\sup_{(2\alpha_k)^{-1} \leqq s,t \leqq k^{-1}} k^{-1}\sqrt{\alpha_k}\,[X_s(\varepsilon_1^{2k}) - X_t(\varepsilon_1^{2k})] \geqq k \Big) .\end{aligned}$$

In the rest of the proof, let us fix k and set $\eta_t = X_t(\varepsilon_1^{2k})$. Then for $0 \leqq s \leqq t \leqq T$,

$$\begin{aligned} E(\eta_s\eta_t) &= \int_0^s \int_{\alpha_k}^{\alpha_k+1} e^{-\lambda(s-u)} e^{-\lambda(t-u)} d\langle E(\lambda)\varepsilon_1^{2k}, \varepsilon_1^{2k}\rangle du \\ &= \int_{\alpha_k}^{\alpha_k+1} e^{-\lambda(t-s)} \frac{1}{2\lambda}(1 - e^{-2\lambda s}) d\langle E(\lambda)\varepsilon_1^{2k}, \varepsilon_1^{2k}\rangle .\end{aligned}$$

Thus, for $1/2\alpha_k \leqq s \leqq t \leqq 1/k$, and for some $C > 1$,

$$C^{-1}\alpha_k^{-1}\exp(-\alpha_k(t-s)) \leqq E(\eta_s\eta_t) \leqq C\alpha_k^{-1}\exp(-\alpha_k(t-s)) .$$

Let us put $Z_t = C^{\frac{1}{2}}\alpha_k^{\frac{1}{2}}\eta_t$. Then

$$1 \leqq E(Z_t^2) \leqq C^2 \quad \text{and} \quad 0 < E(Z_s Z_t) \leqq C^2 \exp(-\alpha_k(t-s)) .$$

Let us fix $n > 1$, set $\delta = 1/2kn$, and $\xi_i = Z_{(2k)^{-1}+(i-1)\delta}$ for $1 \leqq i \leqq n$. Then if

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-x}^{x} e^{-(t^2/2)} \mathrm{d}t ,$$

one has that

$$P\left(\sup_{1 \leqq i \leqq n} |\xi_i| \leqq M \right) \leqq P\left(\sup_{1 \leqq i \leqq n} \frac{|\xi_i|}{\sqrt{E(|\xi_i|^2)}} \leqq M \right) \xrightarrow[\alpha_k \to \infty]{} (\phi(M))^n .$$

Similarly,

$$P\left(\inf_{1 \leqq i \leqq n} |\xi_i| \geqq m \right) \leqq P\left(\inf_{1 \leqq i \leqq n} \frac{|\xi_i|}{\sqrt{E(|\xi_i|^2)}} \geqq \frac{m}{C} \right) \xrightarrow[\alpha_k \to \infty]{} \left(1 - \phi\left(\frac{m}{C}\right)\right)^n .$$

Let us recall that $\alpha_k k^{-1} > t_k$, and that we are free to choose t_k as large as we wish (since we may impose that $t_k \to +\infty$ as $k \to +\infty$). Of course,

$$\begin{aligned} \bar{P}_k &= P\left(\sup_{1 \leqq i,j \leqq n} (|\xi_i - \xi_j|) > M - m \right) \\ &\geqq P\left(\sup_{1 \leqq i,j \leqq n} (|\xi_i| - |\xi_j|) > M - m \right) \\ &\geqq P\left(\left\{ \sup_{1 \leqq i \leqq n} |\xi_i| > M \right\} \cap \left\{ \inf_{1 \leqq i \leqq n} |\xi_i| < m \right\} \right) \\ &\geqq 1 - \left[P\left(\sup_{1 \leqq i \leqq n} |\xi_i| \leqq M \right) + P\left(\inf_{1 \leqq i \leqq n} |\xi_i| \geqq m \right) \right] . \end{aligned}$$

Hence, for fixed k, M, m and n, there exists $A(k, M, m, n)$ such that for $\alpha_k > A(k, M, m, n)$,

$$\bar{P}_k > 1 - [\phi(M)]^n - \left[1 - \phi\left(\frac{m}{C}\right)\right]^n - 2^{-(k+1)} .$$

Choose $m = \sqrt{C}$, $M = (k^2 + 1)\sqrt{C}$, and then n such that

$$[\phi((k^2 + 1)\sqrt{C})]^n + \left[1 - \phi\left(\frac{1}{\sqrt{C}}\right)\right]^n \leqq 2^{-(k+1)} .$$

Now choose t_k such that for $\alpha_k > kt_k$, $\bar{P}_k \geqq 1 - 2^{-k}$. Obviously, $p_k \geqq \bar{P}_k$, which completes the proof of (4.9), and hence that of the theorem. □

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