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The packing measure of a general subordinator*

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Summary. Precise conditions are obtained for the packing measure of an arbitrary subordinator to be zero, positive and finite, or infinite. It develops that the packing measure problem for a subordinator $X(t)$ is equivalent to the upper local growth problem for $Y(t) = \min(Y_1(t), Y_2(t))$, where Y_1 and Y_2 are independent copies of X. A finite and positive packing measure is possible for subordinators "close to Cauchy"; for such a subordinator there is non-random concave upwards function that exactly describes the upper local growth of Y (although, as is well known, there is no such function for the subordinator X itself).

1 Introduction

Fristedt and Pruitt [1] showed that for each subordinator $X(t)$ there is an appropriate measure function $\varphi(s)$ such that the trajectory up to time t has Hausdorff measure

$$\varphi\text{-}mX[0, t] = t \quad \text{for all } t \geqq 0 \quad \text{a.s.}$$

Taylor and Tricot [4] defined packing measure and showed that the trajectory of a transient Brownian motion has finite positive φ-packing measure for $\varphi(s) = s^2/\log|\log s|$. In [6] Taylor gave a criterion on φ which determines whether the sample path of a strictly stable process in $\mathbf{R}^d$ of index $\alpha < d$, $0 < \alpha < 2$ has zero or infinite φ-packing measure. In this case there is no function φ such that

$$0 < \varphi\text{-}pX[0, 1] < +\infty , \tag{1.1}$$

and the reason for this is that efficient packing comes from using points on the path where there are unusually large jumps. One might suspect that this behavior would persist for every Lévy process with infinite Lévy measure. However, the strictly

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asymmetric Cauchy process in $\mathbf{R}^d$ ($d \geqq 2$) satisfies (1.1) with $\varphi(s) = s/|\log s|$, as shown by Rezakhanlou and Taylor [2]. We also believe that there are symmetric Lévy processes in $\mathbf{R}^d$ ($d \geqq 3$) which are sufficiently close to Brownian motion to have an exact packing measure function.

The purpose of the present paper is to analyze the φ-packing measure of an arbitrary subordinator. The results extend immediately to a type B Lévy process in $\mathbf{R}^d$, but we state them only in $\mathbf{R}$. It is well known that no subordinator has an exact upper growth rate, so we were surprised to discover that there is a class of subordinators for which there is an exact φ satisfying (1.1). The reason for this will become clear as we explore precise analytic conditions on the large tail of the distribution of $X(t)$ and translate these into conditions on the growth rate of the Lévy measure near 0. Efficient packing comes from two large excursions rather than just one. For subordinators close to Cauchy the rare event of two large excursions is more likely to be realized by a large number of small jumps rather than two large jumps. Thus, in giving a complete solution of Problem 2 on page 392 of [5], we discover a third category where (1.1) holds.

In Sect. 2 we first gather the results we need about subordinators and about packing measures. Of some independent interest may be a more powerful auxiliary packing measure with the same class of sets of zero or infinite measure. In Sect. 3 we establish connections between the distribution of $X(t)$ and the Lévy measure near zero, and in Sect. 4 we give examples which show that all possibilities can be realized. Finally in Sect. 5, we obain definitive results on the φ-packing measure of the trajectory of a subordinator. As usual, c or k will stand for a finite positive constant whose value may change from line to line.

2 Preliminaries

By a subordinator we mean a real-valued increasing process $X(t)$ with stationary independent increments and $X(0) \equiv 0$. We take X to be right-continuous. For fixed $s > 0$, the processes

$$Y_1(t) = X(s) - X(s-t)\,,$$

$$Y_2(t) = X(s+t) - X(s)$$

are independent copies of $X(t)$, although $Y_1(t)$ is only defined for $t \leqq s$ and if left-continuous rather than right-continuous. We also need the first passage time process $P(a)$ defined for $a > 0$ by

$$P(a) = \inf\{s > 0 \colon X(s) \geqq a\}\,.$$

Our analysis involves a study of the large tail of the distribution of $X(t)$, so, for $t, y \geqq 0$, we define

$$F(t, y) = P\{X(t) \geqq y\} = P\{P(y) \leqq t\}\,.$$

The Laplace transform of the distribution of $X(t)$ is

$$E\{e^{-\lambda X(t)}\} = e^{-tg(\lambda)}\,,$$

where $g(\lambda) = \int_0^\infty (1 - e^{-\lambda r})\nu(dr)$, and ν is a measure on $(0, \infty)$ such that

$$\int_0^\infty (r \wedge 1)\nu(dr) < \infty \tag{2.1}$$

(where $r \wedge 1$ denotes the minimum of r and 1). If $\varphi(s) = s$ then the φ-packing measure of every subset of the real line coincides with Lebesgue measure. Therefore, it is natural to restrict our attention to measure functions φ which satisfy $\varphi(x)/x \downarrow$ in addition to the usual properties: $\varphi(0) = 0$, φ strictly increasing, and φ continuous. One consequence of these assumptions is that $\varphi(ax) + \varphi(bx) \geqq \varphi(x)$ whenever $a + b = 1$, $0 \leqq a \leqq 1$, $x \geqq 0$. Another consequence is that the inverse function ψ of φ satisfies $\psi(t)/t \uparrow$.

For a collection $\mathscr{I}$ of intervals I, let

$$\varphi(\mathscr{I}) = \sum_{I \in \mathscr{I}} \varphi(|I|),$$

where $|\cdot|$ denotes Legesgue measure, and

$$\|\mathscr{I}\| = \sup\{|I|, I \in \mathscr{I}\}.$$

For each $E \subset \mathbf{R}$, let $\mathscr{I}_E$ denote the family of intervals $(x - r, x + r)$ with center $x \in E$. Define the set function

$$\varphi\text{-}P(E) = \limsup_{\delta \to 0} \{\varphi(\mathscr{I}): \|\mathscr{I}\| < \delta, \mathscr{I} \text{ disjoint, and } \mathscr{I} \subset \mathscr{I}_E\}.$$

In [4, p. 681] Taylor and Tricot showed that φ-P changes by only bounded factors if $\mathscr{I}_E$ is replaced by semidyadic intervals each containing a point of E near the midpoint. In the present paper we need to work with a possibly sparser collection of intervals that depends on φ.

Start with any sequence x_n which decreases to zero slowly enough to ensure

$$\frac{\varphi(5x_n)}{\varphi(5x_{n+1})} \leqq c \tag{2.2}$$

for some constant c, and define the packing class $\Gamma = \Gamma(E, \{x_n\})$ to consist of those intervals $(jx_n, (j + 5)x_n)$, $j \in \mathbf{Z}$, such that there is a point of E in $[(j + 2)x_n, (j + 3)x_n]$. Now put

$$\varphi\text{-}P^+(E) = \limsup_{\delta \to 0} \{\varphi(\mathscr{I}): \|\mathscr{I}\| < \delta, \mathscr{I} \text{ disjoint, and } \mathscr{I} \subset \Gamma(E, \{x_n\})\}.$$

An easy argument, approximating inside an interval of $\mathscr{I}_E$ by one of $\Gamma(E, \{x_n\})$, and conversely, shows that there are finite constants c_1, c_2 such that

$$c_1 \varphi\text{-}P^+(E) \leqq \varphi\text{-}P(E) \leqq c_2 \varphi\text{-}P^+(E) \tag{2.3}$$

for all $E \subset \mathbf{R}$. The final step is to generate outer measures:

$$\varphi\text{-}p(E) = \inf\left\{\sum \varphi\text{-}P(E_i): E \subset \cup E_i\right\};$$

$$\varphi\text{-}p^+(E) = \inf\left\{\sum \varphi\text{-}P^+(E_i): E \subset \cup E_i\right\}.$$

We call φ-p the φ-packing measure and use φ-p^+ as a computational aid since (2.3) ensures that both measures have the same class of sets of zero measure. The key tool for evaluating packing measure is to spread a finite measure uniformly on the set and then evaluate the lower density. We state the relevant theorem from [4].

Theorem 1. *Suppose μ is any finite Borel measure in $\mathbf{R}$ and φ is a measure function. Then there is a $\lambda > 0$ such that, for all Borel $E \subset \mathbf{R}$,*

$$\lambda\mu(E)\inf\{\Lambda(x): x\in E\} \leqq \varphi\text{-}p(E) \leqq \mu(E)\sup\{\Lambda(x): x\in E\},$$

where

$$\Lambda(x) = \limsup_{r\to 0} \frac{\varphi(2r)}{\mu(x-r,x+r)}. \tag{2.4}$$

3 Local growth conditions needed for the density theorem

As usual we define a Borel measure μ concentrated on the trajectory by taking the occupation measure. For any Borel set A put

$$\mu(A) = |\{s\in(0,1): X(s)\in A\}|. \tag{3.1}$$

For $x = X(s)$ we want to pick values of r such that $\mu(x-r,x+r)$ is small: this requires us to choose values of t which make both $[X(s+t)-X(s)]$ and $[X(s)-X(s-t)]$ large. We make this precise in

Lemma A. *If μ is defined by* (3.1) *and $\Lambda(x)$ is the upper density defined by* (2.4), *then*

$$\frac{1}{2}\Lambda(X(s)) \leqq \limsup_{t\downarrow 0} \frac{\varphi([X(s+t)-X(s)]\wedge[X(s)-X(s-t)])}{t} \leqq \Lambda(X(s)).$$

Proof. Let $Y_1(t) = X(s+t)-X(s)$, $Y_2(t) = X(s)-X(s-t)$ and suppose that $\limsup_{t\downarrow 0}\varphi(Y_1(t)\wedge Y_2(t))/t \leqq k < +\infty$. Then for all small $t>0$, $\varepsilon>0$ either $\varphi(Y_1(t)) \leqq (k+\varepsilon)t$ or $\varphi(Y_2(t)) \leqq (k+\varepsilon)t$; that is, either $Y_1(t) \leqq \psi((k+\varepsilon)t)$ or $Y_2(t) \leqq \psi((k+\varepsilon)t)$ so that, either $P_1(\psi(k+\varepsilon)t) \geqq t$ or $P_2(\psi(k+\varepsilon)t) \geqq t$ and thus $P_1(\psi(k+\varepsilon)t) + P_2(\psi(k+\varepsilon)t) \geqq t$. Replacing t by $t/(k+\varepsilon)$ gives, for all small t,

$$\frac{P_1(\psi(t)) + P_2(\psi(t))}{2t} \geqq \frac{1}{2(k+\varepsilon)}$$

or

$$\frac{\varphi(2r)}{\mu(x-r,x+r)} \leqq \frac{2\varphi(r)}{\mu(x-r,x+r)} \leqq 2(k+\varepsilon)$$

for $x = X(s)$ and all $r = \psi(t)$ small enough. Hence

$$\Lambda(x) \leqq 2(k+\varepsilon).$$

Conversely suppose $\limsup_{t\downarrow 0}\varphi(Y_1(t)\wedge Y_2(t))/t \geqq k > 0$. The same argument shows that

$$\Lambda(x) \geqq (k-\varepsilon).$$

Since ε is arbitrary, the lemma is proved. It is clear that a zero–one law applies to this lim sup so that there is a constant $m \in [0, \infty]$ such that

$$\limsup_{t \downarrow 0} \frac{\varphi(Y_1(t) \wedge Y_2(t))}{t} = m \quad \text{a.s.} \tag{3.2}$$

Since clearly the value of m is related to the φ-packing measure by Theorem 1 our main task is now to determine conditions which will make m zero, infinite, or finite and positive.

Lemma B. *If k is such that*

$$\int_0^1 \frac{1}{t} F(t, \psi(kt))^2 \, dt < \infty ,$$

then

$$\limsup_{t \downarrow 0} \frac{\varphi(Y_1(t) \wedge Y_2(t))}{t} \leqq k \quad \text{a.s.} ,$$

where Y_1, Y_2 are independent copies of the subordinator X.

Proof. Suppose $0 < b < 1$. By monotonicity

$$\sum_{n=1}^{\infty} P\left\{\frac{\varphi(Y_1(t) \wedge Y_2(t))}{t} \geqq kb^{-2} \quad \text{for some } t \in (b^{n+1}, b^n]\right\}$$

$$\leqq \sum_{n=1}^{\infty} P\{\varphi(Y_1(b^n)) \wedge \varphi(Y_2(b^n)) \geqq kb^{n-1}\}$$

$$\leqq \frac{1}{|\log b|} \sum_{n=1}^{\infty} \int_{b^n}^{b^{n-1}} P\{\varphi(Y_1(t)) \wedge \varphi(Y_2(t)) \geqq kt\} \frac{dt}{t}$$

$$= \frac{1}{|\log b|} \int_0^1 \frac{1}{t} F(t, \psi(kt))^2 \, dt < \infty .$$

By Borel–Cantelli, $\limsup_{t \downarrow 0} \varphi(Y_1(t) \wedge Y_2(t))/t \leqq kb^{-2}$ a.s. Since b can be arbitrarily close to 1, the lemma is proved.

Lemma C. *If k is such that*

$$\int_0^1 \frac{1}{t} F(t, \psi(kt))^2 \, dt = +\infty ,$$

then

$$\limsup_{t \downarrow 0} \frac{\varphi(Y_1(t) \wedge Y_2(t))}{t} \geqq \frac{1}{4} k \quad \text{a.s.}$$

Proof.

$$P\left\{\left[\frac{\varphi(Y_1(t)\wedge Y_2(t))}{t}\geqq \tfrac{1}{4}k \text{ for some } t\in\left(\frac{1}{2^n},\frac{1}{2^{n-1}}\right]\right]\text{i.o.}\right\}$$

$$\geqq P\left\{\left[\frac{\varphi\left[\left(Y_1(t)-Y_1\left(\frac{1}{2^n}\right)\right)\wedge\left(Y_2(t)-Y_2\left(\frac{1}{2^n}\right)\right)\right]}{t}\right.\right.$$

$$\left.\left.\geqq \tfrac{1}{4}k \text{ for some } t\in\left(\frac{1}{2^n},\frac{1}{2^{n-1}}\right]\right]\text{ i.o.}\right\}$$

(where "i.o." denotes "infinitely often"). Since the events are independent, this probability will be 1 by Borel–Cantelli if the following series diverges:

$$\sum_{n=1}^{\infty} P\left\{\frac{\varphi\left(\left(Y_1\left(\frac{1}{2^{n-1}}\right)-Y_1\left(\frac{1}{2^n}\right)\right)\wedge\left(Y_2\left(\frac{1}{2^{n-1}}\right)-Y_2\left(\frac{1}{2^n}\right)\right)\right)}{\left(\frac{1}{2^{n-1}}\right)}\geqq\frac{k}{4}\right\}$$

$$=\sum_{n=1}^{\infty} P\left\{\frac{\varphi\left(Y_1\left(\frac{1}{2^n}\right)\wedge Y_2\left(\frac{1}{2^n}\right)\right)}{\frac{1}{2^{n-1}}}\geqq\frac{k}{4}\right\}$$

$$\geqq\frac{1}{\log 2}\sum_{n=1}^{\infty}\int_{1/2^{n+1}}^{1/2^n} P\{\varphi(Y_1(t)\wedge Y_2(t))\geqq kt\}\frac{dt}{t}$$

$$=\frac{1}{\log 2}\int_0^{1/2}\frac{F(t,\psi(kt))^2}{t}dt=+\infty\,.$$

It is now an obvious question as to whether the integral

$$\int_0^1 \frac{1}{t}F(t,\psi(kt))^2\,dt \tag{3.3}$$

can converge for some values of k and diverge for others. In order to decide this we connect (3.3) with the Lévy measure.

Lemma D. *If there is a finite k making* (3.3) *or*

$$\int_0^1\frac{1}{t}F(t,k\psi(t))^2\,dt$$

finite, then

$$\int_0^1 t\nu[\psi(t),\infty)^2\,dt<\infty\,.$$

Proof. We may assume $k \geqq 1$ so that $\psi(kt) \leqq k\psi(t)$ and $F(t, \psi(kt)) \geqq F(t, k\psi(t))$ and the convergence of (3.3) implies that

$$\int_0^1 \frac{1}{t} F(t, k\psi(t))^2 \, dt < \infty . \tag{3.4}$$

But $F(t, k\psi(t))$ is not less than the probability that by time t there is at least one jump of size greater than $k\psi(t)$. Thus (3.4) requires

$$\int_0^1 \frac{1}{t}(1 - e^{-t\nu[k\psi(t), \infty)})^2 \, dt < \infty ,$$

which implies

$$\int_0^1 \frac{1}{t}(1 \wedge t\nu[k\psi(t), \infty))^2 \, dt < \infty . \tag{3.5}$$

But

$$\begin{aligned}(1 - e^{-\frac{1}{2t}\nu[k\psi(t), \infty)})^2 &\leqq F(\tfrac{1}{2}t, k\psi(t))^2 \\ &\leqq \frac{1}{\log 2} \int_{t/2}^{t} \frac{1}{s} F(s, k\psi(s))^2 \, ds \\ &\to 0 \text{ as } t \downarrow 0 .\end{aligned}$$

Hence,

$$\lim_{t \downarrow 0} t\nu[k\psi(t), \infty) = 0 , \tag{3.6}$$

and (3.5) becomes

$$\int_0^1 t\nu[k\psi(t), \infty)^2 \, dt < \infty .$$

But, for $k > 1$ we have

$$\begin{aligned}k^2 \int_0^{k^{-1}} u\nu[\psi(u), \infty)^2 \, du &= \int_0^1 t\nu[\psi(kt), \infty)^2 \, dt \\ &\leqq \int_0^1 t\nu[k\psi(t), \infty)^2 \, dt < \infty ;\end{aligned}$$

so the lemma is proved.

It will be helpful to state a number of conditions equivalent to the growth condition on the Lévy measure obtained in Lemma D.

Lemma E. *The following statements are equivalent*:

(i) $$\int_0^1 t\nu[\psi(t), \infty)^2 \, dt < \infty ;$$

(ii) $$\int_0^1 t\nu[c\psi(kt), \infty)^2 \, dt < \infty ;$$

(iii) $$\int_0^{\psi(1)} \nu[x, \infty)^2 \varphi(x) \varphi'(x) \, dx < \infty ;$$

(iv) $$\int_0^1 \varphi(x)^2 \nu[x,\infty)\,\nu(\mathrm{d}x) < \infty\,;$$

(v) $$\int_0^{\psi(1)} \nu[cx,\infty)^2 \varphi(x)\varphi'(x)\,\mathrm{d}x < \infty\,.$$

The following statements follow from (i):

(vi) $$t\nu[c\psi(kt),\infty) \to 0 \quad \textit{as } t \downarrow 0\,;$$

(vii) $$\varphi(x)\nu[cx,\infty) \to 0 \quad \textit{as } x \downarrow 0\,;$$

(viii) $$\int_0^1 \frac{t}{\psi(t)^4}\left[\int_0^{\psi(t)} x^2\,\nu(\mathrm{d}x)\right]^2 \mathrm{d}t < \infty\,;$$

(ix) $$\frac{t}{\psi(t)^2}\int_0^{\psi(t)} x^2\,\nu(\mathrm{d}x) \to 0 \quad \textit{as } t \downarrow 0\,.$$

Proof. (i) $\Leftrightarrow$ (ii): For $c = 1$, the equivalence follows immediately from the substitution $s = kt$. Then, general c is easily treated using

$$\psi(kt) \leqq c\psi(kt) \leqq \psi(ckt) \quad \text{if } c \geqq 1\,,$$

$$\psi(kt) \geqq c\psi(kt) \geqq \psi(ckt) \quad \text{if } c \leqq 1\,.$$

(i) $\Leftrightarrow$ (iii) and (ii) $\Leftrightarrow$ (v) come from the change of variables $t = \varphi(x)$ (using $k = 1$ in (ii)), as does (vi) $\Leftrightarrow$ (vii). (ii) $\Rightarrow$ (vi) is done in the portion of the proof of Lemma C leading to (3.6).

(iii) $\Leftrightarrow$ (iv):

$$\begin{aligned}\int_\delta^1 \nu[x,\infty)^2\varphi(x)\varphi'(x)\,\mathrm{d}x &= \frac{1}{2}\int_\delta^1 \nu[x,\infty)^2\,\mathrm{d}(\varphi(x)^2)\\ &= \frac{1}{2}\nu[1,\infty)^2\varphi(1)^2 - \frac{1}{2}\nu[\delta,\infty)^2\varphi(\delta)^2\\ &\quad + \int_\delta^1 \varphi(x)^2\nu[x,\infty)\,\mathrm{d}x\,.\end{aligned}$$

(vii) $\Rightarrow$ (ix): An argument similar to, but easier than, that leading to (3.6) gives $\delta\nu[\delta,\infty) \to 0$ as $\delta \downarrow 0$ as a consequence of (2.1). Thus

$$\begin{aligned}\int_0^{\psi(t)} x^2\,\nu(\mathrm{d}x) &= \lim_{\delta\downarrow 0} - \int_0^{\psi(t)} x^2\,\mathrm{d}\nu[x,\infty)\\ &= \lim_{\delta\downarrow 0}\left[-\psi(t)^2\nu[\psi(t),\infty) + \delta^2\nu[\delta,\infty) + 2\int_\delta^{\psi(t)} x\,\nu[x,\infty)\,\mathrm{d}x\right]\\ &\leqq 2\int_0^{\psi(t)} x\,\nu[x,\infty)\,\mathrm{d}x\,.\end{aligned}$$

But

$$\frac{t}{\psi(t)^2}\int_0^{\psi(t)} x\,\nu[x,\infty)\,\mathrm{d}x \leqq \frac{1}{\psi(t)}\int_0^{\psi(t)} \varphi(x)\nu[x,\infty)\,\mathrm{d}x \to 0$$

since $\varphi(x)\nu[x,\infty)\to 0$ as $x\downarrow 0$.

(iv)$\Rightarrow$(viii): The integral in (viii) can be written

$$2\int_0^1 \frac{t}{\psi(t)^4}\int_0^{\psi(t)} y^2 \int_y^{\psi(t)} z^2\,\nu(\mathrm{d}z)\,\nu(\mathrm{d}y)\,\mathrm{d}t$$

which, by interchange of order of integration, equals

$$2\int_0^{\psi(1)}\int_y^{\psi(1)}\int_{\varphi(z)}^1 \frac{ty^2z^2}{(\psi(t))^4}\mathrm{d}t\,\nu(\mathrm{d}z)\,\nu(\mathrm{d}y)$$

$$\leqq 2\int_0^{\psi(1)}\int_y^{\psi(1)} \frac{y^2z^2\varphi(z)}{z\psi'(\varphi(z))}\int_{\varphi(z)}^1 \frac{\psi'(t)}{\psi(t)^3}\mathrm{d}t\,\nu(\mathrm{d}z)\,\nu(\mathrm{d}y)$$

$$\leqq \int_0^{\psi(1)}\int_y^{\psi(1)} \frac{y^2\varphi(z)\varphi'(z)}{z}\nu(\mathrm{d}z)\,\nu(\mathrm{d}y) \leqq \int_0^{\psi(1)}\int_y^{\psi(1)} \frac{\varphi(y)^2 z\varphi'(z)}{\varphi(z)}\nu(\mathrm{d}z)\,\nu(\mathrm{d}y)$$

$$\leqq \int_0^{\psi(1)}\int_y^1 \varphi(y)^2\,\nu(\mathrm{d}z)\,\nu(\mathrm{d}y) = \int_0^{\psi(1)} \varphi(y)^2\,\nu(y,\infty)\,\nu(\mathrm{d}y) < \infty\,.$$

We now derive another conclusion from the integrability condition on the large tail of $F(t,y)$.

Lemma F. *If c is such that*

$$\int_0^1 \frac{1}{t}F(t,c\psi(t))^2\,\mathrm{d}t < \infty\,,$$

then

$$\limsup_{t\downarrow 0} \frac{t}{\psi(t)}\int_0^{\psi(t)} x\,\nu(\mathrm{d}x) \leqq 4c\,. \tag{3.7}$$

Proof. We split $X(t)$ into two independent pieces $X_1(t)$ and $X_2(t)$, where

$$E(\mathrm{e}^{-\lambda X_1(t)}) = \mathrm{e}^{-t\int_0^{\psi(2t)}(1-\mathrm{e}^{-\lambda x})\nu(\mathrm{d}x)}\,,$$

$$E(\mathrm{e}^{-\lambda X_2(t)}) = \mathrm{e}^{-t\int_{\psi(2t)}^{\infty}(1-\mathrm{e}^{-\lambda x})\nu(\mathrm{d}x)}\,.$$

Then $F(t,c\psi(t)) \geqq F_1(t,c\psi(t))$, where $F_1(t,y) = P\{X_1(t)\geqq y\}$. Thus, if $t_n\downarrow 0$ so that

$$1 \geqq t_1 \geqq 2t_2 \geqq 4t_3 \geqq \cdots > 0\,,$$

then

$$\sum_{n=1}^{\infty}\int_{t_n/2}^{t_n} \frac{F_1(t,c\psi(t))^2}{t}\mathrm{d}t < \infty\,.$$

This implies that $\sum F_1(\frac{1}{2}t_n, c\psi(t_n))^2 < \infty$ so that

$$F_1(\tfrac{1}{2}t_n, c\psi(t_n)) \to 0 \quad \text{as } n\to\infty\,. \tag{3.8}$$

If (3.7) fails we can find such a sequence t_n with

$$\frac{t_n}{\psi(t_n)}\int_0^{\psi(t_n)} x\,\nu(\mathrm{d}x) > 4c \quad \text{for all } n\,.$$

Now the mean and variance of $X_1(t_n/2)$ are

$$\frac{t_n}{2}\int_0^{\psi(t_n)} x\,\nu(\mathrm{d}x) > 2c\psi(t_n)$$

and

$$\frac{t_n}{2}\int_0^{\psi(t_n)} x^2\,\nu(\mathrm{d}x)\,.$$

By Chebyshev's inequality,

$$\begin{aligned} P\left\{X_1\left(\frac{t_n}{2}\right) < c\psi(t_n)\right\} &= 1 - F_1\left(\frac{t_n}{2}, c\psi(t_n)\right) \\ &\leqq \frac{\frac{1}{2}t_n \int_0^{\psi(t_n)} x^2\,\nu(\mathrm{d}x)}{[c\psi(t_n)]^2} \\ &= \frac{1}{2c^2}\frac{t_n}{\psi(t_n)^2}\int_0^{\psi(t_n)} x^2\,\nu(\mathrm{d}x) \to 0 \end{aligned}$$

by Lemma E(ix). But this contradicts (3.8).

We are now ready to prove a converse.

Lemma G. *Suppose there is a constant c such that*

$$\int_0^1 t\,\nu[\psi(t),\infty)^2\,\mathrm{d}t < \infty$$

and

$$\limsup \frac{t}{\psi(t)}\int_0^{\psi(t)} x\,\nu(\mathrm{d}x) \leqq \frac{1}{4}c\,.$$

Then

$$\int_0^1 \frac{1}{t}F(t, c\psi(t))^2\,\mathrm{d}t < \infty\,.$$

Proof. This time we write $X(t) = X_1(t) + X_2(t)$ with

$$E(\mathrm{e}^{-\lambda X_1(t)}) = \mathrm{e}^{-t\int_0^{\psi(t)}(1-\mathrm{e}^{-\lambda x})\nu(\mathrm{d}x)}\,,$$

$$E(\mathrm{e}^{-\lambda X_2(t)}) = \mathrm{e}^{-t\int_{\psi(t)}^{\infty}(1-\mathrm{e}^{-\lambda x})\nu(\mathrm{d}x)}\,,$$

and again write $F_i(t,y) = P\{X_i(t) \geqq y\}$. Since $X(t) \geqq y$ implies that either $X_1(t) \geqq \frac{1}{2}y$ or $X_2(t) \geqq \frac{1}{2}y$, we have

$$F(t, c\psi(t)) \leqq F_1(t, \tfrac{1}{2}c\psi(t)) + F_2(t, \tfrac{1}{2}c\psi(t))\,.$$

Our result will now follow if we can show

(a) $$\int_0^1 \frac{1}{t}F_1(t, \tfrac{1}{2}c\psi(t))^2\,\mathrm{d}t < \infty$$

and

(b) $$\int_0^1 \frac{1}{t} F_2(t, \tfrac{1}{2}c\psi(t))^2 \,\mathrm{d}t < \infty ,$$

since (a) and (b) together also imply that

$$\int_0^1 \frac{1}{t} F_1(t, \tfrac{1}{2}c\psi(t)) F_2(t, \tfrac{1}{2}c\psi(t)) \,\mathrm{d}t < \infty .$$

(a): The expectation and variance of $X_1(t)$ are

$$t \int_0^{\psi(t)} x\,\nu(\mathrm{d}x) \leqq \tfrac{1}{3}c\psi(t) \quad \text{for small } t$$

and

$$t \int_0^{\psi(t)} x^2\,\nu(\mathrm{d}x) .$$

Chebyshev gives

$$F_1(t, \tfrac{1}{2}c\psi(t)) \leqq \frac{t \int_0^{\psi(t)} x^2\,\nu(\mathrm{d}x)}{(\tfrac{1}{6}c\psi(t))^2}$$

$$\leqq k \frac{t}{\psi(t)^2} \int_0^{\psi(t)} x^2\,\nu(\mathrm{d}x) .$$

Hence

$$\int_0^1 \frac{1}{t} F_1(t, \tfrac{1}{2}c\psi(t))^2 \,\mathrm{d}t \leqq k^2 \int_0^1 \frac{t}{\psi(t)^4} \int_0^{\psi(t)} x^2\,\nu(\mathrm{d}x)\,\mathrm{d}t < \infty$$

by Lemma E(viii).

(b): $X_2(t) \geqq \tfrac{1}{2}c\psi(t)$ implies that X_2 has at least one jump so that

$$F_2(t, \tfrac{1}{2}c\psi(t)) \leqq t\nu[\psi(t), \infty)$$

and, therefore

$$\int_0^1 \frac{1}{t} F_2(t, \tfrac{1}{2}c\psi(t))^2 \,\mathrm{d}t \leqq \int_0^1 t\,\nu[\psi(t), \infty)^2 \,\mathrm{d}t < \infty .$$

We can now combine the results of Lemmas D, F, G in the form of

Theorem 2. *If $F(t, y) = P\{X(t) \geqq y\}$, there are three disjoint possibilities*:

(i) $$\int_0^1 \frac{1}{t} F(t, c\psi(t))^2 \,\mathrm{d}t < \infty \quad \textit{for all } c > 0$$

if and only if

$$\int_0^1 t\,\nu[\psi(t), \infty)^2 \,\mathrm{d}t < \infty \quad \textit{and} \quad \lim_{t \downarrow 0} \frac{t}{\psi(t)} \int_0^{\psi(t)} x\,\nu(\mathrm{d}x) = 0 ;$$

(ii) $$\int_0^1 \frac{1}{t} F(t, c\psi(t))^2 \,\mathrm{d}t < \infty \quad \textit{for some but not all } c > 0$$

if and only if

$$\int_0^1 t\,\nu[\psi(t),\infty)^2\,\mathrm{d}t < \infty \quad \text{and} \quad 0 < \limsup_{t\downarrow 0} \frac{t}{\psi(t)} \int_0^{\psi(t)} x\,\nu(\mathrm{d}x) < \infty\,;$$

(iii)
$$\int_0^1 \frac{1}{t} F(t, c\psi(t))^2\,\mathrm{d}t = +\infty \quad \text{for all } c > 0$$

if and only if

$$\int_0^1 t\,\nu[\psi(t),\infty)^2\,\mathrm{d}t = \infty \quad \text{or} \quad \limsup_{t\downarrow 0} \frac{t}{\psi(t)} \int_0^{\psi(t)} x\,\nu(\mathrm{d}x) = \infty\,.$$

Remark 1. The case (ii) above cannot occur if $\varphi(x) = x^\alpha$, $0 < \alpha < 1$ or, more generally, if there is a $\gamma < 1$ such that

$$\varphi(x) \leqq c^\gamma \varphi\left(\frac{x}{c}\right) \quad \text{for all } c \geqq 1\,.$$

For ψ corresponding to such a φ,

$$\int_0^1 t\,\nu[\psi(t),\infty)^2\,\mathrm{d}t < \infty \Rightarrow \lim_{t\downarrow 0} \frac{t}{\psi(t)} \int_0^{\psi(t)} x\,\nu(\mathrm{d}x) = 0\,.$$

Proof. Write $t = \varphi(u)$. Then

$$\begin{aligned}
\frac{\varphi(u)}{u}\int_0^u x\,\nu(\mathrm{d}x) &= \frac{\varphi(u)}{u}\sum_{n=1}^{\infty}\int_{u2^{-n}}^{2u2^{-n}} y\,\nu(\mathrm{d}y) \\
&\leqq 2\varphi(u)\sum_{n=1}^{\infty} 2^{-n}\nu[u2^{-n}, 2u2^{-n}) \\
&\leqq 2\varphi(u)\sum_{n=1}^{\infty} \frac{1}{2^n}\nu[u2^{-n}, \infty) \\
&\leqq 2\sum_{n=1}^{\infty} 2^{n\gamma-n}\varphi(u2^{-n})\nu[u2^{-n},\infty) \\
&\leqq 2 \sup_{y\leqq(1/2)u} \{\varphi(y)\nu[y,\infty)\} \sum_{n=1}^{\infty} 2^{n\gamma-n} \\
&\to 0 \quad \text{as } u \to 0
\end{aligned}$$

since $\varphi(y)\nu[y,\infty) \to 0$ by (3.5).

Remark 2. In the statement of Theorem 2, the constant can be moved inside the argument of the ψ function, even though in general $\psi(kt)$ need not be of the same order as $\psi(t)$. Thus, for example

$$\int_0^1 \frac{1}{t} F(t, c\psi(t))^2\,\mathrm{d}t < \infty \quad \text{for some but not all } c > 0$$

if and only if

$$\int_0^1 \frac{1}{t} F(t, \psi(kt))^2 \, dt < \infty \quad \text{for some but not all } k > 0 .$$

Proof. Use the conditions of the theorem and note:

$$\int_0^1 t\,\nu[\psi(kt), \infty)^2 \, dt = \frac{1}{k^2} \int_0^k u\,\nu[\psi(u), \infty)^2 \, du ,$$

$$\limsup \frac{t}{\psi(kt)} \int_0^{\psi(kt)} x\,\nu(dx) = \frac{1}{k} \limsup \frac{u}{\psi(u)} \int_0^{\psi(u)} x\,\nu(dx) .$$

4 Examples

In the last section we have seen that we will get a finite positive density $\Lambda(x)$ for a φ such that

$$\int_0^1 t\,\nu[\psi(t), \infty)^2 \, dt < \infty \quad \text{and} \quad 0 < \limsup \frac{t}{\psi(t)} \int_0^{\psi(t)} x\,\nu(dx) < \infty ,$$

where ψ is the inverse function of φ. Remark 1 shows that it is impossible for these two conditions to both hold if φ is comparable to a power less than 1, so we need to ask whether there is any subordinator for which the condition can be satisfied. We compute several examples and show in Example C that for certain subordinators close to Cauchy the two conditions are satisfied simultaneously.

A. A stable subordinator of index α, $0 < \alpha < 1$ has

$$\nu(dx) = \frac{dx}{x^{1+\alpha}}, \qquad \nu[x, \infty) = \frac{1}{\alpha x^{\alpha}} .$$

Try $\varphi(x) = x^{\alpha}\left(\log\frac{1}{x}\right)^{-\beta}$ which makes

$$\psi(t) \sim t^{1/\alpha}\left(\log\frac{1}{t}\right)^{\beta/\alpha} \quad \text{as } t \downarrow 0 .$$

Then

$$\int_0^1 t\,\nu[\psi(t), \infty)^2 \, dt = \frac{1}{\alpha^2} \int_0^1 \frac{t}{\psi(t)^{2\alpha}} \, dt \begin{cases} = \infty & \text{if } \beta \leqq 1/2 \\ < \infty & \text{if } \beta > 1/2 . \end{cases}$$

Thus, case (iii) of Theorem 2 holds if $\beta \leqq \frac{1}{2}$ and, by Remark 1, case (i) holds if $\beta > \frac{1}{2}$. This example was previously treated in [6].

B. A gamma process is given by $\nu(dx) = (e^{-x}/x)\,dx$, so $\nu[x, \infty) \sim \log\frac{1}{x}$ as $x \downarrow 0$. Now, for any φ, $\frac{t}{\psi(t)} \int_0^{\psi(t)} x\nu(dx) \leqq 2t \to 0$ as $t \downarrow 0$ so again the behavior will be determined by the convergence or divergence of $\int_0^1 t\,\nu[\psi(t),\infty)^2\,dt$ or, equivalently, $\int_0^1 t\left[\log\frac{1}{\psi(t)}\right]^2 dt$.

A critical function φ is

$$\varphi(x) = \frac{1}{\log\frac{1}{x}\left(\log\log\frac{1}{x}\right)^{\alpha}\left(\log\log\log\frac{1}{x}\right)^{\beta}}$$

for which

$$\psi(t) \sim e^{-\left\{\frac{1}{t(\log\frac{1}{t})^{\alpha}(\log\log\frac{1}{t})^{\beta}}\right\}}$$

and

$$\int_0^1 t\left[\log\frac{1}{\psi(t)}\right]^2 dt \begin{cases} = \infty & \text{if } \alpha < 1/2 \text{ or } \alpha = 1/2,\ \beta \leqq 1 \\ < \infty & \text{if } \alpha > 1/2 \text{ or } \alpha = 1/2,\ \beta > 1\,. \end{cases}$$

C. A subordinator close to Cauchy:

$$\nu(dx) = \frac{dx}{x^2\left(\log\frac{1}{x}\right)^{1+\alpha}} \quad \text{for } 0 < x < e^{-1},\ \alpha > 0\,,$$

so

$$\nu[x,\infty) \sim \frac{1}{x\left(\log\frac{1}{x}\right)^{1+\alpha}} \quad \text{as } x \downarrow 0\,.$$

Now we try

$$\varphi(x) = x(\log 1/x)^{\beta}, \quad \beta > 0\,,$$

so

$$\psi(t) \sim t(\log 1/t)^{-\beta}$$

and thus $\int_0^1 t\nu[\psi(t),\infty)^2\,dt$ converges if and only if

$$\int_0^1 \frac{(\log 1/t)^{2\beta}}{t(\log 1/t)^{2+2\alpha}}\,dt < \infty\,.$$

Therefore, we have convergence if and only if $\beta < \frac{1}{2} + \alpha$. We also have

$$\frac{t}{\psi(t)}\int_0^{\psi(t)} x\,\nu(dx) \sim \frac{1}{\alpha}\left(\log\frac{1}{t}\right)^{\beta-\alpha} \to \begin{cases} +\infty & \text{if } \beta > \alpha \\ 1/\alpha & \text{if } \beta = \alpha \\ 0 & \text{if } \beta < \alpha\,. \end{cases}$$

Applying Theorem 2, we see that, for $\beta = \alpha$,

$$\int_0^1 \frac{1}{t}F(t, c\psi(t))^2\,dt$$

converges for some but not all $c > 0$. By Lemmas B and C this implies that there exists a fine positive k such that

$$\limsup_{t \downarrow 0} \frac{\varphi(Y_1(t) \wedge Y_2(t))}{t} = k \quad \text{a.s.} \tag{4.1}$$

We conjecture, but cannot formulate an exact theorem, that only subordinators which are close to Cauchy can have an exact function φ such that (4.1) holds.

5 The packing φ-measure

We can now adapt standard arguments. Lemmas B and C combined with Theorem 2 and Remark 2 following Theorem 2 give

Theorem 3. *For independent subordinators Y_1 and Y_2 having the same distribution and a measure function φ with inverse ψ, there are three possibilities:*

(i) $$\int_0^t t\,\nu[\psi(t), \infty)^2\,\mathrm{d}t < \infty \quad \text{and} \quad \lim_{t \downarrow 0} \frac{t}{\psi(t)} \int_0^{\psi(t)} x\,\nu(\mathrm{d}x) = 0$$

$$\Rightarrow \limsup_{t \downarrow 0} \frac{\varphi(Y_1(t) \wedge Y_2(t))}{t} = 0 \text{ a.s.};$$

(ii) $$\int_0^t t\,\nu[\psi(t), \infty)^2\,\mathrm{d}t < \infty \quad \text{and} \quad 0 < \limsup \frac{t}{\psi(t)} \int_0^{\psi(t)} x\,\nu(\mathrm{d}x) < \infty$$

$$\Rightarrow \limsup_{t \downarrow 0} \frac{\varphi(Y_1(t) \wedge Y_2(t))}{t} = k \quad \text{a.s.};$$

(iii) $$\int_0^t t\,\nu[\psi(t), \infty)^2\,\mathrm{d}t = +\infty \quad \text{or} \quad \limsup_{t \downarrow 0} \frac{t}{\psi(t)} \int_0^{\psi(t)} x\,\nu(\mathrm{d}x) = +\infty$$

$$\Rightarrow \limsup_{t \downarrow 0} \frac{\varphi(Y_1(t) \wedge Y_2(t))}{t} = \infty \quad \text{a.s.}$$

Moreover, $\limsup[Y_1(t) \wedge Y_2(t)]/\psi(t)$ *is* 0, *positive and finite, or infinite according as the same is true for* $\limsup \varphi(Y_1(t) \wedge Y_2(t))/t$.

Remark. The preceding theorem, in combination with Example C of the preceding section, indicates that it is possible to have

$$0 < \limsup_{t \downarrow 0} \frac{Y_1(t) \wedge Y_2(t)}{\psi(t)} < \infty$$

for appropriate Y_1 and Y_2 (independent identically distributed having no drift and convex ψ), in contrast to the known impossibility of

$$0 < \limsup_{t \downarrow 0} \frac{X(t)}{\psi(t)} < \infty$$

for convex ψ and subordinators X having no drift.

We now wish to prove our main

Theorem 4. *For a measure function φ and subordinator $X(t)$*

$$\varphi\text{-}p(X[0,1]) = \begin{cases} 0 & \text{if case (i) holds} \\ c & \text{if case (ii) holds} \\ \infty & \text{if case (iii) holds} \end{cases}$$

in the classification of Theorem 3.

Proof. (iii): Using the occupation measure (3.1), and Lemma A we see that, for fixed $s \in (0,1)$, $x = X(s)$ a.s. $\Lambda(x) = +\infty$, where

$$\Lambda(x) = \limsup_{r \downarrow 0} \frac{\varphi(2r)}{\mu(x-r, x+r)}.$$

By Fubini, if $E = \{s \in (0,1): \Lambda(X(s)) = +\infty\}$ we have $|E| = 1$ so that $\mu(X(E)) = 1$. Applying Theorem 1 to $X(E)$ gives

$$\varphi\text{-}pX[0,1] \geqq \varphi\text{-}p(X(E)) = +\infty. \quad \text{a.s.}$$

(i): In the first case, Lemma A shows that

$$\Lambda(x) = 0 \quad \text{a.s.}$$

for $x = X(s)$, s fixed in $(0,1)$. If

$$E = \{s \in (0,1): \Lambda(X(s)) = 0\}$$

we again have $|E| = 1$ and $\mu(X(E)) = 1$. Applying Theorem 1 gives

$$\varphi\text{-}pX(E) = 0 \quad \text{a.s.}$$

However we now have to worry about $\varphi\text{-}pX(E^c)$, the packing measure of the bad points on the trajectory where $\Lambda(x) > 0$. We deal with these by using the auxiliary measure $\varphi\text{-}p^+$ on the set

$$F_\varepsilon = \left\{x = X(s), 0 < s < 1: \limsup_{t \downarrow 0} \frac{\varphi[X(s+t) - X(s)] \wedge [X(s) - X(s-t)]}{t} > 4\varepsilon\right\}$$

which contains the set of $x \in X[0,1]$ where $\Lambda(x) \geqq 8\varepsilon$. Now let $x_n = \frac{1}{5}\psi(\varepsilon 2^{-n})$, so that (2.1) is clearly satisfied.

If $x \in F_\varepsilon$, there is a sequence $t_i \downarrow 0$ such that

$$X(s+t_i) - X(s) > \psi(4\varepsilon t_i) \quad \text{and} \quad X(s) - X(s-t_i) > \psi(4\varepsilon t_i),$$

which by Cauchy–Schwartz is bounded by

$$\left[\int_0^1 t\, v[\tfrac{1}{5}\psi(t), \infty]^2 \, dt\right]^{1/2} \left[\int_0^1 \frac{1}{t} F(2t, \tfrac{1}{3}\psi(t))^2 \, dt\right]^{1/2}$$

and both of these integrals are finite using the hypothesis and Lemma G.

On the other hand, if $I_{j,n}$ is bad of type K, then $X(t)$ hits $[jx_n, (j+1)x_n]$, moves at least x_n in less than 2^{-n} units from the first hitting place of this interval and then moves a further $2x_n$ in less than 2^{-n} units of time from the first hitting place of $[(j+2)x_n, (j+3)x_n]$. Hence

$$P(I_{j,n} \text{ is bad of type K}) \leqq P(X(t) \text{ hits } [jx_n, (j+1)x_n])F(2^{-n}, x_n)F(2^{-n}, 2x_n)$$

so that

$$\sum_j P(I_{j,n} \text{ is bad of type K}) \leqq F(2^{-n}, x_n)^2 E \text{ (number of intervals hit)}.$$

By Lemma 6.1 of [3], N_n, the expected number of intervals of the form $[jx_n, (j+1)x_n]$ hit by $X(t)$, is bounded by $c[EP(\frac{1}{3}x_n)]^{-1}$. So, using Lemma 6 of [1] we obtain

$$E(N_n) \leqq cg\left(\frac{3}{x_n}\right) \leqq c\nu[\tfrac{1}{3}x_n, \infty) + c \int_0^{x_n/3} \frac{3}{x_n} x\, \nu(\mathrm{d}x) \tag{5.3}$$

since

$$g(\lambda) = \int_0^\infty (1 - \mathrm{e}^{-\lambda r})\, \nu(\mathrm{d}r)$$

$$\leqq \int_{\lambda^{-1}}^\infty \nu(\mathrm{d}r) + \int_0^{\lambda^{-1}} \lambda r\, \nu(\mathrm{d}r)$$

for any $\lambda > 0$. The first of the terms on the right in (5.3) is $o(2^{-n})$ by Lemma E(vi) and the second is $o(2^{-n})$ by the second hypothesis. If we multiply by 2^{-n} and sum on n (5.2) is bounded by $\sum_n F(2^{-n}, \frac{1}{5}\psi(\varepsilon 2^{-n}))^2$ which is

$$\leqq \int_0^1 \frac{1}{t} F(2t, \tfrac{1}{5}\psi(\tfrac{1}{2}\varepsilon t))^2\, \mathrm{d}t < \infty$$

by hypothesis.

(ii): In this case there is a constant $k_0 > 0$ such that $\int_0^1 \frac{1}{t} F(t, \psi(kt))^2\, \mathrm{d}t$ converges for $k > k_0$ and diverges for $0 < k < k_0$. From Lemmas B and C, for each fixed $s \in (0, 1)$,

$$\limsup_{t \downarrow 0} \frac{\varphi[X(s+t) - X(s)] \wedge [X(s) - X(s-t)]}{t} = c \quad \text{a.s.}$$

for some c satisfying $\frac{1}{4}k_0 \leqq c \leqq k_0$. Using a Fubini argument as before gives

$$\varphi\text{-}pX[0, 1] \geqq \tfrac{1}{4}\lambda k_0 > 0$$

after applying Theorem 1 to the good points. We now pick a $c_1 > c$ large enough to ensure that

$$\int_0^1 \frac{1}{t} F(2t, \tfrac{1}{3}\psi(\tfrac{1}{2}c_1 t))^2\, \mathrm{d}t < \infty\,.$$

Let

$$E_1 = \left\{ s \in (0,1) \colon \limsup_{t \downarrow 0} \frac{[X(s+t) - X(s)] \wedge [X(s) - X(s-t)]}{t} \leqq 4c_1 \right\},$$

$$E_2 = \left\{ s \in (0,1) \colon \limsup_{t \downarrow 0} \frac{[X(s+t) - X(s)] \wedge [X(s) - X(s-t)]}{t} > 4c_1 \right\}.$$

Since $|E_1| = 1$, Theorem 1 tells us that

$$\varphi\text{-}p(X(E_1)) \leqq 8c_1 < \infty .$$

The arguments we used in case (i), now apply to show $\varphi\text{-}P^+ X(E_2) = 0$ for the sequence

$$x_n = \tfrac{1}{5}\psi(c_2 2^{-n}) .$$

Putting all this together gives the result that a.s.

$$0 < c_3 s \leqq \varphi\text{-}pX[0,s] \leqq c_4 s < \infty .$$

But $Z(s) = \varphi\text{-}pX[0,s]$ is clearly a continuous subordinator and must, therefore, satisfy $Z(s) = cs$ a.s. for a suitable constant c.

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