Probab. Theory Relat. Fields 92, 423–464 (1992)

Probability Theory and Related Fields

Poisson law for the number of lattice points in a random strip with finite area

Péter Major

Mathematical Institute of the Hungarian Academy of Sciences, P.O.B. 127,
H–1364 Budapest, Hungary

Received March 19, 1991; in revised form November 25, 1991

Summary. Let a smooth curve be given by a function $r = f(\varphi)$ in polar coordinate system in the plane, and let R be a uniformly distributed random variable on the interval $[a_1 L, a_2 L]$ with some $a_2 > a_1 > 0$ and a large $L > 0$. Ya. G. Sinai has conjectured that given some real numbers $c_2 > c_1$, the number of lattice points in the domain between the curves $\left(R + \frac{c_1}{R}\right) f(\varphi)$ and $\left(R + \frac{c_2}{R}\right) f(\varphi)$ is asymptotically Poisson distributed for "good" functions $f(\cdot)$. We cannot prove this conjecture, but we show that if a probability measure with some nice properties is given on the space of smooth functions, then almost all functions with respect to this measure satisfy Sinai's conjecture. This is an improvement of an earlier result of Sinai [9], and actually the proof also contains many ideas of that paper.

1. Introduction

Let us consider a curve on the two-dimensional Euclidean space $\mathbb{R}^2$ which is given by the equation $r = f(\varphi)$, $0 \le \varphi \le \theta$, with some $0 < \theta \le 2\pi$ in polar coordinate system, where $f(\cdot) > 0$ is a continuous Lipschitz one function on $[0, \theta]$. Given some non-zero point $x = (x_1, x_2) \in \mathbb{R}^2$ let $|x| = \sqrt{x_1^2 + x_2^2}$ denote its absolute value and $\varphi(x)$ the angle between the vectors $(1, 0)$ and $x = (x_1, x_2)$. Let us fix two real numbers $c_2 > c_1$ and define for all sufficiently large $R > 0$ (we need that $R + \frac{c_1}{R} > 0$) the domain

$$\mathbb{O}_R = \mathbb{O}_R(f) = \Big\{ x \in \mathbb{R}^2,\ 0 \le \varphi((x)) \le \theta, \\ \left(R + \frac{c_1}{R}\right) f(\varphi(x)) < |x| < \left(R + \frac{c_2}{R}\right) f(\varphi(x)) \Big\} . \tag{1.1}$$

Simple calculation shows that the area of the domain $\mathbb{O}_R$ is

$$\left(1 + \frac{c_1 + c_2}{2R^2}\right)(c_2 - c_1) \int_0^\theta f^2(\varphi)\, d\varphi .$$

We are interested in the number of lattice points in $\mathbb{O}_R$, i.e. in the cardinality of the set $\mathbb{O}_R \cap \mathbb{Z}^2$, where $\mathbb{Z}^2$ denotes the points in $\mathbb{R}^2$ with integer coordinates, if R is a uniformly distributed random variable in an interval $[a_1 L, a_2 L]$. Here $a_2 > a_1 > 0$ are fixed positive numbers, and the parameter $L > 0$ is large. More precisely, we are interested in the limiting behaviour of the number of lattice points in this domain if $L \to \infty$. Ya. G. Sinai has formulated the conjecture that for "typical" nice curves the distribution of the cardinality of this set tends to the Poisson distribution with parameter $\lambda = (c_2 - c_1) \int_0^\theta f^2(\varphi)\, d\varphi$. There is no explicitly defined curve for which we can verify the above conjecture. On the other hand, we can show that if a probability measure is given on the set of continuous Lipschitz one functions with some nice properties, then almost all functions with respect to this measure satisfy Sinai's conjecture. This is a strengthening of a result of Sinai in paper [9], and actually the proof also depends heavily on the ideas of this paper. To formulate our result first we introduce the following notion:

Definition of Property A. *A probability measure P on the set of continuous Lipschitz one functions $f(\varphi)$, $0 < \varphi < \theta$, satisfies Property A if*

1.) There are some positive numbers $0 < b_1 < b_2$ and $b_3 > 0$ such that almost all functions $f(\varphi)$, $0 < \varphi < \theta$, with respect to the measure P satisfy the inequality $b_1 < f(\varphi) < b_2$ and $|f(\varphi_1) - f(\varphi_2)| < b_3|\varphi_1 - \varphi_2|$ for all $0 \le \varphi_1 < \varphi_2 \le \theta$.

2.) Let us fix some integer $k \ge 2$ and $0 \le \varphi_1 < \varphi_2 < \cdots < \varphi_k \le \theta$. The random vector $(f(\varphi_1), \dots, f(\varphi_k))$ has a density function

$$p_k(x_1, \dots, x_k | \varphi_1, \dots, \varphi_k)$$

which satisfies the following properties:

2a.)

$$p_k(x_1, \dots, x_k | \varphi_1, \dots, \varphi_k) < C_k \prod_{i=2}^k |\varphi_i - \varphi_{i-1}|^{-\tau}$$

with some $\tau < 2$ and C_k depending only on k.

2b.) The density function $p_k(x_1, \dots, x_k | \varphi_1, \dots, \varphi_k)$, $0 \le \varphi_1 < \varphi_2 < \cdots < \varphi_k \le \theta$ is a differentiable function of its $2k$ arguments $x_1, \dots, x_k$ and $\varphi_1, \dots, \varphi_k$, and it satisfies the inequality

$$\left.\begin{aligned} &\left|\frac{\partial}{\partial x_j} p_k(x_1, \dots, x_k | \varphi_1, \dots, \varphi_k)\right| \\ &\left|\frac{\partial}{\partial \varphi_j} p_k(x_1, \dots, x_k | \varphi_1, \dots, \varphi_k)\right| \end{aligned}\right\} < C_k \prod_{i=2}^k |\varphi_i - \varphi_{i-1}|^{-D_k}$$

for all $j = 1, \dots, k$ with some $C_k > 0$ and $D_k > 0$ depending only on k.

We shall prove the following

Theorem. *Let P be a probability measure with Property A on the space of continuous functions on the interval $[0, \theta]$, and let R be a uniformly distributed random variable on the interval $[a_1 L, a_2 L]$ with some $a_2 > a_1 > 0$ and a parameter $L > 0$. Given some function $f(\cdot)$ on $[0, \theta]$, consider the set $\mathbb{O}_R(f)$ defined by formula (1.1). Let $\xi_L = \xi_L(f)$ denote the number of lattice points in $\mathbb{O}_R(f)$, i.e. the cardinality of the* *set* $\mathbb{O}_R(f) \cap \mathbb{Z}^2$. *Then for almost all functions* f *with respect to the measure* P *the random variables* ξ_L *tend in distribution to the Poisson distribution with parameter* $\lambda = (c_2 - c_1) \int_0^\theta f^2(\varphi)\, d\varphi$ *as* $L \to \infty$.

Sinai proved in [9] a weaker version of this result. He proved that if the function $f(\cdot)$ is chosen randomly and independently of the radius R with respect to some probability distribution with nice properties, then the distribution of the number of lattice points tends to a mixture of Poisson distributions with different parameters. Sinai expressed the conditions on the distribution of the functions f in a form slightly different from ours, with the help of certain conditional density functions. Let us remark that our conditions are less restrictive, and this is important in such applications as for instance the example given in Section 2.

Most ideas of this work came from paper [9]. The most important step of the proof, the formulation of the Proposition can be traced in a hidden way in [9], and even the Proposition's proof contains several ideas of that paper. The proof of the Proposition is based on the estimate of the second moments of a certain random variable. For Sinai, to prove his weaker result, it was enough to estimate the first moment of a similar random variable. But he also remarked that the higher moments of such variables can be estimated similarly, although some additional technical difficulties appear.

Problems about the number of lattice points have been investigated for a long time in number theory and probabilistic number theory. See e.g. [8] for a classical treatment, [6] for the investigation of number of lattice points in a large circle with random centre or [5] for a modern treatment of the problem. Recently, this problem got even greater importance because of some questions in physics. We are interested in the behaviour of the spectrum of an operator in a quantum system. In particular, we would like to understand whether the quantization of a completely integrable classical mechanical system (which has nice trajectories) gives a different type of spectrum than that of a hyperbolic system with chaotic behaviour. There are certain conjectures about this problem. It is believed that the local behaviour of the spectrum is similar to the realizations of a Poisson process in the case of the quantum counterpart of a "typical" completely integrable system, and the spectrum satisfies Wigner's semicircle law in the case of quantization of hyperbolic systems. Actually, the situation is much more complex. We do not want to discuss this problem in detail, because this is not the subject of the present paper, and we are rather far from its good understanding.

The investigation of the spectrum of certain quantum systems leads to the problem about the number of lattice points in a given domain. An example for completely integrable systems whose quantization leads to such a problem is the free motion of a particle on a periodic rotation surface. (More precisely, we make a factorization of the surface with respect to the period. In such a way we get the motion of a particle on a compact surface resembling to a torus.) The quantization of this model leads to the problem about the eigenvalues of the Laplace–Beltrami operator on this surface. These eigenvalues can be calculated with a sufficiently good accuracy by means of the so-called quasi-classical approximation. (See papers [2] and [10]). Then the problem about the number of eigenvalues in an interval leads to the problem of counting the number of lattice points in a domain in $\mathbb{R}^2$ whose boundary is determined through the rotation surface and the interval. (See [2] or [10]). We are interested both in the local and global behaviour of the spectrum.

The local behaviour of the spectrum, the number of eigenvalues in a randomly chosen interval of fixed constant length, leads to the probabilistic problem investigated in this paper. This is the reason why Sinai formulated his conjecture. We cannot prove this conjecture for any explicitly given curve. Our aim was to show that it holds for typical curves. In the special case of circle, which corresponds to the spectrum of the Laplace operator on the torus $[0,1]\times[0,1]$, this conjecture does not hold. (See Problem 1 in Section 2.) Sinai's conjecture implies that the number of eigenvalues of the Laplace operator on a generic rotation surface is asymptotically Poissonian in a randomly chosen interval of constant length.

The global behaviour of the spectrum, the number of eigenvalues in a large interval $[0,L]$ leads to problems more intensively investigated in classical number theory, namely to the number of lattice points in a large domain. Here again, we are interested in the behaviour of generic curves. An investigation in this direction is done in paper [7].

Other physical models lead to other number theoretical problems. We mention in this direction paper [3] and the references in it, where the physical problem the authors considered led to the investigation of the number of lattice points in a large circle with random center. This problem was studied by means of computer simulation. Both the local and global behaviour of the spectrum was investigated. The computer simulations indicate a Poissonian local behaviour of this model too. A good description of the global behaviour of the spectrum of this model is still an open question.

The theorem formulated above also has the following generalization:

Theorem′. *For all $m=(m_1,m_2)\in\mathbb{Z}^2$ define, with the help of a function f and a random variable R, the (random) mapping*

$$F=F(R,f)\colon m\to\left(\varphi(m),R\left(\frac{|m|}{f(\varphi(m))}-R\right)\right),\quad m\in\mathbb{Z}^2$$

and the random field

$$\mathcal{P}=\{F((m_1,m_2));\ (m_1,m_2)\in\mathbb{Z}^2\}\,.$$

If R is a uniformly distributed random variable on an interval $[a_1L,a_2L]$, then for almost all functions f with respect to a probability measure P with Property A the finite dimensional distributions of the random field $\mathcal{P}$ tend to that of a Poisson process on $[0,\theta]\times[-\infty,\infty]$ with counting measure $f^2(\varphi)\,d\varphi\,dx$ as $L\to\infty$. This convergence means that for any $K\geq1$ and disjoint rectangles $[d_j,\bar{d}_j]\times[e_j,\bar{e}_j]\subset[0,\theta]\times[-\infty,\infty]$, $j=1,\dots,K$, the number of points in these rectangles tend to independent Poissonian random variables with parameters $\lambda_j=(\bar{e}_j-e_j)\int_{d_j}^{\bar{d}_j}f^2(\varphi)\,d\varphi$, $j=1,\dots,K$.

Theorem′ states in particular that the distribution of the number of lattice points which are mapped by the transformation F to the rectangle $[0,\theta]\times[c_1,c_2]$ tends to the Poisson distribution with parameter $\lambda=(c_2-c_1)\int_0^\theta f^2(\varphi)\,d\varphi$. In such a way it contains the statement of the Theorem as a special case. The proof of Theorem′ is based on the same ideas as the proof of the Theorem. But since it is technically complicated we omit it.

2. Some remarks about the Theorem

The conditions of the Theorem can be slightly weakened. The following version of the Theorem may be useful in certain applications.

Stronger version of the Theorem. *The Theorem and Theorem′ remain valid if Part 1.) of Property A is replaced by the following weaker condition 1.′)*

1.′) There are some positive numbers $0 < b_1 < b_2$ such that almost all functions $f(\varphi)$, $0 < \varphi < \theta$, with respect to the measure P satisfy the inequality $b_1 < f(\varphi) < b_2$ and

$$P\left(\sup_{0\le\varphi_1<\varphi_2\le\theta} \frac{|f(\varphi_1)-f(\varphi_2)|}{|\varphi_1-\varphi_2|} > x\right) \le Ke^{-\lambda x} \tag{2.1}$$

for all $x > 0$ with some $K > 0$ and $\lambda > 0$.

At the end of this paper we briefly explain the modifications needed in the proof of this stronger version of the Theorem.

We discuss the content of Property A and give the following example:

Remark 1. Let $W(t) = W(t,\omega)$, $\theta \ge t \ge u$ with some $u < 0$ be a Wiener process, and define the process $B(\varphi) = B(\varphi,\omega) = \int_u^\varphi W(t,\omega)\,dt$, $0 \le \varphi \le \theta$. Then the Theorem holds for almost all trajectories of the process $B(\varphi,\omega)$ if a sufficiently big constant is added to it. More explicitly, $B(\varphi,\omega) + C(\omega)$ satisfies the Theorem if $C(\omega) > -\min_{u\le\varphi\le\theta} B(\varphi,\omega) + c$ with some positive constant c, i.e. the distribution of the number of lattice points in $\mathbb{O}_R(B(\varphi,\omega)+C(\omega))$ tends to the Poisson distribution with parameter

$$(c_2 - c_1)\int_0^\theta (B(\varphi,\omega)+C(\omega))^2\,d\varphi$$

if R is uniformly distributed in the interval $[a_1L, a_2L]$ with some $a_2 > a_1 > 0$, and $L \to \infty$.

We briefly explain the proof of Remark 1 with the help of the Stronger version of the Theorem.

Introduce the sigma algebra $\mathcal{F}_\varphi = \{\mathcal{F}(W(s)),\ s \le \varphi\}$. Then the process $(B(\varphi,\omega),\mathcal{F}_\varphi)$ is a Gaussian Markov process. We show that $B(\varphi,\omega)$ satisfies Part 2) of Property A with $\tau = 3/2$. For this aim fix the parameters $\varphi_1,\dots,\varphi_k$ and the values $W(\varphi_1) = y_1, \dots, W(\varphi_k) = y_k$. We calculate the conditional density function of the random vector $(B(\varphi_1),\dots,B(\varphi_k))$ under this condition. It equals

$$p_k^{(y_1,\dots,y_k)}(x_1,\dots,x_k|\varphi_1,\dots,\varphi_k) = p_1^{y_1}(x_1|\varphi_1)\prod_{i=2}^k p^{(y_{i-1},y_i)}(x_i|\varphi_{i-1},\varphi_i,x_{i-1}),$$

where $p^{(y_{i-1},y_i)}(x_i|\varphi_{i-1},\varphi_i,x_{i-1})$ is the conditional density function of $B(\varphi_i)$ under the condition $B(\varphi_{i-1}) = x_{i-1}$, $W(\varphi_{i-1}) = y_{i-1}$ and $W(\varphi_i) = y_i$, and $p_1^{y_1}(x_1|\varphi_1)$ is the conditional density of $B(\varphi_1)$ under the condition $W(\varphi_1) = y_1$. These conditional density functions are Gaussian with expectation $x_{i-1} + (\varphi_i - \varphi_{i-1})\dfrac{y_{i-1}+y_i}{2}$ and variance

$$D(\varphi_{i-1},\varphi_i)=\int_{\varphi_{i-1}}^{\varphi_i}\int_{\varphi_{i-1}}^{\varphi_i}(\min(s,t)-\varphi_{i-1})(\varphi_i-\max(s,t))\,ds\,dt$$
$$=O\left(|\varphi_i-\varphi_{i-1}|^3\right)$$

for $i\ge 2$, and the density of $p_1^{y_1}(x_1|\varphi_1)$ can be written down similarly. Part 2) of Property A can be proved with the help of the above formulas after integration with respect to the conditions $W(\varphi_s)=y_s$, $s=1,\dots,k$. (The appearance of the parameter $\tau=3/2$ can also be explained with the help of the observation that $B(\varphi)$ and $T^{-3/2}B(T\varphi)$ have the same distribution.) But the distribution of $B(\varphi)$ does not satisfy Part 1) of Property A, since although the derivative $B'(\varphi,\omega)=W(\varphi,\omega)$ is bounded, this bound depends on ω.

A natural way to overcome this difficulty is to make a conditioning of the process $W(t)$ by the condition $\{\sup|W(t)|<A\}$ with some $A>0$ or to consider the process $\bar W(t)$ which is the reflected Wiener process $W(t)$ with reflective barriers $-A$ and A, then to integrate this process and apply the Theorem for the integrated process, (more precisely for the integrated process $+A'$, with some $A'>A$). Then we can exploit that the probability of the event that this new process agrees with $B(\varphi)$ tends to 1 as $A\to\infty$. To carry out this program we should prove that the distribution of this new process satisfies Property A. This statement is probably true, but we cannot check Part 2b) of Property A. Hence we choose a slightly different approach.

Define the function $h_A(t)$,

$$h_A(t)=\begin{cases}t-4kA & \text{if } (4k-1)A\le t<(4k+1)A, \quad k=0,\pm1,\dots\\ (4k+2)A-t & \text{if } (4k+1)A\le t<(4k+3)A, \quad k=0,\pm1,\dots\end{cases}$$

and the random process $B_1(\varphi)=h_A(B(\varphi))$. (The process $B_1(\varphi)$ is actually the process $B(\varphi)$ after reflection with reflective barriers $-A$ and A.) Then the process $B_1(\varphi)+A'$ with $A'>A$ satisfies Property A if Part 1) is replaced by its weaker version Part 1$'$). Part 2) of Property A can be checked in this case, since the density function appearing in it can be written down explicitly. It is not difficult to show that Part 1$'$) of Property A holds, since

$$\sup_{0\le\varphi_1<\varphi_2\le\theta}\frac{|B_1(\varphi_1)-B_1(\varphi_2)|}{|\varphi_1-\varphi_2|}\le\sup_{0\le\varphi\le\theta}|W(t)|\,.$$

Then we get the proof of Remark 1 by letting A tend to infinity.

Although the technically most difficult part in the proof of Remark 1 was to check Part 2b), actually the most restrictive condition of Property A is Part 2a), especially the restriction $\tau<2$. It has the following content. For fixed $0\le\varphi_1<\varphi_2<\dots<\varphi_k$ the density function of the random vector $(f(\varphi_1),\dots,f(\varphi_k))$ is a bounded function with a bound that may depend on $\varphi_1,\dots,\varphi_k$. Since $b_1<f(\varphi_1)<b_2$ and $|f(\varphi_i)-f(\varphi_{i-1})|<b_3|\varphi_i-\varphi_{i-1}|$ for $i=2,\dots,k$, hence the density function can differ from zero only on a set of Lebesgue measure $(b_2-b_1)b_3^{k-1}\prod\limits_{i=2}^k(\varphi_i-\varphi_{i-1})$. Hence

$$\sup_{x_1,\dots,x_k}p_k(x_1,\dots,x_k|\varphi_1,\dots,\varphi_k)\ge\frac{\text{const.}}{\prod\limits_{i=2}^k(\varphi_i-\varphi_{i-1})}\,.$$

The upper bound imposed in Part 2a) of Property A on this density function is a power of $\tau < 2$ of this lower estimate. It also gives a lower bound on the Lebesgue measure of the set where the density function

$$p_k(x_1, \dots, x_k | \varphi_1, \dots, \varphi_k)$$

is not zero. The requirement that the trajectory $f(\varphi)$ is chosen "sufficiently randomly" is hidden in this condition. It is also connected with the smoothness properties of the functions $f(\varphi)$. We do not want to discuss this question in detail, we only prove the following Remark 2, which also indicates the limits of applicability of the Theorem.

Remark 2. Let a probability measure P on the space of continuous functions satisfy Property A or its weaker version. Then the set of all twice differentiable functions with bounded second derivatives has zero P probability.

To prove Remark 2 it is enough to show that

$$P\left(\lim_{h\to 0} \sup_{0\le\varphi\le\theta-2h} \left| \frac{f(\varphi)+f(\varphi+2h)-2f(\varphi+h)}{h^2} \right| = \infty \right) = 1 , \qquad (2.2)$$

because twice differentiable functions with finite second derivatives do not satisfy this relation. To prove (2.2) fix some $0 \le \varphi \le \theta - 2h$, $K > 0$ and integer $k > \dfrac{\tau+1}{2-\tau}$, define the event

$$\begin{aligned} A_h &= A_h(k, \varphi, K) \\ &= \left\{ \sup_{1\le j\le k} \left| \frac{f(\varphi+jh)+f(\varphi+(j+2)h)-2f(\varphi+(j+1)h))}{h^2} \right| < K \right\} \end{aligned}$$

and estimate its probability.

Observe that

$$\sup p(x_1, \dots, x_{k+2} | \varphi+h, \varphi+2h, \dots, \varphi+(k+2)h) \le C_{k+2} h^{-(k+1)\tau} ,$$

by Part 2a) of Property A. Hence

$$P(A_h) \le \text{const.}\, h^{-(k+1)\tau} \lambda(B_h)$$

with

$$\begin{aligned} B_h = B_h(k) = \{(x_1, \dots, x_{k+2}), \quad & b_1 \le x_1, x_2 \le b_2, \\ & |x_j + x_{j+2} - 2x_{j+1}| < Kh^2, \quad j = 1, \dots, k\} , \end{aligned}$$

where $\lambda(\cdot)$ denotes Lebesgue measure. We have

$$\lambda(B_h(k)) = (b_2 - b_1)^2 (2Kh^2)^k ,$$

since for fixed x_1, …, x_j the point x_{j+1} is in an interval of length $2Kh^2$ for $j = 2$, …, $k+1$. Hence,

$$P(A_h) \le \text{const.}\, h^{2k-(k+1)\tau} \le \text{const.}\, h \to 0 \quad \text{as } h \to 0 ,$$

where the const. may depend on K. Hence we get that

$$P\left(\lim_{h\to 0}\sup_{0\le\varphi\le\theta-2h}\left|\frac{f(\varphi)+f(\varphi+2h)-2f(\varphi+h)}{h^2}\right|\le K\right)=0\,.$$

Since this relation holds for all $K>0$, hence relation (2.2) and Remark 2 holds.

We finish these remarks by posing two open problems.

Problem 1. Give explicit curves which satisfy the Theorem. In particular, let us consider the ellipses given by the equations $x^2+ay^2=1$ with some $a>0$. Is it true that these ellipses satisfy Sinai's conjecture for almost all $a>0$? The circle, i.e. the ellipsis with $a=1$ does not satisfy it. In this case $f(\varphi)=1$, and the problem leads to the following number theoretical question. Let $r(n)$ denote the number of integer solutions of the equation $k^2+l^2=n$. What can be said about the distribution of the number theoretical function $r(n)$?

For the sake of simplicity, let us consider only the case when $c_2-c_1<1/2$. Then the interval $\left[R+\frac{c_1}{R},R+\frac{c_2}{R}\right]$ contains the square root of only one integer n, and the number of lattice points in $\mathbb{O}_R$ equals $r(n)$ with this integer n. On the other hand, the probability that this interval contains the square root of a fixed integer n is less than const. L^{-2} if R is uniformly distributed on the interval $[a_1L,a_2L]$. The behaviour of the function $r(n)$ is fairly well-known. (See e.g. [4].) For our purposes it is enough to know that $r(n)=0$ if the prime factorization of n contains a prime factor of the form $4k+3$ on an odd power. We also know that the density of the integers satisfying this property is one. The above facts imply that in the case of circle the probability that $\mathbb{O}_R$ contains no lattice point tends to one as $L\to\infty$. A more detailed analysis also shows that the conditional probability of the event that the number of lattice points in $\mathbb{O}_R$ tends to infinity is almost one under the condition that $\mathbb{O}_R$ is not empty. On the other hand, some computer simulations suggest that this is a degenerate case, and almost all ellipses satisfy Sinai's conjecture (see [1]).

Problem 2. Prove the Theorem for almost all functions with respect to such probability measures which contain very smooth (e.g. analytic) functions with positive probability.

3. Reduction of the proof of the Theorem

In the proof we apply a version of the method of moments. Let us first show that if a sequence of random variables ξ_L satisfies the relation

$$E\binom{\xi_L}{k}\to\frac{\lambda^k}{k!}\quad\text{for all } k=1,2,\dots,\tag{3.1}$$

as $L\to\infty$, then this sequence tends in distribution to the Poisson distribution with parameter λ. To prove this, let us observe that if ξ is a Poisson distributed random variable with parameter λ, then

$$E\binom{\xi}{k}=\sum_{n=k}^{\infty}\frac{\lambda^n}{n!}e^{-\lambda}\binom{n}{k}=e^{-\lambda}\sum_{n=k}^{\infty}\frac{\lambda^n}{k!(n-k)!}=\frac{\lambda^k}{k!}e^{-\lambda}\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}=\frac{\lambda^k}{k!}\,.$$

The moment $E\xi_L^k$ can be expressed as a linear combination of the quantities $E\binom{\xi_L}{p}$, $0\le p\le k$. Hence if formula (3.1) holds, then $E\xi_L^k$ tends to the k-th moment of a Poisson distributed random variable with parameter λ. But if all moments of a sequence of random variables converge to the moments of a Poisson distribution with parameter λ, then this sequence converges in distribution to the Poisson law with parameter λ.

We have chosen this approach, because the following identity holds: For all functions f

$$\binom{\xi_L(f,R)}{k} = \sum_{\{m_1,\dots,m_k\}\in\mathbb{Z}^{2k}} \chi\left(\{m_s \in \mathbb{O}_R(f), \text{ for all } s=1,\dots,k\}\right).$$

Here $\chi(A)$ denotes the indicator function of the event A. The summation is taken for such k-tuples of lattice points where all points $m_1, \dots, m_k$ are different, and two k-tuples are identified if they contain the same lattice points, only in different order. Hence, for all functions f

$$E\binom{\xi_L(f,R)}{k} = \sum_{\{m_1,\dots,m_k\}\in\mathbb{Z}^{2k}} E\chi\left(\{m_s \in \mathbb{O}_R(f) \text{ for all } s=1,\dots,k\}\right), \quad (3.2)$$

where expectation is taken for a random variable R which is uniformly distributed in the interval $[a_1L, a_2L]$. We can handle the terms in the sum (3.2), but only in the case when the differences between the angles $\varphi(m_s)$, $s=1,\dots,k$, are not too small. Hence, first we reduce the proof of the Theorem to the investigation of a sum where only such terms appear. To formulate this statement more explicitly we need some notations. First we explain the strategy of our proof.

We shall split the domain $\mathbb{O}_R(f)$ by means of small sectors $\mathbb{D}_j$ and put even smaller buffer zones $\mathbb{C}_j$ between them. We shall prove that the contribution of the sectors $\mathbb{C}_j$ is negligible. This is the content of Lemma 1. We can show with the help of Lemma 2 that the probability of the event that there is some $\mathbb{D}_j$ which contains two lattice points in $\mathbb{O}_R(f)$ tends to zero. This is a rareness type argument, typical in the proof of Poissonian limit theorems. In our approach however, we need a stronger statement. We shall drop all k-tuples which have two points in the same sector $\mathbb{D}_j$ with some j and count only the remaining k-tuples all of whose elements are in $\mathbb{O}_R(f)$. We show that only a negligible error is committed in this way. This is the content of formula (3.3), and the reduction of the Theorem to this statement is done by means of Lemma 3. The hard step in the proof of the Theorem is the verification of formula (3.3). It states that some moment type expression behaves so as if the number of lattice points of $\mathbb{O}_R$ in different sectors $\mathbb{D}_j$ were independent. There is no such independence in our model, but we shall prove a Proposition which can be considered as a law of large numbers type result (such results are related to some sort of independence) and which implies the Theorem.

In these Lemmas and in the Proposition the random radius R does not appear. These results formulate some properties which almost all functions with respect to a probability measure with Property A satisfy. Lemma 1 is exceptional in this respect. (The random radius R appears in it, but it formulates a property which all positive continouus Lipschitz one functions satisfy.) We shall show that a function with these properties satisfies Sinai's conjecture.

Put

$$0 = \varphi_0(n) < \varphi_1(n) < \cdots < \varphi_{2p+1}(n) \le \theta < \varphi_{2p+2}(n), \quad (p = p(n))$$

in such a way that

$$\varphi_{2j+1} - \varphi_{2j} = (\log n)^{-\alpha}, \quad j = 0, 1, \dots, p,$$

$$\varphi_{2j+2} - \varphi_{2j+1} = (\log n)^{-\beta}, \quad j = 0, 1, \dots, p-1,$$
$$\text{and} \quad \varphi_{2p+2} - \varphi_{2p+1} < (\log n)^{-\alpha}$$

with some $\alpha < \beta$ and $\alpha > \frac{2}{2-\tau}$, where τ is the same number which appears in Part 2a) of Property A. (For the sake of simpler notations in the sequel we denote by log logarithm with base 2.) Clearly, $p(n) < 2\pi(\log n)^{\alpha}$. Define also the sets

$$\mathbb{C}_j = \mathbb{C}_j(n) = \{x \in \mathbb{R}^2, \quad An < |x| < Bn, \ \varphi_{2j+1} < \varphi(x) < \varphi_{2j+2}\},$$
$$\mathbb{D}_j = \mathbb{D}_j(n) = \{x \in \mathbb{R}^2, \quad An < |x| < Bn, \ \varphi_{2j} < \varphi(x) < \varphi_{2j+1}\},$$
$$j = 0, \dots, p(n).$$

In the definition of the sets $\mathbb{C}_j(n)$ and $\mathbb{D}_j(n)$ we choose $A > 0$ as sufficiently small, $B > 0$ as sufficiently large fixed constants.

For all continuous Lipschitz one functions $f(\cdot)$, integers $k = 1, 2, \dots$, and $n > 0$ we define the random variable (depending on R)

$$\zeta_n(k, f, R) = \sum_{0 \le j_1 < \dots < j_k \le p(n)} \sum_{\substack{m_s \in \mathbb{D}_{j_s}(n) \cap \mathbb{Z}^2 \\ s=1,\dots,k}} \chi\left(\{m_s \in \mathbb{O}_R(f) \text{ for all } s = 1, \dots, k\}\right)$$

and the number

$$E_L(k, f) = E\zeta_{2^n}(k, f, R),$$

where the integer n is determined by the relation $2^n < L < 2^{n+1}$, $n = 1, 2, \dots$, and the sign of expectation E means again expectation for the random variable R, distributed uniformly in the interval $[a_1 L, a_2 L]$. We shall prove that if $f(\cdot)$ is chosen randomly with respect to a probability measure satisfying Property A, then

$$\lim_{L \to \infty} E_L(f, k) = \frac{\lambda(f)^k}{k!} \quad \text{for all } k = 1, 2, \dots \text{ for almost all } f \tag{3.3}$$

with $\lambda(f) = (c_2 - c_1) \int_0^{\theta} f^2(\varphi)\, d\varphi$. First we show with the help of three lemmas to be proved in Section 5 that formula (3.3) implies the Theorem. Let us formulate these lemmas. We introduce the following notation. Given a finite or countable set A, let $|A|$ denote its cardinality.

Lemma 1. *Let $f(\varphi)$, $0 \le \varphi \le \theta$, be an arbitrary continuous Lipschitz one function such that $a < f(\varphi) < b$ with some $0 < a < b < \infty$ for all $0 \le \varphi \le \theta$, and $R = R_L$ a uniformly distributed random variable on the interval $[a_1 L, a_2 L]$. Let $2^n \le L < 2^{n+1}$, and define the random variables*

$$\eta_L^{(1)} = \eta_L^{(1)}(f) = \left| \left\{ m;\ m \in \bigcup_{j=0}^{p(2^n)} \mathbb{C}_j(2^n) \cap \mathbb{Z}^2 \cap \mathbb{O}_R(f) \right\} \right|.$$

Then $\eta_L^{(1)} \Rightarrow 0$ *as* $L \to \infty$, *where* $\Rightarrow$ *means convergence in probability.*

Lemma 2. *Let* P *be a probability measure with Property A on the space of continuous Lipschitz one functions. For arbitrary* $K > 0$ *and function* $f(\cdot)$ *define the sets* $A_n(f)$, $n = 1, 2, \ldots,$

$$A_n(f) = \Big\{(m, \bar{m}), \quad m \in \mathbb{Z}^2,\ \bar{m} \in \mathbb{Z}^2,\ m \neq \bar{m},\\ m \in \mathbb{D}_j(n),\ \bar{m} \in \mathbb{D}_j(n) \text{ for some } 0 \le j \le p(n),\\ \left|\frac{|m|}{f(\varphi(m))} - \frac{|\bar{m}|}{f(\varphi(\bar{m}))}\right| < \frac{K}{n}\Big\}.$$

For almost all functions $f(\cdot)$ *with respect to the measure* P *the relation*

$$|A_{2^n}(f)| < \frac{2^{2n}}{n^{\alpha(2-\tau)/2}} \quad \text{if } n > n(f)$$

holds.

The following Lemma 3 is a generalization of Lemma 2.

Lemma 3. *Let the conditions of Lemma 2 be satisfied. For arbitrary* $K > 0$, $k = 0, 1, \ldots,$ *and function* $f(\cdot)$ *define the sets* $B_{n,k}(f)$, $n = 1, 2, \ldots,$

$$B_{n,k}(f) = \Big\{(m, \bar{m}, m_1, \ldots, m_k),\ m \in \mathbb{Z}^2,\ \bar{m} \in \mathbb{Z}^2,\ m_s \in \mathbb{Z}^2 \text{ for } 1 \le s \le k\ ,\\ m \in \mathbb{D}_j(n),\ \bar{m} \in \mathbb{D}_j(n),\ \text{with some } 0 \le j \le p(n)\ ,\\ \text{all lattice points } m, \bar{m} \text{ and } m_s,\ s = 1, \ldots, k \text{ are different},\\ \left|\frac{|m|}{f(\varphi(m))} - \frac{|\bar{m}|}{f(\varphi(\bar{m}))}\right| < \frac{K}{n},\\ \left|\frac{|m|}{f(\varphi(m))} - \frac{|m_s|}{f(\varphi(m_s))}\right| < \frac{K}{n}, \quad 1 \le s \le k\Big\}.$$

For almost all functions $f(\cdot)$ *with respect to the measure* P *the relation*

$$|B_{2^n,k}(f)| < C_k \frac{2^{2n}}{n^{\alpha(2-\tau)/2}} \quad \text{if } n > n(f, k)$$

holds with some $C_k > 0$.

Given some $L > 1$, introduce the integer n such that $2^n \le L < 2^{n+1}$, and define the random variables

$$\xi_L^{(1)} = \xi_L^{(1)}(f, R) = \text{the number of } m \in \mathbb{Z}^2 \text{ such that } m \in \mathbb{O}_R(f) \cap \bigcup_{j=0}^{p(2^n)} \mathbb{D}_j(2^n)$$

and

$$\xi_L^{(2)} = \xi_L^{(2)}(f, R) = \text{the number of indices } j\\ \text{such that } \exists\, m \in \mathbb{Z}^2 \cap \mathbb{O}_R(f) \cap \mathbb{D}_j(2^n)\,.$$

We claim that if the function $f(\cdot)$ is chosen randomly with respect to a probability measure P with Property A, then

$$\xi_L^{(1)}(f,R)-\xi_L(f,R)\Rightarrow 0 \quad \text{as } L\to\infty\,, \tag{3.4}$$

and

$$\xi_L^{(2)}(f,R)-\xi_L(f,R)\Rightarrow 0 \quad \text{as } L\to\infty\,. \tag{3.4'}$$

for almost all $f(\cdot)$.

If $m\in\mathbb{O}_R$, then $m\in\bigcup_{j=0}^{p(2^n)}\mathbb{C}_j(2^n)\cup\bigcup_{j=0}^{p(2^n)}\mathbb{D}_j(2^n)$ if the constants A and B in the definition of $\mathbb{C}_j$ and $\mathbb{D}_j$ are appropriately chosen, since $a_1L<R<a_2L$, and the function f is bounded away both from zero and infinity. Hence Lemma 1 implies (3.4). To prove (3.4′) observe that

$$\left|\frac{|m|}{f(\varphi(m))}-\frac{|\bar m|}{f(\varphi(\bar m))}\right|<\left|\frac{|m|}{f(\varphi(m))}-R\right|+\left|\frac{|\bar m|}{f(\varphi(\bar m))}-R\right|<\frac{K}{2^n}$$

if $m,\bar m\in\mathbb{O}_R$. Hence the random variable

$$\eta_n=\eta_n(f)=\left|\{(m,\bar m),\quad m,\bar m\in\mathbb{Z}^2,\ (m,\bar m)\in A_{2^n}(f),\ m\in\mathbb{O}_R\}\right|$$

satisfies the relation $\left|\xi_L^{(1)}-\xi_L^{(2)}\right|<\eta_n$.

To prove (3.4′) it is enough to show that $\eta_n\Rightarrow 0$. To show this, first we remark that for all positive continuous Lipschitz one functions $f(\cdot)$

$$\sup_m P(m\in\mathbb{O}_R(f))<\text{const.}\,2^{-2n}$$

if R is uniformly distributed in the interval $[a_1L,a_2L]$, and $2^n\le L<2^{n+1}$. Then Lemma 2 yields that

$$E\eta_n<\frac{2^{2n}}{n^{\alpha(2-\tau)/2}}\sup_m P(m\in\mathbb{O}_R(f))<\text{const.}\,n^{\alpha(\tau-2)/2}\to 0\,.$$

Hence to prove the Theorem it is enough to show that

$$E\binom{\xi_L^{(2)}(f,R)}{k}\to\frac{\lambda^k}{k!}\qquad\text{for almost all } f(\cdot) \text{ as } L\to\infty\,. \tag{3.5}$$

We prove that Lemma 3 and formula (3.3) imply this relation. For this aim we introduce the random variables

$$\zeta_n^{(1)}(k,f,R)=\Big|\{(m,\bar m,m_1,\dots,m_k),\quad (m,\bar m,m_1,\dots,m_k)\in B_{n,k}(f)\,,$$
$$m_s\in\bigcup_{j=1}^{p(n)}\mathbb{D}_j,\quad m,\bar m,m_s\in\mathbb{O}_R(f),\ s=1,\dots,k\}\Big|$$

and verify the relation

$$\zeta_{2^n}(k,f,R)-\zeta^{(1)}_{2^n}(k-1,f,R)\le \binom{\xi^{(2)}_L(f,R)}{k}\le \zeta_{2^n}(k,f,R)\quad \text{for } k=1,2,\dots. \tag{3.6}$$

Indeed, if a k-tuple $\mathbb{D}_{j_1},\dots,\mathbb{D}_{j_k}$ is such that $|\mathbb{D}_{j_s}\cap \mathbb{O}_R(f)|=p_s\ge 1$ for all $s=1,\dots,k$, then it is counted once in the middle term of (3.6), $p_1\cdots p_k\ge 1$ times at the right-hand side and $p_1\cdots p_k\left(1-\sum\limits_{s=1}^{k}\frac{p_s-1}{2}\right)\le 1$ times at the left-hand side of (3.6). Taking expectation in (3.6) we get that

$$E_L(f,R)-E\zeta^{(1)}_{2^n}(k-1,f,R)\le E\binom{\xi^{(2)}_L(f,R)}{k}\le E_L(f,R)\quad \text{for } k=1,2,\dots$$

with $2^n\le L<2^{n+1}$. Hence to reduce the proof of the Theorem to formula (3.3) it is enough to show that

$$E\zeta^{(1)}_{2^n}(k-1,f,R)\to 0\quad \text{if } L\to\infty \text{ for almost all } f \text{ and } k=1,2,\dots,$$

where expectation is taken with respect to the random variable R which is uniformly distributed in the interval $[a_1L,a_2L]$, and $2^n\le L<2^{n+1}$. This relation holds, because Lemma 3 implies that

$$E\zeta^{(1)}_{2^n}(k,f,R)\le C_k\frac{2^{2n}}{n^{\alpha(2-\tau)/2}}\sup_{m\in\mathbb{Z}^2}P(m\in\mathbb{O}_R(f))\le \text{const.}\, n^{-\alpha(2-\tau)/2}$$

if $n>n(f,k)$.

4. Proof of the Theorem with the help of some Lemmas

The hardest part of the proof is the justification of formula (3.3). It is based on a Proposition, which will be formulated below. To do this, first we introduce some notations. Define the intervals

$$I_p(f,m,\delta)=\left[\frac{p\delta}{|m|}f(\varphi(m)),\frac{(p+1)\delta}{|m|}f(\varphi(m))\right],$$
$$\delta=\delta_n=(\log n)^{-\eta},\ p=0,\pm1,\dots,\pm\frac{D}{\delta},\quad m\in\mathbb{Z}^2$$

and

$$\tilde I_z=\tilde I_z(n)=\left[\frac{zn}{(\log n)^{\tilde\eta}},\frac{(z+1)n}{(\log n)^{\tilde\eta}}\right],\quad \bar A(\log n)^{\tilde\eta}<z<\bar B(\log n)^{\tilde\eta},$$

where D, $\bar A$ and $\bar B$ are appropriate positive integers, and $\eta>0$ and $\tilde\eta>0$ are also appropriately chosen. We define with their help the sets

$$\begin{aligned}&\mathcal{M}_{k,n}(f,j_1,\dots,j_k,p_2,\dots,p_k,z)\\&=\Big\{(m_1,\dots,m_k),\quad m_s\in\mathbb{D}_{j_s}\cap\mathbb{Z}^2,\quad s=1,\dots,k,\\&\quad |m_1|\in\tilde I_z(n),\\&\quad \frac{|m_s|}{f(\varphi(m_s))}-\frac{|m_1|}{f(\varphi(m_1))}\in I_{p_s}(f,m_1,\delta_n),\ s=2,\dots,k,\Big\}.\end{aligned}\tag{4.1}$$

Put

$$S_{n,k} = \Big\{ (j_1, \dots, j_k, p_2, \dots, p_k, z)\,, \quad 0 \le j_1 < j_2 < \cdots < j_k \le p(n)\,,$$
$$|p_s| < \frac{D}{\delta_n},\ s = 2, \dots, k, \quad \bar{A}(\log n)^{\bar{\eta}} < z < \bar{B}(\log n)^{\bar{\eta}} \Big\}\,.$$

(All numbers $j_1, \dots, j_k$, $p_2, \dots, p_k$ and z in the definition of $S_{n,k}$ are integers.) Now we formulate the following Proposition.

Proposition. *Let a continuous Lipschitz one function $f(\cdot)$ be chosen randomly with respect to a probability measure with Property A. Then for almost all functions $f(\cdot)$ and all $k \ge 1$ the relation*

$$\lim_{n\to\infty} \sup_{(j_1,\dots,j_k,p_2,\dots,p_k,z)\in S_{2^n,k}} \left[\frac{|\mathcal{M}_{k,2^n}(f, j_1, \dots, j_k, p_2, \dots, p_k, z)|}{z 2^{2n} n^{-k\alpha-(k-1)\eta-2\bar{\eta}}} - \prod_{s=2}^{k} f^2(\varphi_{2j_s}) \right] = 0$$

holds with probability one if n takes integer values.

To explain the content of the Proposition define, for fixed k, n and arbitrary integers $j_1, \dots, j_k$, $p_2, \dots, p_k$ and z, the set

$$X(f) = X(f, k, n, j_1, \dots, j_k, p_2, \dots, p_k, z)$$
$$= \Big\{ (x_1, \dots, x_k), \quad x_s \in \mathbb{D}_{j_s}, \quad s = 1, \dots, k,\ |x_1| \in \bar{I}_z(n)\,,$$
$$\frac{|x_s|}{f(\varphi(x_s))} - \frac{|x_1|}{f(\varphi(x_1))} \in I_{p_s}(f, |x_1|, \delta_n), \quad s = 2, \dots, k \Big\}\,.$$

Some calculation shows that the volume of the set $X(f)$ asymptotically equals

$$z n^2 (\log n)^{-2\bar{\eta}-\alpha} \prod_{s=2}^{k} \delta_n (\log n)^{-\alpha} f^2(\varphi_{2j_s})$$
$$= z n^2 (\log n)^{-2\bar{\eta}-k\alpha-(k-1)\eta} \prod_{s=2}^{k} f^2(\varphi_{2j_s})\,.$$

The Proposition states that for almost all functions $f(\cdot)$ with respect to a probability measure with Property A the number of lattice points in the sets $X(f)$ is asymptotically equal to the volume of these sets, at least for an exponentially rare subsequence of indices, $n = 2^l$, $l = 1, 2, \dots$.

The Proposition is useful for the following reason: We split the set of all k-tuples $\mathbf{m} = (m_1, \dots, m_k)$ which give a contribution to the expression $\zeta_{2^n}(k, f, R)$ into relatively few classes $\mathcal{M}_{k,2^n}$ (their number is a power of n if L is of order 2^n). As the subsequent calculation will show, each k-tuple from a class $\mathcal{M}_{k,2^n}$ gives contribution one to the sum $\zeta_{2^n}(k, f)$ with almost the same probability whose asymptotic value can be given explicitly. This is so, because a k-tuple gives a contribution one to this sum if the random radius R falls in the intersection of k intervals, and we know the length and relative position of these intervals with a sufficiently good accuracy if we know which class $\mathcal{M}_{k,2^n}$ this k-tuple belongs to. The Proposition gives the asymptotic size of the sets $\mathcal{M}_{k,2^n}$. Hence, we can estimate the expected value of $E_L(k,f) = E\zeta_{2^n}(k,f,R)$ by multipying the cardinality of $\mathcal{M}_{k,2^n}$ with the probability that the points from a k-tuple of this class fall into $\mathbb{O}_R(f)$ and then by summing up for all classes. In such a way we can prove formula (3.3).

The following heuristic argument may explain why the subsequent calculation yields the desired result. We consider the following auxiliary problem. Let us have a Poisson process, independent of a uniformly distributed random variable R in an interval $[a_1L, a_2L]$. Give the limit distribution of the number of points $\zeta(L)$ of this Poisson process in the domain $\mathbb{O}_R(f)$ if $L \to \infty$, and $\mathbb{O}_R(f)$ is defined in formula (1.1). It is not difficult to see that this limit is Poissonian with the same parameter λ which appeared in Theorem 1. Indeed, the conditional distribution of $\zeta(L)$ under the condition of prescribed R is Poissonian with a parameter (the area of $\mathbb{O}_R(f)$) which tends to λ as $R \to \infty$. We also have $E\binom{\zeta_L}{k} \to \frac{\lambda^k}{k!}$. This expectation could have been calculated in a more complicated way by means of sets defined analogously to the sets $\mathcal{M}_{k,n}$ in (4.1). The only difference in the definition is that now we count the number of points of the underlying Poisson process instead of lattice points in the same domain. Then $E\binom{\zeta_L}{k}$ equals asymptotically the sum of the expected value of the cardinality of these sets multiplied with the probability of the event that all points of a given k-tuple from this set is in $\mathbb{O}_R(f)$. (These probabilities are asymptotically the same for all k-tuples from a prescribed class.) What we do in the subsequent calculation is to show that our model imitates the previous one, and $E_L(k,f)$ can be approximated by the same sum (disregarding some negligible error terms) as $E\binom{\zeta_L}{k}$ in the auxiliary model. Hence this sum has the right asymptotics. This program can be carried out if we know that the asymptotic cardinality of the sets $\mathcal{M}_{k,n}$ is the same as that of the analogous sets in the auxiliary model. But this is the content of the Proposition.

Let us consider the elements of a class $\mathcal{M}_{k,n}$

$$(m_1, \dots, m_k) \in \mathcal{M}_{k,n}(j_1, \dots, j_k, p_2, \dots, p_k, z)$$

with a fixed $(j_1, \dots, j_k, p_2, \dots, p_k, z) \in S_{n,k}$. Then

$$m_s \in \mathbb{O}_R(f) \text{ for all } s = 1, \dots, k \tag{4.2}$$

if and only if

$$R - \frac{|m_1|}{f(\varphi(m_1))} \in [A(m_1, \dots, m_k), B(m_1, \dots, m_k)] \tag{4.3}$$

with some $A(m_1, \dots, m_k)$ and $B(m_1, \dots, m_k)$. The endpoints of this interval satisfy the relation

$$\begin{aligned} &A(m_1,\dots,m_k)\\ &\quad = \frac{(\log n)^{\tilde\eta} f(\varphi_{2j_1})}{zn} \max\{-c_2, \max_{2\le s\le k} -c_2 + p_s\delta_n\} + O\left(\frac{(\log n)^{-\omega}}{n}\right)\\ &B(m_1,\dots,m_k)\\ &\quad = \frac{(\log n)^{\tilde\eta} f(\varphi_{2j_1})}{zn} \min\{-c_1, \min_{2\le s\le k} -c_1 + p_s\delta_n\} + O\left(\frac{(\log n)^{-\omega}}{n}\right), \end{aligned} \tag{4.3'}$$

where $\omega = \min\{\eta, \tilde\eta, \alpha\}$. In the case $A(m_1,\dots,m_k) > B(m_1,\dots,m_k)$ the interval defined in (4.3) is empty. The $O(\cdot)$ in (4.3′) is uniform for

$$(j_1,\dots,j_k,p_2,\dots,p_k,z) \in S_{n,k}\ .$$

We remark that the main term in (4.3′) is of order $\frac{1}{n}$, since z is of order $(\log n)^{\tilde\eta}$. Hence the $O(\cdot)$ term in this formula is a negligible error. This also means that the length of the interval where R has to fall to satisfy relation (4.2) is asymptotically the same for all k-tuples from a fixed class $\mathcal{M}_{k,n}$. This interval is centered around the point $\frac{|m_1|}{f(\varphi(m_1))}$, which place strongly depends on which k-tuple in $\mathcal{M}_{k,n}$ is considered. But this holds only for the position and not the length of the interval.

Let us first remark that (4.2) holds if and only if R satisfies the relation

$$R + \frac{c_1}{R} \le \frac{|m_s|}{f\varphi(m_s))} \le R + \frac{c_2}{R} \quad \text{for all } s = 1,\dots,k\ . \tag{4.4}$$

The left-hand side and right-hand side of (4.4) are monotone functions of R if R is sufficiently large. Hence R is in the intersection of k intervals if $L > L_0$ with some fixed threshold L_0. Therefore (4.3) holds.

If $m_1 \in \mathbb{O}_R(f)$, then

$$\frac{1}{R} = \frac{f(\varphi(|m_1|))}{|m_1|} + O\left(\frac{1}{n^2}\right), \tag{4.5}$$

and if $(m_1,\dots,m_k) \in \mathcal{M}_{k,n}(j_1,\dots,j_k,p_2,\dots,p_k,z)$, then

$$\frac{f(\varphi(m_1))}{|m_1|} = \frac{(\log n)^{\tilde\eta} f(\varphi_{2j_1})}{zn} + r_1 \tag{4.5'}$$

$$\frac{|m_s|}{f(\varphi(m_s))} = \frac{|m_1|}{f(\varphi(m_1))} + \frac{p_s\delta_n(\log n)^{\tilde\eta} f(\varphi_{2j_1})}{zn} + r_s \quad \text{for } 2 \le s \le k \tag{4.5''}$$

hold with some r_s, $1 \le s \le k$, less than const. $\frac{(\log n)^{-\omega}}{n}$. Hence, if $m_s \in \mathbb{O}_R(f)$, $s = 1, \dots, k$, then by (4.4)

$$R + \frac{c_1(\log n)^{\tilde\eta} f(\varphi_{2j_1})}{zn} + \tilde r_1 \le \frac{|m_1|}{f(\varphi(m_1))} \le R + \frac{c_2(\log n)^{\tilde\eta} f(\varphi_{2j_1})}{zn} + \tilde r_1$$

and

$$R+\frac{c_1(\log n)^{\bar\eta}f(\varphi_{2j_1})}{zn}+\bar r_s\le\frac{|m_1|}{f(\varphi(m_1))}+\frac{p_s\delta_n(\log n)^{\bar\eta}f(\varphi_{2j_1})}{zn}$$
$$\le R+\frac{c_2(\log n)^{\tilde\eta}f(\varphi_{2j_1})}{zn}+\tilde r_s$$

for $2\le s\le k$ with $\bar r_s<\text{const.}\,\frac{(\log n)^{-\omega}}{n}$ and $\tilde r_s<\text{const.}\,\frac{(\log n)^{-\omega}}{n}$, $s=1,\dots,k$. These relations imply that

$$A(m_1,\dots,m_k)>\frac{(\log n)^{\tilde\eta}f(\varphi_{2j_1})}{zn}\max\{-c_2,\max_{2\le s\le k}-c_2+p_s\delta_n\}-K\frac{(\log n)^{-\omega}}{n}$$
$$B(m_1,\dots,m_k)<\frac{(\log n)^{\tilde\eta}f(\varphi_{2j_1})}{zn}\min\{-c_1,\min_{2\le s\le k}-c_1+p_s\delta_n\}+K\frac{(\log n)^{-\omega}}{n}$$

with an appropriate $K>0$. To complete the proof of (4.3′) we have to show that (4.4) holds if $(m_1,\dots,m_k)\in\mathcal{M}_{k,n}(j_1,\dots,j_k,p_2,\dots,p_k,z)$ and

$$\frac{(\log n)^{\bar\eta}f(\varphi_{2j_1})}{zn}\max\{-c_2,\max_{2\le s\le k}-c_2+p_s\delta_n\}+K\frac{(\log n)^{-\omega}}{n}$$
$$<R-\frac{|m_1|}{f(\varphi(m_1))}$$
$$<\frac{(\log n)^{\bar\eta}f(\varphi_{2j_1})}{zn}\min\{-c_1,\min_{2\le s\le k}-c_1+p_s\delta_n\}-K\frac{(\log n)^{-\omega}}{n}$$

with a sufficiently large $K>0$. Under these conditions relations (4.5)–(4.5″) hold again, and they imply together with the last relation that

$$R+\frac{c_1}{R}<\frac{|m_1|}{f(\varphi(m_1))}+\frac{(\log n)^{\bar\eta}f(\varphi_{2j_1})}{zn}(-c_1+p_s\delta_n)$$
$$+c_1\frac{(\log n)^{\bar\eta}f(\varphi_{2j_1})}{zn}-\frac{K}{2}\frac{(\log n)^{-\omega}}{n}<\frac{|m_s|}{f(\varphi(m_s))}$$

for all $s=1,\dots,k$. The other inequality in relation (4.4) can be proved similarly.

We can write

$$E_L(k,f)=\sum_{0\le j_1<j_2<\cdots<j_k\le p(2^n)}\sum_{z=\bar An^{\bar\eta}}^{\bar Bn^{\bar\eta}}B_L(f,j_1,\dots,j_k,z)\tag{4.6}$$

with

$$B_L(f,j_1,\dots,j_k,z)=\sum_{\substack{|p_s|<Dn^{\eta}\\ s=2,\dots,k}}\sum_{(m_1,\dots,m_k)\in\mathcal{M}_{k,2^n}(j_1,\dots,j_k,p_2,\dots,p_k,z)}E(\chi\{m_s\in\mathbb{O}_R(f)\text{ for all }s=1,\dots,k\}),\tag{4.6′}$$

where the dependence on L is present, since R is uniformly distributed in the interval $[a_1L,a_2L]$. (We recall that $2^n\le L<2^{n+1}$.) To see the validity of the above relations one has to observe that the condition $p_s < Dn^\eta$ is not a real restriction in formula (4.6′), since the expectation of all events

$$\{\chi(m_s \in \mathbb{O}_R(f) \text{ for all } s = 1, \dots, k\}$$

is considered in it which are non-empty if the constants $\bar{A}$, $\bar{B}$ and D are sufficiently large. We want to estimate the terms $B_L(f, j_1, \dots, j_k, z)$. Let us first observe that

$$B_L(f, j_1, \dots, j_k, z) = 0 \qquad \begin{array}{l} \text{if } z > \dfrac{a_2 L n^{\bar\eta}}{2^n} f(\varphi_{2j_1})(1 + Kn^{-\omega}) \\ \text{or } z < \dfrac{a_1 L n^{\bar\eta}}{2^n} f(\varphi_{2j_1})(1 - Kn^{-\omega}) , \end{array} \tag{4.7}$$

with some appropriate $K > 0$, since by formulas (4.3) and (4.3′) (with their application for 2^n) the event $m_s \in \mathbb{O}_R(f)$ can occur in this case only for such R which are outside of the interval $[a_1 L, a_2 L]$. To estimate B_L in other cases introduce the quantities

$$\begin{aligned} &\mathcal{K}_n^+(j_1, \dots, j_k, p_2, \dots, p_k, z) \\ &\quad = \sup_{(m_1,\dots,m_k) \in \mathcal{M}_{k,2^n}(j_1,\dots,j_k,p_2,\dots,p_k,z)} E\{\chi(m_s \in \mathbb{O}_R(f) \text{ for all } s = 1, \dots, k\} , \\ &\mathcal{K}_n^-(j_1, \dots, j_k, p_2, \dots, p_k, z) \\ &\quad = \inf_{(m_1,\dots,m_k) \in \mathcal{M}_{k,2^n}(j_1,\dots,j_k,p_2,\dots,p_k,z)} E\{\chi(m_s \in \mathbb{O}_R(f) \text{ for all } s = 1, \dots, k\} , \end{aligned}$$

Because of the Proposition we have

$$\begin{aligned} &(1 - \varepsilon_n) z 2^{2n} n^{-k\alpha - (k-1)\eta - 2\bar\eta} \prod_{s=2}^{k} f^2(\varphi_{2j_s}) \\ &\qquad \sum_{\substack{|p_s| < Dn^\eta \\ s=2,\dots,k}} \mathcal{K}_n^-(j_1, \dots, j_k, p_2, \dots, p_k, z) \le B_L(f, j_1, \dots, j_k, z) \\ &\qquad\qquad \le (1 + \varepsilon_n) z 2^{2n} n^{-k\alpha - (k-1)\eta - 2\bar\eta} \prod_{s=2}^{k} f^2(\varphi_{2j_s}) \\ &\qquad\qquad\qquad \sum_{\substack{|p_s| < Dn^\eta \\ s=2,\dots,k}} \mathcal{K}_n^+(j_1, \dots, j_k, p_2, \dots, p_k, z) \end{aligned} \tag{4.8}$$

for almost all functions $f(\cdot)$ with respect to a probability measure with Property A, where $\varepsilon_n \to 0$ uniformly for $(j_1, \dots, j_k, p_2, \dots, p_k, z) \in S_{k,2^n}$ as $n \to \infty$. Introduce also the following notation: Given some interval $A = [a, b]$, integers $p_2, \dots, p_k$ and some number $0 < \quad < 1$, define the interval

$$A(p_2, \dots, p_k,) = [a, b] \cap \bigcap_{s=2}^{k} [a + p_s(b - a), b + p_s(b - a)] ,$$

and let $\ell(A(p_2, \dots, p_k,))$ denote its length. It follows from formula (4.3) and (4.3′) (with their application for 2^n) that

$$\mathcal{K}_n^+(j_1,\dots,j_k,p_2,\dots,p_k,z)=\frac{1}{(a_2-a_1)L}\left[\ell(A(p_2,\dots,p_k,))+O(2^{-n}n^{-\omega})\right]$$
$$\mathcal{K}_n^-(j_1,\dots,j_k,p_2,\dots,p_k,z)=\frac{1}{(a_2-a_1)L}\left[\ell(A(p_2,\dots,p_k,))+O(2^{-n}n^{-\omega})\right] \tag{4.9}$$

with $A=[a,b]$, $a=-c_2\dfrac{f(\varphi_{2j_1})n^{\bar\eta}}{z2^n}$, $b=-c_1\dfrac{f(\varphi_{2j_1})n^{\bar\eta}}{z2^n}$ and $=\dfrac{n^{-\eta}}{c_2-c_1}$ if

$$(m_1,\dots,m_k)\in\mathcal{M}_{k,2^n}(j_1,\dots,j_k,p_2,\dots,p_k,z)\,,$$

and if $\frac{a_1Ln^{\bar\eta}}{2^n}f(\varphi_{2j_1})(1+Kn^{-\omega})<z<\frac{a_2Ln^{\bar\eta}}{2^n}f(\varphi_{2j_1})(1-Kn^{-\omega})$. We have to observe that in this case the interval of R for which $m_s\in\mathbb{O}_R(f)$, $s=1,\dots,k$, is contained in $[a_1L,a_2L]$. Moreover, the right-hand side of the first line in formula (4.9) is an upper bound for $\mathcal{K}_n^+$ for arbitrary z.

We need the following Lemma 4, which is a version of Lemma 3 in [9].

Lemma 4. *Let an interval $A=[a,b]$ and some number $0< \; <1$ be given. Then, using the notation introduced above, the relation*

$$\sum_{\substack{-\infty<p_s<\infty\\ s=2,\dots,k}}\ell(A(p_2,\dots,p_k,))=(b-a)^{1-k}+O\left((b-a)^{2-k}\right)$$

holds.

Since only those terms $\ell(A(p_2,\dots,p_k,))$ are non-zero in the sum appearing in Lemma 4 for which $|p_s|<\text{const.}^{-1}$, $2\le s\le k$, and there are only const.$^{1-k}$ such terms, hence the following estimate holds. By Lemma 4 and (4.9)

$$\sum_{\substack{|p_s|<Dn^{\eta}\\ s=2,\dots,k}}\mathcal{K}_n^{\pm}(j_1,\dots,j_k,p_2,\dots,p_k,z)$$
$$=\frac{(c_2-c_1)^k n^{\bar\eta+(k-1)\eta}f(\varphi_{2j_1})}{L(a_2-a_1)2^nz}\left(1+O(n^{-\omega})\right),$$

and by (4.8)

$$B_L(f,j_1,\dots,j_k,z)=(1+\varepsilon_n)\frac{2^n(c_2-c_1)^kn^{-\bar\eta-k\alpha}f(\varphi_{2j_1})}{L(a_2-a_1)}\prod_{s=2}^k f^2(\varphi_{2j_s}) \tag{4.10}$$

with some $\varepsilon_n=\varepsilon_n(j_1,\dots,j_k,z)\to 0$ uniformly for all

$$\frac{a_1Ln^{\bar\eta}}{2^n}f(\varphi_{2j_1})(1+Kn^{-\omega})<z<\frac{a_2Ln^{\bar\eta}}{2^n}f(\varphi_{2j_1})(1-Kn^{-\omega})$$
$$\text{and } 0\le j_1<\dots<j_k\le p(2^n)\,,$$

for almost all functions $f(\cdot)$. Moreover, the right-hand side of (4.10) is an upper bound for all z. Hence (4.7), (4.10) and (4.6) imply that

$$E_L(k,f) = (c_2-c_1)^k \sum_{0\le j_1<j_2<\cdots<j_k\le p(2^n)} (1+\varepsilon_n) \sum_{z=a_1\frac{L}{2^n}f(\varphi_{2j_1})n^{\tilde\eta}}^{a_2\frac{L}{2^n}f(\varphi_{2j_1})n^{\tilde\eta}} \frac{2^n n^{-\tilde\eta}}{L(a_2-a_1)} f(\varphi_{2j_1})n^{-\alpha} \prod_{s=2}^{k} f^2(\varphi_{2j_s})n^{-\alpha} \tag{4.11}$$
$$= (1+\varepsilon_n')(c_2-c_1)^k \sum_{0\le j_1<j_2<\cdots<j_k\le p(2^n)} \prod_{s=1}^{k} f^2(\varphi_{2j_s})n^{-\alpha}$$

for almost all functions $f(\cdot)$ with some $\varepsilon_n \to 0$ and $\varepsilon_n' \to 0$. The right-hand side of (4.11) tends to the integral

$$\frac{(c_2-c_1)^k}{k!}\int_0^\theta \cdots \int_0^\theta f^2(\varphi_1)\cdots f^2(\varphi_k)\,d\varphi_1\ldots d\varphi_k = \frac{(c_2-c_1)^k}{k!}\left(\int_0^\theta f^2(\varphi)\,d\varphi\right)^k,$$

as $n\to\infty$. These relations imply that the limit of $E_L(k,f)$ is $\dfrac{\lambda(f)^k}{k!}$ for all $k\ge 1$ for almost all functions $f(\cdot)$. Hence relation (3.3) and the Theorem hold.

5. Proof of the Lemmas

Proof of Lemma 1. The cardinality of the set $\bigcup_{j=0}^{p(2^n)} \mathbb{C}_j(2^n)\cap\mathbb{Z}^2$ is less than const. $2^{2n}(n^{-\alpha}+n^{\alpha-\beta})$. Hence

$$E\eta_L^{(1)} \le \text{const.}\, 2^{2n}(n^{-\alpha}+n^{\alpha-\beta}) \max_{2^nA\le|m|\le 2^nB} P(m\in\mathbb{O}_R(f)) \le \text{const.}\,(n^{-\alpha}+n^{\alpha-\beta}), \tag{5.1}$$

since in the case $m\in\mathbb{O}_R(f)$ for $2^nA\le|m|\le 2^nB$ the variable R must be in an interval of length const. 2^{-n}. Its probability is less than const. 2^{-2n} if R is uniformly distributed in the interval $[a_1L, a_2L]$ with $2^n\le L<2^{n+1}$. Relation (5.1) implies Lemma 1. □

Proof of Lemma 2. For fixed $m\in\mathbb{Z}^2$ define the set

$$A_n^m(f) = \{\bar m,\quad (m,\bar m)\in A_n(f)\}\,.$$

We claim that

$$E|A_n^m(f)| < \text{const.}\,(\log n)^{\alpha(\tau-2)} \quad \text{if } An<|m|<Bn\,. \tag{5.2}$$

First we show that (5.2) implies Lemma 2. Indeed, it follows from (5.2) that

$$E|A_n(f)| < \text{const.}\, n^2(\log n)^{\alpha(\tau-2)},$$

hence

$$P\left(|A_{2^n}(f)| > \frac{2^{2n}}{n^{\alpha(2-\tau)/2}}\right) \le \text{const.}\, n^{-\alpha(2-\tau)/2}.$$

Since $\sum n^{-\alpha(2-\tau)/2} < \infty$, the last relation together with the Borel–Cantelli lemma imply Lemma 2.

To prove (5.2) fix some small number $a > 0$ and introduce the sectors

$$U_s = U_{s,n}(m) = \left\{x \in \mathbb{R}^2, \quad \frac{as}{n} \le \varphi(x) - \varphi(m) < \frac{a(s+1)}{n}\right\}, \qquad s = 0, \pm 1, \pm 2, \dots$$

We show that for $\bar{m} \in A_n^m(f) \cap U_s$ there exists some $\bar{K} = \bar{K}(K, A, B, b_1, b_2, b_3)$ such that

$$\big||\bar{m}| - |m|\big| < a\bar{K}(|s| + 1).$$

Indeed, in this case

$$\begin{aligned}\big||\bar{m}| - |m|\big| &< \frac{Kf(\varphi(\bar{m}))}{n} + \frac{|f(\varphi(\bar{m})) - f(\varphi(m))|}{f(\varphi(m))}|m| \\ &< \frac{K'}{n} + K''\frac{a(|s|+1)}{n}|m| < a\bar{K}(|s|+1)\end{aligned}$$

by the Lipschitz one property of the functions $f(\cdot)$ we are considering and the fact that $|m|$ is of order n. The set

$$\bar{U}_s = U_s \cap \left\{\big||\bar{m}| - |m|\big| \le a\bar{K}(|s|+1)\right\}$$

has no more than const. $(|s| + 1)$ elements, and the sets $\bar{U}_0$ and $\bar{U}_{-1}$ are empty if $An \le |m| \le Bn$. The first statement is clear, and the second one holds for the following reason. If there is some $\bar{m} \in \bar{U}_0$ or $\bar{m} \in \bar{U}_{-1}$, and m, $\bar{m}$ are lattice points, then

$$1 \le |m - \bar{m}|^2 = (|m| - |\bar{m}|)^2 + 2|m||\bar{m}|\big(1 - \cos(\varphi(m) - \varphi(\bar{m}))\big)$$

and

$$\big||m| - |\bar{m}|\big| \le a\bar{K}\,.$$

Since $|m| > An$, $|\bar{m}| > An$, the above relations imply that

$$1 \le 2A^2n^2\left(1 - \cos\frac{a}{n}\right) + a^2\bar{K}^2 \le (3A^2 + \bar{K})a^2 \quad \text{if } n > n_0\,.$$

But this is impossible if $a > 0$ is sufficiently small, hence $\bar{U}_0$ and $\bar{U}_{-1}$ are empty.

We can write

$$E\,|A_n^m(f)| \le \text{const.} \sum_{s=1}^{\frac{n}{(\log n)^{\alpha}}} s \sup_{m \in \bar{U}_s \cap \bar{U}_{-s-1}} P\left(\left|\frac{|m|}{f(\varphi(m))} - \frac{|\bar{m}|}{f(\varphi(\bar{m}))}\right| < \frac{\bar{K}}{n}\right). \tag{5.3}$$

To estimate the above sum observe that by Part 2a) of Property A the probability density of the random vector $(f(\varphi(m)), f(\varphi(\bar{m})))$ is less than $\left(\frac{n}{s}\right)^{\tau}$ if $\bar{m} \in \bar{U}_s \cup \bar{U}_{-s-1}$, and the Lebesgue measure of the set

$$\left\{\big(f(\varphi(m)), f(\varphi(\bar{m}))\big), \quad b_1 < f(\varphi(m)) < b_2, \left|\frac{|m|}{f(\varphi(m))} - \frac{|\bar{m}|}{f(\varphi(\bar{m}))}\right| < \frac{K}{n}\right\}$$

is less than const. n^{-2}, since m and $\bar{m}$ are of order n. Hence,

$$P\left(\left|\frac{|m|}{f(\varphi(m))}-\frac{|\bar{m}|}{f(\varphi(\bar{m}))}\right|<\frac{K}{n}\right)\le\frac{\text{const.}}{n^2}\left(\frac{n}{s}\right)^{\tau}\quad \text{if } m\in\mathbb{Z}^2\cap\bar{U}_s\cap\bar{U}_{-s-1}\,,$$

and (5.3) implies that

$$E\left|A_n^m(f)\right|\le\text{const.}\sum_{s=1}^{\frac{n}{(\log n)^{\alpha}}}\frac{n^{\tau-2}}{s^{\tau-1}}\le\text{const.}\,(\log n)^{\alpha(\tau-2)}$$

as we claimed. Lemma 2 is proved. □

Proof of Lemma 3. The proof of Lemma 3 is similar to that of Lemma 2. For all $m\in\mathbb{Z}^2$ define the set

$$B_{n,k}^m(f)=\{(\bar{m},m_1,\dots,m_k),\quad (m,\bar{m},m_1,\dots,m_k)\in B_{n,k}(f)\}\,.$$

Similarly to Lemma 2, to prove Lemma 3 it is enough to show that

$$E|B_{n,k}^m(f)|<\text{const.}\,(\log n)^{\alpha(\tau-2)}\quad\text{if } An<|m|<Bn\,. \tag{5.4}$$

To prove formula (5.4), let us introduce for all integers $s_1,\dots,s_k$ and $\bar{s}$ the set

$$U(s_1,\dots,s_k,\bar{s})=\left\{(\bar{m},m_1,\dots,m_k),\quad m_j\in U_{j_p},\ 1\le j\le k,\ \bar{m}\in U_{\bar{s}}\right\}\,,$$

where U_s is the same as in the proof of Lemma 2. Let $s_1^*\le s_2^*\le\dots\le s_{k+2}^*$ be the monotone ordering of the numbers $s_1,\dots,s_k$, $\bar{s}$ and 0, and let $m_1',\dots,m_{k+2}'$ be the monotone ordering of the lattice points $m_1,\dots,m_k,m,\bar{m}$ by their angles, i.e. let $\varphi(m_1')<\varphi(m_2')<\dots<\varphi(m_{k+2}')$. Put $s_j'=s_{j+1}^*-s_j^*$ for $1\le j\le k+1$. Introduce the set

$$\bar{U}(s_1,\dots,s_k,\bar{s})=\Big\{(\bar{m},m_1,\dots,m_k)\in U(s_1,\dots,s_k,\bar{s})\,,$$
$$\left||m_{j+1}'|-|m_j'|\right|<a\bar{K}(s_j'+1),\quad j=1,\dots,k+1\Big\}\,.$$

We get, similarly to the argument in the proof of Lemma 2 that

$$U(s_1,\dots,s_k,\bar{s})\cap B_{n,k}^m(f)\subset\bar{U}(s_1,\dots,s_k,\bar{s})\,,$$

$$\bar{U}(s_1,\dots,s_k,\bar{s})=\emptyset\quad\text{if } s_j'=0 \text{ for some } 1\le j\le k+1$$

and

$$|\bar{U}(s_1,\dots,s_k,\bar{s})|<\text{const.}\prod_{j=1}^{k+1}s_j'\,.$$

Let us also observe that we have to consider only such sequences s_j', $1\le j\le k+1$, for which $\min_{1\le j\le k+1}s_j'\le n(\log n)^{-\alpha}$, since $|\varphi(m)-\varphi(\bar{m})|\le(\log n)^{-\alpha}$. A similar argument to that in Lemma 2 gives that

$$\sup_{(\bar m, m_1,\dots,m_k)\in\bar U(s_1,\dots,s_k,\bar s)} P\left(\left|\frac{|m|}{f(\varphi(m))}-\frac{|\bar m|}{f(\varphi(\bar m))}\right|<\frac{K}{n},\right.$$
$$\left.\left|\frac{|m|}{f(\varphi(m))}-\frac{|m_s|}{f(\varphi(m_s))}\right|<\frac{K}{n},\quad 1\le s\le k\right) \tag{5.5}$$
$$\le \text{const.}\, n^{-2k-2}\prod_{j=1}^{k+1}\left(\frac{n}{s'_j}\right)^{\tau}.$$

Indeed, we have to integrate a density function bounded by const. $\prod_{j=1}^{k+1}\left(\frac{n}{s'_j}\right)^{\tau}$ on a set of Lebesgue measure const. n^{-2k-2} to calculate a probability term appearing in (5.5). The above estimates yield that

$$E|B^m_{n,k}(f)| < \text{const.} \sum_{\substack{1\le s'_j\le n,\ \text{for all } 1\le j\le k+1\\ s_j\le \frac{n}{(\log n)^\alpha}\ \text{for some } j}} \prod_{j=1}^{k+1}\frac{1}{n}\left(\frac{n}{s'_j}\right)^{\tau-1}$$
$$\le \text{const.}\left(\sum_{s=1}^{n}\frac{1}{n}\left(\frac{n}{s}\right)^{\tau-1}\right)^{k}\sum_{s=1}^{\frac{n}{(\log n)^\alpha}}\frac{1}{n}\left(\frac{n}{s}\right)^{\tau-1}$$
$$\le \text{const.}\,(\log n)^{\alpha(\tau-2)}.$$

Lemma 3 is proved. □

Proof of Lemma 4. The relation

$$((b-a))^{(k-1)}\sum_{\substack{-\infty<p_s<\infty\\ s=2,\dots,k}}\ell(A(p_2,\dots,p_k,))$$
$$=\int\lambda_k^{(a,b)}(x_2,\dots,x_k)\,dx_2\dots dx_k+O\left((b-a)\right) \tag{5.6}$$

holds with

$$\lambda_k^{(a,b)}(x_2,\dots,x_k)=\lambda\left([a,b]\cap\bigcap_{s=2}^{k}[a+x_s,b+x_s]\right),$$

where $\lambda(\cdot)$ denotes Lebesgue measure. This relation holds, since the left-hand side of (5.6) is an approximating sum of the integral at the right-hand side, and $O\left((b-a)\right)$ is the error of the approximation. To complete the proof of Lemma 4 it is enough to show that

$$\mathcal{J}_k(a,b)=\int\lambda_k^{(a,b)}(x_2,\dots,x_k)\,dx_2\dots dx_k=(b-a)^k. \tag{5.7}$$

To prove (5.7) observe that

$$\lambda_k^{(a,b)}(x_2,\dots,x_{k-1},x_k)+\lambda_k^{(a,b)}(x_2,\dots,x_{k-1},x_k+(b-a))$$
$$=\lambda_{k-1}^{(a,b)}(x_2,\dots,x_{k-1})\quad\text{if } x_k\in[a-b,0]$$

and

$$\lambda_k^{(a,b)}(x_2,\dots,x_{k-1},x_k)=0 \quad \text{if } |x_k|>b-a\,.$$

Hence,

$$\mathcal{J}_k(a,b)=\int\limits_{\{a-b<x_k<0\}}\left[\lambda_k^{(a,b)}(x_2,\dots,x_k)+\lambda_k^{(a,b)}(x_2,\dots,x_k+(b-a))\right]\,dx_2\dots dx_k$$

$$=(b-a)\int\lambda_{k-1}^{(a,b)}(x_2,\dots,x_{k-1})\,dx_2\dots dx_{k-1}=(b-a)\mathcal{J}_{k-1}(a,b)\,.$$

The last relation implies (5.7) by induction. Lemma 4 is proved. □

6. Reduction of the proof of the Proposition

Let us introduce the following notation:

$$\mathcal{H}_n(f,j_1,\dots,j_k)=\int_{\varphi_{2j_1}}^{\varphi_{2j_1+1}}1\,d\varphi\prod_{s=2}^{k}\int_{\varphi_{2j_s}}^{\varphi_{2j_s+1}}f^2(\varphi)\,d\varphi\,.$$

We shall prove that

$$E\left[|\mathcal{M}_{k,n}(f,j_1,\dots,j_k,p_2,\dots,p_k,z)| - \frac{\left(z+\frac{1}{2}\right)n^2\delta_n^{k-1}}{(\log n)^{2\bar\eta}}\mathcal{H}_n(f,j_1,\dots,j_k)\right]^2<\frac{C_M n^4}{(\log n)^M} \tag{6.1}$$

for all $(j_1,\dots,j_k,p_2,\dots,p_k,z)\in S_{n,k}$ and arbitrarily large $M>0$.

We make some comments about the content of formula (6.1). The second term at the left-hand side is an estimate of the volume of the domain in $\mathbb{R}^{2k}$ where the k-tuples from $\mathcal{M}_{k,n}$ must fall. This is a better approximation of this volume than that given in the discussion after the formulation of the Proposition. The second moment of $|\mathcal{M}_{k,n}|$ is of order n^4 divided by some power of $\log n$ which depends on the parameters η, $\bar\eta$ and α appearing in the definition of $\mathcal{M}_{k,n}$. The expression at the left-hand side of (6.1) is much smaller, since in its estimate on the right-hand side we can divide by an arbitrary large power of $\log n$. Such an estimate holds only if the second term at the left-hand side is appropriately chosen, i.e. if the volume of the domain where the points of $\mathcal{M}_{k,n}$ must fall is computed with a sufficiently good accuracy. Let us also remark that we have only gained a logarithmic factor on a large negative power by making an appropriate centering of $|\mathcal{M}_{k,n}|$ on the left-hand side of (6.1).

First we show that formula (6.1) implies the Proposition. For this aim we introduce the events

$$A_n(j_1,\dots,j_k,p_2,\dots,p_k,z)=\left\{f(\cdot),\ \left||\mathcal{M}_{k,n}(f,j_1,\dots,j_k,p_2,\dots,p_k,z)| - \frac{\left(z+\frac{1}{2}\right)n^2\delta_n^{k-1}}{(\log n)^{2\bar\eta}}\mathcal{H}_n(f,j_1,\dots,j_k)\right|>\frac{n^2}{(\log n)^{M/3}}\right\}$$

in the space of continuous functions. By (6.1)

$$P(A_n(j_1,\dots,j_k,p_2,\dots,p_k,z)) < \frac{\text{const.}}{(\log n)^{M/3}}\,,$$

and since $M > 0$ can be chosen arbitrary large

$$\sum_{n=1}^{\infty} \sum_{0\le j_1<j_2<\dots<j_k\le p(2^n)} \sum_{\substack{|p_s|<D\delta_n\\ s=2,\dots,k}} \sum_{z=An^{\bar\eta}}^{Bn^{\bar\eta}} P\left(A_{2^n}(j_1,\dots,j_k,p_2,\dots,p_k,z)\right) < \infty\,. \tag{6.2}$$

Among the events

$$A_{2^n}(j_1,\dots,j_k,p_2,\dots,p_k,z)$$

for some $(j_1,\dots,j_k,p_2,\dots,p_k,z)\in S_{2^n,k}$ only finitely many one will occur with probability one by the Borel–Cantelli lemma and relation (6.2). On the other hand, because of the continuity properties of the functions $f(\cdot)$ we are considering, and since z has a value of order $n^{\bar\eta}$ the relation

$$\left(z+\frac{1}{2}\right)\mathcal{H}_{2^n}(f,j_1,\dots,j_k) = zn^{-k\alpha}\prod_{s=2}^{k} f^2(\varphi_{2j_s})\left(1+O(n^{-\alpha}+n^{-\bar\eta})\right)$$

holds. The last relation together with the fact that only finitely many events A_{2^n} occur with probability one imply the Proposition.

Relation (6.1) follows from the following two lemmas:

Lemma 5. *For arbitrary $M>0$ and $(j_1,\dots,j_k,p_2,\dots,p_k,z)\in S_{n,k}$ we have*

$$\begin{aligned} &E\left\{|\mathcal{M}_{k,n}(f,j_1,\dots,j_k,p_2,\dots,p_k,z)|^2\right\} \\ &\quad= \frac{\left(z+\frac{1}{2}\right)^2 n^4\delta_n^{2(k-1)}}{(\log n)^{4\bar\eta}} E\left\{\mathcal{H}_n(f,j_1,\dots,j_k)^2\right\} + O\left(\frac{n^4}{(\log n)^M}\right), \end{aligned}$$

where the $O(\cdot)$ is uniform in $j_1,\dots,j_k$, $p_2,\dots,p_k$ and z.

Lemma 6. *For arbitrary $M>0$ and $(j_1,\dots,j_k,p_2,\dots,p_k,z)\in S_{n,k}$ we have*

$$\begin{aligned} &E|\mathcal{M}_{k,n}(f,j_1,\dots,j_k,p_2,\dots,p_k,z)|\mathcal{H}_n(f,j_1,\dots,j_k) \\ &\quad= \frac{\left(z+\frac{1}{2}\right) n^2\delta_n^{(k-1)}}{(\log n)^{2\bar\eta}} E\left\{\mathcal{H}_n(f,j_1,\dots,j_k)^2\right\} + O\left(\frac{n^2}{(\log n)^M}\right), \end{aligned}$$

where the $O(\cdot)$ is uniform in $j_1,\dots,j_k$, $p_2,\dots,p_k$ and z.

First we give an informal explanation about the proof of Lemmas 5 and 6. The second moment at the left-hand side of the formula in Lemma 5 can be expressed as the sum of the probabilities that two pairs of k-tuples $\mathbf{m}=(m_1,\dots,m_k)$ and $\bar{\mathbf{m}}=(\bar m_1,\dots,\bar m_k)$ fall simultaneously into the set $\mathcal{M}_{k,n}$. This statement is expressed in formula (6.4). All terms in this sum can be written as an integral of the density function (introduced in Part 2 of the definition of Property A)

$$p_{2k}(x_1,\dots,x_k,x_{k+1},\dots,x_{2k}|\varphi(m_1),\dots,\varphi(m_k),\varphi(\bar m_1),\dots,\varphi(\bar m_k)) \qquad (6.3)$$

of the random vector $(f(\varphi(m_1)),\dots,f(\varphi(m_k)),\ f(\varphi(\bar m_1)),\dots,f(\varphi(\bar m_k)))$. The sum of these integrals can be considered as the approximating sum of an integral in an appropriate domain. As the subsequent calculation will show, this integral equals the main term of the right-hand side of the formula in Lemma 5. Lemma 5 gives a bound on the error which is committed when the integral expressing $E\mathcal{H}_n^2$ multiplied with the constant appearing in Lemma 5 is replaced by the sum by which we expressed the left-hand side.

This error is small, because by Part 2b) of Property A the density function (6.3) depends continuously on its arguments $x_1,\dots,x_{2k}$ and $\varphi(m_1),\dots,\varphi(m_k)$, $\varphi(\bar m_1),\dots,\varphi(\bar m_k)$. But this property supplies a good estimate only if all differences between the angles $\varphi(m_s)$ and $\varphi(\bar m_s)$ are not too small. The difference between $\varphi(m_s)$ and $\varphi(m_{s'})$ or $\varphi(\bar m_{s'})$ is bigger than $\log n^{-\beta}$, if $s\neq s'$ because of the existence of the buffer zones $\mathbb{C}_j$, and the same statement holds for $\varphi(\bar m_s)$. But $\varphi(m_s)-\varphi(\bar m_s)$ can be very small. Hence we fix some large positive number γ and split the sum (6.4) which expresses $\mathcal{M}_{k,n}$ into two parts. The first sum contains all pairs such that $|\varphi(m_s)-\varphi(\bar m_s)|>\log n^{-\gamma}$ with some fixed $\gamma>0$ for all $s=1,\dots,k$. This sum can be approximated by an appropriate integral very well because of Part 2b) of Property A, and this is the content of Lemma 7B. The remaining sum can be bounded sufficiently well for our purposes because of Part 2a) of Property A, and this is done in Lemma 7A. The integral appearing in Lemma 7B is not equal to the main term at the right-hand side of the formula in Lemma 5, because the domain of integration was diminished by not taking all terms in the sum (6.4). But we show with the help of formula (6.11) that this change of domain of integration causes only a negligible error. These estimates together imply Lemma 5. The proof of Lemma 6 is analogous. Here Lemmas 8B and 8A give the estimate of the main and the error term if we split the sum expressing the left-hand side of the formula in Lemma 6 in an appropriate way.

To carry out the above program we introduce for fixed numbers k, j_1, …, j_k, p_2, …, p_k and z the notation $\mathcal{M}_n=\mathcal{M}_{k,n}(f,j_1,\dots,j_k,p_2,\dots,p_k,z)$, where $\mathcal{M}_{k,n}$ was defined in (4.1). Let $\mathcal{Z}=\mathcal{Z}_k$ denote the set

$$\mathcal{Z}=\mathcal{Z}_k=\{\mathbf{m}=(m_1,\dots,m_k),\quad m_s\in\mathbb{Z}^2 \text{ for all } s=1,\dots,k\}\,,$$

and put

$$\mathcal{F}_n=\mathcal{M}_n\times\mathcal{M}_n\,.$$

Clearly,

$$E\left\{|\mathcal{M}_{k,n}(f,j_1,\dots,j_k,p_2,\dots,p_k,z)|^2\right\}=\sum_{\mathbf{m}\in\mathcal{Z}}\sum_{\bar{\mathbf{m}}\in\mathcal{Z}}P((\mathbf{m},\bar{\mathbf{m}})\in\mathcal{F}_n)\,. \qquad (6.4)$$

To prove Lemma 5 we have to estimate the sum at the right-hand side of (6.4). We shall split this sum into two parts and handle differently those pairs $(\mathbf{m},\bar{\mathbf{m}})$, $\mathbf{m}=(m_1,\dots,m_k)$ and $\bar{\mathbf{m}}=(\bar m_1,\dots,\bar m_k)$, for which $|\varphi(m_s)-\varphi(\bar m_s)|$ is very small for some $1\le s\le k$ and those pairs for which all these differences are not too small. To formulate this statement in a more explicit way we introduce some notations.

Let us fix some very large $\gamma>0$ which may depend on k, but not on n or on $j_1,\dots,j_k$, $p_2,\dots,p_k$ and z. (This number will be chosen much bigger than α, β, η and $\bar\eta$.) Define the set

$$G_n = \{(t_1,\dots,t_k,\bar t_1,\dots,\bar t_k),\quad t_s,\bar t_s \in \mathbb{R}^1,\quad \varphi_{2j_s} \le t_s,\bar t_s < \varphi_{2j_s+1},\ s=1,\dots,k\},$$

and split it into two disjoint sets $G_n^{(1)}$ and $G_n^{(2)}$ in the following way: For $\varphi_{2j_s} \le t < \varphi_{2j_s+1}$ define $\ell(t)$, $0 \le \ell(t) < n^{\gamma-\alpha}$, as the integer l for which $\varphi_{2j_s} + ln^{-\gamma} \le t < \varphi_{2j_s} + (l+1)n^{-\gamma}$. Put

$$G_n^{(1)} = \{(t_1,\dots,t_k,\bar t_1,\dots,\bar t_k) \in G_n,\quad |\ell(t_s) - \ell(\bar t_s)| > 1 \text{ for all } 1 \le s \le k\}$$

and

$$G_n^{(2)} = \{(t_1,\dots,t_k,\bar t_1,\dots,\bar t_k) \in G_n,\quad |\ell(t_s) - \ell(\bar t_s)| \le 1 \text{ for some } 1 \le s \le k\}.$$

Clearly,

$$G_n = G_n^{(1)} \cup G_n^{(2)}.$$

Given some measurable $B \subset \mathbb{R}^{2k}$ define the integral

$$I(B) = \int\limits_{\substack{(t_1,\dots,t_k,\bar t_1,\dots,\bar t_k)\in B\\ (x_1,\dots,x_k,\bar x_1,\dots,\bar x_k)\in\mathbb{R}^{2k}}} x_2^2\cdots x_k^2 \bar x_2^2 \cdots \bar x_k^2\, p(x_1,\dots,x_k,\bar x_1,\dots,\bar x_k | t_1,\dots,t_k,\bar t_1,\dots,\bar t_k)\, dx_1\dots dx_k\, d\bar x_1\dots d\bar x_k\, dt_1\dots dt_k\, d\bar t_1\dots d\bar t_k. \tag{6.5}$$

Since for fixed $(t_1,\dots,t_k,\bar t_1,\dots,\bar t_k)$

$$\int_{(x_1,\dots,x_k,\bar x_1,\dots,\bar x_k)\in\mathbb{R}^{2k}} x_2^2\cdots x_k^2 \bar x_2^2 \cdots \bar x_k^2\, p(x_1,\dots,x_k,\bar x_1,\dots,\bar x_k | t_1,\dots,t_k,\bar t_1,\dots,\bar t_k)\, dx_1\dots dx_k\, d\bar x_1\dots d\bar x_k = Ef^2(t_2)\cdots f^2(t_k) f^2(\bar t_2)\cdots f^2(\bar t_k), \tag{6.6}$$

hence

$$E\{\mathcal{H}_n(f,j_1,\dots,j_k)^2\} = I(G_n) = I(G_n^{(1)}) + I(G_n^{(2)}). \tag{6.7}$$

Let us also observe that since the right-hand side of (6.6) is bounded, hence (6.5) and (6.6) imply that

$$I(B) \le \text{const.}\, \lambda(B), \tag{6.8}$$

where $\lambda(B)$ denotes the Lebesgue measure of the set B in $\mathbb{R}^{2k}$.

We split the set $\mathcal{F}_n$ into two disjoint sets $\mathcal{F}_n^1$ and $\mathcal{F}_n^2$ in the following way:

$$\mathcal{F}_n^1 = \{(\mathbf{m},\bar{\mathbf{m}}) = ((m_1,\dots,m_k),(\bar m_1,\dots,\bar m_k)) \in \mathcal{F}_n;\ (\varphi(m_1),\dots,\varphi(m_k),\varphi(\bar m_1),\dots,\varphi(\bar m_k)) \in G_n^{(1)}\},$$

$$\mathcal{F}_n^2 = \{(\mathbf{m},\bar{\mathbf{m}}) = ((m_1,\dots,m_k),(\bar m_1,\dots,\bar m_k)) \in \mathcal{F}_n;\ (\varphi(m_1),\dots,\varphi(m_k),\varphi(\bar m_1),\dots,\varphi(\bar m_k)) \in G_n^{(2)}\}.$$

Put

$$\mathcal{I}(\mathcal{F}_n^1) = \sum_{\mathbf{m}\in\mathcal{Z}} \sum_{\bar{\mathbf{m}}\in\mathcal{Z}} P((\mathbf{m},\bar{\mathbf{m}}) \in \mathcal{F}_n^1) \tag{6.9}$$

and

$$\mathcal{I}(\mathcal{F}_n^2) = \sum_{\mathbf{m}\in\mathcal{Z}} \sum_{\bar{\mathbf{m}}\in\mathcal{Z}} P((\mathbf{m}, \bar{\mathbf{m}}) \in \mathcal{F}_n^2) \,. \tag{6.9'}$$

Then we have

$$E\left\{|\mathcal{M}_{k,n}(f, j_1, \dots, j_k, p_2, \dots, p_k, z)|^2\right\} = \mathcal{I}(\mathcal{F}_n^1) + \mathcal{I}(\mathcal{F}_n^2) \,. \tag{6.10}$$

It follows from (6.8) and the observation that $\lambda(G_n^{(2)}) < \text{const.}\,(\log n)^{-\gamma}$ that for sufficiently large $\gamma = \gamma(M, k)$

$$I(G_n^{(2)}) < \text{const.}\,(\log n)^{-\gamma} < \text{const.}\,(\log n)^{-M} . \tag{6.11}$$

Lemma 5 follows from formulas (6.7), (6.10), (6.11) and the following Lemmas 7A and 7B.

Lemma 7A. *If $\gamma = \gamma(M, k)$ is sufficiently large, then*

$$\mathcal{I}(\mathcal{F}_n^2) < \text{const.}\, n^4 (\log n)^{-M}$$

for arbitrarily large $M > 0$.

Lemma 7B. *For all $\gamma > 0$*

$$\left| \mathcal{I}(\mathcal{F}_n^1) - \frac{n^4 \left(z + \frac{1}{2}\right)^2 \delta_n^{2(k-1)}}{\log n^{4\bar{\eta}}} I(G_n^{(1)}) \right| < \text{const.}\, n^3 (\log n)^K$$

with some appropriate $K = K(\gamma) > 0$.

(In our problem the upper bound $\text{const.}\, n^3 (\log n)^K$ in Lemma 7B can be replaced by the weaker estimate $n^4 (\log n)^{-M}$ with a sufficiently large $M > 0$.)

We reduce the proof of Lemma 6, similarly to Lemma 5, to two Lemmas 8A and 8B. To formulate them we introduce the following quantities. Given some $\mathbf{m} = (m_1, \dots, m_k) \in \mathcal{Z} = \mathcal{Z}_k$ define the sets

$$G_n^{(i)}(\mathbf{m}) = \left\{(t_1, \dots, t_k) \in \mathbb{R}^k, \quad (\varphi(m_1), \dots, \varphi(m_k), t_1, \dots, t_k) \in G_n^{(i)}\right\}$$

and the integrals

$$\mathcal{J}(G_n^{(i)}(\mathbf{m})) = \mathcal{J}(G_n^{(i)}(\mathbf{m}), f) = \int_{(t_1, \dots, t_k) \in G_n^{(i)}(\mathbf{m})} f^2(t_2) \cdots f^2(t_k)\, dt_1 \dots dt_k$$

for $i = 1, 2$.

The identity

$$\begin{aligned} &E\mathcal{M}_{k,n}(f, j_1, \dots, j_k, p_2, \dots, p_k, z) \mathcal{H}_n(f, j_1, \dots, j_k) \\ &= \sum_{\mathbf{m}\in\mathcal{Z}} E\chi(\mathbf{m} \in \mathcal{M}_n) \mathcal{J}(G_n^{(1)}(\mathbf{m})) + E\chi(\mathbf{m} \in \mathcal{M}_n) \mathcal{J}(G_n^{(2)}(\mathbf{m})) \end{aligned} \tag{6.12}$$

holds. Hence Lemma 6 follows from formulas (6.7), (6.11), (6.12) and the following lemmas.

Lemma 8A. *If $\gamma = \gamma(M, k)$ is sufficiently large, then*

$$\sum_{\mathbf{m}\in\mathcal{Z}} E\chi(\mathbf{m}\in\mathcal{M}_n)\mathcal{J}(G_n^{(2)}(\mathbf{m})) < \text{const.}\, n^2(\log n)^{-M}$$

for arbitrarily large $M > 0$.

Lemma 8B. *For all* $\gamma > 0$

$$\left|\sum_{\mathbf{m}\in\mathcal{Z}} E\chi(\mathbf{m}\in\mathcal{M}_n)\mathcal{J}(G_n^{(1)}(\mathbf{m})) - \frac{n^2\left(z+\frac{1}{2}\right)\delta_n^{(k-1)}}{\log n^{2\bar\eta}} I(G_n^{(1)})\right| < \text{const.}\, n(\log n)^K$$

with some appropriate $K = K(\gamma) > 0$.

7. Proof of Lemmas 7A and 8A

Proof of Lemma 7A. Fix the numbers $j_1, \dots, j_k$. Let us split the sets $\mathbb{D}_{j_s}$ into smaller sectors $\mathbb{U}_{s,l}$, $l = 1, \dots, \frac{n}{(\log n)^\alpha}$ defined by the formula

$$\mathbb{U}_{s,l} = \left\{x\colon x\in\mathbb{D}_{j_s},\quad \varphi_{2j_s} + \frac{l-1}{n} \le \varphi(x) < \varphi_{2j_s} + \frac{l}{n}\right\}.$$

Fix some positive number $K > 0$, and define the set

$$\begin{aligned}\mathbb{B}(l_1,\dots,l_k,\bar l_1,\dots,\bar l_k) &= \mathbb{B}(l_1,\dots,l_k,\bar l_1,\dots,\bar l_k,j_1,\dots,j_k,f)\\ &= \Bigg\{(\mathbf{m},\bar{\mathbf{m}}) = ((m_1,\dots,m_k),(\bar m_1,\dots,\bar m_k))\in\mathcal{Z}\times\mathcal{Z},\\ &\qquad m_s\in\mathbb{U}_{s,l_s},\ \bar m_s\in\mathbb{U}_{s,\bar l_s},\quad s=1,\dots,k,\\ &\qquad \left|\frac{|m_1|}{f(\varphi(m_1))} - \frac{|m_s|}{f(\varphi(m_s))}\right| < \frac{K}{n},\\ &\qquad \left|\frac{|\bar m_1|}{f(\varphi(\bar m_1))} - \frac{|\bar m_s|}{f(\varphi(\bar m_s))}\right| < \frac{K}{n},\quad s=2,\dots,k\Bigg\}.\end{aligned}$$

Introduce the random variables

$$\zeta(l_1,\dots,l_k,\bar l_1,\dots,\bar l_k) = |\mathbb{B}(l_1,\dots,l_k,\bar l_1,\dots,\bar l_k)|\,.$$

The estimate

$$\mathcal{I}(\mathcal{F}_n^2) \le \sum_{\substack{0\le l_s,\bar l_s\le \frac{n}{(\log n)^\alpha}\text{ for all } s=1,\dots,k,\\ |l_s-\bar l_s|\le\frac{2n}{(\log n)^\gamma}\text{ for some } 1\le s\le k}} E\zeta(l_1,\dots,l_k,\bar l_1,\dots,\bar l_k) \tag{7.1}$$

holds. We prove some bounds on the expressions $E\zeta(l_1,\dots,l_k,\bar l_1,\dots,\bar l_k)$. The cases when $|l_s-\bar l_s| > 1$ for all $s=1,\dots,k$ and when $|l_s-\bar l_s|\le 1$ for some $1\le s\le k$ will be handled differently. First remark that all sets $\mathbb{U}_{s,l}$ contain less than const. n lattice points. We also show that

$$\left| |\bar{m}_s| - |m_s| + |m_1|\frac{f(\varphi(m_s))}{f(\varphi(m_1))} - |\bar{m}_1|\frac{f(\varphi(m_s))}{f(\varphi(\bar{m}_1))} \right| < \text{const.}\,(|l_s - \bar{l}_s| + 1) \tag{7.2}$$

$$\text{for all } s = 2, \ldots, k, \quad \text{if } (\mathbf{m}, \bar{\mathbf{m}}) \in \mathbb{B}(l_1, \ldots, l_k, \bar{l}_1, \ldots, \bar{l}_k)$$

holds with $\mathbf{m} = (m_1, \ldots, m_k)$ and $\bar{\mathbf{m}} = (\bar{m}_1, \ldots, \bar{m}_k)$.

Indeed, we have

$$\left| |m_s| - |m_1|\frac{f(\varphi(m_s))}{f(\varphi(m_1))} - |\bar{m}_s| + |\bar{m}_1|\frac{f(\varphi(\bar{m}_s))}{f(\varphi(\bar{m}_1))} \right| < \frac{\bar{K}}{n}$$

and

$$\left| |\bar{m}_1|\frac{f(\varphi(\bar{m}_s))}{f(\varphi(\bar{m}_1))} - |\bar{m}_1|\frac{f(\varphi(m_s))}{f(\varphi(\bar{m}_1))} \right| < \text{const.}\,(|l_s - \bar{l}_s| + 1)\,,$$

since the function $f(\cdot)$ is Lipschitz one. The last two relations imply (7.2).

The inequalities

$$\begin{aligned} \left| f(\varphi(m_s)) - \frac{|m_s|}{|m_1|} f(\varphi(m_1)) \right| &< \frac{C}{n^2} \\ \left| f(\varphi(\bar{m}_s)) - \frac{|\bar{m}_s|}{|\bar{m}_1|} f(\varphi(\bar{m}_1)) \right| &< \frac{C}{n^2} \end{aligned} \qquad s = 2, \ldots, k \tag{7.3}$$

are also valid if $(\mathbf{m}, \bar{\mathbf{m}}) \in \mathbb{B}(l_1, \ldots, l_k, \bar{l}_1, \ldots, \bar{l}_k)$. Let us fix $l_1, \ldots, l_k$ and $\bar{l}_1, \ldots, \bar{l}_k$. Take some $\mathbf{m} = (m_1, \ldots, m_k)$ and $\bar{m}$ in such a way that $m_s \in \mathbb{U}_{s,l_s}$, $s = 1, \ldots, k$, and $\bar{m} \in \mathbb{U}_{1,\bar{l}_1}$ respectively. Introduce

$$\begin{aligned} \mathbb{B}^{\mathbf{m},\bar{m}}(l_1, \ldots, l_k, \bar{l}_1, \ldots, \bar{l}_k) = \{ &\bar{\mathbf{m}} = (\bar{m}_1, \ldots, \bar{m}_k), \\ &(\mathbf{m}, \bar{\mathbf{m}}) \in \mathbb{B}(l_1, \ldots, l_k, \bar{l}_1, \ldots, \bar{l}_k), \quad \bar{m} = \bar{m}_1 \} \end{aligned}$$

and

$$\zeta^{\mathbf{m},\bar{m}}(l_1, \ldots, l_k, \bar{l}_1, \ldots, \bar{l}_k) = |\mathbb{B}^{\mathbf{m},\bar{m}}(l_1, \ldots, l_k, \bar{l}_1, \ldots, \bar{l}_k)|\,.$$

We estimate the expected value of $\zeta^{\mathbf{m},\bar{m}}$. First we consider the case when $|l_s - \bar{l}_s| > 1$ for all $s = 1, \ldots, k$.

Fix the values of $f(\varphi(m_1)), \ldots, f(\varphi(m_k))$ and $f(\varphi(\bar{m}_1))$ and estimate conditional expectation

$$\begin{aligned} E(\zeta^{\mathbf{m},\bar{m}}(l_1, \ldots, l_k, \bar{l}_1, \ldots, \bar{l}_k) \mid \\ f(\varphi(m_1)) = x_1, \ldots, f(\varphi(m_k)) = x_k, f(\varphi(\bar{m}_1)) = \bar{x}_1)\,. \end{aligned}$$

Because of (7.2) we can determine with the help of the values of $f(\varphi(m_1)), \ldots,$ $f(\varphi(m_k))$ and $f(\varphi(\bar{m}_1))$ a set consisting of at most const. $\prod_{s=2}^{k} |l_s - \bar{l}_s|$ vectors $\bar{\mathbf{m}}$ in such a way that only the vectors $(\mathbf{m}, \bar{\mathbf{m}})$ with these $\bar{\mathbf{m}}$ can be in the set

$$\mathbb{B}^{\mathbf{m},\bar{m}}(l_1, \ldots, l_k, \bar{l}_1, \ldots, \bar{l}_k)\,.$$

Let us estimate the conditional probability of the event that such a vector $\bar{\mathbf{m}}$ really belongs to this set.

The conditional density of the random vector $f(\varphi(\bar{m}_2)), \ldots, f(\varphi(\bar{m}_k))$ with respect to the condition $f(\varphi(m_1)) = x_1$, $\ldots$, $f(\varphi(m_k)) = x_k$, $f(\varphi(\bar{m}_1)) = \bar{x}_1$ can be bounded by

$$C(\log n)^{(k-1)\beta\tau}\frac{\prod_{s=1}^{k}\left(\frac{n}{|l_s-\bar{l}_s|}\right)^{\tau}}{p(x_1,\dots,x_k,\bar{x}_1|\varphi(m_1),\dots,\varphi(m_k),\varphi(\bar{m}_1))}. \tag{7.4}$$

We shall show that this conditional density function has the above estimate for all $f(\varphi(\bar{m}_2))=\bar{x}_2,\dots,f(\varphi(\bar{m}_k))=\bar{x}_k$. Relation (7.4) follows from the inequality

$$\begin{aligned}p(x_1,\dots,x_k,\bar{x}_1,\dots,\bar{x}_k|\varphi(m_1),\dots,\varphi(m_k),\varphi(\bar{m}_1),\dots,\varphi(\bar{m}_k))\\ < C(\log n)^{(k-1)\beta\tau}\prod_{s=1}^{k}\left(\frac{n}{|l_s-\bar{l}_s|}\right)^{\tau}.\end{aligned}$$

The last inequality holds because of Part 2b) of Property A and the following observations: $|\varphi(m_s)-\varphi(\bar{m}_s)|>\frac{|l_s-\bar{l}_s|-1}{n}$, and all other terms $|\varphi(m_s)-\varphi(m_{s'})|$, $|\varphi(\bar{m}_s)-\varphi(m_{s'})|$ and $|\varphi(\bar{m}_s)-\varphi(\bar{m}_{s'})|$ which appear in the upper bound of the density we are considering are greater than $(\log n)^{-\beta}$. (This statement holds, because there is a sector $\mathbb{C}_j$ between these points.)

We claim that

$$\begin{aligned}&P(\bar{\mathbf{m}}\in\mathbb{B}^{\mathbf{m},\bar{m}}(l_1,\dots,l_k,\bar{l}_1,\dots,\bar{l}_k)|\\ &\qquad f(\varphi(m_1))=x_1,\dots,f(\varphi(m_k))=x_k,f(\varphi(\bar{m}_1))=\bar{x}_1)\\ &\qquad< C(\log n)^{(k-1)\beta\tau}\frac{n^{-2k+2}\prod_{s=1}^{k}\left(\frac{n}{|l_s-\bar{l}_s|}\right)^{\tau}}{p(x_1,\dots,x_k,\bar{x}_1|\varphi(m_1),\dots,\varphi(m_k),\varphi(\bar{m}_1))},\end{aligned} \tag{7.5}$$

and the conditional expectation of $\zeta^{\mathbf{m},\bar{m}}$ satisfies the inequality

$$\begin{aligned}&E\big(\zeta^{\mathbf{m},\bar{m}}(l_1,\dots,l_k,\bar{l}_1,\dots,\bar{l}_k)|\\ &\qquad f(\varphi(m_1))=x_1,\dots,f(\varphi(m_k))=x_k,f(\varphi(\bar{m}_1))=\bar{x}_1\big)\\ &\qquad< C(\log n)^{(k-1)\beta\tau}\frac{n^{-2k+2}\prod_{s=1}^{k}\left(\frac{n}{|l_s-\bar{l}_s|}\right)^{\tau}\prod_{s=2}^{k}|l_s-\bar{l}_s|}{p(x_1,\dots,x_k,\bar{x}_1|\varphi(m_1),\dots,\varphi(m_k),\varphi(\bar{m}_1))}.\end{aligned} \tag{7.5'}$$

Indeed, to calculate the conditional probability in (7.5) we have to integrate the conditional density which was bounded in (7.4) with respect to the variables $\bar{x}_2,\dots,\bar{x}_k$ by the Lebesgue measure on an appropriate set. But by the second line of formula (7.3) this set is contained in the set

$$\left\{\left|\bar{x}_s-\frac{|\bar{m}_s|}{|\bar{m}_1|}x_1\right|<\frac{C}{n^2},\ s=2,\dots,k\right\},$$

which is a set with Lebesgue measure less than const. n^{-2k+2}. This fact together with our bound on the conditional density imply the bound on the conditional density, and the estimate on the conditional expectation is obtained if we remark that it is the sum of the conditional probability of at most const. $\prod_{s=2}^{k}|l_s-\bar{l}_s|$ terms.

Finally we show that

$$E\zeta^{\mathbf{m},\bar{m}}(l_1,\dots,l_k,\bar{l}_1,\dots,\bar{l}_k) < C(\log n)^{(k-1)\beta\tau} n^{3-3k} \prod_{s=1}^{k} \left(\frac{n}{|l_s-\bar{l}_s|}\right)^{\tau-1} \quad \text{for all pairs } (\mathbf{m},\bar{m}) . \tag{7.6}$$

To prove (7.6) we make the following observations: The expectation of $\zeta^{\mathbf{m},\bar{m}}$ can be obtained by integrating the left-hand side of (7.5′) with respect to the measure

$$p(x_1,\dots,x_k,\bar{x}_1|\varphi(m_1),\dots,\varphi(m_k),\varphi(\bar{m}_1))\,dx_1\dots dx_k\,d\bar{x}_1$$

on a subset of

$$A = \left\{(x_1,\dots,x_k,\bar{x}_1),\quad c_1 \le x_1 \le c_2, \right.$$
$$\left. \left|x_s - \frac{|m_s|}{|m_1|}x_1\right| < \frac{C}{n^2},\ s=2,\dots,k, \quad \text{and } |x_1-\bar{x}_1| < \frac{C|l_1-\bar{l}_1|}{n} \right\},$$

where $C > 0$ is some appropriate constant. The first inequalities in the definition of the set A appeared because of the definition of $\mathbb{B}(l_1,\dots,l_k,\bar{l}_1,\dots,\bar{l}_k)$, and the last one, since

$$|f(\varphi(\bar{m}_1)) - f(\varphi(m_1))| \le b_3|\varphi(\bar{m}_1)-\varphi(m_1)| \le 2b_3\frac{|l_1-\bar{l}_1|}{n} ,$$

because of the Lipschitz one property of the function $f(\cdot)$. Now formula (7.6) follows from (7.5′) and the fact that the Lebesgue measure of the set A is less than const. $n^{1-2k}|l_1-\bar{l}_1|$.

Let us now consider the case when there are $p \ge 1$ indices s such that $|l_s-\bar{l}_s| \le 1$. We claim that in this case

$$E\zeta^{\mathbf{m},\bar{m}}(l_1,\dots,l_k,\bar{l}_1,\dots,\bar{l}_k) < C(\log n)^{(k-1)\beta\tau} n^{-3k+p+3-\varepsilon} \prod{}' \left(\frac{n}{|l_s-\bar{l}_s|}\right)^{\tau-1} \quad \text{for all pairs } (\mathbf{m},\bar{m}) \tag{7.6′}$$

with $\varepsilon = \dfrac{2-\tau}{\tau}$, where $\prod'$ denotes product with indices $s \in V$ with

$$V = \{s;\quad 1 \le s \le k \text{ and } |l_s-\bar{l}_s| \ge 2\} .$$

We prove (7.6′) with some refinement of the proof of (7.6). We may assume that $1 \notin V$, i.e. $|l_1-\bar{l}_1| \le 1$ with the help of the following observation. The set $\mathbb{B}(l_1,\dots,l_k,\bar{l}_1,\dots,\bar{l}_k)$ becomes smaller if we make an arbitrary permutation of the indices s and choose $K/2$ instead of K in its definition. On the other hand, the order of the angles $\varphi(m_s)$ has no importance in the subsequent arguments. We shall consider the following two cases separately:

a) $|\varphi(m_1)-\varphi(\bar{m}_1)| > n^{-1-\varepsilon}$,
b) $|\varphi(m_1)-\varphi(\bar{m}_1)| \le n^{-1-\varepsilon}$.

In case a) we bound the conditional expectation of $\zeta^{\mathbf{m},\bar{m}}$ under the condition $f(\varphi(m_1)) = x_1,\dots,f(\varphi(m_k)) = x_k, f(\varphi(\bar{m}_1)) = \bar{x}_1$ similarly to the already investigated case. Because of (7.2) we can determine a set of vectors $\bar{\mathbf{m}}$ with cardinality const. $\prod' |l_s-\bar{l}_s|$ in such a way that only the pairs $(\mathbf{m},\bar{\mathbf{m}})$ with these $\bar{\mathbf{m}}$ can be in the set $\mathbb{B}^{\bar{\mathbf{m}},\bar{m}}(l_1,\dots,l_k,\bar{l}_1,\dots,\bar{l}_k)$. Arguing similarly as before, with the difference that now the conditional density of the vector $(\varphi(f(\bar{m}_s)),\ s\in V)$ is estimated, we get that

$$E(\zeta^{\mathbf{m},\bar{m}}(l_1,\dots,l_k,\bar{l}_1,\dots,\bar{l}_k)|\\ f(\varphi(m_1))=x_1,\dots,f(\varphi(m_k))=x_k,f(\varphi(\bar{m}_1))=\bar{x}_1)\\ <\frac{C(\log n)^{(k-1)\beta\tau}n^{-2k+2p}\prod'\left(\frac{n}{|l_s-\bar{l}_s|}\right)^{\tau}|l_s-\bar{l}_s|}{p(x_1,\dots,x_k,\bar{x}_1|\varphi(m_1),\dots,\varphi(m_k),\varphi(\bar{m}_1))}|\varphi(m_1)-\varphi(\bar{m}_1)|^{-\tau}.$$

Now we get similarly to the previous case by integrating the conditional expectation with respect to the distribution of the condition

$$f(\varphi(m_1))=x_1,\dots,f(\varphi(m_k))=x_k,f(\varphi(\bar{m}_1))=\bar{x}_1$$

that

$$E\zeta^{\mathbf{m},\bar{m}}(l_1,\dots,l_k,\bar{l}_1,\dots,\bar{l}_k)<C(\log n)^{(k-1)\beta\tau}n^{2-3k+p}\\ \prod'\left(\frac{n}{|l_s-\bar{l}_s|}\right)^{\tau-1}|\varphi(m_1)-\varphi(\bar{m}_1)|^{1-\tau}\\ \text{for all pairs } (\mathbf{m},\bar{m})\ .$$

We only have to observe that in the present case the Lebesgue measure of the set where we integrate the conditional expectation is less than

$$\text{const.}\,n^{-2k+2}|\varphi(m_1)-\varphi(\bar{m}_1)|\ .$$

Indeed, it is contained in a set defined analogously to the set A defined in the previous case, only the last inequality in its definition must be replaced with the inequality $|x_1-\bar{x}_1|<C|\varphi(m_1)-\varphi(\bar{m}_1)|$. This estimate implies (7.6′), since $|\varphi(m_1)-\varphi(\bar{m}_1)|^{1-\tau}>n^{1-\varepsilon}$ in case a).

In case b) we show that

$$E(\zeta^{\mathbf{m},\bar{m}}\left(l_1,\dots,l_k,\bar{l}_1,\dots,\bar{l}_k)|f(\varphi(m_1))=x_1,\dots,f(\varphi(m_k))=x_k\right)\\ <C(\log n)^{(k-1)\beta\tau}\frac{n^{-2k+2p+1-\varepsilon}\prod'\left(\frac{n}{|l_s-\bar{l}_s|}\right)^{\tau}|l_s-\bar{l}_s|}{p(x_1,\dots,x_k|\varphi(m_1),\dots,\varphi(m_k))}\ . \tag{7.7}$$

In this case we estimate the conditional expectation when the value of $f(\varphi(\bar{m}_1))$ is not prescribed in the condition. Nevertheless, the value of $f(\varphi(\bar{m}_1))$ can be determined by means of the conditioning terms appearing at the left-hand side of (7.7) with a precision of const. $n^{-1-\varepsilon}$ because of the Lipschitz one property of the function $f(\cdot)$. Hence we can determine, with the help of relation (7.2), const. $\prod'|l_s-\bar{l}_s|$ elements $\bar{\mathbf{m}}$ in such a way that under the conditioning at the left-hand side of (7.7) the event

$$(\mathbf{m},\bar{\mathbf{m}})\in\mathbb{B}^{\mathbf{m},\bar{m}}(l_1,\dots,l_k,\bar{l}_1,\dots,\bar{l}_k)$$

can take place only with these elements $\bar{\mathbf{m}}$.

To prove relation (7.7) we remark that the conditional density function of the vector $\{f(\varphi(\bar{m}_s))=\bar{x}_s,\ s\in V\}$ with respect to the condition

$$\{f(\varphi(m_1)) = x_1, \dots, f(\varphi(m_k)) = x_k\}$$

is bounded by

$$C(\log n)^{(k-1)\beta\tau} \frac{\prod' \left(\frac{n}{|l_s - \bar{l}_s|}\right)^\tau}{p(x_1, \dots, x_k|\varphi(m_1), \dots, \varphi(m_k))}, \tag{7.8}$$

and for any $\mathbb{B}^{\mathbf{m},\bar{m}}$

$$\begin{aligned} &P(\bar{\mathbf{m}} \in \mathbb{B}^{\mathbf{m},\bar{m}}(l_1, \dots, l_k, \bar{l}_1, \dots, \bar{l}_k)|f(\varphi(m_1)) = x_1, \dots, f(\varphi(m_k)) = x_k) \\ &\qquad < C(\log n)^{(k-1)\beta\tau} \frac{n^{-2k+2p+1-\varepsilon} \prod' \left(\frac{n}{|l_s - \bar{l}_s|}\right)^\tau}{p(x_1, \dots, x_k|\varphi(m_1), \dots, \varphi(m_k))}. \end{aligned} \tag{7.8'}$$

The estimate (7.8′) follows from the estimate (7.8) on the conditional density and the following observation. To calculate the conditional probability of the event $\bar{\mathbf{m}} \in \mathbb{B}^{\mathbf{m},\bar{m}}(l_1, \dots, l_k, \bar{l}_1, \dots, \bar{l}_k)$ one has to integrate the conditional density bounded by formula (7.8) on the set

$$\begin{aligned} A^* = A^*(x) = \Bigg\{ (\bar{x}_s,\ s \in V), \quad & \left| \bar{x}_s - \frac{|\bar{m}_s|}{|\bar{m}_{s^*}|} \bar{x}_{s^*} \right| < \frac{C}{n^2}, \\ & \text{for all } s \in V \text{ and } \left| \bar{x}_{s^*} - \frac{|\bar{m}_{s^*}|}{|\bar{m}_1|} x_1 \right| < Cn^{-1-\varepsilon} \Bigg\} \end{aligned}$$

with an appropriate constant $C > 0$ and arbitrary $s^* \in V$. (We may assume that V is non-empty, i.e. $p < k$. In the case $p = k$ relation (7.8′) obviously holds, since the right-hand side of (7.8′) in this case is more than $n^{1-\varepsilon}$ divided by a power of $\log n$. To see this, observe that the denominator in (7.8′) is less than a power of $\log n$.) The last inequality may be imposed in the definition of A^*, because

$$\left| x_{s^*} - \frac{|\bar{m}_{s^*}|}{|\bar{m}_1|} x_1 \right| < \left| \bar{x}_{s^*} - \frac{|\bar{m}_{s^*}|}{|\bar{m}_1|} \bar{x}_1 \right| + \frac{|\bar{m}_{s^*}|}{|\bar{m}_1|} |\bar{x}_1 - x_1| < Cn^{-1-\varepsilon}$$

in case b). Now (7.8′) follows from the bound in (7.8) and the fact that the Lebesgue measure of the set A^* is less than const. $n^{-2k+2p+1-\varepsilon}$. Relation (7.8′) implies (7.7), since there are $\prod' |l_s - \bar{l}_s|$ possibilities for choosing $\bar{\mathbf{m}}$. Formula (7.6′) follows from (7.7) and the observation that we have to integrate the conditional expectation on a subset of the set

$$\left\{ (x_1, \dots, x_k), \quad c_1 \le x_1 \le c_2, \quad \left| x_s - \frac{|m_s|}{|m_1|} x_1 \right| < \frac{C}{n^2},\ s = 2, \dots, k, \right\},$$

which is a set of Lebesgue measure less than const. n^{-2k+2}.

Since $\mathbf{m} = (m_1, \dots, m_k)$ and $\bar{\mathbf{m}}$ can be chosen with their coordinates m_s in prescribed sectors $\mathbb{U}_{s,l}$ in const. n^{k+1} ways, formulas (7.6) and (7.6′) imply that

$$E\zeta(l_1,\dots,l_k,\bar{l}_1,\dots,\bar{l}_k) < C(\log n)^{(k-1)\beta\tau} n^{4-2k} \prod_{s=1}^{k} \left(\frac{n}{|l_s-\bar{l}_s|}\right)^{\tau-1}$$
$$\text{if } |l_s-\bar{l}_s| > 1 \text{ for all } s=1,\dots,k\,, \tag{7.9}$$
$$< C(\log n)^{(k-1)\beta\tau} n^{4-2k+p-\varepsilon} \prod{}' \left(\frac{n}{|l_s-\bar{l}_s|}\right)^{\tau-1}$$
$$\text{if there are } p \ge 1 \text{ indices } s \text{ such that } |l_s-\bar{l}_s| \le 1\,.$$

Let us remark that for all $\bar{l}$ the equation $l_s - \bar{l}_s = \bar{l}$ has less than n solutions. Hence relations (7.1) and (7.9) imply that

$$\mathcal{I}(\mathcal{F}_n^2) \le C(\log n)^{(k-1)\beta\tau} \left(\sum{}^{(0)} + \sum_{p=1}^{k} \sum{}^{(p)}\right)$$

with

$$\sum{}^{(0)} = n^{4-k} \sum_{\substack{1<|\bar{l}_s|\le \frac{n}{(\log n)^{\alpha}},\ s=1,\dots,k \\ |\bar{l}_s|<\frac{n}{(\log n)^{\gamma}} \text{ for some } 1\le s\le k}} \prod_{s=1}^{k} \left(\frac{n}{\bar{l}_s}\right)^{\tau-1}$$

and

$$\sum{}^{(p)} = n^{4-k+p-\varepsilon} \sum_{1<|\bar{l}_s|\le \frac{n}{(\log n)^{\alpha}},\ s=1,\dots,k-p} \prod_{s=1}^{k-p} \left(\frac{n}{\bar{l}_s}\right)^{\tau-1}.$$

We have

$$\sum{}^{(0)} \le \text{const.}\, n^{4-k} \left(\sum_{p=1}^{\frac{n}{(\log n)^{\alpha}}} \left(\frac{n}{p}\right)^{\tau-1}\right)^{k-1} \sum_{p=1}^{\frac{n}{(\log n)^{\gamma}}} \left(\frac{n}{p}\right)^{\tau-1}$$
$$\le \text{const.}\, n^4 (\log n)^{-\gamma(2-\tau)}$$

and

$$\sum{}^{(p)} \le \text{const.}\, n^{4-k+p-\varepsilon} \left(\sum_{p=1}^{\frac{n}{(\log n)^{\alpha}}} \left(\frac{n}{p}\right)^{\tau-1}\right)^{k-p} < \text{const.}\, n^{4-\varepsilon}$$

for $1 \le p \le k$. Hence

$$\mathcal{I}(\mathcal{F}_n^2) \le C(\log n)^{(k-1)\beta\tau} n^4 \left(\log n^{-(2-\tau)\gamma} + n^{-\varepsilon}\right) \le n^4 (\log n)^{-M}$$

if $\gamma > 0$ is sufficiently large. Lemma 7A is proved. □

Proof of Lemma 8A. Since $An < |m_s| < Bn$ for all $s = 1, \dots, k$ if $\mathbf{m} = (m_1,\dots,m_k)$, and there are less than const. n^{2k} vectors $\mathbf{m} \in \mathcal{Z}$ satisfying this condition, it is enough to show that

$$E\chi(\mathbf{m} \in \mathcal{M}_n)\mathcal{J}(G_n^{(2)}(\mathbf{m})) < \text{const.}\, n^{-2k+2} (\log n)^{-M}$$
$$\text{for all } m \in \mathcal{Z} \text{ if } \gamma > \gamma(M,k) \tag{7.10}$$

to prove Lemma 8A. Let us introduce the notation $\bar t_s = \varphi(m_s))$, $s = 1, \dots, k$. We can rewrite the expression in formula (7.10) in the following form:

$$E\chi(\mathbf{m} \in \mathcal{M}_n)\mathcal{J}(G_n^{(2)}(\mathbf{m})) = \int_{\substack{(\bar x_1,\dots,\bar x_k)\in D\\ (x_1,\dots,x_k)\in\mathbb{R}^k\\ (t_1,\dots,t_k)\in G_n^{(2)}(\mathbf{m})}} x_2^2\cdots x_k^2 \, p(x_1,\dots,x_k,\bar x_1,\dots,\bar x_k|t_1,\dots,t_k,\bar t_1,\dots,\bar t_k)\, dt_1\dots dt_k\, dx_1\dots dx_k\, d\bar x_1\dots d\bar x_k$$

with

$$D = \left\{(\bar x_1,\dots,\bar x_k),\quad x_1\in I_z,\ \left|\frac{|m_s|}{\bar x_s} - \frac{|m_1|}{\bar x_1}\right| \in I_{p_{js}}\right\}.$$

Let us first fix $t_1,\dots,t_k$ and $\bar x_1,\dots,\bar x_k$, and integrate with respect to the variables $x_1,\dots,x_k$. Because of the Lipschitz one property of the function $f(\cdot)$, the density $p(\cdot|\cdot)$ is concentrated on the set $|x_i - \bar x_i| \le b_3|t_i - \bar t_i|$, and $|x_i| < b_2$. Hence we get after integration with respect to the variables $x_1,\dots,x_k$ that

$$E\chi(\mathbf{m}\in\mathcal{M}_n)\mathcal{J}(G_n^{(2)}(\mathbf{m})) < \text{const.} \int_{\substack{(\bar x_1,\dots,\bar x_k)\in D\\ (t_1,\dots,t_k)\in G_n^{(2)}(\mathbf{m})}} \prod_{s=1}^k |t_s - \bar t_s| \sup_{x_1,\dots,x_k} p(x_1,\dots,x_k,\bar x_1,\dots,\bar x_k|t_1,\dots,t_k,\bar t_1,\dots,\bar t_k)\, dt_1\dots dt_k\, d\bar x_1\dots d\bar x_k\,.$$

Then integrating with respect to the variables $\bar x_s$, $s = 1,\dots,k$ and exploiting that the Lebesgue measure of D is less than n^{-2k+2} (this is so, because for fixed $\bar x_1$, $\bar x_s$ is in an interval of length const. n^{-2} for all $s = 2,\dots,k$ if $(\bar x_1,\dots,\bar x_k)\in D$) we get that

$$E\chi(\mathbf{m}\in\mathcal{M}_n)\mathcal{J}(G_n^{(2)}(\mathbf{m})) < \text{const.}\, n^{-2k+2} \int_{(t_1,\dots,t_k)\in G_n^{(2)}(\mathbf{m})} \prod_{s=1}^k |t_s - \bar t_s| \sup_{\substack{x_1,\dots,x_k\\ \bar x_1,\dots,\bar x_k}} p(x_1,\dots,x_k,\bar x_1,\dots,\bar x_k|t_1,\dots,t_k,\bar t_1,\dots,\bar t_k)\, dt_1\dots dt_k\,. \tag{7.11}$$

Part 2a) of Property A implies that

$$p(x_1,\dots,x_k,\bar x_1,\dots,\bar x_k|t_1,\dots,t_k,\bar t_1,\dots,\bar t_k) \le \prod_{s=2}^{2k} |\pi(t_s) - \pi(t_{s-1})|^{-\tau} \le C(\log n)^{(k-1)\beta\tau} \prod_{s=2}^k |t_s - \bar t_s|^{-\tau}, \tag{7.12}$$

where $\{\pi(t_1),\dots,\pi(t_{2k})\}$ is the monotone ordering of the numbers $t_1,\dots,t_k,\bar t_1,\dots,\bar t_k$. The last inequality in (7.12) holds, because $|t_s - t_{s'}|$, $|t_s - \bar t_{s'}|$ and $|\bar t_s - \bar t_{s'}|$ are greater than $(\log n)^{-\beta}$ if $s \ne s'$. (There is a sector $\mathbb{C}_j$ between them.) Relations (7.11) and (7.12) imply that

$$E\chi(\mathbf{m} \in \mathcal{M}_n)\mathcal{J}(G_n^{(2)}(\mathbf{m})) < \text{const.}\, n^{-2k+2} C(\log n)^{(k-1)\beta\tau} \int_{(t_1,\dots,t_k)\in G_n^{(1)}(\mathbf{m})} \prod_{s-1}^{k} |t_s - \bar{t}_s|^{1-\tau} dt_1 \dots dt_k \,.$$

Since $\min |t_s - \bar{t}_s| < (\log n)^{-\gamma}$ by the definition of the set $G_n^{(2)}(\mathbf{m})$, the last inequality implies that

$$E\chi(\mathbf{m} \in \mathcal{M}_n)\mathcal{J}(G_n^{(2)}(\mathbf{m})) < \text{const.}\, n^{-2k+2}(\log n)^{-M},$$

if $\gamma > \gamma(k, M)$. Lemma 8A is proved. □

8. The proof of Lemmas 7B and 8B

Proof of Lemma 7B. Let us first observe that

$$\begin{gathered}|p(\tilde{x}_1,\dots,\tilde{x}_{2k}|\tilde{t}_1,\dots,\tilde{t}_{2k}) - p(x_1,\dots,x_{2k}|t_1,\dots,t_{2k})| < \text{const.}\, \frac{(\log n)^K}{n} \\ \text{if } (t_1,\dots t_{2k}) \in G_n^{(1)}, \text{ and } |\tilde{x}_s - x_s| < \frac{C}{n},\ |\tilde{t}_s - t_s| < \frac{C}{n},\ s = 1,\dots,2k\end{gathered} \tag{8.1}$$

with some $K = K(\gamma, C)$. This statement follows from Part 2b) of Property A and the fact that $\pi(t_s) - \pi(t_{s-1}) > (\log n)^{-\gamma}$, if $(t_1, \dots, t_{2k}) \in G_n^{(1)}$. (Here $\pi(t_s)$, $s = 1, \dots, 2k$, denotes again the monotone ordering of the sequence t_s, $s = 1, \dots, 2k$.)

The relation $\mathbf{m} = (m_1, \dots, m_k) \in \mathcal{M}_n$ holds if and only if

$$\begin{gathered}f(\varphi(m_s)) \in I\left(m_1, m_s, f(\varphi(m_1))\right) \\ = \left[\frac{|m_s|}{\frac{|m_1|}{f(\varphi(m_1))} + \frac{(p_{j_s}+1)\delta_n}{|m_1|} f(\varphi(m_1))},\ \frac{|m_s|}{\frac{|m_1|}{f(\varphi(m_1))} + \frac{p_{j_s}\delta_n}{|m_1|} f(\varphi(m_1))} \right], \\ s = 2,\dots,k, \quad \varphi(m_s) \in [\varphi_{2j_s}, \varphi_{2j_s+1}], \quad s = 1,\dots,k\,, \\ \text{and } |m_1| \in \tilde{I}_z\,.\end{gathered} \tag{8.2}$$

Since $\bar{A}n < |m_1| < \bar{B}n$ and $0 < b_1 < f(\varphi(m_1)) < b_2 < \infty$, hence the interval $I(m_1, m_s, f(\varphi(m_1)))$ can be non-empty only if $|m_s|$ is of order n for all $s = 1, \dots, n$. Using this fact, we get with the help of standard calculation that

$$\begin{aligned}&I\left(m_1, m_s, f(\varphi(m_1))\right) \\ &= \left[\frac{|m_s|}{|m_1|} f(\varphi(m_1)) - (p_{j_s}+1)\delta_n \frac{|m_s|}{|m_1|^3} f^3(\varphi(m_1)) + O\left(\frac{1}{n^4}\right), \right. \\ &\qquad \left. \frac{|m_s|}{|m_1|} f(\varphi(m_1)) - p_{j_s}\delta_n \frac{|m_s|}{|m_1|^3} f^3(\varphi(m_1)) + O\left(\frac{1}{n^4}\right) \right].\end{aligned}$$

We claim that

$$
\begin{aligned}
P((\mathbf{m}, \bar{\mathbf{m}}) \in \mathcal{F}_n^1) \\
&= \int p\Big(x, \frac{|m_2|}{|m_1|}x, \dots, \frac{|m_k|}{|m_1|}x, \bar{x}, \frac{|\bar{m}_2|}{|\bar{m}_1|}\bar{x}, \dots, \frac{|\bar{m}_k|}{|\bar{m}_1|}\bar{x} \Big| \\
&\qquad \Big| \varphi(m_1), \dots, \varphi(m_k), \varphi(\bar{m}_1), \dots, \varphi(\bar{m}_k)\Big) \\
&\qquad x^{3(k-1)}\bar{x}^{3(k-1)}\, dx\, d\bar{x} \delta_n^{2(k-1)} \prod_{s=2}^{k} \frac{|m_s||\bar{m}_s|}{|m_1|^3|\bar{m}_1|^3} + O\left(\frac{(\log n)^K}{n^{4k-3}}\right)
\end{aligned}
\tag{8.3}
$$

if $(\varphi(m_1), \dots, \varphi(m_k), \varphi(\bar{m}_1), \dots, \varphi(\bar{m}_k)) \in G_n^{(1)}$ and $|m_1|, |\bar{m}_1| \in \tilde{I}_z$. Otherwise, this probability equals zero. Indeed, we get the above probability by fixing first the values $f(\varphi(m_1)) = x$ and $f(\varphi(\bar{m}_1)) = \bar{x}$, integrating the density function

$$
p(x, y_2, \dots, y_k, \bar{x}, \bar{y}_2, \dots, \bar{y}_k | \varphi(m_1), \dots, \varphi(m_k), \varphi(\bar{m}_1), \dots, \varphi(\bar{m}_k))
$$

on the set

$$
\begin{aligned}
&(y_2, \dots, y_k, \bar{y}_2, \dots, \bar{y}_k) \\
&\quad \in I(m_1, m_2, x) \times \dots \times I(m_1, m_k, x) \times I(\bar{m}_1, \bar{m}_2, \bar{x}) \times \dots \times I(\bar{m}_1, \bar{m}_k, \bar{x})
\end{aligned}
$$

and then integrating with respect to x and $\bar{x}$.

It follows from (8.1) and the definition of the interval $I(m_1, m_s, f(\varphi(m_1)))$ that we commit an error of order $O\left(\frac{(\log n)^K}{n^{4k-3}}\right)$ by replacing the argument y_s by $\frac{|m_s|}{|m_1|}x$ and $\bar{y}_s$ by $\frac{|\bar{m}_s|}{|\bar{m}_1|}\bar{x}$, the length of the intervals $I(m_1, m_s, x)$ and $I(\bar{m}_1, \bar{m}_s, \bar{x})$ by $\delta_n \frac{|m_s|}{|m_1|^3}x^3$ and $\delta_n \frac{|\bar{m}_s|}{|\bar{m}_1|^3}\bar{x}^3$ when integrating with fixed x and $\bar{x}$. (Observe that we may assume that $b_1 \le x, \bar{x} \le b_2$, otherwise the density function $p(\cdot|\cdot)$ equals zero. Let us also observe that the density $p(\cdot|\cdot)$ in the integrand of (8.3) is less than a power of $\log n$, because of Part 2a) of Property A and the fact that the difference between the angles $\varphi(m_s)$ and $\varphi(\bar{m}_s)$ is greater than $(\log n)^{-\gamma}$. We need this observation to show that the approximation of the length of the intervals $I(m_1, m_s, x)$ and $I(\bar{m}_1, \bar{m}_s, \bar{x})$ we have made causes an error of order $O\left(\frac{(\log n)^K}{n^{4k-2}}\right)$.) In such a way we get formula (8.3).

Fix some $m \in \mathbb{D}_{j_1} \cap \mathbb{Z}^2$ and $\bar{m} \in \mathbb{D}_{j_1} \cap \mathbb{Z}^2$. Relations (8.1) and (8.3) imply that

$$
\begin{aligned}
&\sum\nolimits^{(m,\bar{m})} P\left((\mathbf{m}, \bar{\mathbf{m}}) \in \mathcal{F}_n^1\right) \\
&\quad = \delta_n^{2(k-1)} \int_{\{\varphi(m_1), \varphi(y_2), \dots, \varphi(y_k), \varphi(\bar{m}_1), \varphi(\bar{y}_2), \dots, \varphi(\bar{y}_k)\} \in G_n^{(1)}} |y_2||\bar{y}_2| \dots |y_k||\bar{y}_k| \\
&\qquad p\big(x, |y_2|, \dots, |y_k|, \bar{x}, |\bar{y}_2|, \dots, |\bar{y}_k| \big| \\
&\qquad\quad \big| \varphi(m), \varphi(y_2), \dots, \varphi(y_k), \varphi(\bar{m}), \varphi(\bar{y}_2), \dots, \varphi(\bar{y}_k)\big) \\
&\qquad\quad dy_2\, d\bar{y}_2 \dots dy_k\, d\bar{y}_k\, dx\, d\bar{x} + O\left(\frac{(\log n)^K}{n}\right),
\end{aligned}
\tag{8.4}
$$

where $y_s, \bar y_s \in \mathbb{R}^2$, for $s = 2, \dots, k$, $x, \bar x \in \mathbb{R}^1$ and $\sum^{(m,\bar m)}$ denotes summation for such pairs $(\mathbf{m}, \bar{\mathbf{m}}) \in \mathcal{F}_n^1$, $\mathbf{m} = (m_1, \dots, m_k)$, $\bar{\mathbf{m}} = (\bar m_1, \dots, \bar m_k)$ for which $m_1 = m$ and $\bar m_1 = \bar m$. (We remark that the dependence of the last integral on m and $\bar m$ appears only in the dependence of the density function $p(\cdot|\cdot)$ on $\varphi(m)$ and $\varphi(\bar m)$.)

Indeed, let us estimate each term of the sum at the left-hand side of (8.4) with the help of the integral (8.3). Let us fix the values x and $\bar x$ in these integrals and consider the sum of the integrands at the right-hand side of (8.3). This is an approximating sum of the integral on the right-hand side of (8.4) with fixed (x and $\bar x$) on a lattice of span $\left[0, \dfrac{x}{|m_1|}\right]^{2(k-1)} \times \left[0, \dfrac{\bar x}{|\bar m_1|}\right]^{2(k-1)}$, and the difference between the sum and the approximating integral is $O\left(\dfrac{(\log n)^K}{n}\right)$ by formula (8.1). Then integrating with respect to the arguments x and $\bar x$ we get formula (8.4).

Summing up relation (8.4) for m and $\bar m$ we get with the help of relation (8.1) that

$$\begin{aligned}\mathcal{I}(\mathcal{F}_n^1) = \delta_n^{2(k-1)} \frac{n^4}{(\log n)^{4\bar\eta}} \int_{\{(x,y_1,\dots,y_k;\bar x,\bar y_1,\dots,\bar y_k)\in D_k\}} & |y_2||\bar y_2| \dots |y_k||\bar y_k| \\ p\big(x, |y_2|, \dots, |y_k|, \bar x, |\bar y_2|, \dots, |\bar y_k| \,\big|\, & \varphi(y_1), \dots, \varphi(y_k), \varphi(\bar y_1), \dots, \varphi(\bar y_k)\big) \\ & dy_1 \dots d\bar y_k \, dx \, d\bar x + O\left(n^3 (\log n)^K\right),\end{aligned} \tag{8.5}$$

where the set D_k is defined as

$$\begin{aligned}D_k = \Big\{ & (x, y_1, \dots, y_k, \bar x, \bar y_1, \dots, \bar y_k), \quad x, \bar x \in \mathbb{R}^1, \quad y_s, \bar y_s \in \mathbb{R}^2,\ s = 1, \dots, k \\ & (\varphi(y_1), \dots, \varphi(y_k), \varphi(\bar y_1), \dots, \varphi(\bar y_k)) \in G_n^{(1)}, \quad |y_1|, |\bar y_1| \in [z, z+1] \Big\}.\end{aligned}$$

Here we exploit that $\mathcal{I}(\mathcal{F}_n^1)$ is the sum of the expressions at the left-hand side of formula (8.4), and $\dfrac{(\log n)^{4\bar\eta}}{n^4}$ times the sum of the integrals at the right-hand side of (8.4) is an approximating sum of the integral at the right-hand side in (8.5) on a lattice of span $\left[0, \dfrac{(\log n)^{\bar\eta}}{n}\right]^4$. Rewriting the integral in (8.5) in polar coordinate system, i.e. making the change of variables $r_s = |y_s|$, $\varphi_s = \varphi(y_s)$ and $\bar r_s = |\bar y_s|$, $\bar\varphi_s = \varphi(\bar y_s)$ for all $s = 1, \dots, k$ and then integrating with respect to r_1 and $\bar r_1$ we get that

$$\mathcal{I}(\mathcal{F}_n^1) = \delta_n^{2(k-1)} \frac{n^4}{(\log n)^{4\bar\eta}} \left(z + \frac{1}{2}\right)^2 I(G_n^{(1)}) + O\left(n^3 (\log n)^K\right).$$

The last relation implies Lemma 7B. □

The proof of Theorem 8B is similar, hence we only give a brief sketch of it. We get, arguing similarly to the proof of formula (8.3) that

$$E\chi(\mathbf{m} \in \mathcal{M}_n)\mathcal{J}(G_n^{(2)}(\mathbf{m}))$$

$$= \int\limits_{\substack{x\in\mathbb{R}^1\ (y_1,\dots,y_k)\in\mathbb{R}^k \\ (\bar t_2,\dots,\bar t_k)\in\mathcal{J}(G_n^{(2)}(\mathbf{m}))}} p\Bigg(x, \frac{|m_2|}{|m_1|}x,\dots,\frac{|m_k|}{|m_1|}x, |\bar y_1|, |\bar y_2|,\dots,|\bar y_k|\Bigg|$$

$$\varphi(m_1),\dots,\varphi(m_k),\bar t_1,\dots,\bar t_k\Bigg) x^{3(k-1)}|\bar y_2|^2\dots|\bar y_k|^2\,dx\,d\bar y_1\dots,\ d\bar y_k$$

$$dt_1\dots dt_k\,\delta_n^{(k-1)}\prod_{s=2}^k\frac{|m_s|}{|m_1|^3}+O\left(\frac{(\log n)^K}{n^{2k-1}}\right).$$

To prove Theorem 8B first we sum up this formula for all $\mathbf{m}=(m_1,\dots,m_k)$ with prescribed m_1 then for m_1 and observe that these sums approximate well certain integrals. In such a way we get formulas analogous to (8.4) and (8.5). Then rewriting the formula analogous to formula (8.5) in polar coordinate system (in the variables corresponding to the points $\mathbf{m}$) we get Theorem 8B.

9. On the proof of the Stronger version of the Theorem

We have used Part 1) of Property A in the proof of Lemmas 7A, 8A and in Lemmas 2 and 3. If only its weaker version Part 1′) is satisfied, then the following observations help us to prove the Theorem.

If the estimate $|f(\varphi_2)-f(\varphi_1)|\le D|\varphi_2-\varphi_1|\log n$ holds with some $D>0$ instead of the inequality $|f(\varphi_2)-f(\varphi_1|\le \text{const.}\,|\varphi_2-\varphi_1|$, then the bounds we get are worse with a multiplying factor which is a power of $\log n$. Such estimates are appropriate in the proofs of Lemmas 7A and 8A, if the exponent γ is chosen sufficiently large in them. Hence we make the following approach. Let us consider the event

$$F(D,n)=\left\{\sup_{0\le\varphi_1<\varphi_2\le\theta}\frac{|f(\varphi_1)-f(\varphi_2)|}{|\varphi_1-\varphi_2|}<D\log n\right\},$$

where $D=D(k)$ may depend on the number k appearing in Lemmas 7A and 8A. We get the necessary bound on this set and show that the contribution of the complementary set is negligible.

In Lemma 7A we have to bound the sum (6.9′). There are only const. n^{2k} non-zero terms in this sum, and each of them can be bounded well by means of formula (2.1) on the complementary set of $F(D,n)$. Hence, it is enough to estimate the sum

$$\sum_{\mathbf{m}\in\mathcal{Z}}\sum_{\bar{\mathbf{m}}\in\mathcal{Z}}P\left(\{(\mathbf{m},\bar{\mathbf{m}})\in\mathcal{F}_n^2\}\cap F(D,n)\right)$$

with sufficiently large $D>0$. The estimates given in the proof of Section 7 work in this case too, the only difference is that now an additional multiplying factor $(\log n)^{2k}$ appears. But this term causes no problem if $\gamma>0$ is chosen sufficiently large. The same argument works in the proof of Lemma 8A and Lemma 2, but in the proof of Lemma 3 we have to be more careful. The problem which arises in this case is that although α can be chosen large in the definition of the sets $\mathbb{D}_j(n)$, but it cannot depend on the number k appearing in this lemma. We have to bound the expression in formula (5.4) more carefully. Actually, it is enough to bound the expression

$$E|B_{n,k}^m(f)|\chi(F'(\varepsilon_k,D_k)) \tag{9.1}$$

with some appropriate $\varepsilon_k > 0$ and $D_k > 0$, where

$$F'(\varepsilon, D) = \left\{ f; \quad \sup_{0 \le \varphi_1 < \varphi_2 \le \theta} (\log n)^{\varepsilon} \le \frac{|f(\varphi_1) - f(\varphi_2)|}{|\varphi_1 - \varphi_2|} < D \log n \right\}.$$

The estimate on the complementary set of F'can be done by modifying the argument of the proof in the same way as it was done in the case of Lemma 7A.

We estimate the expression in (9.1) with the help of the Schwarz inequality and the following observation:

$$\left| B^m_{n,k}(f) \right|^2 < \binom{2k+2}{k+1} \sum_{s=k}^{2k+1} \left| B^m_{n,s}(f) \right|. \tag{9.2}$$

The estimate (9.2) holds, since at the left-hand side we have counted the number of pairs $(\bar{m}, m_1, \ldots, m_k) \in B^m_{n,k}(f)$ and $(\bar{m}', m'_1, \ldots, m'_k) \in B^m_{n,k}(f)$, the union of these two sets is contained in one of the sets $B^m_{n,s}(f)$, and at most $\binom{2k+2}{k+1}$ different pairs can give the same union.

In such a way we get the inequality

$$E|B^m_{n,k}(f)|\chi(F'(\varepsilon_k, D_k) < \left[\binom{2k+2}{k+1} \sum_{s=k}^{2k+1} E\left(\left| B^m_{n,s}(f) \right| \chi(F'(\varepsilon_k, D_k)) \right) \right]^{1/2}$$

$$P\left(\sup_{0 \le \varphi_1 < \varphi_2 \le \theta} \frac{|f(\varphi_1) - f(\varphi_2)|}{|\varphi_1 - \varphi_2|} > (\log n)^{\varepsilon_k} \right)^{1/2}$$

$$< \text{const.} (\log n)^{k+1} \exp\left\{ -\lambda (\log n)^{\varepsilon_k} \right\}.$$

The right-hand side estimate of the last inequality holds. Indeed, the argument of the proof of Lemma 3 yields that the first term of the right-hand side is an upper bound on the sum of expectations, since $\frac{|f(\varphi_1) - f(\varphi_2)|}{|\varphi_1 - \varphi_2|} < D_k \log n$ on the set $F'(\varepsilon_k, D_k)$. On the other hand, formula (2.1) gives the bound $\exp\{-\lambda(\log n)^{\varepsilon_k}\}$ on the probability in the middle term. Since this is the dominating term in the expression at the right-hand side, we have got a sufficiently good upper bound on the expression (9.1). In such a way we can prove Lemma 4 under the weaker condition Part 1′) instead of Part 1) in Property A.

References

1. Berry, M. V., Tabor, M.: Level clustering in the regular spectrum. Proc. R. Soc. Lond. A **356** (1977), 375–394.
2. Bleher, P. M.: Quasi-classical expansion and the theory of quantum chaos. Lecture Notes in Mathematics. to appear
3. Cheng, Z., Lebowitz, L. J.: Statistics of energy levels in integrable quantum systems. Preprint
4. Hardy, G. H., Wright, E. M.: An introduction to the theory of numbers. Clarendon Press, Oxford (1960)
5. Huxley, M. N.: Exponential sums and lattice points. Proc. London Math. Soc. **60** (3) (1990) 470–502

6. Kendall, D. G.: On the number of lattice points inside a random oval. Quart. J. Math. Oxford **19** (1948) 1–26
7. Kosygin, D. V., Sinai, Ya. G.: On the number of lattice points inside a random domain. In From Phase Transition to Chaos. eds. G. Györgyi et al., World Scientific, Singapore (1992)
8. Landau, E.: Aus der analytischen und geometrischen Zahlentheorie, Bd. 2. Verlag von S. Hirzel in Leipzig (1927)
9. Sinai, Ya. G.: Poisson distribution in a geometrical problem. Advances in Soviet Mathematics, Publications of AMS
10. Sinai, Ya. G.: Mathematical problems in the theory of quantum chaos. (1990) to appear

This article was processed by the author
using the Springer-Verlag TEX ProbTh macro package 1991.