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Heat content and Brownian motion for some regions with a fractal boundary

M. van den Berg

Department of Mathematics, Heriot-Watt University, Riccarton, Edinburgh, EH14 4AS, UK

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Summary. Let D be an open, bounded set in euclidean space $\mathbb{R}^m$ $(m = 2, 3, \ldots)$ with boundary ∂D. Suppose D has temperature 0 at time $t = 0$, while ∂D is kept at temperature 1 for all $t > 0$. We use brownian motion to obtain estimates for the solution of corresponding heat equation and to obtain results for the asymptotic behaviour of $E_D(t)$, the amount of heat in D at time t, as $t \to 0^+$. For the triadic von Koch snowflake K our results imply that

$$c^{-1} t^{1-(\log 2)/\log 3} \leqq E_K(t) \leqq c t^{1-(\log 2)/\log 3}, \quad 0 \leqq t \leqq c^{-1},$$

for some constant $c > 1$.

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1 Introduction

Let D be an open set in euclidean space $\mathbb{R}^m$ $(m = 2, 3, \ldots)$ with finite volume $|D|_m$ and with boundary ∂D. Let $v : D \times [0, \infty) \to R$ be the unique solution of

$$\Delta v = \frac{\partial v}{\partial t}, \quad x \in D, \quad t > 0, \tag{1.1}$$

$$v = 0, \quad x \in D, \quad t = 0, \tag{1.2}$$

$$v = 1, \quad x \in \partial D, \quad t > 0, \tag{1.3}$$

where Δ is the Laplace operator; $v(x; t)$ represents the temperature at time t at a point $x \in D$ when D has initial temperature 0, and ∂D is kept at temperature 1 for all $t > 0$. Let

$$E_D(t) = \int_D v(x; t)\, dx \tag{1.4}$$

represent the total amount of heat contained in D at time t.

The asymptotic behaviour of $E_D(t)$ for $t \to 0^+$ has been investigated in a variety of situations [4, 7, 8]. For example if ∂D is compact and of class C^3, then there exists a constant $C > 0$ such that for all $t > 0$

$$|E_D(t) - 2(t/\pi)^{1/2}|\partial D|_{m-1} + 2^{-1}(m-1)t \int_{\partial D} H(x)\,dx| \leqq Ct^{3/2}, \tag{1.5}$$

where $|\partial D|_{m-1}$ is the $(m-1)$-dimensional measure of ∂D and $H(x)$ is the mean curvature at a point $x \in \partial D$, where ∂D is oriented with a smooth, inward unit normal vector field $\mathbb{N}$. For the proof of (1.5) we refer to [7]. For refinements of (1.5) and extensions to riemannian manifolds with a compact C^∞ boundary we refer to [2, 5, 6].

In this paper we consider open sets D in $\mathbb{R}^m$ ($m = 2, 3, \dots$) with finite volume $|D|_m$ which do not necessarily satisfy the smoothness conditions on ∂D which were assumed in [2, 5–8]. Instead we assume that the boundary satisfies a uniform capacitary density condition. This condition on one hand enables us to obtain a non-trivial pointwise lower bound for the solution v of (1.1)–(1.3) near the boundary. On the other hand this condition is satisfied by many open sets D in $\mathbb{R}^m$ with a non-smooth boundary, and is always satisfied if D is open, bounded and simply connected in $\mathbb{R}^2$. For example if K is a triadic von Koch snowflake [11, pp. 120–121] then there exists (by Corollary 1.5) a constant $c > 1$ such that for $0 \leqq t \leqq c^{-1}$

$$c^{-1}t^{1-(\log 2)/\log 3} \leqq E_k(t) \leqq ct^{1-(\log 2)/\log 3}. \tag{1.6}$$

Denote by $\mathrm{Cap}(A)$ the newtonian capacity of a compact set $A \subset \mathbb{R}^m$ ($m = 3, 4, \dots$) or the logarithmic capacity of a compact set $A \subset \mathbb{R}^2$. For $x \in \mathbb{R}^m$, and $r > 0$ we define

$$B(x; r) = \{y \in \mathbb{R}^m \colon |y - x| \leqq r\}, \tag{1.7}$$

and for a non-empty set $G \subset \mathbb{R}^m$

$$\mathrm{diam}\,(G) = \sup\{|x_1 - x_2| \colon x_1 \in G,\, x_2 \in G\}. \tag{1.8}$$

Definition 1.1 *Let $D \subset \mathbb{R}^m$ ($m = 2, 3, \dots$) be an open set with boundary ∂D. The capacitary density of ∂D is bounded away from zero if there exists a constant $c_0 > 0$ such that*

$$\mathrm{Cap}(\partial D \cap B(x; r)) \geqq c_0\,\mathrm{Cap}(B(x; r)), \quad x \in \partial D, \quad 0 < r < \mathrm{diam}(D). \tag{1.9}$$

Definition 1.1 has been introduced in [9] in a study of the partition function of the Dirichlet laplacian on open sets with a non-smooth or fractal boundary. Note that if (1.9) is satisfied for some $c_0 > 0$ then there exists a constant $k > 0$ such that the Dirichlet laplacian $-\Delta_D$ for D satisfies a quadratic form inequality $-\Delta_D \geqq k/d^2(x)$, where $d \colon D \to \mathbb{R}$ denotes the distance function defined by

$$d(x) = \inf\{|y - x| \colon y \in \partial D\}. \tag{1.10}$$

See [1, 3] for details and applications respectively.

Brossard and Carmona [9] showed in their study of the asymptotic behaviour of the partition function that the relevant measure of roughness of the boundary is based on the Minkowski dimension and on Minkowski contents. Motivated by Definition (2.3) in [9] we define the function $\mu: \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\mu(\varepsilon) = |\{x \in D: d(x) < \varepsilon\}|_m. \tag{1.11}$$

Note that μ is monotone increasing, and its behaviour near 0 determines the interior Minkowski dimension d of ∂D:

$$d = \inf\{u > 0: \limsup_{\varepsilon \to 0} \varepsilon^{u-m} \mu(\varepsilon) = 0\}. \tag{1.12}$$

We recall that ∂D has finite upper Minkowski content if

$$M_d^+(\partial D) = \limsup_{\varepsilon \to 0} \varepsilon^{d-m} \mu(\varepsilon) < \infty, \tag{1.13}$$

that ∂D has positive lower Minkowski content if

$$M_d^-(\partial D) = \limsup_{\varepsilon \to 0} \varepsilon^{d-m} \mu(\varepsilon) > 0, \tag{1.14}$$

and that ∂D is said to be d-Minkowski measurable if

$$0 < M_d^-(\partial D) = M_d(\partial D) = M_d^+(\partial D) < \infty. \tag{1.15}$$

It is elementary that $d \in [m-1, m]$ for any open, bounded set in D in $\mathbb{R}^m$. However, in general, open bounded sets D need not have a Minkowski measurable boundary. For example for the triadic von Koch snowflake K we have $d_K = (\log 4)/\log 3$, and

$$0 < M_{d_K}^-(\partial K) < M_{d_K}^+(\partial K) < \infty. \tag{1.16}$$

The bounds on the heat content $E_D(t)$ will be expressed in terms of μ rather than the upper and lower Minkowski contents of ∂D. This allows for more general situations where $\mu(\varepsilon)$ is not bounded within multiplicative constants of ε^{m-d}. For example Theorems 1.2 and 1.3, 1.4 below still give bounds of the same order if $c_0 > 0$, $m - 1 < d < m$, $\alpha \in \mathbb{R}$, and for some $\varepsilon_0 > 0$

$$\mu(\varepsilon) = \varepsilon^{m-d}\{\log(2 + 1/\varepsilon)\}^\alpha, \quad 0 < \varepsilon \leqq \varepsilon_0, \tag{1.17}$$

even though $M_d^+(\partial D) = \infty$ for $\alpha > 0$ and $M_d^-(\partial D) = 0$ for $\alpha < 0$.

The main results of this paper are the following.

Theorem 1.2 *Let D be an open set in $\mathbb{R}^m$ $(m = 2, 3, \dots)$ with finite volume $|D|_m$. Then for all $t > 0$,*

$$E_D(t) \leqq 2^{(m-2)/2} t^{-1} \int_0^\infty e^{-\varepsilon^2/(8t)} \varepsilon \mu(\varepsilon)\, d\varepsilon. \tag{1.18}$$

Theorem 1.3 *Let D be an open, bounded set in $\mathbb{R}^2$ and suppose that* (1.9) *holds for some $c_0 > 0$. Then for all $t > 0$,*

$$E_D(t) \geqq (4 - 4\pi(\log c_0)/\log 2)^{-1} \mu((\pi t)^{1/2}/(24 - 24\pi(\log c_0)/\log 2)). \tag{1.19}$$

Theorem 1.4 *Let D be an open, bounded set in $\mathbb{R}^m (m = 3, 4, \ldots)$ and suppose that* (1.9) *holds for some $c_0 > 0$. Then for all $t > 0$,*

$$E_D(t) \geqq (2/3)^m c_0 \mu((2/3)^m c_0 (\pi t)^{1/2}/72). \tag{1.20}$$

We have the following immediate Corollary to the Theorems 1.2–1.4.

Corollary 1.5 *Let D be an open, bounded set in $\mathbb{R}^m (m = 2, 3, \ldots)$. Suppose that* (1.9) *holds for some $c_0 > 0$ and there exist constants $c_1 > 1$ and $\varepsilon_0 > 0$ such that*

$$c_1^{-1} \varepsilon^{m-d} \leqq \mu(\varepsilon) \leqq c_1 \varepsilon^{m-d}, \quad 0 \leqq \varepsilon \leqq \varepsilon_0. \tag{1.21}$$

Then there exist constants $c_2 > 1$ and $t_0 > 0$ such that

$$c_2^{-1} t^{(m-d)/2} \leqq E_D(t) \leqq c_2 t^{(m-d)/2}, \quad 0 \leqq t \leqq t_0. \tag{1.22}$$

Proposition 1.6 *Let D be open, bounded and simply connected in $\mathbb{R}^2$. Then* (1.9) *holds with*

$$c_0 = 2^{-3/(2\pi)}. \tag{1.23}$$

Note that (1.16) implies (1.21) with $m = 2$, $d = d_K$. Moreover since K is open, bounded and simply connected the capacitary density of ∂K is bounded away from zero (by Proposition 1.6). Then Corollary 1.5 implies estimate (1.6) for the heat content of the von Koch snowflake.

The asymptotic behaviour of $E_K(t) t^{(\log 2/\log 3)-1}$ as $t \to 0^+$ has been investigated by Fleckinger et al. [12]. They proved the existence of a $\log 9$ periodic function $\psi_K : \mathbb{R} \to \mathbb{R}^+$ such that for $t \to 0^+$

$$E_K(t) = \psi_K(\log t) t^{1-(\log 2)/\log 3}(1 + o(1)). \tag{1.24}$$

However, it is still an open question whether or not ψ_K is a constant function. Below we give an example of an open, planar set F_s, with finite volume $|F_s|_2$ and Minkowski dimension $d_s \in (1, 2)$ for which $E_{F_s}(t) t^{(d_s - 2)/2}$ is a non-constant periodic function of $\log t$.

Define for $n = 0, 1, 2, \ldots, m = 0, 1, 2, \ldots$

$$N_{m,n} = \binom{m+n}{n} 3^n 2^{m+2}, \tag{1.25}$$

and for $0 < s < (1 + \sqrt{8})/7$

$$a_{m,n} = s^{n+1} ((1 - s)/2)^m. \tag{1.26}$$

The set F_s consists of the union over $m = 0, 1, 2, \ldots$, $n = 0, 1, 2, \ldots$ of $N_{m,n}$ disjoint open squares of sidelength $a_{m,n}$. The volume of F_s is finite and is given by

$$|F_s|_2 = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} N_{m,n} a_{m,n}^2 = \frac{8s^2}{1 - 7s^2 + 2s}. \tag{1.27}$$

The length of ∂F_s is given by

$$|\partial F_s|_1 = 4 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} N_{m,n} a_{m,n} = +\infty, \tag{1.28}$$

and the interior Minkowski dimension d_s of F_s is given by the unique positive real root of (see [10, Chapter 8.3])

$$3s^d + 2((1-s)/2)^d = 1. \tag{1.29}$$

For positive integers p and q we let (p, q) be the greatest common divisor of p and q. Furthermore let

$$I = \{s \in (0, (1+\sqrt{8})/7) \colon \log((1-s)/2)/\log s = p/q,\ p \in \mathbb{Z}^+, q \in \mathbb{Z}^+, (p,q) = 1\}. \tag{1.30}$$

Theorem 1.7 (i) *Let $s \in I$. Then there exists a constant $\alpha_1 > (2-d_s)/2$ such that for $t \to 0^+$*

$$E_{F_s}(t) = \frac{s^{d_s}}{3qs^{d_s} + 2p((1-s)/2)^{d_s}} t^{(2-d_s)/2} \sum_{j \in \mathbb{Z}} s^{(2-d_s)(y+j)/q} E(s^{-2(y+j)/q}) + O(t^{\alpha_1}), \tag{1.31}$$

where $E(t)$ is the heat content of a square in $\mathbb{R}^2$ with volume 1 given by

$$E(t) = 16\pi^{-4} \sum_{k \in \mathbb{Z}} \sum_{\ell \in \mathbb{Z}} \{1 - e^{-t\pi^2\{(2k+1)^2 + (2l+1)^2\}}\}(2k+1)^{-2}(2l+1)^{-2}, \tag{1.32}$$

and

$$y = \frac{q}{2} \frac{\log t}{\log 1/s}, \tag{1.33}$$

and p, q are the positive integers corresponding to the choice of $s \in I$ (so $(p,q) = 1$).

(ii) *Let $s \in (0, (1+\sqrt{8})/7) \backslash I$. Then for $t \to 0^+$,*

$$E_{F_s}(t) = \frac{2^7}{\pi^2}\left(\frac{s}{\pi}\right)^{d_s} \frac{1}{2-d_s} \Gamma(d_s/2) t^{(2-d_s)/2} \left\{3s^{d_s} \log\frac{1}{s} + 2\left(\frac{1-s}{2}\right)^{d_s} \log\left(\frac{2}{1-s}\right)\right\}^{-1}$$
$$\cdot \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} (2k+1)^{-2}\{(2k+1)^2 + (2l+1)^2\}^{-d_s/2} + o(t^{(2-d_s)/2}). \tag{1.34}$$

Note that the set I is dense in $(0, (1+\sqrt{8})/7)$, and that for $s \in I$, $E_{F_s} t^{(d_s-2)/2}$ is a periodic function of $\log t$ (as $t \to 0^+$) with period $(2/q)\log(1/s)$, while for $s \in (0, (1+\sqrt{8})/7) \backslash I$, $E_{F_s}(t) t^{(d_s-2)/2}$ converges to a constant as $t \to 0^+$.

We defer the proof of Theorem 1.7 to Sect. 3. The set F_s is not connected. In Sect. 4 we construct an open, bounded, simply connected set G_s, $0 < s \leqq \frac{1}{3}$, such that

$$E_{G_s}(t) = E_{F_s}(t) + O(t^{1/2}), \quad t \to 0^+. \tag{1.35}$$

Remark 1.8 Let $s \in (0, (1+\sqrt{8})/7)$. Then F_s is Minkowski measurable if and only if $s \notin I$. Moreover, for $s \in (0, (1+\sqrt{8})/7) \backslash I$,

$$M_{d_s}(\partial F_s) = 2^5 (s/2)^{d_s} \{d_s(d_s-1)(2-d_s)\}^{-1} \left\{3s^{d_s} \log\frac{1}{s} + 2\left(\frac{1-s}{2}\right)^{d_s} \log\frac{2}{1-s}\right\}^{-1}. \tag{1.36}$$

The proof of Remark 1.8 is very similar to the proof of Theorem 1.7(ii). We will not give its details.

2 Proofs of Theorems 1.2, 1.3, 1.4 and Proposition 1.6

The proofs in this section (and in Section 4) use the probabilistic solution of the initial-boundary value problem (1.1)–(1.3). Let $(B(t), t \geqq 0; \mathbb{P}_x, x \in \mathbb{R}^m)$ be a brownian motion associated to $-\Delta + \partial/\partial t$. For $x \in D$ we define

$$T_D = \inf\{t \geqq 0: B(t) \in \mathbb{R}^m \backslash D\}. \tag{2.1}$$

Then the solution of (1.1)–(1.3) is given by

$$v(x; t) = \mathbb{P}_x[T_D \leqq t]. \tag{2.2}$$

Proof of Theorem 1.2 Let

$$B^0(x; d(x)) = \{y \in \mathbb{R}^m: |y - x| < d(x)\}. \tag{2.3}$$

Then by Levy's maximal inequality ([17, Theorem 3.6.5])

$$\begin{aligned}\mathbb{P}_x[T_D \leqq t] &\leqq \mathbb{P}_x[T_{B^0(x;d(x))} \leqq t] = \mathbb{P}_0\left[\max_{0 \leqq s \leqq t} |B(s)| \geqq d(x)\right] \\ &\leqq 2\mathbb{P}_0[|B(t)| \geqq d(x)] = 2(4\pi t)^{-m/2} \int_{\{y \in \mathbb{R}^m: |y| \geqq d(x)\}} e^{-|y|^2/(4t)}\, dy \\ &\leqq 2^{(m+2)/2} e^{-d^2(x)/(8t)}.\end{aligned} \tag{2.4}$$

Hence

$$\begin{aligned}E_D(t) &\leqq \int_D dx\, 2^{(m+2)/2} e^{-d^2(x)/(8t)} \\ &= 2^{(m+2)/2} \int e^{-\varepsilon^2/(8t)}\, d\mu(\varepsilon),\end{aligned} \tag{2.5}$$

and (1.18) follows after an integration by parts.

The ideas in the proofs of Theorems 1.3 and 1.4 below are due to Carmona and Brossard. Their Lemma 3.5 in [9] is the first step in obtaining a lower bound for the conditional probability of a brownian bridge hitting the boundary (Lemma 3.6). Since we only require a lower bound for the unconditional probability $\mathbb{P}_x[T_D \leqq t]$, Lemma 3.5 suffices.

We closely follow its proof of pp. 117, 118 in [9], keeping careful track of the numerical constants involved, and in particular of the dependence of the lower bound on c_0. See in particular the estimate (2.23)–(2.24) and (2.34)–(2.36) below for the cases $m \geqq 3$ and $m = 2$ respectively.

Proof of Theorem 1.4 For any closed set $C \subset \mathbb{R}^m, x \notin C$, we define the first entry time τ_C by

$$\tau_C = \inf\{t \geqq 0: B(t) \in C\}. \tag{2.6}$$

Let $m = 3, 4, \ldots$ and define

$$g(x, y) = 4^{-1}\pi^{-m/2}\Gamma((m-2)/2)|x - y|^{2-m}. \tag{2.7}$$

The newtonian capacity of a compact set K is defined by

$$\mathrm{Cap}(K) = \left\{ \inf_{\mu \in P(K)} \iint \mu(dx)\mu(dy)\, g(x, y) \right\}^{-1}, \tag{2.8}$$

where $P(K)$ is the set of all probability measures supported on K. For compact sets K_1, K_2 with $K_1 \subset K_2$ we have by (2.8)

$$\mathrm{Cap}(K_1) \leqq \mathrm{Cap}(K_2), \tag{2.9}$$

and in particular ([16, Proposition 1.9]) that

$$\mathrm{Cap}(B(x; r)) = 4\pi^{m/2}(\Gamma((m-2)/2))^{-1} r^{m-2}. \tag{2.10}$$

Let μ_C be the equilibrium measure of C (i.e. the unique minimizer in the right hand side of (2.8)). Then

$$\mathbb{P}_x[\tau_C < \infty] = \int_C g(x, y)\mu_C(dy), \tag{2.11}$$

and one can show that (see [16])

$$\mathrm{Cap}(K) = \int_K \mu_K(dy). \tag{2.12}$$

We have the following estimates:

$$\begin{aligned}\mathbb{P}_x[T_D \leqq t] &\geqq \mathbb{P}_x[\tau_{\partial D \cap B(x;\, 3d(x))} \leqq t] \\ &\geqq \mathbb{P}_x[\tau_{\partial D \cap B(x;\, 3d(x))} \leqq T_{B^0(x;\, 9d(x))}] - \mathbb{P}_x[T_{B^0(x;\, 9d(x))} > t]. \end{aligned}\tag{2.13}$$

Let H be an open half space, containing $B^0(x; 9d(x))$, such that ∂H is tangent to $\partial B^0(x; 9d(x))$. Then

$$\begin{aligned}\mathbb{P}_x[T_{B^0(x;\, 9d(x))} > t] &\leqq \mathbb{P}_x[T_H > t] \\ &= (\pi t)^{-1/2} \int_{[0,\, 9d(x))} e^{-u^2/(4t)}\, du \leqq 9\, d(x)(\pi t)^{-1/2}. \end{aligned}\tag{2.14}$$

By the strong Markov property we obtain a lower bound for the first term in the right hand side of (2.13).

$$\begin{aligned}\mathbb{P}_x[\tau_{\partial D \cap B(x;\, 3d(x))} < \infty] &= \mathbb{P}_x[\tau_{\partial D \cap B(x;\, 3d(x))} \leqq T_{B^0(x;\, 9d(x))}] \\ &\quad + \mathbb{P}_x[\tau_{\partial D \cap B(x;\, 3d(x))} > T_{B^0(x;\, 9d(x))}, \tau_{\partial D \cap B(x;\, 3d(x))} < \infty] \\ &= \mathbb{P}_x[\tau_{\partial D \cap B(x;\, 3d(x))} \leqq T_{B^0(x;\, 9d(x))}] \\ &\quad + \mathbb{E}_x[\mathbb{P}_{B(T_{B^0(x;\, 9d(x))})}[\tau_{\partial D \cap B(x;\, 3d(x))} < \infty]]. \end{aligned}\tag{2.15}$$

Let y be such that $|y - x| = 9d(x)$ and let z be such that $|z - x| = 3d(x)$. Then $|z - y| \geqq 2|z - x|$. Hence by (2.7) and (2.11)

$$\begin{aligned}\mathbb{P}_y[\tau_{\partial D \cap B(x;\, 3d(x))} < \infty] &\leqq 4^{-1}\pi^{-m/2}\Gamma((m-2)/2)\int |y - z|^{2-m}\mu_{\partial D \cap B(x;\, 3d(x))}(dz) \\ &\leqq 4^{-1}\pi^{-m/2}2^{2-m}\Gamma((m-2)/2)\int |x - z|^{2-m}\mu_{\partial D \cap B(x;\, 3d(x))}(dz) \\ &\leqq 2^{-1}\mathbb{P}_x[\tau_{\partial D \cap B(x;\, 3d(x))} < \infty]. \end{aligned}\tag{2.16}$$

By (2.15) and (2.16)

$$\mathbb{P}_x[\tau_{\partial D \cap B(x;\,3d(x))} < T_{B^0(x;\,9d(x))}] \geqq 2^{-1}\mathbb{P}_x[\tau_{\partial D \cap B(x;\,3d(x))} < \infty]. \quad (2.17)$$

By (2.7), (2.11), (2.12) we obtain

$$\begin{aligned}\mathbb{P}_x[\tau_{\partial D \cap B(x;\,3d(x))} < \infty] &= 4^{-1}\pi^{-m/2}\Gamma((m-2)/2)\int |x-y|^{2-m}\mu_{\partial D \cap B(x;\,3d(x))}(dy)\\ &\geqq 4^{-1}\pi^{-m/2}\Gamma((m-2)/2)(3d(x))^{2-m}\int \mu_{\partial D \cap B(x;\,3d(x))}(dy).\\ &\geqq 4^{-1}\pi^{-m/2}\Gamma((m-2)/2)(3d(x))^{2-m}\,\mathrm{Cap}(\partial D \cap B(x;\,3d(x))).\end{aligned} \quad (2.18)$$

Let $w \in \partial D$ be such that $|w-x| = d(x)$. Then $B(w;\,2d(x)) \subset B(x;\,3d(x))$ and by (2.9)

$$\mathrm{Cap}(\partial D \cap B(x;\,3d(x))) \geqq \mathrm{Cap}(\partial D \cap B(w;\,2d(x))). \quad (2.19)$$

Note that for any $x \in D$, $2d(x) \leqq \mathrm{diam}(D)$. Hence by (1.9), (2.10)

$$\begin{aligned}\mathrm{Cap}(\partial D \cap B(w;\,2d(x))) &\geqq c_0\,\mathrm{Cap}(B(w;\,2d(x)))\\ &= 4c_0\pi^{m/2}(\Gamma((m-2)/2))^{-1}(2d(x))^{m-2}.\end{aligned} \quad (2.20)$$

From (2.17)–(2.20) we have

$$\mathbb{P}_x[\tau_{\partial D \cap B(x;\,3d(x))} < T_{B^0(x;\,9d(x))}] \geqq 2^{-1}(2/3)^{m-2}c_0. \quad (2.21)$$

Finally by (2.13), (2.14) and (2.21) we obtain for any $x \in D$

$$\mathbb{P}_x[T_D \leqq t] \geqq 2^{-1}(2/3)^{m-2}c_0 - 9d(x)(\pi t)^{-1/2}. \quad (2.22)$$

Let $x \in D$ be such that

$$d(x) \leqq (2/3)^m c_0(\pi t)^{1/2}/(72). \quad (2.23)$$

For $x \in D$ satisfying (2.23) we have by (2.22)

$$\mathbb{P}_x[T_D \leqq t] \geqq (2/3)^m c_0. \quad (2.24)$$

Integrating (2.24) over the set of $x \in D$ satisfying (2.23) yields (1.20).

Proof of Theorem 1.3 Let $m = 2$ and define

$$g(x, y) = -(2\pi)^{-1}\log|x-y|. \quad (2.25)$$

The equilibrium measure on a compact $K \subset \mathbb{R}^2$ is the unique probability measure μ_K, concentrated on K for which

$$u_K(x) = \int_K g(x, y)\mu_K(dy) \quad (2.26)$$

is constant on the regular points of K. Define the logarithmic capacity for a compact set K by

$$\mathrm{Cap}(K) = \exp\left\{-\inf_{\mu \in P(K)}\int_K\int_K \mu(dx)\mu(dy)g(x, y)\right\}, \quad (2.27)$$

where $P(K)$ is the set of all probability measures supported on K. Let

$$R(K) = -\log \mathrm{Cap}(K), \tag{2.28}$$

be the Robin constant for K. Then one can show that [16]

$$R(K) = \int_K g(x, y)\mu_K(dy), \tag{2.29}$$

on the regular points of K. Moreover, for compact sets K_1, K_2 with $K_1 \subset K_2$ we have by (2.27)

$$\mathrm{Cap}(K_1) \leqq \mathrm{Cap}(K_2), \tag{2.30}$$

and in particular ([16, Proposition 4.11, Chapter 3])

$$\mathrm{Cap}(B(x; r)) = r^{1/(2\pi)}. \tag{2.31}$$

We have the following estimates:

$$\begin{aligned} \mathbb{P}_x[T_D \leqq t] &\geqq \mathbb{P}_x[\tau_{\partial D \cap B(x;\, 2d(x))} < t] \\ &= \mathbb{P}_x[\tau_{\partial D \cap B(x;\, 2d(x))} < T_{B^0(x;\, 6d(x))}] - \mathbb{P}_x[T_{B^0(x;\, 6d(x))} > t]. \end{aligned} \tag{2.32}$$

Let H be an open half space containing $B^0(x; 6d(x))$ such that ∂H is tangent to $\partial B^0(x; 6d(x))$. Then

$$\begin{aligned} \mathbb{P}_x[T_{B^0(x;\, 6d(x))} > t] &\leqq \mathbb{P}_x[T_H > t] \\ &= (\pi t)^{-1/2} \int_{[0,\, 6d(x))} e^{-u^2/(4t)}\, du \leqq 6d(x)(\pi t)^{-1/2}. \end{aligned} \tag{2.33}$$

Let $x \in D$ be such that

$$d(x) \leqq (24 - 24\pi(\log c_0)/\log 2)^{-1}(\pi t)^{1/2}. \tag{2.34}$$

For $x \in D$ satisfying (2.34) we have by (2.32), (2.33)

$$\mathbb{P}_x[T_D \leqq t] \geqq \mathbb{P}_x[\tau_{\partial D \cap B(x;\, 2d(x))} < T_{B^0(x;\, 6d(x))}] - (4 - 4\pi(\log c_0)/\log 2)^{-1}. \tag{2.35}$$

We will show that for $x \in D$ satisfying (2.34)

$$\mathbb{P}_x[\tau_{\partial D \cap B(x;\, 2d(x))} < T_{B^0(x;\, 6d(x))}] \geqq (2 - 2\pi(\log c_0)/\log 2)^{-1}. \tag{2.36}$$

This proves Theorem 1.3 since by (2.35) and (2.36)

$$\begin{aligned} E_D(t) &\geqq \int_{\{x \in D:\ d(x) \leqq (24 - 24\pi(\log c_0)/\log 2)^{-1}(\pi t)^{1/2}\}} dx\, \mathbb{P}_x[T_D \leqq t] \\ &\geqq \int_{\{x \in D:\ d(x) \leqq (24 - 24\pi(\log c_0)/\log 2)^{-1}(\pi t)^{1/2}\}} dx\, (4 - 4\pi(\log c_0)/\log 2)^{-1} \\ &= (4 - 4\pi(\log c_0)/\log 2)^{-1} \mu(\pi t)^{1/2}/(24 - 24\pi(\log c_0)/\log 2)). \end{aligned} \tag{2.37}$$

It remains to prove (2.36). Let $w \in \partial D$ be such that $|w - x| = d(x)$. Then $B(x; 2d(x)) \supset B(w; d(x))$ and by (2.30), (1.9) and (2.31)

$$\begin{aligned} \mathrm{Cap}(\partial D \cap B(x; 2d(x))) &\geqq \mathrm{Cap}(\partial D \cap B(w; d(x))) \\ &\geqq c_0 \mathrm{Cap}(B(w; d(x))) = c_0 (d(x))^{1/(2\pi)}. \end{aligned} \tag{2.38}$$

By (2.28) and (2.38)

$$R(\partial D \cap B(x; 2d(x)) \leqq -(2\pi)^{-1}\log(d(x)) - c_0, \tag{2.39}$$

and by (2.28) and (2.30)

$$R(\partial D \cap B(x; 2d(x)) \geqq R(B(x; 2d(x))) = -(2\pi)^{-1}\log(2d(x)). \tag{2.40}$$

Moreover by (2.25) and (2.26)

$$\begin{aligned} &\sup\{u_{\partial D \cap B(x;\, 2d(x))}(y) \colon y \in \partial B(x; 6d(x))\} \\ &\leqq -(2\pi)^{-1}\log(4d(x)) \int_{\partial D \cap B(x;\, 2d(x))} \mu_{\partial D \cap B(x;\, 2d(x))}(dy) \\ &= -(2\pi)^{-1}\log(4d(x)). \end{aligned} \tag{2.41}$$

Following the proof of Lemma 3.5 for $n = 2$ in [9] we define for $r > 0$

$$m(r) = -(2\pi)^{-1}\log(4r). \tag{2.42}$$

Note that

$$\sup\{u_{\partial D \cap B(x;\, 2r)}(y) \colon y \in \partial B(x; 6r)\} \leqq m(r) < R(\partial D \cap B(x; 2r)). \tag{2.43}$$

Define $h \colon \mathbb{R}^m \to \mathbb{R}$ by

$$h(y) = (R(\partial D \cap B(x; 2d(x))) - m(d(x)))^{-1}(u_{\partial D \cap B(x;\, 2d(x))}(y) - m(d(x))). \tag{2.44}$$

Then h is superharmonic, harmonic on $B(x; 6d(x)) \backslash (\partial D \cap B(x; 2d(x)))$, equal to 1 on the regular points of $\partial D \cap B(x; 2d(x))$, and negative on $\partial B(x; 6d(x))$. Hence

$$\mathbb{P}_x[\tau_{\partial D \cap B(x;\, 2d(x))} < T_{B^0(x;\, 6d(x))}] \geqq h(x) \geqq (\log 2)/((\log 4) - 2\pi \log c_0), \tag{2.45}$$

by (2.39), (2.42) and the trivial estimate

$$u_{\partial D \cap B(x;\, 2d(x))}(x) \geqq -(2\pi)^{-1}\log(2d(x)). \tag{2.46}$$

Proof of Proposition 1.6 Let $x_0 \in \partial D$ be arbitrary and let $r \in (0, \operatorname{diam}(D))$ be arbitrary. Since D is simply connected there exists a one to one continuous map $\gamma \colon [0, 1] \to \partial D$ such that $\gamma(0) = x_0$, $|\gamma(1) - x_0| = r/2$ and Image $(\gamma) \subset B(x_0; r/2)$. Let L be the line through x_0 and $\gamma(1)$. Then the orthogonal projection of Image(γ) onto L contains the closed line segment $[x_0, \gamma(1)]$. By (2.4.4) on p.173 in [15], (2.30) and (2.31)

$$\begin{aligned} \operatorname{Cap}(\partial D \cap B(x_0; r)) &\geqq \operatorname{Cap}(\operatorname{Image}(\gamma)) \\ &\geqq \left(\frac{1}{4}|x_1 - x_0|\right)^{1/(2\pi)} = 2^{-1/\pi}(r/2)^{1/(2\pi)} \\ &= 2^{-3/(2\pi)}\operatorname{Cap}(B_0(x; r)). \end{aligned} \tag{2.47}$$

3 Proof of Theorem 1.7

Let Q_a, $a > 0$ be a square in $\mathbb{R}^2$ of sidelength a. By definition of the function E we have $E(t) = E_{Q_1}(t)$. The solution of (1.1)–(1.3) for $D = Q_1$ can be obtained by separation of variables. Integration of v with respect to x yields the well-known expression (1.32). For the square Q_a we have by scaling

$$E_{Q_a}(t) = a^2 E(t/a^2). \tag{3.1}$$

Since the squares in F_s are disjoint we have by (3.1)

$$E_{F_s}(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} N_{m,n} a_{m,n}^2 E(t/a_{m,n}^2). \tag{3.2}$$

Proof of Theorem 1.7 (i). Let $s \in I$ and let p and q be the corresponding positive integers. Then

$$\frac{1-s}{2} = s^{p/q}, \tag{3.3}$$

and hence

$$a_{m,n} = s(s^{1/q})^{mp+nq}. \tag{3.4}$$

Define for $j \in \mathbb{Z}$

$$a_j = s^{1+j/q}, \tag{3.5}$$

and we can rewrite (3.2) as follows:

$$E_{F_s}(t) = \sum_{j=0}^{\infty} C_j a_j^2 E(t/a_j^2), \tag{3.6}$$

where

$$C_j = \sum_{\{(m,n)\in \mathbb{Z}_0^+ \times \mathbb{Z}_0^+ :\, mp+nq=j\}} \binom{m+n}{n} 3^n 2^{m+2}, \tag{3.7}$$

and $\mathbb{Z}_0^+ = \{0, 1, 2, \dots\}$.

Lemma 3.1 *There exist constants $J_s \in (0, \infty)$ and $j_s > 0$ such that*

$$|C_j - 4\{3qs^{d_s} + 2p((1-s)/2)^{d_s}\}^{-1} s^{-d_s j/q}| \leqq J_s s^{(j_s - d_s)j/q}. \tag{3.8}$$

Proof. Define

$$z_s = s^{d_s/q}, \tag{3.9}$$

and compute the generating function of C_j:

$$\sum_{j=0}^{\infty} z^j C_j = 4 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} z^{mp+nq} \binom{m+n}{n} 3^n 2^m$$

$$= 4 \sum_{m=0}^{\infty} (2z^p)^m (1-3z^q)^{-m-1} = 4(1-2z^p-3z^q)^{-1}. \tag{3.10}$$

We note that by (1.29) and (3.9)

$$2z_s^p + 3z_s^q = 1, \tag{3.11}$$

and the generating function in (3.10) converges for $|z| < z_s$. Define

$$f_{p,q}(z) = 1 - 2z^p - 3z^q. \tag{3.12}$$

Then $f'_{p,q}(z_s) < 0$. Hence z_s is a simple zero of $f_{p,q}$. Let $z_0 \neq z_s$ be a zero of $f_{p,q}$. Suppose $z_0 = e^{i\theta} z_s$, where $0 < \theta < 2\pi$. Then

$$1 = 3z_s^q e^{qi\theta} + 2z_s^p e^{pi\theta}. \tag{3.13}$$

Taking real parts in (3.13) gives

$$1 = 3z_s^q\cos(q\theta) + 2z_s^p\cos(p\theta) \leqq 3z_s^q + 2z_s^p = 1, \tag{3.14}$$

and hence $\cos(q\theta) = \cos(p\theta) = 1$. So $q\theta = 2m\pi$, $p\theta = 2n\pi$, for integers m, n. Then $0 < m < q$, $0 < n < p$, and $p/q = n/m$. We conclude $(p, q) > 1$, contradiction. Hence $|z_0| > z_s$. Inverting (3.10) gives

$$C_j = \frac{2}{\pi i}\int_\gamma dz\, z^{-1-j}\{f_{p,q}(z)\}^{-1}, \tag{3.15}$$

where γ is the contour parameterized by $\gamma(\theta) = z_s e^{i\theta}/2$, $0 \leqq \theta < 2\pi$. It follows that

$$C_j = -4\sum \text{residue}\,\{f_{p,q}^{-1}(z)z^{-1-j}\}, \tag{3.16}$$

where the sum is taken over all zeros of f. The main contribution comes from the residue at z_s. This proves (3.8) for some constants $j_s > 0$ and $0 < J_s < \infty$.

Lemma 3.2 *For $\alpha \in (0, \frac{1}{2})$ there exists a constant $C(\alpha) \in (0, \infty)$ such that for all $t > 0$*

$$E(t) \leqq C(\alpha)t^\alpha, \tag{3.17}$$

$$E(t) \leqq 16(2\pi t)^{1/2}. \tag{3.18}$$

Proof. For $x \geqq 0$, $1 - e^{-x} \leqq x^\alpha$. Hence by (1.32) we conclude (3.17) with

$$C(\alpha) = 16\pi^{2\alpha-4}\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}\frac{\{(2k+1)^2 + (2l+1)^2\}^\alpha}{(2k+1)^2(2l+1)^2}. \tag{3.19}$$

The double sum in (3.19) converges for $0 < \alpha < \frac{1}{2}$. Moreover for the unit square in $\mathbb{R}^2$, $\mu(\varepsilon) \leqq 4\varepsilon$ for all $\varepsilon < 0$. Estimate (3.18) follows by Theorem 1.2. To prove Theorem 1.7(i) we choose

$$\alpha_1 = \min\{1 - d_s/2 + j_s/4, 1/2\}. \tag{3.20}$$

By Lemma 3.2 and (3.5)

$$\sum_{j=0}^{\infty} s^{(j_s-d_s)j/q}a_j^2 E(t/a_j^2) \leqq C_1(\alpha_1)s^{2-2\alpha_1}t^{\alpha_1}\sum_{j=0}^{\infty} s^{j\{2-2\alpha_1+j_s-d_s\}/q}, \tag{3.21}$$

where $C_1(\alpha_1)$ is given by

$$C_1(\alpha_1) = \begin{cases} C(\alpha_1), & \alpha_1 < \frac{1}{2}, \\ 16(2\pi)^{1/2} & \alpha_1 = \frac{1}{2}. \end{cases} \tag{3.22}$$

The sum in the right hand side of (3.21) converges by (3.20). The following estimate is the key for the proof of the periodicity in $\log t$. See also [14, Section II.B]. By (3.5) and (3.18)

$$\sum_{j=-\infty}^{-1} s^{-d_s j/q}a_j^2 E(t/a_j^2) \leqq 16s(2\pi t)^{1/2}\sum_{j=1}^{\infty} s^{j(d_s-1)/q}. \tag{3.23}$$

The sum in (3.23) converges since $d_s > 1$. By (3.6), (3.8), (3.21), (3.23) we conclude

$$E_{F_s}(t) = 4\{3qs^{d_s} + 2p((1-s)/2)^{d_s}\}^{-1} \sum_{j\in\mathbb{Z}} s^{(2-d_s)j/q+2} E(ts^{-2-2j/q}) + O(t^{\alpha_1})$$
$$= 4s^{d_s}\{3qs^{d_s} + 2p((1-s)/2)^{d_s}\}^{-1} t^{(2-d_s)/2} \cdot \sum_{j\in\mathbb{Z}} s^{(2-d_s)(y+j)/q} E(s^{-2(y+j)/q}) + O(t^{\alpha_1}), \tag{3.24}$$

where y is given by (1.33).

The proof of Theorem 1.7(ii) relies on Ikehara's theorem which we state here without proof (see [19, pp. 127–130]).

Ikehara's theorem. *Let $e:[0,\infty) \to [0,\infty)$ be a continuous, monotone increasing function and let*

$$\hat{e}(y) = \int_0^\infty t^{y-1} e(t)\, dt \tag{3.25}$$

converge for $y_1 < \operatorname{Re} y < y_2$. Let $\hat{e}(y) - A(y-y_1)^{-1}$ converge uniformly over compact sets of the line $\operatorname{Re} y = y_1$ to a finite limit as $\operatorname{Re} y \to y_1^+$. Then for $t \to 0^+$

$$e(t) = At^{-y_1} + o(t^{-y_1}). \tag{3.26}$$

Proof of Theorem 1.7(ii). Let $\hat{E}_{F_s}(y)$ denote the Mellin transform of $E_{F_s}(t)$. Then by (3.2)

$$\hat{E}_{F_s}(y) = \int_0^\infty t^{y-1} E_{F_s}(t)\, dt$$
$$= \sum_{n=0}^\infty \sum_{m=0}^\infty N_{m,n} a_{m,n}^2 \int_0^\infty t^{y-1} E(t/a_{m,n}^2)\, dt$$
$$= \sum_{n=0}^\infty \sum_{m=0}^\infty N_{m,n} (a_{m,n})^{2y+2} \int_0^\infty u^{y-1} E(u)\, du$$
$$= 4s^{2y+2}\{1 - 3s^{2y+2} - 2((1-s)/2)^{2y+2}\}^{-1} \int_0^\infty u^{y-1} E(u)\, du, \tag{3.27}$$

for $(d_s - 2)/2 < \operatorname{Re} y < 0$. Let $s \in (0,(1+\sqrt{8})/7)\backslash I$. Then $\{1 - 3s^{2y+2} - 2((1-s)/2\}^{-1}$ has one (simple) pole at $y_1 = (d_s-2)/2$ on the line $\operatorname{Re} y = y_1$. Let

$$A_s = 2s^{d_s}\left\{3s^{d_s}\log\frac{1}{s} + 2((1-s)/2)^{d_s}\log\left(\frac{2}{1-s}\right)\right\}^{-1} \int_0^\infty u^{(d_s-4)/2} E(u)\, du. \tag{3.28}$$

Let K be a compact subset of the line $\operatorname{Re} y = y_1$. The function $y \to \int_0^\infty u^{y-1} E(u)\, du$ is analytic in the strip $-1 < \operatorname{Re} y < 0$ and hence does not have any singularities in a compact neighbourhood of K. The function $y \to \{1 - 3s^{2y+2} - 2((1-s)/2)^{2y+2}\}$ is analytic on $\mathbb{C}$ and hence its zero set does not have limit points in $\mathbb{C}$. So there exists a compact neighbourhood N of K on which $\hat{E}_{F_s}(y) - A_s(y-y_1)^{-1}$ is continuous. This implies the uniform convergence of $\hat{E}_{F_s}(y) - A_s(y - y_1)^{-1}$ to a finite limit as $\operatorname{Re} y \to y_1^+$. Theorem 1.7(ii) follows by Ikehara's theorem.

Note that for $s \in I$ there are simple poles at $y_k = (d_s - 2)/2 + kq\pi i$, $k \in \mathbb{Z}$, which gives rise to the periodic behaviour obtained in (i). The remaining poles have real part strictly less than $(d_s - 2)/2$.

4 Heat content asymptotics for a simply connected set

In this section we construct for $0 < s \leqq \frac{1}{3}$, an open, bounded, simply connected set G_s satisfying (1.31). However, first we construct an open, bounded set H_s in $\mathbb{R}^2$ with a fractal boundary ∂H_s. The set H_s consists of the squares of the set F_s patched together with one additional square of sidelength 1. The construction is as follows: Let Q_0 be the open square in $\mathbb{R}^2$ with sidelength 1, centre $(0, 0)$ and its boundary parallel to the x, y axes respectively. We attach four open squares $Q_1, \ldots, Q_4$ with sidelength s, onto the middles of the four sides of Q_0. We call the boundary of $\overline{Q_0 \cup \ldots \cup Q_4}$, the "outer boundary" of generation one. This "outer boundary" is polygonal and consists of 12 line segments of length s and 8 line segments of length $(1 - s)/2$. We attach 12 open squares of length s^2 onto the middles of the 12 line segments of length s, and 8 open squares of length $s(1 - s)/2$ onto the middles of the 8 line segments of length $(1 - s)/2$. The outer boundary of generation two consists of 100 line segments. To each of these line segments we attach an open square of sidelength s times the length of the line segment to which it is attached. Let H_s be the induction limit of this process. We conclude that H_s consists of the square Q_0 together with the $N_{m,n}$ squares of sidelength $a_{m,n}$, $m \in \mathbb{Z}_0^+$, $n \in \mathbb{Z}_0^+$, and hence

$$E_{H_s}(t) = E(t) + E_{F_s}(t). \tag{4.1}$$

The condition $0 < s \leqq \frac{1}{3}$ is necessary and sufficient to guarantee that the open squares in the set H_s are disjoint.

Next we construct G_s. Consider any square Q in H_s different from Q_0. Then Q has precisely one edge which is part of the boundary of a larger square in H_s. Let $a_{m,n}$ be the length of this edge. We enlarge H_s by adding in the middle of this edge a relatively open interval $I_{m,n}$ of length $\varepsilon_{m,n} a_{m,n}$, where $0 < \varepsilon_{m,n} < 1$. We do this for all squares in H_s except for Q_0. Let

$$G_s = H_s \cup \{\bigcup I_{m,n}: m \in \mathbb{Z}_0^+, n \in \mathbb{Z}_0^+\}. \tag{4.2}$$

The set G_s is an open, simply connected, bounded subset of $\mathbb{R}^2$, with $|G_s|_2 = 1 + |F_s|_2$. The Minkowski dimension of G_s is equal to its Hausdorff dimension, and is equal to d_s. Moreover,

$$|\partial H_s \backslash \partial G_s|_1 = 8 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \binom{m+n}{n} 3^n 2^m \varepsilon_{m,n} a_{m,n}. \tag{4.3}$$

The main result of this section is the following.

Theorem 4.1 *Let* $a > 5$, $0 < s \leqq \frac{1}{3}$, *and*

$$\varepsilon_{m,n} = e^{-(a^{m+n})}, \quad m \in \mathbb{Z}_0^+, \quad n \in \mathbb{Z}_0^+, \tag{4.4}$$

then there exists a constant $0 < \beta < \infty$ *such that for* $0 \leqq t \leqq 1$

$$\beta t^{1/2} \geqq E_{F_s}(t) - E_{G_s}(t) \geqq 0, \tag{4.5}$$

so that

$$\lim_{t \to 0^+} E_{F_s}(t)/E_{G_s}(t) = 1. \tag{4.6}$$

Proof. By Lemma 3.2 $E(t) \leqq 16(2\pi t)^{1/2}$. So it is sufficient to prove that for $0 \leqq t \leqq 1$

$$\beta t^{1/2} \geqq E_{H_s}(t) - E_{G_s}(t) \geqq 0. \tag{4.7}$$

The right hand side of (4.7) is trivial since $\partial G_s \subset \partial H_s$ and $H_s \subset G_s$. Let T, and τ be as in Section 2. Then

$$\begin{aligned} E_{H_s}(t) &= \int_{H_s} \mathbb{P}_x[T_{H_s} \leqq t]\, dx \\ &\leqq \int_{G_s} \{\mathbb{P}_x[T_{G_s} \leqq t] + \mathbb{P}_x[T_{G_s} > t, T_{H_s} \leqq t]\}\, dx \\ &= E_{G_s}(t) + \int_{G_s} \mathbb{P}_x[T_{G_s} > t, T_{H_s} \leqq t]\, dx \\ &\leqq E_{G_s}(t) + \int_{G_s} \mathbb{P}_x[\tau_{\bigcup I_{m,n}} \leqq t]\, dx \\ &\leqq E_{G_s}(t) + 4 \int_{G_s} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \binom{m+n}{n} 3^n 2^m \mathbb{P}_x[\tau_{\bar{I}_{m,n}} \leqq t]\, dx \\ &\leqq E_{G_s}(t) + 4 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \binom{m+n}{n} 3^n 2^m \int_{\mathbb{R}^2} \mathbb{P}_x[\tau_{\bar{I}_{m,n}} \leqq t]\, dx \end{aligned} \tag{4.8}$$

Note that

$$\int_{\mathbb{R}^2} \mathbb{P}_x[\tau_{\bar{I}_{m,n}} \leqq t]\, dx$$

is precisely the expected volume of the Wiener sausage associated to the compact set $\bar{I}_{m,n}$ up to time t. Let I be a closed line segment of length 1. Then by (1.6) in [8]

$$\int_{\mathbb{R}^2} \mathbb{P}_x[\tau_I \leqq t]\, dx = \frac{4t^{1/2}}{\pi^{1/2}}(1 + o(1)), \quad t \to 0^+. \tag{4.9}$$

Hence there exists a constant $k_1 < \infty$ such that

$$\int_{\mathbb{R}^2} \mathbb{P}_x[\tau_I \leqq t]\, dx \leqq k_1 t^{1/2}, \quad 0 \leqq t \leqq 1. \tag{4.10}$$

Furthermore, since I has positive logarithmic capacity we have by Theorem 2 in [18] for $t \to \infty$

$$\int_{\mathbb{R}^2} \mathbb{P}_x[\tau_I \leqq t]\, dx = \frac{4\pi t}{\log t}(1 + o(1)). \tag{4.11}$$

Hence there exists a constant $k_2 < \infty$ such that

$$\int_{\mathbb{R}^2} \mathbb{P}_x[\tau_I \leqq t]\,dx \leqq \frac{k_2 t}{\log(1+t)}, \quad t \geqq 1. \tag{4.12}$$

Finally we recall the scaling property. Let I_ε be a line segment of length ε. Then for all $t > 0$

$$\int_{\mathbb{R}^2} \mathbb{P}_x[\tau_{I_\varepsilon} \leqq t]\,dx = \varepsilon^2 \int_{\mathbb{R}^2} \mathbb{P}_x[\tau_I \leqq t/\varepsilon^2]\,dx. \tag{4.13}$$

So by (4.10), (4.12) and (4.13)

$$\int_{\mathbb{R}^2} \mathbb{P}_x[\tau_{\bar{I}_{m,n}} \leqq t]\,dx \leqq \begin{cases} k_1 b_{m,n} t^{1/2}, & t \leqq b_{m,n}^2 \\ k_2 t\{\log(1+t/b_{m,n}^2)\}^{-1}, & t > b_{m,n}^2, \end{cases} \tag{4.14}$$

where $b_{m,n} = \varepsilon_{m,n} a_{m,n}$. We have the following estimates for $0 \leqq t < 1$:

$$\begin{aligned} &\sum_{\{m,n:\, b_{m,n} \geqq \sqrt{t}\}} \binom{m+n}{n} 3^n 2^m \int_{\mathbb{R}^2} dx \mathbb{P}_x[\tau_{\bar{I}_{m,n}} \leqq t] \\ &\leqq \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \binom{m+n}{n} 3^n 2^m k_1 b_{m,n} t^{1/2} \\ &\leqq k_1 t^{1/2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \binom{m+n}{n} 3^n 2^m a^{-(m+n)} = \frac{a}{a-5} k_1 t^{1/2}, \end{aligned} \tag{4.15}$$

$$\begin{aligned} &\sum_{\{m,n:\, b_{m,n} \leqq t\}} \binom{m+n}{n} 3^n 2^m \int_{\mathbb{R}^2} dx \mathbb{P}_x[\tau_{\bar{I}_{m,n}} \leqq t] \\ &\leqq \sum_{\{m,n:\, b_{m,n} \leqq t\}} \binom{m+n}{n} 3^n 2^m k_2 t\{\log(1+tb_{m,n}^{-2})\}^{-1} \\ &\leqq \sum_{\{m,n:\, b_{m,n} \leqq t\}} \binom{m+n}{n} 3^n 2^m k_2 t\{\log(1+b_{m,n}^{-1})\}^{-1} \\ &\leqq \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \binom{m+n}{n} 3^n 2^m k_2 t \left\{\log\left(\frac{1}{\varepsilon_{m,n}}\right)\right\}^{-1} \\ &\leqq \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \binom{m+n}{n} 3^n 2^m k_2 t a^{-(m+n)} \\ &= \frac{a}{a-5} k_2 t \leqq \frac{a}{a-5} k_2 t^{1/2}, \end{aligned} \tag{4.16}$$

and finally

$$\begin{aligned} &\sum_{\{m,n:\, t < b_{m,n} < t^{1/2}\}} \binom{m+n}{n} 3^n 2^m \int_{\mathbb{R}^2} dx \mathbb{P}_x[\tau_{\bar{I}_{m,n}} \leqq t] \\ &\leqq \sum_{\{m,n:\, t < b_{m,n} < t^{1/2}\}} \binom{m+n}{n} 3^n 2^m k_2 t\{\log(1+tb_{m,n}^{-2})\}^{-1} \end{aligned}$$

$$\leqq \sum_{\{m,n:\, t<b_{m,n}<t^{1/2}\}} \binom{m+n}{n} 3^n 2^m k_2 t/(\log 2)$$

$$\leqq \sum_{\{m,n:\, t<b_{m,n}\}} \binom{m+n}{n} 3^n 2^m k_2 t/(\log 2)$$

$$\leqq \sum_{\{m,n:\, t<\varepsilon_{m,n}\}} \binom{m+n}{n} 3^n 2^m k_2 t/(\log 2)$$

$$\leqq \sum_{\{m,n:\, a^{m+n}<\log\frac{1}{t}\}} \binom{m+n}{n} 3^n 2^m k_2 t/(\log 2)$$

$$\leqq \sum_{\{m,n:\, a^{m+n}<\log\frac{1}{t}\}} \binom{m+n}{n} 3^n 2^m \cdot \frac{t\log\frac{1}{t}}{a^{m+n}} \cdot \frac{k_2}{\log 2}$$

$$\leqq \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \binom{m+n}{n} \left(\frac{3}{a}\right)^n \left(\frac{2}{a}\right)^m \frac{k_2 t \log\frac{1}{t}}{\log 2} \leqq \frac{a}{a-5}(k_2 t^{1/2})/\log 2, \quad (4.17)$$

which proves (4.7) by (4.8), (4.15–4.17).

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