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Probability Theory and Related Fields

Some remarks on a question of Strassen

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Summary. Strassen's original functional law of the iterated logarithm for partial sums and Brownian motion examined convergence and clustering in the sup-norm. Here we address what happens if we use the much larger H-norm. We provide the answer to a query which appeared at the end of Strassen's original paper, and also present several contrasting results which are shown to be essentially best possible.

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1 Introduction

At the very end of his seminal paper on the law of the iterated logarithm [S], Strassen mentions a couple of things he would find interesting to know. One of these seems to have gone completely unnoticed, and our Theorem 1 provides the answer for this question. We also present some contrasting results motivated by more recent work.

Throughout $X, X_1, X_2, \ldots$ denote i.i.d. random variables with $E(X) = 0$, $E(X^2) = 1$, $S_0 = 0$ and $S_k = X_1 + \ldots + X_k$ for $k \geqq 1$. If $\{a_n : n \geqq 0\}$ is a strictly increasing sequence of integers with $a_0 = 0$, we define for $n \geqq 1$ the processes

$$(1.1) \qquad \eta_n(t) = \begin{cases} S_{a_k}/(2a_n L_2 a_n)^{1/2} & t = a_k/a_n,\ k = 0, 1, \ldots, n\,, \\ \text{linearly interpolated elsewhere on } [0,1]\,. \end{cases}$$

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In (1.1) and elsewhere $L_2x = L(Lx)$ with $Lx = \max(1, \log_e x)$. Of course, $\{\eta_n\}$ depends on $\{a_n\colon n \geqq 0\}$, but we suppress that to simplify notation. Let

$$(1.2) \qquad H = \left\{ f(t) = \int_0^t g(s)\,ds\colon 0 \leqq t \leqq 1,\ \int_0^1 |g(s)|^2\,ds < \infty \right\},$$

with norm given by the inner product

$$(1.3) \qquad \langle f_1, f_2 \rangle_H = \int_0^1 f_1'(s) f_2'(s)\,ds\,.$$

Then H is a Hilbert space with unit ball

$$(1.4) \qquad K = \left\{ f(t) = \int_0^t f'(s)\,ds\colon 0 \leqq t \leqq 1,\ \int_0^1 |f'(s)|^2\,ds \leqq 1 \right\},$$

and when $a_n = n$, Strassen's fundamental result is that $\{\eta_n\}$ converges to K and clusters throughout K in the sup-norm topology with probability one. It is an easy calculation when $a_n = n$ to show that

$$(1.5) \qquad \lim_n \langle \eta_n, \eta_n \rangle_H = \infty \quad \text{w.p.1}\,,$$

and Strassen questioned what the situation might be for other strictly increasing sequences. More precisely, if $\{a_n\colon n \geqq 0\}$ is a strictly increasing sequence of integers with $a_0 = 0$ and $\lim_n a_{n+1}/a_n = 1$, Strassen pointed out that it would be interesting to know for which sequences does $\{\eta_n\}$ cluster throughout K in the H-inner product norm.

The answer is given in Theorem 1 below, and Theorem 2 is motivated by Theorem 1 and some recent work in [GK, G91 and KLT]. Theorem 3 clarifies the assumptions used in Theorem 2, while Theorem 4 provides a contrasting result to Theorem 1 in a related situation.

If $\{f_n\}$ is a sequence of functions, we let $C(\{f_n\})_U$ denote all subsequential limits of $\{f_n\}$ in the sup-norm, and $C(\{f_n\})_H$ is the corresponding cluster set for the H-norm. We write $\{f_n\} \overset{H}{\twoheadrightarrow} K$ if both

$$(1.6) \qquad C(\{f_n\})_H = K$$

and

$$(1.7) \qquad \overline{\lim_n} \inf_{h \in K} \|f_n - h\|_H = 0\,.$$

Thus $\{f_n\} \overset{H}{\twoheadrightarrow} K$ denotes convergence to K and throughout K by the sequence $\{f_n\}$ when distances are computed in the H-norm. If $\{f_n\} \overset{U}{\twoheadrightarrow} K$ denotes the analogue for the sup-norm, then Strassen's result can be expressed as

$$(1.8) \qquad P(\{\eta_n\} \overset{U}{\twoheadrightarrow} K) = 1\,,$$

where in (1.8) we are assuming $a_n = n$ and $\{\eta_n\}$ is as in (1.1)

Theorem 1. *Let $\{a_n: n \geqq 0\}$ be a strictly increasing sequence of integers with $a_0 = 0$, and assume $\{\eta_n\}$ is defined as in* (1.1). *Then,*

$$P(C(\{\eta_n\})_H = \phi) = 1 \tag{1.9}$$

or

$$P(C(\{\eta_n\})_H = \{0\}) = 1 , \tag{1.10}$$

and (1.10) *holds iff*

$$\overline{\lim_n}(L_2 a_n)/n = \infty . \tag{1.11}$$

Remark. (I) If $\overline{\lim}_n a_{n+1}/a_n = \rho < \infty$, then $\rho \geqq 1$ and eventually $a_n \leqq (2\rho)^n$. Hence (1.11) fails for such $\{a_n\}$, and (1.9) must hold. Thus in the setting of Strassen's question we always have

$$P(C(\{\eta_n\})_H = \phi) = 1 .$$

Of course, we never have $P(C(\{\eta_n\})_H = K) = 1$, regardless of the sequence $\{a_n\}$.

(II) Lemma 2 below shows that if $\{a_n\}$ is such that $L_2 a_n \leqq n^\beta$ for some $\beta \in (0, 1)$, then (1.5) holds, and hence we do not have (1.7) for the related $\{\eta_n\}$.

2 Proof of Theorem 1

First we establish several lemmas. Lemma 1 is an elementary observation to be used later. Its proof follows almost immediately from the central limit theorem, and hence will not be included.

Lemma 1. *If $X, X_1, X_2, \ldots$ are i.i.d. with $E(X) = 0$, $E(X^2) = 1$, and $S_k = X_1 + \ldots + X_k$ for $k \geqq 1$, then*

$$\lim_{\lambda\to\infty} \inf_{k\geqq 1} E((S_k/\sqrt{k})^2 I(|S_k/\sqrt{k}| \leqq \lambda)) = 1 , \tag{2.1}$$

and

$$\inf_{k\geqq 1} P(|S_k/\sqrt{k}| \geqq 1) = c > 0 . \tag{2.2}$$

Lemma 2. *Let $\{a_n: n \geqq 0\}$ be as in Theorem* 1, *and*

$$I = \{n \geqq 1: 2L_2 a_n \leqq n^\beta\} \tag{2.3}$$

where $0 < \beta < 1$. If card $(I) = \infty$, *then with probability one*

$$\lim_{n\in I} \langle \eta_n, \eta_n\rangle_H = \infty , \tag{2.4}$$

and hence

$$P(C(\{\eta_n: n \in I\})_H = \phi) = 1 . \tag{2.5}$$

Proof. Let $N_i = (S_{a_i} - S_{a_{i-1}})/(a_i - a_{i-1})^{1/2}$ for $i \geqq 1$. Then $N_1, N_2, \ldots$ are independent with $E(N_i) = 0$, $E(N_i^2) = 1$, and

$$\langle \eta_n, \eta_n\rangle_H = (2L_2 a_n)^{-1} \sum_{i=1}^n N_i^2 . \tag{2.6}$$

Since $a_n \uparrow \infty$ and $\mathrm{card}(I) = \infty$, Kolmogorov's zero–one law implies that

$$(2.7) \qquad \lim_{n \in I} \langle \eta_n, \eta_n \rangle_H = \infty \quad \text{w.p.1} ,$$

or

$$(2.8) \qquad \underline{\lim}_{n \in I} \langle \eta_n, \eta_n \rangle_H < M , \quad \text{w.p.1}$$

where M is a finite constant. Hence if (2.8) holds we have

$$P\left(\sum_{i=1}^{n} N_i^2 < M(2L_2 a_n) \quad \text{i.o. in } n \in I \right) = 1 ,$$

and by the Borel–Cantelli lemma this implies

$$(2.9) \qquad \theta = \sum_{n \in I} P\left(\sum_{i=1}^{n} N_i^2 < M(2L_2 a_n) \right) = \infty .$$

Now let

$$(2.10) \qquad p_n = P\left(\sum_{i=1}^{n} N_i^2 < Mn^{\beta} \right) .$$

Then the definitions of I and θ imply that

$$(2.11) \qquad \sum_{n \in I} p_n = \infty .$$

Furthermore, for $\lambda_n \geqq 0$

$$(2.12) \qquad p_n \leqq E\left(\exp\left\{ -\lambda_n \sum_{i=1}^{n} N_i^2 + \lambda_n M n^{\beta} \right\} \right) = e^{\lambda_n M n^{\beta}} \prod_{i=1}^{n} E(e^{-\lambda_n N_i^2}) .$$

Since $e^{-x} \leqq 1 - x/2$ if $0 \leqq x \leqq 1$, and $e^{-x} \leqq 1$ for $x \geqq 0$, we have

$$(2.13) \qquad E(e^{-\lambda_n N_i^2}) \leqq E((1 - \lambda_n N_i^2/2) I(\lambda_n N_i^2 \leqq 1) + I(\lambda_n N_i^2 > 1)) = 1 - \frac{\lambda_n}{2} E(N_i^2 I(\lambda_n N_i^2 \leqq 1)) .$$

If $\lambda_n \to 0$, $\lambda_n \geqq 0$, then (2.1) implies that for all $n \geqq n_0$

$$(2.14) \qquad \inf_{i \geqq 1} E(N_i^2 I(\lambda_n N_i^2 \leqq 1)) \geqq \tfrac{1}{2} .$$

Thus for $n \geqq n_0$ (2.12)–(2.14) combine to show

$$p_n \leqq e^{\lambda_n M n^{\beta}} \prod_{i=1}^{n} (1 - \lambda_n/4) \leqq e^{\lambda_n M n^{\beta}} e^{-n\lambda_n/4}$$

since $1 - x \leqq e^{-x}$ for $x \geqq 0$. Taking $\lambda_n = n^{-\beta}$ we get for all $n \geqq n_0$ that

$$p_n \leqq \exp\left\{ M - \tfrac{1}{4} n^{1-\beta} \right\} ,$$

so (2.11) fails as $0 < \beta < 1$. Thus (2.4) holds w.p.1.

If $f \in C(\{\eta_n: n \in I\})_H$, then $\underline{\lim}_{n\in I}\|\eta_n - f\|_H = 0$ and hence $\underline{\lim}_{n\in I}$ $\langle\eta_n, \eta_n\rangle_H < \infty$. Thus (2.4) holding w.p.1 implies (2.5), and the lemma is proven. □

Lemma 3. *Let* $\{a_n: n \geqq 0\}$ *be as in Theorem* 1 *and*

$$J = \{n \geqq 1: 2L_2a_n > n^\beta\}, \tag{2.15}$$

where $0 < \beta < 1$. *If* $\mathrm{card}(J) = \infty$ *and* $\|\cdot\|_\infty$ *denotes the sup-norm, then*

$$\lim_{n\in J}\|\eta_n\|_\infty = 0 \quad \text{w.p.1}, \tag{2.16}$$

and hence

$$P(C(\{\eta_n: n \in J\})_U = \{0\}) = 1. \tag{2.17}$$

Proof. Let $Y_1, Y_2, \ldots$ be independent centered Gaussian random variables with

$$E(Y_j^2) = \sigma_j^2 \quad 2^r \leqq j \leqq 2^{r+1},$$

where

$$\sigma_j^2 = E(X^2I(X^2 \leqq 2^r)) - (E(XI(X^2 \leqq 2^r)))^2 \leqq 1$$

for $r = 0, 1, 2, \ldots$. Hence with $T_0 = 0$ and $T_k = \sum_{j=1}^k Y_j$ for $k \geqq 1$, we define the polygonal processes

$$\theta_n(t) = \begin{cases} T_{a_k}/(2a_nL_2a_n)^{1/2} & t = a_k/a_n,\ k = 0, 1, \ldots, n, \\ \text{linearly interpolated elsewhere on } [0, 1]. \end{cases} \tag{2.18}$$

Then by [M] there is a probability space on which we can define copies of $\{X_j: j \geqq 1\}$ and $\{Y_j: j \geqq 1\}$ such that

$$\|\theta_n - \eta_n\|_\infty = o((L_2a_n)^{-1/2}) \quad \text{w.p.1}. \tag{2.19}$$

Hence (2.16) will follow if we show that

$$\lim_{n\in J}\|\theta_n\|_\infty = 0 \quad \text{w.p.1}. \tag{2.20}$$

Using Levy's inequality we have

$$P(\|\theta_n\|_\infty > \varepsilon) \leqq 2P(|T_{a_n}| > (2a_nL_2a_n)^{1/2}\varepsilon),$$

and since T_{a_n} is centered Gaussian with $E(T_{a_n}^2) \leqq a_n$ we see for $G \overset{d}{=} N(0, 1)$ that

$$P(\|\theta_n\|_\infty > \varepsilon) \leqq 2P(|G| > \varepsilon(2L_2a_n)^{1/2}) \leqq 2\exp\{-\varepsilon^2n^\beta/2\}$$

for $n \in J$. Hence

$$\sum_{n\in J}P(\|\theta_n\|_\infty > \varepsilon) < \infty$$

for any $\varepsilon > 0$, and thus (2.20) holds with probability one by a standard application of the Borel–Cantelli lemma. Now (2.17) follows immediately, so Lemma 3 is proven. □

Proof of Theorem 1. Let A be any infinite subset of the positive integers. Since the H-norm is larger than the sup-norm, we have pointwise (and hence w.p.1) that

$$C(\{\eta_n: n \in A\})_H \subset C(\{\eta_n: n \in A\})_U . \tag{2.21}$$

Let $\{a_n: n \geqq 0\}$ be given. Fix $0 < \beta < 1$ and define I and J as in Lemmas 2 and 3, respectively. By Lemma 2

$$C(\{\eta_n: n \geqq 1\})_H = C(\{\eta_n: n \in J\})_H , \tag{2.22}$$

while by Lemma 3

$$C(\{\eta_n: n \in J\})_U \subseteq \{0\} \tag{2.23}$$

with equality in (2.23) iff card $(J) = \infty$. Combining (2.21)–(2.23) shows that precisely one of (1.9) or (1.10) must hold.

Now $0 \in C(\{\eta_n\})_H$ w.p.1. iff $\underline{\lim}_n \langle \eta_n, \eta_n \rangle_H = 0$ w.p.1. Applying Fatou's lemma we have

$$\underline{\lim}_n E(\langle \eta_n, \eta_n \rangle_H) \geqq E\left(\underline{\lim}_n \langle \eta_n, \eta_n \rangle_H\right) ,$$

and since $E(\langle \eta_n, \eta_n \rangle_H) = (n/(2L_2 a_n))$ we have

$$\underline{\lim} \langle \eta_n, \eta_n \rangle_H = 0 \quad \text{w.p.1} \tag{2.24}$$

whenever (1.11) holds. Since (1.9) and (1.10) must hold, it follows that (1.11) implies (1.10). It remains to be shown that the converse implication holds.

If (1.10) holds, then

$$\underline{\lim}_n \sum_{i=1}^{n} N_i^2/(2L_2 a_n) = 0 \quad \text{w.p.1} , \tag{2.25}$$

and hence if (1.11) fails we have

$$\underline{\lim}_n \sum_{i=1}^{n} N_i^2/n = 0 \quad \text{w.p.1} . \tag{2.26}$$

Now $N_i^2 \geqq I(N_i^2 \geqq 1)$, so if (2.26) holds we have

$$\underline{\lim}_n \sum_{i=1}^{n} I(N_i^2 \geqq 1)/n = 0 \quad \text{w.p.1} . \tag{2.27}$$

Let $V_n = \sum_{i=1}^n I(N_i^2 \geqq 1)$ for $n \geqq 1$ and fix $\varepsilon > 0$. Then independence of $N_1, N_2, \ldots$ and Chebyshev's inequality implies

(2.28)
$$\sum_{r \geqq 1} P(|V_{2^r} - E(V_{2^r})| > \varepsilon 2^r) \leqq \sum_{r \geqq 1} (\varepsilon 2^r)^{-2} \mathrm{Var}(V_{2^r})$$
$$= \sum_{r \geqq 1} (\varepsilon 2^r)^{-2} \sum_{i \geqq 1}^{2^r} \mathrm{Var}(I(N_i^2 \geqq 1)) \leqq \varepsilon^{-2}$$

since $\mathrm{Var}(I(N_i^2 \geqq 1)) \leqq 1$. Hence

$$\lim_r |V_{2^r} - E(V_{2^r})|/2^r = 0 \quad \text{w.p.1} , \tag{2.29}$$

and for all $r \geqq 1$

$$E(V_{2^r}/2^r) = \sum_{i=1}^{2^r} P(N_i^2 \geqq 1)/2^r \geqq c > 0 \tag{2.30}$$

where $c > 0$ is given as in Lemma 1. Combining (2.29) and (2.30) we have that

$$\varliminf_r V_{2^r}/2^r \geqq c > 0 \quad \text{w.p.1}\,.$$

If $2^r \leqq j \leqq 2^{r+1}$, then

$$V_j/j \geqq V_{2^r}/2^{r+1} \geqq 2^{-1}V_{2^r}/2^r\,,$$

and hence

$$\varliminf_j V_j/j \geqq c/2 \quad \text{w.p.1}\,. \tag{2.31}$$

Now (2.31) contradicts (2.27), and hence (1.11) must hold. Thus Theorem 1 is proven. □

3 Some contrasting results

If $\varliminf_n a_{n+1}/a_n > 1$, then it is fairly immediate that $C(\{\eta_n\})_U$ is a strict subset of K, and since $C(\{\eta_n\})_H \subseteq C(\{\eta_n\})_U$, this is why Strassen assumed $\lim_n a_{n+1}/a_n = 1$. Of course, the remark following Theorem 1 implies that in this case we always have $C(\{\eta_n\})_H = \phi$ w.p.1, but we will see from our next theorem that this can be drastically changed if we perturb $\{\eta_n\}$ ever so slightly. We only consider the case $a_n = n$ in Theorem 2, as we apply the intricate results in [deA] and [EG]. When $\lim a_{n+1}/a_n = 1$, one still has $\{\eta_n\} \overset{U}{\twoheadrightarrow} K$ w.p.1, but the rates at which this takes place have not been determined. They undoubtedly depend on the sequence $\{a_n\}$, so not to be taken too far afield, we restrict our attention to $a_n = n$ in the remainder of the paper. Now we need some additional notation.

Let $C_0[0,1]$ denote the continuous functions on $[0,1]$ which are zero at zero, and for $g \in C_0[0,1]$ and $\varepsilon > 0$ let

$$I(g,\varepsilon) = \inf_{\|g-h\|_\infty \leqq \varepsilon} \langle h,h\rangle_H\,. \tag{3.1}$$

We naturally assume $\langle h,h\rangle_H = \infty$ for $h \notin H$ and, of course,

$$I(g,0) = \langle g,g\rangle_H\,,$$

which is finite iff $g \in H$. Furthermore, H is dense in $C_0[0,1]$ so if $\varepsilon > 0$, then $I(g,\varepsilon) < \infty$ for all $g \in C_0[0,1]$, and by Lemma 1 in [G91] we know there is a unique function $h \in H$ such that $\|g-h\|_\infty \leqq \varepsilon$ and

$$I(g,\varepsilon) = \langle h,h\rangle_H\,. \tag{3.2}$$

We will denote this functional relationship by writing

$$h = (g)^\varepsilon\,. \tag{3.3}$$

Theorem 2. *Let $a_n = n$ for $n \geqq 0$ and assume $\{\eta_n\}$ is given as in* (1.1). *Let $\{\varepsilon_n\}$ be a positive sequence such that $\varepsilon_n = d/(L_2 n)$ where $d > 0$, and for $n \geqq 1$ let*

$$f_n = (\eta_n)^{\varepsilon_n}, \tag{3.4}$$

where $(\eta_n)^{\varepsilon_n}$ is given as in (3.2) *and* (3.3). *Then, for $0 \leqq \rho < 1$, with probability one*

$$C(\{f_n\})_H \supseteq \rho K \tag{3.5}$$

provided $d > \frac{\pi}{4}(1-\rho^2)^{-1/2}$. In particular, if we replace the constant d in $\varepsilon_n = d/(L_2 n)$ by d_n where $\{d_n\}$ is such that

$$\lim_n \varepsilon_n = 0 \quad \textit{and} \quad \lim_n (L_2 n)\varepsilon_n = \infty, \tag{3.6}$$

then with probability one

$$C(\{f_n\})_H = K. \tag{3.7}$$

Furthermore, if (3.6) *holds and*

$$E(X^2(L_2|X|)) < \infty, \tag{3.8}$$

then we also have

$$\lim_n \inf_{h \in K} \|f_n - h\|_H = 0 \quad \text{w.p.1}. \tag{3.9}$$

Remarks. (I) Theorem 2 follows rather easily once one has the intricate results in [deA] and [EG], but it is somewhat surprising in view of Theorem 1. Of course, Theorem 2 was the result of trying to improve the lack of convergence and clustering in the H-norm established in Theorem 1. The integrability condition on X used to establish (3.9) can perhaps be weakened slightly (see [EG] and [EM]), but for ease of exposition we have used the above.

(II) If one perturbs the sequence $\{\eta_n\}$ in the H-norm by taking H-neighborhoods of radius ε_n instead of sup-norm neighborhoods, then for $\eta_n \neq 0$ and ε_n sufficiently small $f_n = (1-\varepsilon_n)\eta_n$. Hence $\varepsilon_n \to 0$ implies $\lim_n \langle f_n, f_n\rangle_H = \lim_n \langle \eta_n, \eta_n\rangle_H = \infty$ w.p.1 and $C(\{f_n\})_H = \phi$ w.p.1 (recall we are assuming $a_n = n$). Of course, (3.9) also fails.

(III) If one sets $w_n(t) = B(nt)/(2nL_2 n)^{1/2}$ for $n \geqq 1$ and $0 \leqq t \leqq 1$, where $\{B(t)\colon t \geqq 0\}$ is a Brownian motion, then $w_n \notin H$ w.p.1 and hence $\langle w_n, w_n\rangle_H = \infty$ w.p.1. However, the ε_n-perturbations of $\{w_n\}$ reside in H provided $\varepsilon_n > 0$, and if $\{\varepsilon_n\}$ satisfies (3.6), then the proof of Theorem 2 and the results in [deA] for Brownian motion combine to show that

$$\{(w_n)^{\varepsilon_n}\} \overset{H}{\twoheadrightarrow} K \quad \text{w.p.1}.$$

Proof of Theorem 2. To verify (3.5) fix ρ, $0 \leqq \rho < 1$, take $h \in \rho K$, and set $\varepsilon_n = d/(L_2 n)$ where $d > \frac{\pi}{4}(1-\rho^2)^{-1/2}$. Then by Corollary 1-(a) of [EG] we have a random subsequence n'' such that

$$\|\eta_{n''} - h\|_\infty < d/(L_2 n'') \quad \text{w.p.1}. \tag{3.10}$$

Hence for f_n given by (3.4),

$$\|f_{n''} - h\|_\infty < 2d/(L_2 n'') \tag{3.11}$$

and

$$\langle f_{n''}, f_{n''}\rangle_H \leqq \langle h, h\rangle_H \leqq \rho^2 . \tag{3.12}$$

Thus by applying the following lemma we have

$$\lim_{n''} \|f_{n''} - h\|_H = 0 \quad \text{w.p.1} \tag{3.13}$$

provided $d > \frac{\pi}{4}(1-\rho^2)^{-1/2}$.

Lemma 4. *Let* $\{f_n\}$, f *be absolutely continuous on* $[0, 1]$ *with* $f_n(0) = f(0) = 0$ *and such that*

$$\lim_n \|f_n - f\|_\infty = 0 . \tag{3.14}$$

If

$$\overline{\lim_n} \int_0^1 |f_n'(s)|^2\, ds \leqq \int_0^1 |f'(s)|^2\, ds < \infty , \tag{3.15}$$

then

$$\lim_n \int_0^1 |f_n'(s) - f'(s)|^2\, ds = 0 . \tag{3.16}$$

Proof. Since (3.15) holds, there exists $g \in L^2[0,1]$ such that for some subsequence $\{n_k\}$ of $\{n\}$ we have

$$f_{n_k}' \overset{\text{weakly}}{\longrightarrow} g$$

in $L^2[0,1]$. Thus for $0 \leqq t \leqq 1$

$$f_{n_k}(t) = \int_0^1 I_{[0,t]}(s) f_{n_k}'(s)\, ds \underset{n_k}{\rightarrow} \int_0^1 I_{[0,t]}(s) g(s)\, ds ,$$

and hence

$$\lim_{n_k} f_{n_k}(t) = \int_0^t g(s)\, ds$$

for all $t \in [0, 1]$. Hence (3.14) implies that

$$f(t) = \int_0^t g(s)\, ds \quad (0 \leqq t \leqq 1) ,$$

and $f' = g$ a.e. on $[0,1]$. Furthermore, the previous argument shows that $f_n' \overset{\text{weakly}}{\longrightarrow} f'$ in $L^2[0,1]$ since every subsequence of $\{f_n'\}$ has a further subsequence which converges to f'. Expanding the square in (3.16), and using (3.15), we have,

$$\lim_n \int_0^1 (|f'(s)|^2 - 2 f_n'(s) f'(s) + |f_n'(s)|^2)\, ds = 0 .$$

Thus (3.16) holds, and Lemma 4 is proven.

Returning to the proof of Theorem 2 we have (3.13) for every $h \in \rho K$ provided $d > \frac{\pi}{4}(1-\rho^2)^{-1/2}$. Hence if $\{h_j\colon j \geqq 1\}$ is a countable dense subset of ρK in the H-norm (H is separable), then

$$P\left(\underline{\lim}_n \|f_n - h_j\|_H = 0 \text{ for all } j \geqq 1\right) = 1\,.$$

This implies that

$$P(C(\{f_n\})_H \supseteq \{h_j\}) = 1\,,$$

and since $C(\{f_n\})_H$ is closed (by definition), we have $C(\{f_n\})_H \supseteq \rho K$ w.p.1. Furthermore, if ε_n satisfies (3.6), then the previous argument implies (3.5) holds for all ρ, $0 \leqq \rho < 1$, with probability one. Hence (3.6) implies $P(C(\{f_n\})_H \supseteq K) = 1$, and since

$$C(\{f_n\})_H \subseteq C(\{f_n\})_U\,,$$

with

$$C(\{f_n\})_U = K \quad \text{w.p.1}$$

by Strassen and $\lim_n \varepsilon_n = 0$, (3.7) holds w.p.1.

To prove (3.9) we observe that when $E(X^2(L_2|X|)) < \infty$, then the strong invariance result in Theorem 2 of [E] implies that it suffices to prove

$$\lim_n \inf_{h \in K} \|(w_n)^{\varepsilon_n} - h\|_H = 0 \quad \text{w.p.1}\,,$$

when $w_n(t) = B(nt)/(2nL_2n)^{1/2}$, $0 \leqq t \leqq 1$, and $\{B(t)\colon t \geqq 0\}$ is a standard Brownian motion. This follows from (4.3) in Theorem 3 below. Hence Theorem 2 is proven. □

4 Some results for Brownian motion

In this section we will examine how close to necessary the assumption (3.6) is in the setting of Brownian motion. We will show that (3.6) is near best possible in that (3.7) and (3.9) both fail when $\varepsilon_n = d/(L_2n)$ and $d > 0$ is sufficiently small. For further details see the remarks following Theorem 3.

Theorem 3. *Let* $\{B(t)\colon t \geqq 0\}$ *be a sample continuous Brownian motion with*

$$w_n(t) = B(nt)/(2nL_2n)^{1/2}$$

for $n \geqq 1$ *and* $0 \leqq t \leqq 1$. *Let* $I(\cdot,\cdot)$ *be defined by* (3.1), $d(f,K) = \inf_{g \in K} \|f-g\|_\infty$, *and set*

$$J(f,\delta) = \inf_{\|f-h\|_2 \leqq \delta} \langle h,h\rangle_H\,,$$

where $\|\cdot\|_2$ *denotes the usual* L^2-norm *on* $C_0[0,1]$. *Then*:

$$0 < \underline{\lim}_n L_2n\, d(w_n,K) \leqq \pi/8 \quad \text{w.p.1}\,, \tag{4.1}$$

and for each $M > 0$ there exists $\gamma > 0$ sufficiently small so that

(4.2) $$\varliminf_n J(w_n, \gamma/L_2 n) \geqq M \quad \text{w.p.1}.$$

Furthermore, for each $M > 1$ there exists $\gamma > 0$ sufficiently large such that

(4.3) $$\varlimsup_n I(w_n, \gamma/L_2 n) \leqq M \quad \text{w.p.1}.$$

Remarks. (I) It is known from [G92] and [T], and the earlier lower bound in [GK], that

(4.4) $$0 < \varlimsup_n (L_2 n)^{2/3} d(w_n, K) < \infty \quad \text{w.p.1}.$$

Hence (4.1) is the lim inf analogue of (4.4). Of course, the functional form of Chung's LIL given in [deA], Theorems 6.1 and 6.2, implies that

(4.5) $$\inf_{h \in K} \varliminf_n L_2 n \|w_n - h\|_\infty = \pi/4 \quad \text{w.p.1}.$$

Thus (4.1) implies that taking the inf inside the liminf in (4.5) reduces the constant, but does not change the rate. Finally, if one combines (4.3) and (4.4) it follows that for $\gamma > 0$ sufficiently small

(4.6) $$\varlimsup_n I(w_n, \gamma/(L_2 n)^{2/3}) = 1 \quad \text{w.p.1}.$$

Thus the analogue of (4.2) fails when $M > 1$ if $\gamma/L_2 n$ is replaced by $\gamma/(L_2 n)^{2/3}$, even though (4.4) holds.

(II) Since $\|f\|_\infty \geqq \|f\|_2$ for all $f \in C_0[0,1]$, it follows immediately that $I(f,\delta) \geqq J(f,\delta)$, and hence (4.2) holds if J is replaced by I. Similarly (4.3) holds if I is replaced by J. With a bit more thought one can also obtain (4.6) for J as well (see [GK, pp. 305–309]).

(III) By the strong approximation in Theorem 2 of [E], if $E(X^2(L_2|X|)) < \infty$ and $a_n = n$, then (4.1)–(4.3), and (4.6) hold with w_n replaced by η_n.

(IV) The statement in (4.1) with the right-hand side equal to $\pi/4$ was obtained earlier in [G91]. We provide a smaller upper bound, and the lower bound is achieved by applying the more delicate result in (4.2). Our approach also implies the same sort of result when one computes the distance from w_n to K in the L^2-norm. Then the upper bound will be 1/8, rather than $\pi/8$.

Proof of Theorem 3. Using rescaling ideas which are now fairly standard (see [GK] for details and other references), it suffices to prove analogues of (4.1), (4.2), and (4.3) for i.i.d. Brownian motion samples $B, B_1, B_2, \dots$ with continuous sample paths restricted to $[0,1]$. That is, we prove

(4.1′) $$0 < \varliminf_n Ln\, d(B_n/(2Ln)^{1/2}, K) \leqq \pi/8 \quad \text{w.p.1},$$

(4.2′) $$\varliminf_n J(B_n/(2Ln)^{1/2}, \gamma/Ln) \geqq M \quad \text{w.p.1}$$

for each $M > 0$ provided $\gamma > 0$ is sufficiently small, and for $M > 1$

(4.3′) $$\varlimsup_n I(B_n/(2Ln)^{1/2}, \gamma/Ln) \leqq M \quad \text{w.p.1}$$

provided $\gamma > 0$ is sufficiently large.

The first step of the proof will be to verify the right-hand side of (4.1′). After that we proceed to the proof of (4.2′), since by taking $M > 1$ it easily follows from $J \leqq I$ and the definition of I that the left-hand side of (4.1′) holds.

Let $p_n = P(d(B_n/(2Ln)^{1/2}, K) \leqq \gamma/Ln)$. Then

$$p_n = P(B \in (2Ln)^{1/2}K + (\sqrt{2}\gamma/(Ln)^{1/2})U)$$

where U is the closed unit ball of $C_0[0,1]$ in the sup-norm. Hence the isoperimentric property of Gaussian measures implies

$$p_n \geqq \Phi(\alpha_n + (2Ln)^{1/2})$$

where

$$\Phi(\alpha_n) = P(B \in (\sqrt{2}\gamma/(Ln)^{1/2})U)$$

and $\Phi(t) = (2\pi)^{-1/2} \int_{-\infty}^{t} e^{-x^2/2} dx$. Now as $\varepsilon \downarrow 0$

$$P(B \in \varepsilon U) \sim \frac{4}{\pi} \exp\left\{-\frac{\pi^2}{8}\varepsilon^{-2}\right\},$$

so as $n \to \infty$, $\alpha_n \to -\infty$, and

$$((2\pi)^{1/2}|\alpha_n|)^{-1} \exp\{-\alpha_n^2/2\} \sim \frac{4}{\pi} \exp\{-\pi^2 Ln/(16\gamma^2)\}.$$

Hence as $n \to \infty$

$$\alpha_n \sim -\pi(Ln)^{1/2}/(8^{1/2}\gamma),$$

and $\sum_n p_n = \infty$ if $(\pi/(8^{1/2}\gamma) - \sqrt{2})^2/2 < 1$. Thus taking $\gamma > \pi/8$, the Borel–Cantelli lemma easily implies the right side of (4.1′) with limiting value at most $\pi/8$.

Turning to (4.2′) we fix $M \geqq 1$, and point out this suffices for the general result. Now $\{B_n(t)\colon 0 \leqq t \leqq 1\}$ can be written as

$$B_n(t) = \sum_{k \geqq 0} \lambda_k g_{k,n} \psi_k(t), \tag{4.7}$$

where $\{g_{k,n}\colon k \geqq 0\}$ are i.i.d. sequences of i.i.d. $N(0,1)$ random variables, and

$$\lambda_k = 2(\pi(2k+1))^{-1}, \qquad \psi_k(t) = 2^{1/2} \sin((2k+1)\pi t/2) \tag{4.8}$$

for $k \geqq 0$ and $0 \leqq t \leqq 1$. Then for each $x \in C_0[0,1]$ and $\delta > 0$ the arguments in Lemma 1 in [G91] can be used to show that there exists a unique $h \in H$ such that $\|x - h\|_2 \leqq \delta$ and

$$J(x,\delta) = \langle h, h \rangle_H .$$

Lemma 1 in [KL] provides a general result of this type, so we do not include details. To determine h we use the method of Lagrange multipliers as applied in [GK, pp. 305–306] to a somewhat similar problem. That is, if $x \in C_0[0,1]$, then with Wiener measure one $x = \sum_{k \geqq 0} x_k \psi_k$, where the series converges uniformly and in $L^2[0,1]$. Thus as in [GK] we have

$$J(x,\delta) = \inf_{\sum_{k \geqq 0} \delta_k^2 \leqq \delta^2} \sum_{k \geqq 0} (x_k + \delta_k)^2/\lambda_k^2, \tag{4.9}$$

where

$$\delta_k = -x_k/(1+\lambda_k^2 t), \tag{4.10}$$

and $t = t(\delta)$ satisfies

$$\sum_{k \geqq 0} x_k^2/(1+\lambda_k^2 t)^2 = \delta^2 . \tag{4.11}$$

Hence

$$J(x, \delta) = t^2 \sum_{k \geqq 0} \lambda_k^2 x_k^2/(1+\lambda_k^2 t)^2 . \tag{4.12}$$

Thus (4.7) implies that

$$J(B_n/(2Ln)^{1/2}, \delta) = t_n^2/(2Ln) \cdot \sum_{k \geqq 0} \lambda_k^4 g_{k,n}^2/(1+\lambda_k^2 t_n)^2 , \tag{4.13}$$

where $t_n = t_n(\delta)$ satisfies

$$\sum_{k \geqq 0} \lambda_k^2 g_{k,n}^2/((2Ln)(1+\lambda_k^2 t_n)^2) = \delta^2 \tag{4.14}$$

for all $n \geqq 1$.

Now take $\delta = \gamma/Ln$ and let $J_n = J(B_n/(2Ln)^{1/2}, \gamma/Ln)$. Then for $c > 0$

$$\begin{aligned} P(J_n \leqq M) &= P(J_n \leqq M, t_n \geqq (Ln/c)^2) \\ &\quad + P(J_n \leqq M, t_n < (Ln/c)^2) \\ &\equiv I_n + II_n . \end{aligned} \tag{4.15}$$

Thus

$$\begin{aligned} I_n &= P\left(\sum_{k \geqq 0} \lambda_k^4 g_{k,n}^2/((1/t_n)+\lambda_k^2)^2 \leqq 2MLn, t_n \geqq (Ln/c)^2\right) \\ &\leqq P\left(\sum_{k \geqq 0} \lambda_k^4 g_{k,n}^2/((c/Ln)^2+\lambda_k^2)^2 \leqq 2MLn\right) \\ &\leqq P\left(\sum_{\lambda_k^2 > (c/Ln)^2} g_{k,n}^2 \leqq 8MLn\right) \end{aligned} \tag{4.16}$$

since $\inf_{x^2 \geqq a^2} x^4/(a^2+x^2)^2 = 1/4$. To estimate the last term in (4.16) we digress to a lemma which also is used to estimate II_n.

Lemma 5. *If* $g_1, g_2, \ldots$ *are* i.i.d. $N(0,1)$, *then for all* $m \geqq 1$ *and for* τ *small enough so that* $0 < \tau \leqq e^{-3}$

$$P\left(\sum_{j=1}^{m} g_j^2 \leqq \tau m\right) \leqq \exp\{-(1+\tau/2)m\} . \tag{4.17}$$

Remark. Perhaps the bound in (4.17) looks strange, but it applies only for $\tau > 0$ small, namely $0 < \tau \leqq e^{-3}$. This is adequate for our purposes; a sharper bound appears in the proof.

Proof. First observe for $u \geqq 0$

$$P\left(\sum_{j=1}^{m} g_j^2 \leqq \tau m\right) \leqq E\left(e^{-u\sum_{j=1}^{m} g_j^2 + u\tau m}\right)$$
$$= \exp\{u\tau m - (m/2)\log(1+2u)\} .$$

Minimizing $f(u) = \tau u - \frac{1}{2}\log(1+2u)$ for $u \geqq 0$ we take $u = (1-\tau)/(2\tau)$, so

$$P\left(\sum_{j=1}^{m} g_j^2 \leqq \tau m\right) \leqq \exp\{m(1-\tau-\log 1/\tau)/2\} .$$

Hence $0 < \tau \leqq e^{-3}$ implies (4.17) for all $m \geqq 1$, and the lemma is proven. □

Returning to the proof of (4.2′) we apply Lemma 5 to (4.16). Let $[\,\cdot\,]$ denote the greatest integer function and fix τ, $0 < \tau \leqq e^{-3}$. Hence since $\lambda_k^2 = 4(\pi(2k+1))^{-2}$ we have $\lambda_k^2 > (c/Ln)^2$ iff $k < Ln/\pi c - \frac{1}{2}$. Therefore, for $c > 0$ sufficiently small so that $[Ln/\pi c - \frac{1}{2}] > (8MLn)/\tau$ we have (by applying (4.17) with $m = [Ln/\pi c - \frac{1}{2}] + 1$) that

$$I_n \leqq P\left(\sum_{k=0}^{[Ln/\pi c - \frac{1}{2}]} g_{k,n}^2 \leqq 8MLn\right) \tag{4.18}$$
$$\leqq \exp\left\{-(1+\tau/2)\left[\frac{Ln}{\pi c} - \frac{1}{2}\right]\right\}$$
$$\leqq \exp\{-(1+\tau/2)Ln\} .$$

Thus $\sum_{n\geqq 1} I_n < \infty$ for $c > 0$ sufficiently small, so it suffices to prove $\sum_{n\geqq 1} II_n < \infty$.

Now

(4.19)

$$II_n \leqq P\left(t_n < \left(\frac{Ln}{c}\right)^2\right)$$
$$= P\left(\sum_{k\geqq 0}\left(\frac{1}{t_n}\right)^2 \lambda_k^2 g_{k,n}^2/((1/t_n)+\lambda_k^2)^2 \leqq 2\gamma^2/Ln, (1/t_n) > (c/Ln)^2\right)$$
$$\leqq P\left(\sum_{\lambda_k^2 \leqq \frac{1}{t_n}} \lambda_k^2 g_{k,n}^2 \leqq 8\gamma^2/Ln, \frac{1}{t_n} \geqq (c/Ln)^2\right)$$
$$\text{since } \inf_{x^2\leqq a} \frac{a^2}{(a+x^2)^2} = \frac{1}{4}$$

$$\leqq P\left(\sum_{\lambda_k^2 \leqq (c/Ln)^2} \lambda_k^2 g_{k,n}^2 \leqq 8\gamma^2/Ln\right)$$

$$\leqq P\left(\sum_{k \geqq \frac{Ln}{\pi c} - \frac{1}{2}} 4(\pi(2k+1))^{-2} g_{k,n}^2 \leqq 8\gamma^2/Ln\right)$$

$$\leqq P\left(\sum_{\frac{Ln}{c} \leqq k \leqq \frac{Ln}{d}} g_{k,n}^2 \leqq 16\pi^2(\gamma/d)^2 Ln\right),$$

where $d > 0$ is taken such that $(1/d) - (1/c) = 2$. Now fix $0 < \tau < e^{-3}$. Then for $\gamma > 0$ sufficiently small so that $16(\pi\gamma/d)^2 < 2\tau$, we have from Lemma 5 that

$$II_n \leqq \exp\{-(1 + \tau/2)Ln\}$$

for all n sufficiently large. Hence we also have $\sum_{n \geqq 1} II_n < \infty$, and (4.5) and the Borel–Cantelli lemma together imply (4.2′). Hence it remains to prove (4.3′).

To verify (4.3′) fix $M > 1$ and recall $U = \{f\colon \|f\|_\infty \leqq 1\}$. Then

$$\begin{aligned} (4.20) \qquad & P(I(B_n/(2Ln)^{1/2}, \gamma/Ln) > M^2) \\ &= P(B_n/(2Ln)^{1/2} \notin MK + \gamma U/Ln) \\ &= P(B \notin M(2Ln)^{1/2}K + \sqrt{2}\,\gamma(Ln)^{-1/2}U) \\ &\leqq 1 - \Phi(M(2Ln)^{1/2} + \alpha_n) \end{aligned}$$

by the isoperimetric property of Gaussian measures, where

$$\Phi(\alpha_n) = P(B \in \sqrt{2}\,\gamma(Ln)^{-1/2}U) \sim 4/\pi \exp\{-\pi^2 Ln/(16\gamma^2)\}$$

as in the proof of (4.1′). Thus as before $\alpha_n \sim -\pi(Ln/8\gamma^2)^{1/2}$ as $n \to \infty$. Hence (4.20) implies that

$$P(I(B_n(2Ln)^{1/2}, \gamma/Ln) > M^2) \leqq P(Z > (Ln)^{1/2}(M\sqrt{2} - \pi/(2\gamma))),$$

where Z is $N(0,1)$. Since $M > 1$, by taking $\gamma > 0$ sufficiently large we have

$$\sum_{n \geqq 1} P(I(B_n/(2Ln)^{1/2}, \gamma/Ln) > M^2) < \infty.$$

Thus the Borel–Cantelli lemma implies (4.3′), and Theorem 3 is proven.

5 Convergence and clustering in H-norm via interpolation

Our next result shows that non-trivial H-convergence and clustering can be obtained for independent Gaussian samples when they are suitably interpolated.

As before we assume $B_1, B_2, \ldots$ are i.i.d. Brownian motion samples with continuous paths restricted to $[0,1]$. Let p_n be a non-decreasing sequence of integers such that $\lim_n p_n = \infty$, $\lim_n p_n/Ln = 0$, and let

$$(5.1)\qquad Y_n(t) = \begin{cases} B_n(k/p_n)/(2Ln)^{1/2} & t = k/p_n \; k = 0,1,\ldots,p_n\,, \\ \text{linearly interpolated otherwise}\,. \end{cases}$$

Theorem 4. *Let $B_1, B_2, \ldots, \{p_n: n \geqq 1\}$, and $\{Y_n: n \geqq 1\}$ be as above. Then*

$$(5.2)\qquad \{Y_n: n \geqq 1\} \xrightarrow{H} K \quad \text{w.p.1}\,.$$

Proof. Let

$$\pi_n(f)(t) = \begin{cases} f(k/p_n) & t = k/p_n \; k = 0,1,\ldots,p_n\,, \\ \text{linearly interpolated elsewhere}\,. \end{cases}$$

Then for $f \in K$, when f' is Lip(1) on $[0,1]$, we have by the mean-value theorem that

$$\begin{aligned} \|f - \pi_n(f)\|_H^2 &\leqq \sum_{k=1}^{p_n} \int_{\frac{k-1}{p_n}}^{k/p_n} (f'(s) - f'(c_{k,n}))^2\, ds \\ &\leqq \sum_{k=1}^{p_n} M_f \int_{\frac{(k-1)}{p_n}}^{k/p_n} |s - c_{k,n}|^2\, ds\,, \end{aligned}$$

where $(k-1)/p_n < c_{k,n} < k/p_n$ for $k = 1,\ldots,p_n$. Hence for such an f we have

$$\lim_n \|f - \pi_n f\|_H^2 = 0\,.$$

Thus

$$\{Y_n: n \geqq 1\} \xrightarrow{H} K \quad \text{w.p.1}$$

if we show for all $\varepsilon > 0$ that

$$(5.3)\qquad \overline{\lim_n}\, \|Y_n\|_H^2 \leqq 1 \quad \text{w.p.1}\,,$$

and

$$(5.4)\qquad \underline{\lim_n}\, \|Y_n - f\|_H = 0 \quad \text{w.p.1}$$

for all $f \in K$ when f' is Lip(1) on $[0,1]$. That (5.4) suffices follows from the fact that these f are dense in K in the H-norm.

Fix $\varepsilon > 0$. Then

$$\begin{aligned} q_n &\equiv P(\|Y_n\|_H^2 > 1 + \varepsilon) \\ &= P\left(\sum_{k=1}^{p_n} (B_n(k/p_n) - B_n((k-1)/p_n))^2 p_n > 2(1+\varepsilon)Ln\right) \\ &= P\left(\sum_{k=1}^{p_n} g_{k,n}^2 > 2(1+\varepsilon)Ln\right)\,, \end{aligned}$$

where $\{g_{k,n}\colon k = 1, \ldots, p_n\}$ are i.i.d. $N(0,1)$ for $n \geqq 1$. Thus by Markov's inequality, for any $u > 0$

$$q_n \leqq \exp\left\{-2(1+\varepsilon)uLn - \frac{p_n}{2}\log(1-2u)\right\}$$
$$= \exp\left\{-\tfrac{1}{2}(2(1+\varepsilon)Ln - p_n) - \frac{p_n}{2}\log(p_n/(2(1+\varepsilon)Ln))\right\}$$

by setting $u = \frac{1}{2}(1 - p_n/(2(1+\varepsilon)Ln))$. Hence

$$q_n \leqq \frac{1}{n^{(1+\varepsilon)}}\exp\left\{\tfrac{1}{2}p_n + p_n\log\left(\frac{Ln}{p_n}\right)\right\}$$
$$= \frac{1}{n^{(1+\varepsilon)}}\exp\{\tfrac{1}{2}(Ln)g(n) - (Ln)g(n)\log g(n)\},$$

where $g(n) = p_n/Ln \to 0$ as $n \to \infty$. Thus $\sum_{n\geqq 1} q_n < \infty$, and (5.3) holds.

To verify (5.4) first observe that

$$\|Y_n - f\|_H \leqq \|Y_n - \pi_n f\|_H + \|\pi_n f - f\|_H .$$

If $f \in K$ and f' is Lip(1) on $[0,1]$, we have $\lim_n\|\pi_n f - f\|_H = 0$. Hence it suffices to prove $\underline{\lim}_n \|Y_n - \pi_n f\|_H = 0$ for $f \in K$. Set $\gamma_k = (f(k/p_n) - f(k-1/p_n))/p_n^{-1/2}$, $k = 1, \ldots, p_n$. Then

$$\|Y_n - \pi_n f\|_H^2 = \sum_{k=1}^{p_n}\left(\frac{((B_n(\frac{k}{p_n}) - B_n\left(\frac{(k-1)}{p_n}\right))/p_n^{-1})}{(2Ln)^{1/2}}\right.$$
$$\left. - \left(f\left(\frac{k}{p_n}\right) - f\left(\frac{(k-1)}{p_n}\right)\right)\Big/ p_n^{-1}\right)^2 p_n^{-1}$$
$$= \sum_{k=1}^{p_n}\left\{\frac{((B_n(\frac{k}{p_n}) - B_n(\frac{(k-1)}{p_n}))/p_n^{-1/2})}{(2Ln)^{1/2}}\right.$$
$$\left. - \left(\left(f\left(\frac{k}{p_n}\right) - f\left(\frac{(k-1)}{p_n}\right)\right)\Big/ p_n^{-1/2}\right)\right\}^2$$
$$= \sum_{k=1}^{p_n}(g_{k,n}/(2Ln)^{1/2} - \gamma_k)^2,$$

where $g_{k,n}$ are i.i.d. $N(0,1)$ for $k = 1, \ldots, p_n$, $n \geqq 1$. Also observe that $\sum_{k=1}^{p_n}\gamma_k^2 = \sum_{k=1}^{p_n}(\int_{\frac{(k-1)}{p_n}}^{k/p_n} f'(s)\,ds)^2/p_n^{-1} \leqq \sum_{k=1}^{p_n}\int_{\frac{(k-1)}{p_n}}^{k/p_n}(f'(s))^2\,ds \leqq 1$. Using the

Cameron–Martin formula in $\mathbb{R}^{p_n}$ we see

$$\begin{aligned} &P(\|Y_n - \pi_n f\|_H^2 \leqq \varepsilon) \\ &= P\left(\sum_{k=1}^{p_n} (g_{k,n}/(2Ln)^{1/2} - \gamma_k)^2 \leqq \varepsilon\right) \\ &= \exp\left\{-2Ln \sum_{k=1}^{p_n} \gamma_k^2/2\right\} \int_{\{\sum_{k=1}^{p_n} g_{k,n}^2 \leqq 2\varepsilon Ln\}} \exp\left\{-\sum_{k=1}^{p_n} g_{k,n}\gamma_k(2Ln)^{1/2}\right\} dP \\ &\geqq \exp\{-Ln \sum_{k=1}^{p_n} \gamma_k^2\} P\left(\sum_{k=1}^{p_n} g_{k,n}^2 \leqq 2\varepsilon Ln\right), \end{aligned}$$

where the inequality follows from the symmetry of the Gaussian measure and Jensen's inequality. Now $P(\Sigma_{k=1}^{p_n} g_{k,n}^2 > 2\varepsilon Ln) \leqq p_n/(2\varepsilon Ln) \to 0$ as $n \to \infty$ so $\Sigma_{k=1}^{p_n} \gamma_k^2 \leqq 1$ implies for n sufficiently large

$$P(\|Y_n - \pi_n f\|_H^2 \leqq \varepsilon) \geqq \frac{1}{2n}.$$

Applying the Borel–Cantelli lemma, we thus have for all $f \in K$, with f' being Lip(1), that

$$\varliminf_n \|Y_n - f\|_H^2 = 0 \quad \text{w.p.1}.$$

Thus the theorem is proven.

Remark. As before, assume $\{p_n\}$ is a non-decreasing sequence of positive integers. If $\{p_n\}$ is bounded, then eventually $p_n = d$, a positive integer, and it is easily seen that $\{Y_n\} \overset{H}{\twoheadrightarrow} \pi_d K$ w.p.1. Thus (5.2) fails in this case. On the other hand, suppose that $\varliminf_n p_n/Ln > \beta > 0$. Fix $\varepsilon > 0$ and $0 < \sigma < 1$. Then it follows from (6.20) of [GK] that for some absolute constant $C_1 > 0$

$$\begin{aligned} P(\|Y_n\|_H^2 > 1 + \varepsilon) &= P\left(\sum_1^{p_n} g_{k,n}^2 > 2(1+\varepsilon)Ln\right) \\ &\geqq P\left(g_{1,n}^2 + \sum_2^{p_n} \sigma^2 g_{k,n}^2 > 2(1+\varepsilon)Ln\right) \\ &\geqq \frac{C_1 \exp(-(1+\varepsilon)Ln + (p_n - 1)L(1-\sigma^2)^{-1/2})}{(2(1+\varepsilon)Ln)^{1/2}} \end{aligned}$$

provided

$$2(1+\varepsilon)Ln \geqq \frac{25(1 + p_n\sigma^2)}{1 - \sigma^2}. \tag{5.4}$$

Thus first choose σ so that equality holds in (5.4), hence

$$\sigma^2 = \frac{2(1+\varepsilon)Ln - 25}{25 p_n + 2(1+\varepsilon)Ln} = \frac{Ln/p_n + O(1/Pn)}{C_\varepsilon + Ln/p_n}$$

where $C_\varepsilon = 25/(2(1+\varepsilon))$. In particular

$$1 - \sigma^2 = \frac{C_\varepsilon + O(1/p_n)}{C_\varepsilon + Ln/p_n}$$

which means that

$$\begin{aligned}(1+\varepsilon)Ln - (p_n - 1)L(1-\sigma^2)^{-1/2} \\ = Ln\left[(1+\varepsilon) - \frac{1}{2}\left(\frac{p_n - 1}{Ln}\right)L\left(\frac{C_\varepsilon + Ln/p_n}{C_\varepsilon + O(1/p_n)}\right)\right].\end{aligned}$$

Since $\underline{\lim}\, p_n/Ln > \beta > 0$, this enables us to choose $\varepsilon > 0$ so that

$$P(\|Y_n\|_H^2 > 1+\varepsilon) \geqq \frac{C_1 \exp(-Ln)}{(2(1+\varepsilon)Ln)^{1/2}}$$

for all large n. Hence by the Borel-Cantelli lemma

$$\overline{\lim}\|Y_n\|_H > 1 \quad \text{w.p.1.}$$

Thus Y_n does not converge to K, and again (5.2) fails. This shows that the assumptions in Theorem 4 are relatively "sharp".

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