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The cut-off phenomenon for random reflections II: Complex and quaternionic cases

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Summary. In this paper we generalize the random reflections problem on $O(N)$ considered in an earlier paper to the complex and quaternionic cases. We give precise estimates on the speed of convergence to stationarity for specific examples of random walks on $U(N)$ and $Sp(N)$ for which the one-step distribution is a certain probability measure concentrated on reflections. Our results show that in both cases the so-called cut-off phenomenon occurs at $k_0 = \frac{1}{2}N \log N$.

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1 Introduction and statement of results

This paper is a sequel to [P1] where we studied the random walk on $O(N)$ whose step distribution is uniform measure on the set of reflections. Here we extend our results for "random orthogonal reflections" to the complex and quaternionic cases. We suggest suitable analogues to the orthogonal case for the unitary group $U(N)$ and the symplectic group $Sp(N)$ and prove precise estimates on the distance to stationarity with respect to *total variation distance* (Theorems C1–2 and Theorems Q1–2) for these random walks. The results show that, in the large N limit, the so-called *cut-off phenomenon* occurs (see Remark 1.1). Since $Sp(n) \subset U(Zn)$ we henceforth use the notation $Sp(n)$ and $U(N)$ for these groups.

By a complex (quaternionic) reflection we mean a norm preserving automorphism of $\mathbf{C}^N$ ($\mathbf{H}^n$) that leaves exactly one hyperplane pointwise fixed. We denote the set of complex (quaternionic) reflections of $\mathbf{C}^N$ ($\mathbf{H}^n$) by $\mathscr{R}_{\mathbf{C}^N}$ ($\mathscr{R}_{\mathbf{H}^n}$).

[1] This paper is based on parts of the author's doctoral dissertation written at The Johns Hopkins University

Notice that the sets $\mathscr{R}_{\mathbf{C}^N}$ and $\mathscr{R}_{\mathbf{H}^n}$ are unions of conjugacy classes:

$$\mathscr{R}_{\mathbf{C}^N} = \{B\,\mathrm{diag}(e^{i\varphi}, 1, \dots, 1)B^*\colon B \in U(N),\ \varphi \in [0, 2\pi)\}$$

and

$$\mathscr{R}_{\mathbf{H}^n} = \{B\,\mathrm{diag}(h, 1, \dots, 1)B^*\colon B \in Sp(n),\ h \in Sp(1)\}\,.$$

The one-step distributions for the random walks we consider are certain delta-functions on $\mathscr{R}_{\mathbf{C}^N}$ and $\mathscr{R}_{\mathbf{H}^n}$.

It can be shown that $\mathscr{R}_{\mathbf{C}^N}$ and $\mathscr{R}_{\mathbf{H}^n}$ generate $U(N)$ and $Sp(n)$, respectively. In fact, each element in $U(N)$ (resp. $Sp(n)$) can be written as a product of at most N (resp. n) complex (resp. quaternionic) reflections. We omit the proof.

Random complex reflections: As a natural analogue to random orthogonal reflections, we choose the following random walk on the unitary group $U(N)$. The random walk starts at the identity $I \in U(N)$ and takes steps according to the following step distribution μ. μ is concentrated on the set $\mathscr{R}_{\mathbf{C}^N}$ and induced by the product of Haar measure ϑ on $U(N)$ and the probability measure on $[0, 2\pi)$ with density proportional to $(\sin(\varphi/2))^{N-1}$. In [P1] we considered a random walk on $O(N)$ which, at even times, can be viewed as a random walk on $SO(N)$. This random walk on $SO(N)$ has step distribution concentrated on the 2-dimensional rotations and induced from Haar measure on $SO(N)$ and the probability measure on $[0, 2\pi)$ with density proportional to $(\sin(\varphi/2))^{N-2}$. See Remark 1.4 in [P1].

Notation: We denote the distribution of a random walk (with step distribution μ) after k steps, i.e., the k-fold convolution power of μ, by μ_k.

Theorem C1 *Let μ be the probability measure on $U(N)$ defined above. There exist universal positive constants α, β, c_0 such that for any integer $N \geqq 16$ and any positive number $c \geqq c_0$, we have: if $k = \frac{1}{2}N\log N + cN$ is an integer, then*

$$\|\mu_k - \vartheta\|_{TV} \leqq \alpha e^{-\beta c}\,.$$

Theorem C2 *Let μ be the probability measure on $U(N)$ defined above. For any integer $N \geqq 6$ and any positive number c, we have: if $k = \frac{1}{2}N\log N - cN$ is an integer,*

$$\|\mu_k - \vartheta\|_{TV} \geqq 1 - 6e^{-2c}\,.$$

Random quaternionic reflections: We further extend the notion of "random reflections" to the quaternionic case. In order to explain how our random walk on the symplectic group $Sp(n)$ proceeds, we start out by defining a probability measure (call it ε) on $Sp(1)$.

For any $h \in Sp(1)$ the eigenvalues are $e^{\pm i\varphi_h}$ for some $\varphi_h \in [0, 2\pi)$ (we identify $Sp(1)$ with $SU(2)$). We take the probability measure ε to be the measure of mass 1 that has density proportional to $(\sin(\varphi_h/2))^{2n-2}$ with respect to Haar measure on $Sp(1)$.

Our random walk on $Sp(n)$ starts at the identity $I \in Sp(n)$ and takes steps according to the probability measure η which is concentrated on $\mathscr{R}_{\mathbf{H}^n}$ and induced from Haar measure ϑ on $Sp(n)$ and the measure ε on $Sp(1)$.

Theorem Q1 *Let η be the probability measure on $Sp(n)$ defined above. For any integer $n \geqq 8$ and any positive number $c \geqq 11.6$, we have: if*

$k = \frac{1}{2} n \log n + cn$ is an integer, then

$$\|\eta_k - \vartheta\|_{TV} \leqq 3.6 e^{-c/5} .$$

Theorem Q2 *Let η be the probability measure on $Sp(n)$ defined above. For any integer $n \geqq 2$ and any positive number c, we have: if $k = \frac{1}{2} n \log n - cn$ is an integer, then*

$$\|\eta_k - \vartheta\|_{TV} \geqq 1 - 6e^{-4c} - 75 \frac{\log n}{n^{1/3}} .$$

The constants appearing in the statements of Theorems C2 and Q1–2 are not sharp. The methods of proofs for our main theorems rely on previous work by Diaconis, Rosenthal, and others and are similar in spirit to the ones used in [P1]. Our main tool throughout will be Fourier analysis.

Remark. 1.1. Together, Theorems C1 and C2 (and Theorems Q1 and Q2) show that the cut-off phenomenon occurs: For large N (n), there exists a critical number k_0 depending on a size parameter for the group (here, $k_0 = \frac{1}{2} N \log N$ for $U(N)$ and $k_0 = \frac{1}{2} n \log n$ for $Sp(n)$) such that k_0 steps are both necessary and sufficient to be close to stationarity (Haar measure ϑ).

For background on the cut-off phenomenon see, for example, [AD]. Together with Rosenthal's example on $SO(N)$ [R] and "random orthogonal reflections" in [P1], the random walks here considered are, to the author's knowledge, the only existing examples of random walks on the *classical compact Lie groups* for which the occurrence of a cut-off phenomenon has been proved.

Remark. 1.2. It turns out that the speed of convergence of a random walk depends rather sensitively (at least with respect to L_2-distance) on the step distribution chosen: If we change the step distribution μ by changing the probability measure on $[0, 2\pi)$ from density proportional to $(\sin(\varphi/2))^{N-1}$ to *uniform* on $[0, 2\pi)$, k needs to be at least of order N^2 for the k-fold convolution power of this new step distribution to have an L_2-density. The proof involves some delicate but interesting multiplicity problems in representation theory (see [P2], Example 2(a) in Sect. 4, for details). On the other hand (and as the proof of Theorem C1 will show), the L_2-norm of μ_k is already close to 1 for $k = \frac{1}{2} N \log N + cN$ and $c > c_0$, where c_0 is some universal positive constant. As a possible explanation for this discrepancy in convergence behavior, note that $\mathscr{R}_{\mathbb{C}^N}$ is a manifold with a conic singularity at the identity I. Our probability measure μ, as opposed to the one with uniform measure on $[0, 2\pi)$, obviously "sufficiently smoothens out" this singularity. We do not know how to make this rigorous, however.

Also, see Example 4 in Sect. 4 in [P2] for a different example of "random quaternionic reflections" on $Sp(n)$ with rather "slow" (if at all) convergence to stationarity with respect to L_2-distance.

Organization: This paper is organized as follows. In Sect. 2 we present basics on random walks and Fourier analysis used throughout. The necessary background on the representation theory of $U(N)$ and $Sp(n)$, together with the computation of the required Fourier coefficients, are presented in Sects. 3 and 5, respectively. We prove Theorem C1 in Sect. 4 and Theorem Q1 in Sect. 6.

The proofs of Theorems C2 and Q2 are very similar; they are both presented in Sect. 7.

2 Random walks and Fourier analysis

A random walk on a group G is determined by its one-step probability distribution μ. The random walk starts at the identity (say) and takes steps according to the measure μ. Thus at time $t = 0$ the distribution of the walk is the measure concentrated at the identity, at time $t = 1$ it is μ, and at time $t = 2$ it is the *convolution* of μ with itself:

$$\mu_2(S) = (\mu \star \mu)(S) = \int_G \mu(g^{-1}S)\, d\mu(g)$$

for any Borel set $S \in \mathscr{B}(G)$. In general, at time $t = k$ the distribution is μ_k, the k-fold convolution $\mu\star\mu\star\ldots\star\mu$ of μ:

$$\mu_k = \mu \star \mu_{k-1}\,.$$

Our focus of interest is the speed of convergence to stationarity (Haar measure) with respect to *total variation distance* (and L_2-distance) for the random walks under consideration. We recall the definition of total variation distance:

Definition 2.1 Let μ and ϑ be two Borel probability measures on a topological space M and let $\mathscr{B}(M)$ be the Borel sigma-field of M. The *total variation distance* is defined by

$$\|\mu - \vartheta\|_{TV} := \sup_{S\in\mathscr{B}(M)} |\mu(S) - \vartheta(S)| = \frac{1}{2}|\mu - \vartheta|(M)\,.$$

If μ has density f with respect to ϑ, we have

$$\|\mu - \vartheta\|_{TV} = \frac{1}{2}\int_M |f - 1|\, d\vartheta\,.$$

(Notice that $0 \leqq \|\mu - \vartheta\|_{TV} \leqq 1$.)

We now present basic background on Fourier transforms and the *Upper Bound Lemma* of Diaconis and Shahshahani [D]. For background on representation theory see, for example, [BtD, FH, Ž]. Let G be a compact Lie group; $\rho_0, \rho_1, \rho_2, \ldots$ its irreducible unitary representations; and $\chi_0, \chi_1, \chi_2, \ldots$ the corresponding characters.

Definition 2.2 Let ν be a finite measure on G.
(a) The Fourier transform of ν at ρ_i is defined by

$$\hat{\nu}(\rho_i) := \int_G \rho_i(g)\, d\nu(g)\,.$$

(b) The Fourier coefficient of ν at ρ_i is defined by

$$\hat{\nu}(\chi_i) := \operatorname{trace} \hat{\nu}(\rho_i) = \int_G \chi_i(g)\, d\nu(g)\,.$$

Fourier transforms convert convolution to multiplication:

$$\widehat{\nu^1 \star \nu^2}(\rho_i) = \widehat{\nu^1}(\rho_i) \cdot \widehat{\nu^2}(\rho_i) .$$

If ν is conjugate-invariant, i.e., if $\nu(gSg^{-1}) = \nu(S)$ for all measurable sets $S \subseteq G$ and for all $g \in G$, a simplification occurs: $\hat{\nu}(\rho_i)$ commutes with $\rho_i(g)$ for all $g \in G$. By Schur's lemma, $\hat{\nu}(\rho_i) = r_i I$ for certain r_i for any irreducible representation ρ_i, $i = 0, 1, 2, \ldots$ Clearly, $r_i = \hat{\nu}(\chi_i)/d_i$ where d_i denotes the dimension of the irreducible representation ρ_i. Furthermore, we have

$$\hat{\nu}_k(\rho_i) = r_i^k I = \left(\frac{\hat{\nu}(\chi_i)}{d_i}\right)^k I$$

and

$$\hat{\nu}_k(\chi_i) = d_i r_i^k = d_i \left(\frac{\hat{\nu}(\chi_i)}{d_i}\right)^k . \tag{1}$$

A fundamental property of the Fourier transform is the following theorem.

Theorem 2.3 *A finite positive measure ν on a compact Lie group G is uniquely determined by its Fourier transform $(\hat{\nu}(\rho_i),\ i = 0, 1, 2, \ldots)$.*

As an immediate corollary we have

Corollary 2.4 *If a finite positive measure ν on a compact Lie group G is conjugate-invariant, it is uniquely determined by its Fourier coefficients $(\hat{\nu}(\chi_i),\ i = 0, 1, 2, \ldots)$.*

For a given irreducible representation ρ_s of G, let

$$\phi_{jk}^{(s)}, \quad j, k = 1, 2, \ldots, d_s ,$$

denote the entry functions, i.e., $\rho_s(g) = (\phi_{jk}^{(s)}(g))$. The Schur orthogonality relations assert that, with respect to the usual inner product in $L_2(G)$, the functions $\phi_{jk}^{(s)}$ are orthogonal to each other and of norm $d_s^{-1/2}$, i.e.,

$$\int_G \phi_{jk}^{(s)} \cdot \overline{\phi_{lm}^{(t)}}\, d\vartheta = \delta_{st}\delta_{jl}\delta_{km} d_s^{-1}$$

where ϑ is normalized Haar measure on G and the bar denotes complex conjugation. It follows that the irreducible characters $\chi_0, \chi_1, \chi_2, \ldots$ form an orthonormal set of functions in the Hilbert space $L_2(G)$:

$$\int_G \chi_i \overline{\chi}_j\, d\vartheta = \delta_{ij} .$$

The following version of the Upper Bound Lemma can be found in [R].

Lemma 2.5 (Upper Bound Lemma) *Let G be a compact Lie group, ϑ its normalized Haar measure, and ν a conjugate-invariant probability measure on G. Set $l_i := \hat{\nu}(\chi_i)$. Then*

$$\|\nu - \vartheta\|_{TV}^2 \leqq \frac{1}{4}\left(\sum_{i=0}^{\infty} |l_i|^2 - 1\right) .$$

3 Fourier analysis on $U(N)$

We briefly summarize, without giving proofs, basic facts from the representation theory of $U(N)$ to the extent necessary for our computations. For background see [BtD, FH, Ž]. The main goal of this section is to compute the eigenvalues of the Fourier transforms for each irreducible representation of $U(N)$ (Proposition 3.3).

Recall that the irreducible representations of a compact connected Lie group are in one-to-one correspondence with the integer lattice points in a fundamental Weyl chamber (the *highest weights*). The set of possible highest weights for $U(N)$ can be chosen to be

$$\omega \in \mathbf{Z}^N \quad \text{with } \omega_1 \leqq \omega_2 \leqq \ldots \leqq \omega_N .$$

We can thus index the irreducibles of $U(N)$ by N-tuples $\lambda = \omega + \psi$ of strictly increasing (pos. or neg.) integers for $N = 2n+1$ odd, and of strictly increasing half-integers (odd multiples of $\frac{1}{2}$) for $N = 2n$ even. Here ψ stands for

$$\psi = \frac{1}{2} \sum_{\alpha \in R^+} \alpha = \begin{cases} (-n, -n+1, \ldots, -1, 0, 1, \ldots, n) & \text{for } N = 2n+1 , \\ (-n+\frac{1}{2}, \ldots, -\frac{1}{2}, \frac{1}{2}, \ldots, n-\frac{1}{2}) & \text{for } N = 2n , \end{cases}$$

where R^+ denotes the (chosen) set of *positive roots* of $U(N)$ and can be taken to be $R^+ = \{e_j - e_i \colon 1 \leqq i < j \leqq N\}$ (e_i denotes the ith basis vector $(0, \ldots, 0, 1, 0, \ldots, 0)$).

We will need the dimension of each irreducible of $U(N)$.

Proposition 3.1 *Let d_λ denote the dimension of the irreducible representation ρ_λ of $U(N)$ corresponding to the index $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$. Then*

$$d_\lambda = \frac{\Pi_{1 \leqq r < s \leqq N} (\lambda_s - \lambda_r)}{0!1! \ldots (N-1)!} . \tag{2}$$

Proof. We use the *Weyl dimension polynomial*:

$$d_\lambda = \prod_{\alpha \in R^+} \frac{\langle \alpha, \lambda \rangle}{\langle \alpha, \psi \rangle} . \tag{3}$$

Here $\langle \cdot , \cdot \rangle$ denotes the usual Euclidian inner product and R^+ and ψ are as given above. An easy calculation yields (2). We omit the details. □

The Weyl character formula allows us to compute the irreducible characters of a compact connected Lie group at any element in the (chosen) maximal torus T. For $U(N)$, the standard choice for maximal torus is the subgroup of diagonal matrices.

The Weyl character formula for $U(N)$

Let χ_λ denote the character of the irreducible representation ρ_λ of $U(N)$ corresponding to the index $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$. The value of the character χ_λ at the element $\operatorname{diag}(e^{i\varphi_1},\dots,e^{i\varphi_N})$ is

$$\chi_\lambda(\operatorname{diag}(e^{i\varphi_1},\dots,e^{i\varphi_N})) = \frac{\sum_{w\in S_N}\operatorname{sgn}(w)\exp\left(i\sum_{j=1}^N \lambda_{w(j)}\varphi_j\right)}{\Pi_{1\leqq r<s\leqq N} 2i\sin\left(\frac{1}{2}(\varphi_s-\varphi_r)\right)}. \tag{4}$$

Here S_N denotes the symmetric group and $\operatorname{sgn}(w)$ denotes the sign of the permutation w.

Proposition 3.2 *Fix the reflection* $B = \operatorname{diag}(e^{i\theta},1,\dots,1)\in U(N)$. *Let* $\lambda = (\lambda_1,\lambda_2,\dots,\lambda_N)$ *be the index of an irreducible representation of* $U(N)$. *Then,*

$$\chi_\lambda(B) = \frac{1}{(2i)^{N-1}}\sum_{j=1}^N(-1)^{N-j}\frac{e^{i\theta\lambda_j}}{(\sin(\theta/2))^{N-1}}d_{\widehat{\lambda_j}}^{(N-1)}. \tag{5}$$

Here $d_{\widehat{\lambda_j}}^{(N-1)}$ *denotes the dimension of the irreducible representation of* $U(N-1)$ *of index* $(\lambda_1,\dots,\widehat{\lambda_j},\dots,\lambda_N)$ *(the hat symbol means deletion).*

Proof. Use formula (4) and set $\varphi_1 = \theta$. Eventually we will let $\varphi_s\to 0$ for $2\leqq s\leqq N$. We can rewrite the numerator in (4) as

$$\sum_{j=1}^N(-1)^{j-1}\exp(i\theta\lambda_j)\cdot\sum_{\sigma\in S_{N-1}}\operatorname{sgn}(\sigma)\exp\left(i\sum_{s=2}^N\lambda_{\sigma(s)}\varphi_s\right)$$

where, for brevity, we have written S_{N-1} for the set of maps from $\{2,\dots,N\}$ onto $\{1,\dots,\hat{j},\dots,N\}$. We can view such a map σ as a permutation of $\{2,\dots,N\}$ under the order preserving identification of $\{2,\dots,N\}$ with $\{1,\dots,\hat{j},\dots,N\}$, and $\operatorname{sgn}(\sigma)$ denotes the sign of this permutation. Furthermore, we can rewrite the denominator in (4) as

$$\prod_{s=2}^N 2i\sin\left(\frac{1}{2}(\varphi_s-\theta)\right)\prod_{2\leqq r<s\leqq N}2i\sin\left(\frac{1}{2}(\varphi_s-\varphi_r)\right).$$

Taking the quotient of these two expressions and letting $\varphi_s\to 0$ for $2\leqq s\leqq N$ yields (5). □

Notation: From now on we will write $a!!$ for $a(a-2)(a-4)\dots 1$ (a odd).

Proposition 3.3 (a) *For* $N = 2n+1$ *and any index* λ,

$$\frac{\hat{\mu}(\chi_\lambda)}{d_\lambda} = \frac{(-1)^{N-i-n}(n!)^2}{\Pi_{j\neq i}|\lambda_j|} \quad \text{if } \lambda_i = 0 \tag{6}$$

and

$$\frac{\hat{\mu}(\chi_\lambda)}{d_\lambda} = 0 \quad \text{if } \lambda_j\neq 0 \text{ for all } j = 1,2,\dots,N.$$

(b) *For* $N = 2n$ *and any* λ,

$$\frac{\hat{\mu}(\chi_\lambda)}{d_\lambda} = \frac{((2n-1)!!)^2}{2^N\cdot\Pi_{j=1}^N\lambda_j}\cdot(-1)^n. \tag{7}$$

Proof. By Propositions 3.1 and 3.2,

$$\chi_\lambda(\theta) = \frac{\sum_{j=1}^{N}(-1)^{N-j}\exp(i\lambda_j\theta)\prod_{\substack{1\leqq r<s\leqq N\\ r,s\neq j}}(\lambda_s-\lambda_r)}{0!1!\dots(N-2)!(2i)^{N-1}(\sin(\theta/2))^{N-1}}\,.$$

We need to compute

$$\hat{\mu}(\chi_\lambda) = C_N\int_0^\pi(\sin\theta)^{N-1}\chi_\lambda(2\theta)\,d\theta$$

where $C_N = ((N-1)(N-3)\dots2)/((N-2)(N-4)\dots1)(1/\pi)$ for $N = 2n+1$ odd and $C_N = ((N-1)(N-3)\dots3)/((N-2)(N-4)\dots2)(1/2)$ for $N = 2n$ even.

(a) In case $N = 2n+1$, the λ_j's are integers and we have

$$\int_0^\pi e^{i\lambda_j 2\theta}\,d\theta = \begin{cases} 0 & \text{for } \lambda_j\neq0\,,\\ \pi & \text{for } \lambda_j = 0\,.\end{cases}$$

This, together with (2), yields (6).

(b) In case $N = 2n$, the λ_j's are half integers and we have

$$\int_0^\pi e^{i\lambda_j 2\theta}\,d\theta = \frac{-1}{i\lambda_j}\quad\text{for all } \lambda_j\,.$$

Therefore,

$$\frac{\hat{\mu}(\chi_\lambda)}{d_\lambda} = \frac{((N-1)!!)^2}{2^N i^N}\sum_{j=1}^{N}(-1)^{N-j+1}\frac{\prod_{\substack{1\leqq r<s\leqq N\\ r,s\neq j}}(\lambda_s-\lambda_r)}{\lambda_j\prod_{1\leqq r<s\leqq N}(\lambda_s-\lambda_r)}\,.$$

Since

$$\prod_{1\leqq r<s\leqq N}(\lambda_s-\lambda_r) = \sum_{j=1}^{N}(-1)^{j-1}\prod_{i\neq j}\lambda_i\cdot\prod_{\substack{1\leqq r<s\leqq N\\ r,s\neq j}}(\lambda_s-\lambda_r)$$

(this is the Vandermonde determinant expanded along the first column), we get (7). □

4 Proof of Theorem C1

By Lemma 2.5 and (1) it suffices to show that there exist positive constants $\tilde{\alpha}, \tilde{\beta}$ such that for any integer $N \geqq 16$ and any positive real number c larger than some universal constant c_0,

$$\sum_{i=0}^{\infty}\left|\frac{\hat{\mu}(\chi_i)}{d_i}\right|^{2k} d_i^2 - 1 \leqq \tilde{\alpha}e^{-\tilde{\beta}c} \tag{8}$$

for $k = \frac{1}{2}N\log N + cN$.

For simplicity we restrict the proof of Theorem C1 to the case $N = 2n+1$ odd. For the case $N = 2n$ even the proof is extremely similar, so we can safely omit the details.

From now on we will always assume $N = 2n+1$. We therefore need to estimate

$$\sum_{\substack{\lambda:\ \lambda_i=0 \\ \text{for some } i}} \left(\frac{(n!)^2}{\Pi_{j\neq i}|\lambda_j|}\right)^{2k} d_\lambda^2 \tag{9}$$

where $k = n\log n + cn$ and the λ's are N-tuples of strictly increasing integers. This will be done in two parts: In Part I we estimate

$$\sum_{\substack{\lambda:\ \lambda_i=0 \text{ for some } i \\ -8n<\lambda_1,\ \lambda_N<8n}} \left(\frac{(n!)^2}{\Pi_{j\neq i}|\lambda_j|}\right)^{2k} d_\lambda^2, \tag{9a}$$

and in Part II we complete the proof of Theorem C1 by estimating the tail sum

$$\sum_{\substack{\lambda:\ \lambda_i=0 \text{ for some } i \\ \lambda_1 \leqq -8n \text{ or } \lambda_N \geqq 8n}} \left(\frac{(n!)^2}{\Pi_{j\neq i}|\lambda_j|}\right)^{2k} d_\lambda^2. \tag{9b}$$

Part I

Write λ^x instead of λ if $\lambda_1 < \lambda_2 < \ldots < \lambda_x < \lambda_{x+1} = 0$ to indicate the number $x \leqq 2n$ of negative integers in the N-tuple. Set

$$\mathcal{S}_x := \sum_{\substack{\lambda^x: \\ -8n<\lambda_1,\ \lambda_N<8n}} \left|\frac{\hat{\mu}(\chi_{\lambda^x})}{d_{\lambda^x}}\right|^{2k} d_{\lambda^x}^2 = \sum_{\substack{\lambda^x: \\ -8n<\lambda_1,\ \lambda_N<8n}} \left(\frac{(n!)^2}{\Pi_{i\neq x+1}|\lambda_i|}\right)^{2k} d_{\lambda^x}^2 .$$

Clearly, $\mathcal{S}_x = \mathcal{S}_{2n-x}$ for $0 \leqq x \leqq 2n$.
Therefore, (9a) equals

$$2\sum_{x=0}^{n-1} \mathcal{S}_x + \mathcal{S}_n . \tag{10}$$

In the following we will estimate

$$\tilde{\mathcal{S}}_x := \left(\frac{x!(2n-x)!}{(n!)^2}\right)^{2k} \mathcal{S}_x = \sum_{\substack{\lambda^x: \\ -8n<\lambda_1,\ \lambda_N<8n}} \left(\frac{x!(2n-x)!}{\Pi_{i\neq x+1}|\lambda_i|}\right)^{2k} d_{\lambda^x}^2$$

for each x, $0 \leqq x \leqq n$, separately and then use (10) in order to estimate (9a).

Consider $\lambda^x = (\lambda_1, \lambda_2, \ldots, \lambda_x, 0, \lambda_{x+2}, \ldots, \lambda_N)$ with $\lambda_i < 0$ for $1 \leqq i \leqq x$ and $\lambda_i > 0$ for $x+2 \leqq i \leqq N$. Set $\lambda_0^x := (-x, -x+1, \ldots, -1, 0, 1, 2, \ldots, 2n-x)$ and $\lambda^x - \lambda_0^x =:\ b = (-b_1, -b_2, \ldots, -b_x, 0, b_{x+2}, \ldots, b_N)$.

Proposition 4.1 *There exist two universal positive constants K and a such that for any x with $0 \leqq x \leqq n$ we have for all indices λ^x with $-8n < \lambda_1$ and $\lambda_N < 8n$*

$$\left(\frac{x!(2n-x)!}{\Pi_{i\neq x+1}|\lambda_i|}\right)^k d_{\lambda^x} \leqq (K e^{-ac})^{b_1+b_2+\ldots+b_N} (7n)^{(n-x)+(n-x-1)+\ldots+1} .$$

Proof. We need the following two lemmas.

Lemma 4.2 *Let* $d(T) := d_{\lambda^x}$ *with* $\lambda^x = (-x, \ldots, -1, 0, 1, \ldots, 2n - x - 1, T)$. *Then*

$$\frac{d(T+1)}{d(T)} = \frac{T+1+x}{T+1+x-2n}. \tag{11}$$

Lemma 4.3 *Set* $r(T) := \frac{x!(2n-x)!}{\prod_{i \neq x+1} |\lambda_i|}$ *with* λ^x *as in Lemma* 4.2. *Then*

$$\frac{r(T+1)}{r(T)} = 1 - \frac{1}{T+1} \leqq e^{-1/(T+1)}.$$

We now split the proof of Proposition 4.1 into three parts. In Part A we prove the statement for all λ^x for which $b = (0, \ldots, 0, b_N)$, in Part B we prove the statement for all λ^x for which $b = (0, \ldots, 0, b_{x+2}, \ldots, b_N)$, and in Part C we prove the full statement.

(A) First we analyze

$$q := \frac{r(T+1)^k d(T+1)}{r(T)^k d(T)}.$$

From the above results we have

$$\begin{aligned} q &\leqq \frac{T+1+x}{T+1+x-2n} e^{-(1/(T+1))(n \log n + cn)} = \frac{T+1+x}{T+1+x-2n} \cdot \frac{T+1-n}{T+1+n} \tilde{q} \\ &\leqq \frac{T+1-n}{T+1+x-2n} \tilde{q} \end{aligned}$$

with

$$\tilde{q} = \left(1 + \frac{2n}{T+1-n}\right) e^{-(1/(T+1))(n \log n + cn)}.$$

We have already essentially estimated $\tilde{q}$ in the proof of Proposition 4.1, Part A in [P1] and will therefore not repeat the details here. From there, changing $b_n^* = T + \frac{3}{2} - n$ to $b_n^* = T + 1 - n$ we get

$$\tilde{q} \leqq 5e^{-c/9} \quad \text{for } n \geqq 8.$$

Now start with the index λ_0^x (for which $T = T_{\text{beg}} = 2n - x$) and, step by step, increase the last number in this N-tuple by 1 until the desired $T = T_{\text{end}} = \lambda_N$ is reached. Clearly, we have picked up a factor less than or equal to

$$\left[\frac{T_{\text{beg}} + 1 - n}{T_{\text{beg}} + 1 + x - 2n} \cdot \frac{(T_{\text{beg}} + 1) + 1 - n}{(T_{\text{beg}} + 1) + 1 + x - 2n} \cdots \right. \\ \left. \frac{(T_{\text{end}} - 1) + 1 - n}{(T_{\text{end}} - 1) + 1 + x - 2n}\right] (5e^{-c/9})^{b_N} \tag{12}$$

in going from $r(T_{\text{beg}})^k d(T_{\text{beg}})$ to $r(T_{\text{end}})^k d(T_{\text{end}})$. This yields the desired result for $x = n$. We now investigate the expression in square brackets in (12) more closely for the case $0 \leqq x \leqq n - 1$. Each factor $(T + 1 - n)/(T + 1 + x - 2n)$

(where $T_{\text{beg}} \leqq T < T_{\text{end}}$) in this expression can be written as

$$\frac{T+1-n}{T-n} \cdot \frac{T-n}{T-1-n} \cdots \frac{T+2+x-2n}{T+1+x-2n}$$

so that the whole expression becomes

$$\frac{T_{\text{end}}-n}{T_{\text{beg}}-n} \cdot \frac{T_{\text{end}}-1-n}{T_{\text{beg}}-1-n} \cdots \frac{T_{\text{end}}+1+x-2n}{T_{\text{beg}}+1+x-2n} .$$

Since we assume $T_{\text{end}} < 8n$ and $T_{\text{beg}} = 2n-x$, we see that this product is less than or equal to $(7n)^{n-x}$. In total we have picked up a factor of at most

$$(5e^{-c/9})^{b_N}(7n)^{n-x} .$$

It follows that the statement of Proposition 4.1 holds with $K=5$, $a=1/9$, and the last factor reduced to $(7n)^{n-x}$.
(B) For λ^x with $b=(0,\ldots,0,b_{x+2},\ldots,b_N)$, repeat the procedure in Part A for the $(N-1)$st number in λ^x until the desired λ_{N-1} is reached, and so on. We now estimate the total factor picked up in this procedure.
We use the following notation for $i \geqq x+1$:

$$r(T_i) := \frac{x!(2n-x)!}{\Pi_{j \neq x+1}|\delta_j|} , \quad d(T_i) := d_{\delta^x} ,$$

where $\delta^x = (-x,\ldots,-1,0,1,\ldots,i-x-2,T_i,\lambda_{i+1},\ldots,\lambda_N)$. We now have

$$\frac{r(T_i+1)}{r(T_i)} = 1 - \frac{1}{T_i+1}$$

and

$$\begin{aligned}\frac{d(T_i+1)}{d(T_i)} &= \left(\prod_{j=i+1}^{N} \frac{(\lambda_j-(T_i+1))}{(\lambda_j-T_i)}\right) \cdot \frac{(T_i+1-(i-x-1))\ldots(T_i+1+x)}{(T_i-(i-x-2))\ldots(T_i+x)} \\ &\leqq \frac{T_i+1+x}{T_i-(i-x-2)} . \end{aligned} \tag{13}$$

For $n+x+2 \leqq i \leqq N-1$, the same ideas as used in Part A can be applied here with T replaced by T_i. For example, we can easily see that for $i=N-1$ we pick up a total factor of at most $(5e^{-c/9})^{b_{N-1}}(7n)^{n-x-1}$ in going from

$$(-x,\ldots,-1,0,1,\ldots,2n-x-1,\lambda_N)$$

to

$$(-x,\ldots,-1,0,1,\ldots,2n-x-2,\lambda_{N-1},\lambda_N)$$

and for $i=n+x+2$ we pick up a factor of at most $(5e^{-c/9})^{b_{n+x+2}}(7n)$ in going from

$$(-x,\ldots,-1,0,1,\ldots,n+1,\lambda_{n+x+3},\ldots,\lambda_N)$$

to

$$(-x, \ldots, -1, 0, 1, \ldots, n, \lambda_{n+x+2}, \ldots, \lambda_N)\,.$$

For $x+2 \leqq i \leqq n+x+1$, use

$$\frac{r(T_i+1)}{r(T_i)} = 1 - \frac{1}{T_i+1} \leqq 1 - \frac{1}{T_i+1+(n+x+1-i)}$$

and an upper bound for $d(T_i+1)/d(T_i)$ as in (13). We wish to estimate

$$q_i := \frac{r(T_i+1)^k d(T_i+1)}{r(T_i)^k d(T_i)}\,.$$

Again, our previously performed estimate in the proof of Proposition 4.1, Part A in [P1] goes through with b_n^* replaced by $b_i^* := (T_i+1)-(i-x-1)$. In each step we pick up a factor $\tilde{q}_i \leqq 5e^{-c/9}$.

We summarize: In going from λ_0^x, for which $((x!(2n-x)!)/(\Pi_{i \neq x+1}|\lambda_i|))^k d_{\lambda^x} = 1$, to λ^x with $b = (0, \ldots, 0, b_{x+2}, \ldots, b_N)$ we have picked up a factor of at most

$$(5e^{-c/9})^{b_{x+2}+\ldots+b_N}(7n)^{(n-x)+(n-x-1)+\ldots+1}\,. \tag{14}$$

(C) We now also allow negative numbers to occur in b. Again, start with the index λ_0^x and now successively decrease the first number $-x$ in steps by 1 until the desired λ_1 is reached. The total factor picked up in this procedure is the same as the one picked up in starting with $\lambda_0^{2n-x} = (-(2n-x), \ldots, -1, 0, 1, \ldots, x)$ and successively increasing the last number x in steps by one until $|\lambda_1|$ is reached. From Lemmas 4.2 and 4.3 (with x replaced by $2n-x$) we see that

$$q := \frac{r(T+1)^k d(T+1)}{r(T)^k d(T)} \leqq \frac{T+1+2n-x}{T+1-x} e^{-k/(T+1)} \leqq \frac{T+1+n}{T+1-n} e^{-k/(T+1)}$$

for $0 \leqq x \leqq n$. Thus we get the same estimate for q as in Part A, namely $q \leqq 5e^{-c/9}$.

We can now see that, after having decreased all the negative numbers in λ_0^x in steps by 1 to reach λ_1, then λ_2, and so on, until we have reached the index $(\lambda_1, \lambda_2, \ldots, \lambda_x, 0, 1, 2, \ldots, 2n-x)$, we have picked up a factor less than or equal to

$$(5e^{-c/9})^{b_1+\ldots+b_x}\,. \tag{15}$$

We can now complete the proof of Proposition 4.1 by taking the product of (14) and (15) to yield

$$\left(\frac{x!(2n-x)!}{\Pi_{i \neq x+1}|\lambda_i|}\right)^k d_{\lambda^x} \leqq (5e^{-c/9})^{b_1+\ldots+b_N}(7n)^{(n-x)+(n-x-1)+\ldots+1}$$

for all λ^x with $-8n < \lambda_1$ and $\lambda_N < 8n$. Indeed, taking the product is valid for the following reason:

The dimension d_{λ^x} is the same as the dimension of the irreducible representation of $U(N)$ corresponding to the highest weight $(-b_1, -b_2, \ldots, -b_x, 0, b_{x+2}, \ldots, b_N)$. Therefore, by Lemma 7.2, d_{λ^x} is less than or equal to the product of the dimension of the irreducible representations of $U(n)$ corresponding to the highest weights $(-b_1, \ldots, -b_x, 0, \ldots, 0)$ and $(0, \ldots, 0, b_{x+2}, \ldots, b_N)$.

But the dimension of the irreducible representation of highest weight $(-b_1,\dots,-b_x,0,\dots,0)$ is the same as the dimension of the irreducible representation of index $(\lambda_1,\dots,\lambda_x,0,1,2,\dots,2n-x)$, and the dimension of the irreducible representation of highest weight $(0,\dots,0,b_{x+2},\dots,b_N)$ is the same as the dimension of the irreducible representation of index $(-x,\dots,-1,0,\lambda_{x+2},\dots,\lambda_N)$.

This completes the proof of Proposition 4.1. □

Writing Q for $(Ke^{-ac})^2$, we can apply the same argument as in the proof of Theorem 1.1 in [P1] to yield

$$\tilde{\mathscr{S}}_n \leqq (1+3Q) = 1+75e^{-c/4.5}$$

and

$$\begin{aligned}\tilde{\mathscr{S}}_x &\leqq (1+3Q)(7n)^{(1+n-x)(n-x)}\\ &= (1+75e^{-c/4.5})(7n)^{(1+n-x)(n-x)} \quad \text{for } 0 \leqq x \leqq n-1\,,\end{aligned}$$

provided we take c larger than some universal constant c_0.
We therefore have

$$\begin{aligned}\sum_{\substack{\lambda:\ \lambda_i=0 \text{ for some } i\\ -8n<\lambda_1,\,\lambda_N<8n}} \left(\frac{(n!)^2}{\Pi_{j\neq i}|\lambda_j|}\right)^{2k} d_\lambda^2 &= 2\sum_{x=0}^{n-1}\left(\frac{(n!)^2}{x!(2n-x)!}\right)^{2k}\tilde{\mathscr{S}}_x + \tilde{\mathscr{S}}_n\\ &\leqq 2\sum_{x=0}^{n-1}\left(\frac{(n!)^2}{x!(2n-x)!}\right)^{2k}(1+75e^{-c/4.5})(7n)^{(1+n-x)(n-x)}\\ &\quad + 1+75e^{-c/4.5}\,. \end{aligned} \tag{16}$$

We now take a closer look at the expression

$$\left(\frac{(n!)^2}{x!(2n-x)!}\right)^{k}(7n)^{(1+n-x)(n-x)/2} \quad \text{for } 0 \leqq x \leqq n-1\,. \tag{17}$$

An upper bound for (17) is easily obtained:

$$\begin{aligned}&e^{-((n-x)^2/(n+(n-x)))^k}(7n)^{(1+n-x)(n-x)/2}\\ &\quad = e^{-(n-x)^2(n\log n+cn)/(n+(n-x))}(7n)^{(1+n-x)(n-x)/2}\,.\end{aligned} \tag{18}$$

Set $s := (n-x)$, so that $1 \leqq s \leqq n$. For $s=1$, there exist two constants c_1 and c_2 (not depending on c) such that

$$e^{-(n\log n+cn)/(n+1)}(7n) \leqq c_1e^{-c_2c}\,.$$

For $2 \leqq s \leqq n$, we will now show that

$$e^{-s^2(n\log n+cn)/(n+s)}(7n)^{s+s^2/2} \leqq \frac{e^{-c}}{n}\,.$$

To do so, we only need to show that

$$-\frac{s^2(n\log n+cn)}{n+s} + (2+\log n)\frac{s+s^2}{2} + \log n \leqq -c \tag{19}$$

for $2 \leqq s \leqq n$ (we have used $\log 7 < 2$). Rewriting (19) yields

$$(\log n)(-s^2 n + s^3 + sn + s^2 + 2n + 2s) + 2(-s^2 cn + s^3 + s^2 n + sn + s^2 + cn + cs) \leqq 0\,,$$

from which one can verify that (19) holds for all $2 \leqq s \leqq n$ provided we take c larger than some universal constant.
This estimate, together with (16), proves that there exists a universal constant, call it B, such that

$$\sum_{\substack{\lambda:\, \lambda_i=0 \text{ for some } i \\ -8n<\lambda_1,\, \lambda_N<8n}} \left(\frac{(n!)^2}{\Pi_{j\neq i}|\lambda_j|}\right)^{2k} d_\lambda^2 \leqq 1 + Be^{-c/4.5}$$

for c larger than some universal constant c_0.

Part II

We now need to find an upper bound of similar form for the tail sum

$$\sum_{\substack{\lambda:\, \lambda_i=0 \text{ for some } i \\ \lambda_1 \leqq -8n \text{ or } \lambda_N \geqq 8n}} \left(\frac{(n!)^2}{\Pi_{j\neq i}|\lambda_j|}\right)^{2k} d_\lambda^2\,.$$

From now on we will denote the m-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ with $\lambda_i = 0$ for some $1 \leqq i \leqq m$ by $\lambda^{(m)}$ for $2 \leqq m \leqq N$. Accordingly, we will use the notation

$$d_{\lambda^{(m)}} := \frac{\Pi_{1\leqq r<s\leqq m}(\lambda_s - \lambda_r)}{0!1!\ldots(m-1)!}$$

and

$$r_{\lambda^{(m)}} := \begin{cases} \dfrac{(u!)^2}{\Pi_{j\neq i}|\lambda_j|} & \text{if } m = 2u+1\,, \\[2ex] \dfrac{u!(u-1)!}{\Pi_{j\neq i}|\lambda_j|} & \text{if } m = 2u\,. \end{cases}$$

The following lemma will complete the proof of Theorem C1.

Lemma 4.4 *We have*

$$\sum_{\lambda^{(m)}} (r_{\lambda^{(m)}})^{2k} (d_{\lambda^{(m)}})^2 \leqq 1 + \tilde{B}e^{-c/4.5}$$

for $3 \leqq m \leqq N$, *m odd, and* $k = n\log n + cn$, *where* $\tilde{B} = B + 1$ *with B from Part* I.

Proof. We use induction on m. For $m = 3$,

$$d_{\lambda^{(3)}} = \begin{cases} \frac{1}{2}xy(y+x) & \text{for } \lambda^{(3)} = (-x, 0, y)\,, \\ \frac{1}{2}xy(y-x) & \text{for } \lambda^{(3)} = (-y, -x, 0) \text{ or } \lambda^{(3)} = (0, x, y)\,, \end{cases}$$

$$r_{\lambda^{(3)}} = \frac{1}{xy}\,.$$

Therefore,

$$\sum_{\lambda(3)} (r_{\lambda(3)})^{2k}(d_{\lambda(3)})^2 = \frac{1}{4}\sum_{x,y=1}^{\infty} \frac{1}{(xy)^{2k}}(xy(x+y))^2 + \frac{1}{2}\sum_{y=2}^{\infty}\sum_{x=1}^{y-1} \frac{1}{(xy)^{2k}}(xy(y-x))^2 . \tag{20}$$

The first sum on the right hand side of (20) is less than or equal to

$$\sum_{x,y=1}^{\infty} \frac{1}{(xy)^{2k-4}}$$

and the second sum is less than or equal to

$$\frac{1}{2}\sum_{y=2}^{\infty}\sum_{x=1}^{y-1} \frac{1}{(xy)^{2k-4}} \leqq \frac{1}{2}\sum_{y=2}^{\infty}\sum_{x=1}^{\infty} \frac{1}{(xy)^{2k-4}} .$$

Both double sums can be estimated by double integrals. From this it is obvious that an upper bound for (20) can be chosen much smaller than the statement of Lemma 4.4 requires.

Notice that in order to prove Lemma 4.4, we need only show that

$$\sum_{\lambda^{(m)}:\ \lambda_1 \leqq -8n \text{ or } \lambda_m \geqq 8n} (r_{\lambda(m)})^{2k}(d_{\lambda(m)})^2 \leqq e^{-c/4.5}$$

for $3 \leqq m \leqq N$, m odd.

Indeed, the first part of our proof clearly goes through with $r_{\lambda(N)}$ replaced by $r_{\lambda(m)}$ (the statement of Lemma 4.3 is unchanged) and $d_{\lambda(N)}$ replaced by $d_{\lambda(m)}$ (the right hand side of (11) in Lemma 4.2 clearly becomes smaller).

Also, we can apply the very same analysis used in the proof of Proposition 4.1 to show that

$$\sum_{\lambda^{(m)}:\ -8n<\lambda_1,\, \lambda_m<8n} \left(\frac{u!(u-1)!}{\Pi_{j\neq i}|\lambda_j|}\right)^{2k} (d_{\lambda(m)})^2 \leqq 2 + Be^{-c/4.5}$$

for $3 \leqq m \leqq N$, m even, and $k = n\log n + cn$.

From the definitions of $d_{\lambda(m)}$ and $r_{\lambda(m)}$ we also see that

$$d_{\lambda(m+1)} \leqq \frac{1}{m!}(2|\lambda|)^m d_{\lambda(m)} \leqq (2|\lambda|)^m d_{\lambda(m)}$$

and

$$r_{\lambda(m+1)} = \frac{u+1}{|\lambda|} r_{\lambda(m)}$$

for $m = 2u+1$, $m \geqq 3$. Here $|\lambda| := \max\{|\lambda_1|, |\lambda_{m+1}|\}$ and $\lambda^{(m)}$ is the remaining m-tuple.

We now have

$$\sum_{\substack{\lambda^{(m+1)}:\\ \lambda_1 \leqq -8n \text{ or } \lambda_{m+1} \geqq 8n}} (r_{\lambda^{(m+1)}})^{2k}(d_{\lambda^{(m+1)}})^2 \leqq 2 \sum_{\lambda \geqq 8n} \left(\frac{u+1}{|\lambda|}\right)^{2k} (2|\lambda|)^{2m} \times \sum_{\lambda^{(m)}} (r_{\lambda^{(m)}})^{2k}(d_{\lambda^{(m)}})^2 \tag{21}$$

for $m = 2u+1$ odd. By our induction hypothesis,

$$\sum_{\lambda^{(m)}} (r_{\lambda^{(m)}})^{2k}(d_{\lambda^{(m)}})^2 \leqq 1 + \tilde{B}e^{-c/4.5} \leqq 2, \text{ say (for } c \geqq c_0) .$$

Thus the left hand side of (21) is less than or equal to

$$4 \sum_{y \geqq 8n} \left(\frac{n}{y}\right)^{2k} (2y)^{4n} \leqq 2^{4n+2} n^{2k} \int_{8n-1}^{\infty} \frac{1}{y^{2k-4n}}\, dy .$$

Since $2k - 4n = 2n \log n + 2cn - 4n > 4$ for $c \geqq c_0$,

$$\begin{aligned} 2^{4n+2} n^{2k} \int_{8n-1}^{\infty} \frac{1}{y^{2k-4n}}\, dy &= 2^{4n+2} n^{2k} \frac{1}{2k-4n-1} (8n-1)^{-2k+4n+1} \\ &< 2^{4n} n^{2k} (8n-1)^{-2k+4n+1} \\ &= \exp[-2n \log n \cdot \log(8 - 1/n) + 4n \log n + 4n \log 2 \\ &\quad + 4n \log(8 - 1/n) + \log n + \log(8 - 1/n) \\ &\quad - 2nc \log(8 - 1/n)] . \end{aligned}$$

Since $2.06 < \log(8 - 1/n) < 2.08$ for $n \geqq 8$, we get

$$\text{LHS}(21) \leqq \exp[-0.12n \log n + 12n + \log n + 2.08 - 4.12nc] \leqq e^{-c/4.5}$$

for $c \geqq c_0$, $n \geqq 8$.
This proves that

$$\sum_{\lambda^{(m+1)}} (r_{\lambda^{(m+1)}})^{2k}(d_{\lambda^{(m+1)}})^2 \leqq 2 + \tilde{B}e^{-c/4.5} \leqq 3, \text{ say} .$$

Since we are interested in the case $m+2$ odd, we need to perform this same step once more, which turns out to yield

$$\sum_{\lambda^{(m+2)}} (r_{\lambda^{(m+2)}})^{2k}(d_{\lambda^{(m+2)}})^2 \leqq 1 + \tilde{B}e^{-c/4.5} .$$

This concludes the proof of Lemma 4.4 and, by setting $m = N$, the proof of Theorem C1. □

5 Fourier analysis on $Sp(n)$

The main goal of this section is to compute the Fourier transform for each irreducible representation of $Sp(n)$ (Proposition 5.3). First, we briefly recall a few basic facts from the representation theory of $Sp(n)$. For more details see [BtD, FH, Ž].

From now on we will identify elements in $Sp(n)$ with $(2n) \times (2n)$ unitary matrices (via the natural representation). Thus the element $\mathrm{diag}(h, 1, \dots, 1) \in M_n(\mathbf{H})$ corresponds to

$$\left(\begin{array}{cccc|cccc} z_1 & 0 & \dots & 0 & z_2 & 0 & \dots & 0 \\ 0 & & & & 0 & & & \\ \vdots & & I_{n-1} & & \vdots & & O_{n-1} & \\ 0 & & & & 0 & & & \\ \hline -\bar{z}_2 & 0 & \dots & 0 & \bar{z}_1 & 0 & \dots & 0 \\ 0 & & & & 0 & & & \\ \vdots & & O_{n-1} & & \vdots & & I_{n-1} & \\ 0 & & & & 0 & & & \end{array}\right) \quad \text{for } h = z_1 + z_2 j\,, \tag{22}$$

where I_{n-1} is the $(n-1) \times (n-1)$ identity matrix and O_{n-1} is the $(n-1) \times (n-1)$ zero matrix. The standard choice for maximal torus $T \subset Sp(n)$ is the subgroup of diagonal matrices

$$\mathrm{diag}\,(e^{i\varphi_1}, \dots, e^{i\varphi_n}, e^{-i\varphi_1}, \dots, e^{-i\varphi_n}) \tag{23}$$

and the following is a set of positive roots:

$$R^+ = \{e_j \pm e_i \colon 1 \leqq i < j \leqq n\} \cup \{2e_j \colon 1 \leqq j \leqq n\}\,.$$

The irreducible representations of $Sp(n)$ can be indexed by strictly increasing n-tuples of integers greater than or equal to one. We call such an index λ. Then $\lambda = \omega + \psi$ with $\psi = (1, 2, \dots, n)$ (half the sum of the positive roots) and ω being the highest weight of the irreducible representation ρ_λ. The weight ω is thus an n-tuple of nondecreasing, nonnegative integers.

The Weyl character formula for $Sp(n)$

Let χ_λ denote the character of the irreducible representation ρ_λ of $Sp(n)$ corresponding to the index $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$. For every element $t \in T$ of the form (23), the value of the charecter χ_λ is

$$\chi_\lambda(t) = \frac{\Sigma_{\sigma \in S_n} \Sigma_{\varepsilon_m = \pm 1} \mathrm{sgn}(\sigma) \left(\Pi_{i=1}^n \varepsilon_i\right) \exp\left(i \Sigma_{j=1}^n \varepsilon_j \lambda_{\sigma(j)} \varphi_j\right)}{\Pi_{1 \leqq r < s \leqq n} 2i \sin\left(\frac{1}{2}(\varphi_s \pm \varphi_r)\right) \Pi_{1 \leqq r \leqq n} 2i \sin \varphi_r}\,. \tag{24}$$

Here S_n denotes the symmetric group, $\mathrm{sgn}(\sigma)$ denotes the sign of the permutation σ, and $\Sigma_{\varepsilon_m = \pm 1}$ indicates summation over all choices of $\varepsilon_1 = \pm 1, \dots, \varepsilon_n = \pm 1$.

Proposition 5.1 *Let d_λ denote the dimension of the irreducible representation ρ_λ of $Sp(n)$ corresponding to the index $\lambda = (\lambda_1, \dots, \lambda_n)$. Then*

$$d_\lambda = \frac{1}{1!3! \dots (2n-1)!} \prod_{j=1}^n \lambda_j \prod_{1 \leqq r < s \leqq n} (\lambda_s^2 - \lambda_r^2)\,. \tag{25}$$

Proof. Use the Weyl dimension polynomial (3) quoted in the proof of Proposition 3.1 and ψ and R^+ as given above. We omit the details. $\square$

We need to evaluate χ_λ at the specific element

$$t_\varphi = \mathrm{diag}(e^{i\varphi}, 1, \ldots, 1, e^{-i\varphi}, 1, \ldots, 1)\,.$$

Notation: From now on we will write $\chi_\lambda(\varphi)$ for $\chi_\lambda(t_\varphi)$.

Proposition 5.2 *For any index* λ,

$$\chi_\lambda(\varphi) = \sum_{j=1}^{n} (-1)^{j-1} \frac{\sin(\lambda_j \varphi)}{2^{2n-2}(\sin(\varphi/2))^{2n-2} \sin\varphi} d_{\widehat{\lambda_j}}^{(n-1)} \tag{26}$$

where $d_{\widehat{\lambda_j}}^{(n-1)}$ *denotes the dimension of the irreducible representation of* $Sp(n-1)$ *of index* $(\lambda_1, \ldots, \widehat{\lambda_j}, \ldots, \lambda_n)$ (*and the hat symbol indicates deletion*).

Proof. Use (24) and set $\varphi_1 = \varphi$. Eventually we will let $\varphi_r \to 0$ for $2 \leqq r \leqq n$. We can rewrite the numerator in (24) as

$$\sum_{j=1}^{n} (-1)^{j-1}[\exp(i\lambda_j\varphi) - \exp(-i\lambda_j\varphi)] \sum_{\sigma \in S_{n-1}} \mathrm{sgn}(\sigma) \left(\prod_{s=2}^{n} \varepsilon_s\right) \exp\left(i \sum_{r=2}^{n} \varepsilon_r \lambda_{\sigma(r)} \varphi_r\right)$$

where we have used the same notational convention as in the proof of Proposition 3.2. Furthermore, we can rewrite the denominator in (24) as

$$2i \sin\varphi \prod_{2 \leqq s \leqq n} 2i \sin\left(\frac{1}{2}(\varphi_s \pm \varphi)\right) \prod_{2 \leqq k < l \leqq n} 2i \sin\left(\frac{1}{2}(\varphi_l \pm \varphi_k)\right)$$
$$\times \prod_{2 \leqq k \leqq n} 2i \sin \varphi_k\,.$$

Taking the quotient of these two expressions and letting $\varphi_r \to 0$ for $2 \leqq r \leqq n$ yields (26). $\square$

Next, let us cite a necessary tool for the computation of the Fourier transform $\hat{\eta}(\chi_\lambda)$ for each irreducible representation ρ_λ.

The Weyl integral formula

Let G be a compact connected Lie group, T a maximal torus, and f a continuous function on G. The *adjoint representation* Ad acts on the Lie algebra $\mathbf{g}$ of G. We can choose an inner product on $\mathbf{g}$ that is invariant under the action of Ad. The Lie algebra $\mathbf{t}$ of T is a subspace of $\mathbf{g}$ and we denote its orthogonal complement by $L(G/T)$. This vector space $L(G/T)$ is isomorphic to the tangent space of G/T at eT. We now restrict the adjoint representation from G to T. The resulting representation $\mathrm{res}_T^G(\mathrm{Ad})$ of T decomposes into two subrepresentations according to the decomposition

$$\mathbf{g} = L(G/T) \oplus \mathbf{t}\,.$$

The representation of T on the first summand is denoted by

$$\mathrm{Ad}_{G/T} : T \to \mathrm{Gl}(L(G/T))\,.$$

We are now able to state the Weyl integral formula:

$$|W_G| \cdot \int_G f(g)\, d_G(g) = \int_T [\det(I_{G/T} - \mathrm{Ad}_{G/T}(t^{-1})) \int_G f(gtg^{-1})\, d\vartheta_G(g)]\, d\vartheta_T(t) \tag{27}$$

where $I_{G/T}$ denotes the identity map on $L(G/T)$, ϑ_G and ϑ_T denote Haar measure on G and T, respectively, and $|W_G|$ denotes the order of the *Weyl group* W_G of G. For $Sp(n)$, the Weyl group is the hyperoctahedral group (permutations and sign changes).

Proposition 5.3 *Let λ be an admissible index and η as defined in Sect. 1. Then*

$$R_\lambda := \frac{\hat{\eta}(\chi_\lambda)}{d_\lambda} = \frac{(2n)(2n+2)((n-1)!)^2}{8\prod_{j=2}^n(\lambda_j^2 - 1)} \tag{28}$$

for $\lambda = (1, \lambda_2, \ldots, \lambda_n)$ and $R_\lambda = 0$ otherwise.

Proof. Write $\chi_\lambda(h)$ for χ_λ $(\mathrm{diag}(h, 1, \ldots, 1))$. We need to compute $\hat{\eta}(\chi_\lambda) = \int_{Sp(1)} \chi_\lambda(h)\, d\varepsilon(h)$. Now χ_λ, regarded as a function on $Sp(1) \cong SU(2)$ in this way, is a class function and is thus determined by its values on elements of the form

$$t = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}.$$

(These elements form a maximal torus T in $Sp(1)$.) For any $h \in Sp(1)$, let $e^{\pm i\varphi_h}$ with $\varphi_h \in [0, 2\pi)$ be the eigenvalues of h. Let ϑ^0 denote Haar measure on $Sp(1)$. Then, from the definition of ε, we have

$$\hat{\eta}(\chi_\lambda) = \int_{Sp(1)} \chi_\lambda(h)\, d\varepsilon(h) = c_n \int_{Sp(1)} \chi_\lambda(h) \left(\sin \frac{\varphi_h}{2}\right)^{2n-2} d\vartheta^0(h) \tag{29}$$

with c_n a normalizing constant.

We will compute the right hand side in (29) by using the Weyl integral formula: $\chi_\lambda(h)(\sin(\varphi_h/2))^{2n-2}$ is a continuous function on $Sp(1)$, and the order of the Weyl group of $Sp(1)$ is 2. Thus we get

$$\begin{aligned} &2 \int_{Sp(1)} \chi_\lambda(h) \left(\sin \frac{\varphi_h}{2}\right)^{2n-2} d\vartheta^0(h) \\ &= \int_T \Bigg[\det(I_{Sp(1)/T} - \mathrm{Ad}_{Sp(1)/T}(t^{-1})) \\ &\quad \times \int_{Sp(1)} \chi_\lambda(hth^{-1}) \left(\sin \frac{\varphi_{hth^{-1}}}{2}\right)^{2n-2} d\vartheta^0(h)\Bigg]\, d\vartheta^1(t)\,, \end{aligned} \tag{30}$$

where ϑ^1 denotes Haar measure on T.

The integrand in the integral over $Sp(1)$ on the right hand side of (30) is conjugacy-invariant and therefore (30) becomes

$$\int_T \det(I_{Sp(1)/T} - \mathrm{Ad}_{Sp(1)/T}(t^{-1}))\chi_\lambda(t) \left(\sin \frac{\varphi_t}{2}\right)^{2n-2} d\vartheta^1(t)\,. \tag{31}$$

Recall that $\mathrm{Ad}_{Sp(1)/T}$ is the representation of T on the tangent space of $Sp(1)/T$ at eT (where e is the identity element in $Sp(1)$) derived from the adjoint representation. Now, the tangent space of $Sp(1)$ at e (i.e., the Lie algebra of $Sp(1) \cong SU(2)$) is a 3-dimensional real vector space spanned by

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

(i.e., the skew-Hermitian 2×2 matrices of trace 0). Therefore, the tangent space of $Sp(1)/T$ at eT is isomorphic to the 2-dimensional real vector space with basis

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

The complexification of this vector space is the 2-dimensional complex vector space with basis

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Set

$$t = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}.$$

We then can verify that

$$\mathrm{Ad}_{Sp(1)/T}(t^{-1})\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & e^{-2i\varphi} \\ 0 & 0 \end{pmatrix},$$

$$\mathrm{Ad}_{Sp(1)/T}(t^{-1})\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ e^{2i\varphi} & 0 \end{pmatrix}.$$

Thus we get

$$I_{Sp(1)/T} - \mathrm{Ad}_{Sp(1)/T}(t^{-1}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} e^{-2i\varphi} & 0 \\ 0 & e^{2i\varphi} \end{pmatrix}$$

and

$$\det(I_{Sp(1)/T}) - \mathrm{Ad}_{Sp(1)/T}(t^{-1})) = 2 - 2\cos 2\varphi = 4\sin^2\varphi .$$

We can now rewrite (31) as

$$\frac{2}{\pi}\int_0^{2\pi} \sin^2\varphi \left(\sin\frac{\varphi}{2}\right)^{2n-2} \chi_\lambda(\varphi)\, d\varphi \tag{32}$$

with $\chi_\lambda(\varphi)$ given by (26). To compute (32), we first note that for integer $a \geqq 1$

$$\int_0^{2\pi} \sin\varphi \cdot \sin(a\varphi)\, d\varphi = \begin{cases} \pi & \text{if } a = 1 , \\ 0 & \text{otherwise} . \end{cases}$$

Altogether, we now have from (29)–(32) and (26) that

$$\hat{\eta}(\chi_\lambda) = \begin{cases} c_n \dfrac{1}{2^{2n-2}} d_{\hat{\lambda}_1}^{(n-1)} & \text{for } \lambda \text{ with } \lambda_1 = 1\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

The normalizing constant c_n can be computed from $1 = \hat{\eta}(\chi_{\lambda^0})$ (where $\lambda^0 = (1,2,\ldots,n)$ is the index of the trivial representation) and turns out to be

$$4^{n-1}(n+1)/\binom{2n}{n}\,.$$

Finally, with the use of (25) we get

$$\begin{aligned} R_\lambda &= \frac{(2n-1)!4^{n-1}(n+1)}{2^{2n-2}\cdot\binom{2n}{n}} \cdot \frac{1}{\Pi_{j=2}^n(\lambda_j^2-1)} \\ &= \frac{(2n+2)(2n)((n-1)!)^2}{8\Pi_{j=2}^n(\lambda_j^2-1)} \end{aligned}$$

for λ with $\lambda_1 = 1$ and $R_\lambda = 0$ otherwise. □

6 Proof of Theorem Q1

We will follow very closely the proof of Theorem 1.1 in [P1] and need to make only a few adaptations. By Lemma 2.5 and (1) it suffices to show that for any integer $n \geqq 8$ and any positive real number $c \geqq 11.6$

$$\sum_\lambda \left|\frac{\hat{\eta}(\chi_\lambda)}{d_\lambda}\right|^{2k} d_\lambda^2 - 1 \leqq 51\, e^{-c/2.5} \tag{33}$$

for $k = \frac{1}{2}n\log n + cn$.

By Proposition 5.3, we only need to consider indices λ for which $\lambda_1 = 1$ in order to estimate (33). From now on, we will assume this condition on λ. Set $\lambda^0 := (1,2,\ldots,n)$ and $b := \lambda - \lambda^0$.
We can see that

$$\frac{\hat{\eta}(\chi_\lambda)}{d_\lambda} = \frac{(2n+2)(2n)((n-1)!)^2}{8\,\Pi_{j=2}^n(\lambda_j^2-1)} \leqq \left(\frac{4}{3}\right)^{n-1}\frac{(n!)^2}{\Pi_{j=2}^n\lambda_j^2}\,.$$

Set

$$r_\lambda := \left(\frac{4}{3}\right)^{n-1}\frac{(n!)^2}{\Pi_{j=2}^n\lambda_j^2}\,.$$

Proposition 6.1 *For $n \geqq 8$ and all indices λ with $\lambda_n \leqq 9n$, we have*

$$(r_\lambda)^k d_\lambda \leqq (5e^{-c/5})^{b_2+b_3+\cdots+b_n}\,.$$

Proof. We need the following two lemmas.

Lemma 6.2 *Let $d(T) := d_\lambda$ with $\lambda = (1, 2, \ldots, n-1, T)$. Then*

$$\frac{d(T+1)}{d(T)} = 1 + \frac{2n-1}{T+1-n} \leqq 1 + \frac{2n}{T+1-n}\,.$$

Lemma 6.3 *Let $r(T) := r_\lambda$ with $\lambda = (1, 2, \ldots, n-1, T)$. Then*

$$\frac{r(T+1)}{r(T)} = \frac{T^2}{(T+1)^2} \leqq e^{-2/(T+1)}\,.$$

From this point on we can almost duplicate the proof of Proposition 1.1 in [P1]. The only two changes are as follows:

(1) We have a slightly different definition for the b_i^*'s: here we use $b_i^* := T + 1 - i$ for $2 \leqq i \leqq n$.
(2) In (c) of Part A of the proof, take $0.5n \leqq b_n^* \leqq 9n$. □

Using the same argument as in the proof of Theorem 1.1 in [P1], we can now conclude that

$$\sum_{\lambda:\, \lambda_n \leqq 9n} (r_\lambda)^{2k} d_\lambda^2 - 1 \leqq 2(5e^{-c/5})^2 = 50e^{-c/2.5}$$

provided that $c \geqq c_0$, some universal constant. (An estimate for c_0 yields $c_0 = 11.6$; we omit the details.)
We still need to bound the tail of the sum in (33). Indeed, we will show that

$$\sum_{\lambda:\, \lambda_n > 9n} (r_\lambda)^{2k} d_\lambda^2 \leqq e^{-c/2.5}$$

for $c \geqq 11.6$ and $k = \frac{1}{2} n \log n + cn$.
From now on we will denote the m-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ with $\lambda_1 = 1$ by $\lambda^{(m)}$ for $1 \leqq m \leqq n$. Accordingly, we will use

$$d_{\lambda^{(m)}} := \frac{1}{1!3!\ldots(2m-1)!} \prod_{j=1}^{m} \lambda_j \prod_{1 \leqq r < s \leqq m} (\lambda_s^2 - \lambda_r^2) \tag{34}$$

and

$$r_{\lambda^{(m)}} := \left(\frac{4}{3}\right)^{m-1} \frac{(m!)^2}{\prod_{j=2}^{m} \lambda_j^2}\,. \tag{35}$$

The following lemma completes the proof of Theorem Q1.

Lemma 6.4 *We have*

$$\sum_{\lambda^{(m)}} (r_{\lambda^{(m)}})^{2k} d_{\lambda^{(m)}}^2 \leqq 1 + 51e^{-c/2.5} \tag{36}$$

for $1 \leqq m \leqq n$ and integer $k = \frac{1}{2} n \log n + cn$.

Proof. We use induction on m. For $m = 1$:

$$d_{\lambda^{(1)}} = 1, \quad r_{\lambda^{(1)}} = 1$$

and

$$\sum_{\lambda^{(1)}} (r_{\lambda^{(1)}})^{2k} d^2_{\lambda^{(1)}} = 1 .$$

Notice that in order to prove (36) for $m > 1$, we need only show that

$$\sum_{\lambda^{(m)}:\, \lambda_m > 9n} (r_{\lambda^{(m)}})^{2k} d^2_{\lambda^{(m)}} \leqq e^{-c/2.5} .$$

Indeed, we can apply the very same reasoning as in the proof of Theorem 1.1 in [P1] to show that

$$\sum_{\lambda^{(m)}:\, \lambda_m \leqq 9n} (r_{\lambda^{(m)}})^{2k} d^2_{\lambda^{(m)}} \leqq 1 + 50e^{-c/2.5}$$

follows from the first part of our proof.
From (34) and (35) we get

$$d_{\lambda^{(m)}} \leqq \lambda_m^{2m-1} d_{\lambda^{(m-1)}}$$

and

$$r_{\lambda^{(m)}} = \frac{4}{3}\frac{m^2}{\lambda_m^2} r_{\lambda^{(m-1)}}$$

for $2 \leqq m \leqq n$. We now have

$$\sum_{\lambda^{(m)}:\, \lambda_m > 9n} (r_{\lambda^{(m)}})^{2k} d^2_{\lambda^{(m)}} \leqq \left[\sum_{\lambda_m > 9n} \left(\frac{4}{3}\frac{m^2}{\lambda_m^2} \right)^{2k} \lambda_m^{4m-2} \right] \sum_{\lambda^{(m-1)}} (r_{\lambda^{(m-1)}})^{2k} d^2_{\lambda^{(m-1)}} . \tag{37}$$

By our induction hypothesis, the sum over $\lambda^{(m-1)}$ on the right hand side of (37) is less than or equal to $1 + 51e^{-c/2.5}$, which is smaller than (say) 2 for $c \geqq 11.6$. Then

$$\begin{aligned}\sum_{\lambda^{(m)}:\, \lambda_m > 9n} (r_{\lambda^{(m)}})^{2k} d^2_{\lambda^{(m)}} &\leqq 2 \sum_{\lambda_m > 9n} \left(\frac{4}{3}\frac{m^2}{\lambda_m^2} \right)^{2k} \lambda_m^{4m-2} \\ &\leqq 2 \left(\frac{4}{3} n^2 \right)^{2k} \int_{9n}^{\infty} \frac{1}{x^{4k-4n+2}}\, dx . \end{aligned} \tag{38}$$

This integral can easily be evaluated, and by estimates extremely similar to the ones used in the proof of Theorem 1.1 in [P1] we see that (38) is less than or equal to $e^{-c/2.5}$ for $n \geqq 8$, $c \geqq 11.6$. We omit the details. □

7 Lower bounds

Recall that $\|\mu_k - \vartheta\|_{TV} = \sup_{S \in \mathscr{B}(G)} |\mu_k(S) - \vartheta(S)|$. We will construct a suitable test set S to prove our lower bound results (Theorem C2 and Theorem Q2). Briefly, under Haar measure ϑ, $\operatorname{Re}\chi_1$, where χ_1 is the character of the natural representation, is with high probability close to 0 (in fact, $\operatorname{Re}\chi_1$ is almost distributed as a standard normal random variable for large N). On the other hand, we show that under μ_k with $k = \frac{1}{2}N \log N - cN$, with high probability

$\mathrm{Re}\,\chi_1$ is still large (close to N). For a suitable positive value B, our test set S can then be chosen to be $S = \{g \in U(N)\colon\ -N \leqq \mathrm{Re}\,\chi_1(g) \leqq B\}$. The idea of this proof is not new. It has been developed by Diaconis and has been applied since then by various authors ([D, P1, R, S-C]).

Proof of Theorem C2. Here we can choose S to be $S = \{g \in U(N)\ \mathrm{Re}\,\chi_1(g) \leqq e^c/\sqrt{3}\}$. Recall that by the orthonormality of the irreducible characters for any compact Lie group we have

$$E_\vartheta(\mathrm{Re}\,\chi_1) = \mathrm{Re}(E_\vartheta(\chi_1)) = 0\,,$$
$$E_\vartheta((\mathrm{Re}\,\chi_1)^2) \leqq E_\vartheta(|\chi_1|^2) = 1$$

and hence $\mathrm{Var}_\vartheta(\mathrm{Re}\,\chi_1) \leqq 1$.

Proposition 7.1 *For $N \geqq 6$ and integer $k = n\log n - cn$, where $c > 0$,*
(a) $E_{\mu_k}(\mathrm{Re}\,\chi_1) \geqq (2/\sqrt{3})e^c$;
(b) $\mathrm{Var}_{\mu_k}(\mathrm{Re}\,\chi_1) \leqq 1$.

Proof. (a) The natural representation ρ_1 has highest weight $\omega = (0,\ldots,0,1)$. Thus we can assign the index $\lambda = (-n, -n+1, \ldots, -1, 0, 1, \ldots, n-1, n+1)$ to ρ_1 for $N = 2n+1$ odd, and the index $\lambda = (-(2n-1)/2, \ldots, -\frac{1}{2}, \frac{1}{2}, \ldots, (2n-3)/2, (2n+1)/2)$ to ρ_1 for $N = 2n$ even. The dimension d_1 of ρ_1 is of course N, and thus, by Proposition 3.3,

$$\frac{\hat{\mu}(\chi_1)}{d_1} = \frac{n}{n+1} \quad \text{for } N = 2n+1\,,$$
$$\frac{\hat{\mu}(\chi_1)}{d_1} = \frac{2n-1}{2n+1} \quad \text{for } N = 2n\,, \tag{39}$$

and therefore

$$E_{\mu_k}(\mathrm{Re}\,\chi_1) = \left(1 - \frac{1}{n+1}\right)^k (2n+1) \geqq \left(1 - \frac{1}{n}\right)^k 2n \quad \text{for } N = 2n+1\,,$$
$$E_{\mu_k}(\mathrm{Re}\,\chi_1) = \left(1 - \frac{2}{2n+1}\right)^k 2n \geqq \left(1 - \frac{1}{n}\right)^k 2n \quad \text{for } N = 2n\,.$$

Now

$$\left(1 - \frac{1}{n}\right)^k 2n \geqq 2n\left(1 - \frac{1}{n}\right)^{n\log n} e^c$$

for $n \geqq 2$ since $1 - x \leqq e^{-x}$ for $x \leqq 1$. Also,

$$\log\left(1 - \frac{1}{n}\right) = -\frac{1}{n} - \frac{1}{2n^2} - \frac{1}{3n^3} - \ldots \geqq -\frac{1}{n}\left(\frac{1}{1 - 1/n}\right)\,.$$

It follows that

$$n\left(1 - \frac{1}{n}\right)^{n\log n} = ne^{\log(1-1/n)\cdot n\log n} \geqq ne^{-(n/(n-1))\log n} = n^{-1/(n-1)}\,.$$

But $f(x) = x^{-1/(x-1)}$ is strictly increasing for $x \geqq 3$, with $f(3) = 1/\sqrt{3}$. It follows that $E_{\mu_k}(\operatorname{Re}\chi_1) \geqq (2/\sqrt{3})e^c$ for $N \geqq 6$.

(b) We now consider $\rho_1 \otimes \bar{\rho}_1$, where $\bar{\rho}_1$ is the complex conjugate representation which assigns to each $g \in U(N)$ the complex conjugate of its defining matrix. $\rho_1 \otimes \bar{\rho}_1$ is a new representation of $U(N)$ of dimension N^2 and with character $|\chi_1|^2$. We can easily establish the decomposition of $\rho_1 \otimes \bar{\rho}_1$ into its irreducible constituents with the use of the following lemma (see [BtD, FH]).

Lemma 7.2 *Let χ_γ denote the character of the irreducible representation corresponding to highest weight γ. Then*

$$\chi_\gamma \cdot \chi_v = \chi_{\gamma+v} + \sum_\omega n_\omega \chi_\omega$$

where the sum is over $\omega < \gamma + v$ with respect to the usual ordering of weights and the coefficients n_ω are all in $\mathbf{N}_0$.

Since $\overline{\rho_1}$ has highest weight $(-1, 0, \ldots, 0)$, by Lemma 7.2, $\rho_1 \otimes \overline{\rho_1}$ contains exactly one copy of the irreducible representation, call it ρ_2, of highest weight $(-1, 0, \ldots, 0, 1)$. We assign to ρ_2 the index

$$\lambda = (-n-1, -n+1, \ldots, -1, 0, 1, \ldots, n-1, n+1) \quad \text{for } N = 2n+1 \text{ odd}$$

and

$$\lambda = \left(-\frac{2n+1}{2}, -\frac{2n-3}{2}, \ldots, -\frac{1}{2}, \frac{1}{2}, \ldots, \frac{2n-3}{2}, \frac{2n+1}{2}\right) \quad \text{for } N = 2n \text{ even}\,.$$

By using the dimension formula (2) we can verify that ρ_2 has dimension $d_2 = N^2 - 1$.

Furthermore, from $E_\vartheta(|\chi_1|^2) = 1$ and the orthonormality of the irreducible characters it follows that $\rho_1 \otimes \overline{\rho_1}$ contains exactly one copy of the trivial representation ρ_0.

We thus have established the decomposition

$$\rho_1 \otimes \bar{\rho}_1 = \rho_0 \oplus \rho_2$$

into irreducibles, and hence

$$|\chi_1|^2 = 1 + \chi_2\,,$$

$$E_{\mu_k}(|\chi_1|^2) = 1 + E_{\mu_k}(\chi_2)\,.$$

Remark 7.3. $\rho_1 \otimes \bar{\rho}_1$ is in fact the *adjoint representation* of $U(N)$ on its complexified Lie algebra.

From Proposition 3.3 we get

$$\frac{\hat{\mu}(\chi_2)}{d_2} = \left(\frac{n}{n+1}\right)^2 \quad \text{for } N = 2n+1 \text{ and } \quad \frac{\hat{\delta}(\chi_2)}{d_2} = \left(\frac{2n-1}{2n+1}\right)^2 \quad \text{for } N = 2n. \tag{40}$$

Therefore, by (39) and (40),

$$\begin{aligned}\mathrm{Var}_{\mu_k}(\mathrm{Re}\,\chi_1) &\leqq 1+\left(\frac{n}{n+1}\right)^{2k}((2n+1)^2-1)-\left(\frac{n}{n+1}\right)^{2k}(2n+1)^2\\ &=1-\left(\frac{n}{n+1}\right)^{2k}\leqq 1 \quad \text{for } N=2n+1\end{aligned}$$

and similarly

$$\begin{aligned}\mathrm{Var}_{\mu_k}(\mathrm{Re}\,\chi_1) &\leqq 1+\left(\frac{2n-1}{2n+1}\right)^{2k}((2n)^2-1)-\left(\frac{2n-1}{2n+1}\right)^{2k}(2n)^2\\ &=1-\left(\frac{2n-1}{2n+1}\right)^{2k}\leqq 1 \quad \text{for } N=2n\,. \qquad \square\end{aligned}$$

We can now conclude the proof of Theorem C2. Using Chebychev's inequality we get

$$P_\vartheta\left(\mathrm{Re}\,\chi_1 > \frac{1}{\sqrt{3}}e^c\right) \leqq 3e^{-2c}$$

and

$$P_{\vartheta_k}\left(\mathrm{Re}\,\chi_1 \leqq \frac{1}{\sqrt{3}}e^c\right) \leqq 3e^{-2c}\,,$$

so

$$\begin{aligned}\|\mu_k-\vartheta\|_{TV} &\geqq |\,\mu_k(S)-\vartheta(S)|\\ &=\left|1-P_\vartheta\left(\mathrm{Re}\,\chi_1>\frac{1}{\sqrt{3}}e^c\right)-P_{\mu_k}\left(\mathrm{Re}\,\chi_1\leqq\frac{1}{\sqrt{3}}e^c\right)\right|\\ &\geqq 1-6e^{-2c}\,. \qquad \square\end{aligned}$$

Proof of Theorem Q2. We will use the same overall outline of proof as for the complex case.

Let χ_1 denote the character of the ((2n)-dimensional) natural representation of $Sp(n)$. Notice that χ_1 is a real-valued function. We will choose S to be $S=\{g\in Sp(n)\colon \chi_1(g)\leqq e^{2c}/\sqrt{3}\}$. Again, by the orthonormality of the irreducible characters we have $E_\vartheta(\chi_1)=0$, $E_\vartheta(\chi_1^2)=1$ and hence $\mathrm{Var}_\vartheta(\chi_1)=1$.

Proposition 7.4 *For integer* $k=\frac{1}{2}n\log n-cn$ *with* $c>0$,

(a) $E_{\eta_k}(\chi_1)\geqq(2/\sqrt{3})e^{2c}$ if $n\geqq 3$;

(b) $\mathrm{Var}_{\eta_k}(\chi_1)\leqq 1+25e^{4c}\log n/n^{1/3}$ if $n\geqq 6$.

Proof. (a) The natural representation ρ_1 has highest weight $\omega=(0,\dots,0,1)$ and dimension $d_1=2n$. We assign the index $\lambda=(1,\dots,n-1,n+1)$ to ρ_1. By Proposition 5.3, $\hat{\eta}(\chi_1)/d_1=(n^2-1)/((n+1)^2-1)$, and therefore

$$E_{\eta_k}(\chi_1)=\left(\frac{n^2-1}{(n+1)^2-1}\right)^k 2n\geqq\left(\frac{n-1}{n}\right)^{2k}2n=\left(\frac{n-1}{n}\right)^{n\log n-2cn}2n\,.$$

In the proof of Proposition 7.1(a) we have shown that

$$\left(1-\frac{1}{n}\right)^{n\log n-cn} 2n \geqq \frac{2}{\sqrt{3}}e^{c} ,$$

from which the statement of Proposition 7.4(a) directly follows.

(b) $\mathrm{Var}_{\eta_k}(\chi_1) = E_{\eta_k}(\chi_1^2) - (E_{\eta_k}(\chi_1))^2$.

We now consider the representation $\rho_1 \otimes \rho_1$, which is of dimension $4n^2$ and has character χ_1^2, and study its decomposition into its irreducible constituents. By Lemma 7.2, $\rho_1 \otimes \rho_1$ contains exactly one copy of the irreducible representation, call it ρ_2, of highest weight $(0,\dots,0,2)$. ρ_2 corresponds to the index $\lambda = (1,2,\dots,n-1,n+2)$ and is of dimension $d_2 = 2n^2+n$. Secondly, from $E_\vartheta(\chi_1^2) = 1$ it is clear that $\rho_1 \otimes \rho_1$ contains exactly one copy of the trivial representation ρ_0. We will determine the third and last irreducible constituent of $\rho_1 \otimes \rho_1$ with the use of the following lemma (see [BtD, FH]).

Proposition 7.5 *Let χ_γ denote the character of the irreducible representation corresponding to dominant weight γ. Let α be a simple root. If $\langle\gamma,\alpha\rangle$ and $\langle\omega,\alpha\rangle$ are both not zero, where $\langle\gamma,\alpha\rangle := \sum_{j=1}^n \gamma_j\alpha_j$, then $\gamma+\omega-\alpha$ is a dominant weight,*

$$\chi_\gamma \cdot \chi_\omega = \chi_{\gamma+\omega} + \chi_{\gamma+\omega-\alpha} + \textit{others} ,$$

and $\chi_{\gamma+\omega-\alpha}$ occurs with multiplicity 1.

For $Sp(n)$, our choice for a system of simple roots S is $S = \{e_{j+1} - e_j \colon 1 \leqq j < n\} \cup \{2e_n\}$. Clearly, $\alpha = (0,\dots,0,-1,1)$ fulfills the condition of Lemma 7.5 when $\gamma = \omega = (0,\dots,0,1)$. Thus we conclude that $\rho_1 \otimes \rho_1$ contains exactly one copy of the irreducible representation, call it ρ_3, corresponding to highest weight $(0,\dots,0,1,1)$. ρ_3 corresponds to the index $\lambda = (1,2,\dots,n-2,n+1,n+2)$ and is of dimension $d_3 = 2n^2-n-1$.

We thus have established the decomposition

$$\rho_1 \otimes \rho_1 = \rho_0 \oplus \rho_2 \oplus \rho_3$$

into its irreducibles and hence

$$\chi_1^2 = 1 + \chi_2 + \chi_3 ,$$

$$E_{\eta_k}(\chi_1^2) = 1 + E_{\eta_k}(\chi_2) + E_{\eta_k}(\chi_3) .$$

From Proposition 5.3 we get

$$\frac{\hat{\eta}(\chi_2)}{d_2} = \frac{n-1}{n+3} \quad\text{and}\quad \frac{\hat{\eta}(\chi_3)}{d_3} = \frac{n-2}{n+2} .$$

Therefore,

$$\begin{aligned}
\mathrm{Var}_{\eta_k}(\chi_1) &= 1 + \left(\frac{n-1}{n+3}\right)^k (2n^2+n) \\
&\quad + \left(\frac{n-2}{n+2}\right)^k (2n^2-n-1) - \left(\frac{n^2-1}{(n+1)^2-1}\right)^{2k} 4n^2 \\
&\leqq 1 + \left(\frac{n-1}{n+3}\right)^k (2n^2+n) + \left(\frac{n-2}{n+2}\right)^k (2n^2-n) - \left(\frac{n-1}{n}\right)^{4k} 4n^2
\end{aligned}$$

$$\leqq 1 + 4n^2 \left[\left(1 - \frac{4}{n+3}\right)^k - \left(1 - \frac{1}{n}\right)^{4k} \right]$$

$$+ n \left[\left(1 - \frac{4}{n+3}\right)^k - \left(1 - \frac{4}{n+2}\right)^k \right]$$

$$\leqq 1 + 4n^2 \left[\left(1 - \frac{4}{n+3}\right)^k - \left(1 - \frac{4}{n}\right)^k \right]$$

$$+ n \left[\left(1 - \frac{4}{n+3}\right)^k - \left(1 - \frac{4}{n+2}\right)^k \right]$$

$$\leqq 1 + (4n^2 + n) \left[\left(1 - \frac{4}{n+3}\right)^k - \left(1 - \frac{4}{n}\right)^k \right] .$$

By the mean value theorem,

$$\left(1 - \frac{4}{n+3}\right)^k - \left(1 - \frac{4}{n}\right)^k$$

$$\leqq k \left(1 - \frac{4}{n+3}\right)^{k-1} \left[\left(1 - \frac{4}{n+3}\right) - \left(1 - \frac{4}{n}\right) \right]$$

$$= k \left(1 - \frac{4}{n+3}\right)^k \frac{12}{n(n-1)}$$

$$\leqq n \log n \exp \left(- \frac{4}{n+3} \left(\frac{1}{2} n \log n - cn \right) \right) \frac{6}{n(n-1)}$$

$$= \frac{\log n}{n} \frac{6n}{n-1} n^{-2n/(n+3)} e^{4cn/(n+3)}$$

$$\leqq 6 \frac{\log n}{n} n^{-4/3} e^{4c} \quad \text{for } n \geqq 6$$

$$= 6e^{4c} \frac{\log n}{n^{7/3}} .$$

From this it follows that

$$\mathrm{Var}_{\eta_k}(\chi_1) \leqq 1 + 24e^{4c} \frac{\log n}{n^{1/3}} + 6e^{4c} \frac{\log n}{n^{4/3}} \leqq 1 + 25e^{4c} \frac{\log n}{n^{1/3}} . \qquad \square$$

We now have by Chebychev's inequality

$$P_\vartheta \left(\chi_1 \leqq \frac{e^{2c}}{\sqrt{3}} \right) \geqq 1 - 3e^{-4c}$$

and

$$P_{\eta_k} \left(\chi_1 \leqq \frac{e^{2c}}{\sqrt{3}} \right) \leqq 3e^{-4c} \left(1 + 25e^{4c} \frac{\log n}{n^{1/3}} \right) ,$$

so

$$\|\eta_k - \vartheta\|_{TV} \geqq \left| P_\vartheta\left(\chi_1 \leqq \frac{e^{2c}}{\sqrt{3}}\right) - P_{\eta_k}\left(\chi_1 \leqq \frac{e^{2c}}{\sqrt{3}}\right)\right|$$
$$\geqq 1 - 6e^{-4c} - 75\frac{\log n}{n^{1/3}} \quad \text{for } n \geqq 6\,. \qquad \square$$

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