Probab. Theory Relat. Fields 103, 553–570 (1995)

Probability Theory and Related Fields

A generalized notion of variation applied to Markov chains and Anosov maps

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Received: 26 April 1994 / In revised form: 1 November 1994

Summary. Extending the operator formalism of [3] we show that there exists a large class of functions which possess an exponential decay of correlations and fulfill a central limit theorem under a certain type of Markov chains. This result can be applied to the symbolic dynamics of Anosov maps, showing that in the case of a absolutely continuous invariant measure there is a large class of functions with good ergodic properties – larger than the usual class of Hölder continuous functions.

Mathematics Subject Classifications: 60F05, 58F15

1 Definitions

Let $r \in \mathbb{N}$ be given. For sequences $\{\xi_s\}_{s\in\mathbb{Z}}$ of symbols $\xi_s \in \{1,\dots,r\}$ define the shift-by-one-digit map T by $\{\xi_s\}_{s\in\mathbb{Z}} \mapsto \{\xi_{s+1}\}_{s\in\mathbb{Z}}$. We look at a T-invariant subset $\Sigma \subset \{1,\dots,r\}^{\mathbb{Z}}$ of "admissible" two-sided symbol sequences. Σ is defined with the help of a fixed $r\times r$-matrix $\mathscr{T}$ whose entries are either 0 or 1: A sequence $\{\xi_s\}_{s\in\mathbb{Z}}$ is admissible if and only if $\mathscr{T}_{\xi_s,\xi_{s+1}} = 1$ for all $s \in \mathbb{Z}$. The admissible "future" symbol sequences form the subset Σ^+ of $\{1,\dots,r\}^{\mathbb{N}_0}$ which consists of all $\{\xi_s\}_{s\in\mathbb{N}_0}$ with $\mathscr{T}_{\xi_s,\xi_{s+1}} = 1$ for all $s \in \mathbb{N}_0$.

Σ and Σ^+ possess a natural measurable structure. Assume that we are given a probability measure $d\mu$ which is invariant and weakly mixing w.r.t. the shift T. (The further assumptions which will be introduced about $d\mu$ demand implicitly that T is topologically mixing, i.e. there exists $m \in \mathbb{N}$ such that $\mathscr{T}^m > 0$.) In the following, the norm of the space $L^p(d\mu)$, $p > 0$, will be denoted $\|\cdot\|_p$. The measure $d\mu$ induces a measure $d\mu^+$ on Σ^+ in a natural way.
Define a distance function on Σ by

* work supported by Studienstiftung des deutschen Volkes

$$d_\Sigma(\xi^1, \xi^2) := \exp(-\max\{n \in \mathbb{N}_0 : \xi^1_s = \xi^2_s \text{ for all } s = -n, \dots, n\}).$$

A similar distance function d_{Σ^+} can be introduced on Σ^+.
We assume that the conditional expectation $\mu(\xi_0|\xi_1, \xi_2, \dots)$ is strictly positive on Σ^+ and define a function $h : \{1, \dots, r\}^{\mathbb{N}_0} \to \mathbb{R}^+_0$ by

$$h(\nu_0, \nu_1, \nu_2, \dots) := \mu(\xi_0 = \nu_0 | \xi_1 = \nu_1, \xi_2 = \nu_2, \dots)$$

if $(\nu_0, \nu_1, \nu_2, \dots)$ is admissible and set $h(\nu_0, \nu_1, \nu_2, \dots) := 0$ else. We assume furthermore that there exist $\alpha_h > 0$ and $C_h < \infty$ such that for all $n \in \mathbb{N}_0$:

$$\sup_{\substack{\xi^1, \xi^2 \in \Sigma^+ \\ \xi^1_s = \xi_s = \xi^2_s, 0 \le s \le n}} |h(\xi^1) - h(\xi^2)| \le C_h e^{-\alpha_h n} \qquad \text{for all } \xi \in \Sigma^+. \tag{1}$$

Thus, h is continuous w.r.t. d and possesses a strictly positive infimum H on the compact space (Σ^+, d_{Σ^+}).

The aim of this paper is to find a large class of functions f such that $\int d\mu \bar{f} f \circ T^N$ decays exponentially and $N^{-1/2} \sum_{n=0}^{N-1} f \circ T^n$ converges to a Gaussian law as $N \to \infty$.

2 Generalized Hölder continuous functions

Now we generalize the notion of Hölder continuity w.r.t. the distance function d_Σ. To this end, we carry over the notion of generalized variation employed in the study of multidimensional piecewise expanding maps (see [2], [5], and the references therein.)

Definition. For $n \in \mathbb{N}_0$, $\xi \in \Sigma$, and $\tilde{f} : \Sigma \to \mathbb{C}$, define an *oscillation* by

$$\operatorname{osc}_n(\tilde{f}, \xi) := \sup\{|\tilde{f}(\xi^1) - \tilde{f}(\xi^2)| : \xi^1, \xi^2 \in \Sigma; \xi^1_s = \xi_s = \xi^2_s, -n \le s \le n\}.$$

For $\alpha > 0$ and $m \in \mathbb{N}_0$ the set $GH_{\alpha,m}$ consists by definition of all functions $f \in L^1(\Sigma \to \mathbb{C}, d\mu)$ with finite *variation*

$$\operatorname{var}_{\alpha,m}(f) := \inf_{\tilde{f} = f \text{ a.e.}} \sup_{n \ge m} e^{\alpha n} \int d\mu(\xi) \operatorname{osc}_n(\tilde{f}, \xi).$$

$GH_{\alpha,m}$ is equipped with the norm $\|\cdot\|_{\alpha,m} := \|\cdot\|_1 + \operatorname{var}_{\alpha,m}(\cdot)$.

The following *properties* of $GH_{\alpha,m}$ are obvious:

1. For $\alpha' \le \alpha$ and $m' \ge m$ every function $f \in GH_{\alpha,m}$ is contained in $GH_{\alpha',m'}$, too.
2. For all $f \in GH_{\alpha,m}$ and all Lipshitz continuous $\phi : \operatorname{img} f \to \mathbb{C}$, also $\phi \circ f \in GH_{\alpha,m}$ with $\operatorname{var}_{\alpha,m}(\phi \circ f) \le \operatorname{Lip}(\phi) \operatorname{var}_{\alpha,m}(f)$.
3. There exist $C < \infty$ such that $\|f\|_\infty \le C\|f\|_{\alpha,m}$ for every $f \in GH_{\alpha,m}$.
4. $\operatorname{var}_{\alpha,m}(fg) \le \operatorname{var}_{\alpha,m}(f)\|g\|_\infty + \operatorname{var}_{\alpha,m}(g)\|f\|_\infty$ for all f, $g \in GH_{\alpha,m}$.

Let $L^1(d\mu^+)$ denote the subspace of $L^1(d\mu)$ consisting of the functions that possess a $d\mu$-version which only depends on the symbols ξ_s with $s \geq 0$. Analogously, call $GH^+_{\alpha,m}$ the class of functions $f \in GH_{\alpha,m}$ possessing a $d\mu$-version f_0 which only depends on the symbols ξ_s with $s \geq 0$ such that

$$\infty > \mathrm{var}^+_{\alpha,m}(f) := \inf_{\widetilde{f} = f_0 \text{ a.e.}} \sup_{n \geq m} e^{\alpha n} \int d\mu^+(\xi)\, \mathrm{osc}^+_n(\widetilde{f}, \xi),$$

where the infimum runs over all functions $\widetilde{f} : \Sigma^+ \to \mathbb{C}$ and where

$$\mathrm{osc}^+_n(\widetilde{f}, \xi) := \sup\{|\widetilde{f}(\xi^1) - \widetilde{f}(\xi^2)| : \xi^1, \xi^2 \in \Sigma^+; \xi^1_s = \xi_s = \xi^2_s, 0 \leq s \leq n\}.$$

On $GH^+_{\alpha,m}$, a norm can be defined by $\|\cdot\|_{+,\alpha,m} := \|\cdot\|_1 + \mathrm{var}^+_{\alpha,m}(\cdot)$.

Theorem 1 *$GH_{\alpha,m}$ with the norm $\|\cdot\|_{\alpha,m}$ and $GH^+_{\alpha,m}$ with the norm $\|\cdot\|_{+,\alpha,m}$ are Banach spaces.*

Proof. (Only the $GH_{\alpha,m}$-part.) Let $\{f_l\}_{l\in\mathbb{N}}$ be a Cauchy sequence in $GH_{\alpha,m}$. Due to Property 3 of $GH_{\alpha,m}$ it is also a Cauchy sequence in $L^\infty(d\mu)$, so that it possesses a limit $f \in L^\infty(d\mu)$ with respect to $\|\cdot\|_\infty$.

Fix $\epsilon > 0$. By the Cauchy property for $\{f_l\}_{l\in\mathbb{N}}$ in $GH_{\alpha,m}$ we find: There exists $L \in \mathbb{N}$ such that for all $l \geq L$ there is some $d\mu$-version $\widetilde{f}_l$ of f_l with

$$\int d\mu(\xi)\, \mathrm{osc}_n(\widetilde{f}_l - \widetilde{f}_L, \xi) \leq \epsilon\, e^{-\alpha n} \tag{2}$$

for all $n \geq m$. We can assume that $|\widetilde{f}_l(\xi) - \widetilde{f}_L(\xi)|$ is bounded for all ξ and all $l \geq L$ by $C := 2\sup_l \|f_l\|_\infty < \infty$. There exists a set A with $\mu(A) = 1$ such that for all $x \in A$ we have $f(x) = \lim_l \widetilde{f}_l(x)$ and on A the modulus of all $\widetilde{f}_l$, $l \in \mathbb{N}$, is bounded by $C/2$.

Given these building blocks, we construct a $d\mu$-version $\widetilde{f}$ of f such that

$$\int d\mu(\xi)\, \mathrm{osc}_n(\widetilde{f} - \widetilde{f}_L, \xi) \leq \epsilon\, e^{-\alpha n} \tag{3}$$

for all $n \geq m$. This implies $\mathrm{var}_{\alpha,m}(f - f_L) < \epsilon$, hence $f = (f - f_L) + f_L \in GH_{\alpha,m}$, and f is $GH_{\alpha,m}$-limit of f_L, $L \to \infty$, which proves the theorem.

Construction of $\widetilde{f}$: Let $\widetilde{f}|_A := f|_A$. For ξ not in A, take some sequence $\{\xi^k(\xi)\}_{k\in\mathbb{N}}$ in A which converges to ξ with respect to d_Σ. Such a sequence exists, because otherwise ξ would possess a neighborhood disjoint from A, in contradiction to $\mu(A) = 1$. The sequence $\{\widetilde{f}(\xi^k(\xi)) - \widetilde{f}_L(\xi^k(\xi))\}_{k\in\mathbb{N}}$ is bounded and, hence, possesses at least one point of accumulation. Define $\widetilde{f}(\xi)$ in such a way that one of these points equals $\widetilde{f}(\xi) - \widetilde{f}_L(\xi)$.

This implies that for all $\xi^{1/2} \in \Sigma$, $\xi^{1/2}_s = \xi_s$ for $-n \leq s \leq n$, the value $|\widetilde{f}(\xi^1) - \widetilde{f}_L(\xi^1) - \widetilde{f}(\xi^2) + \widetilde{f}_L(\xi^2)|$, and hence $\mathrm{osc}_n(\widetilde{f} - \widetilde{f}_L, \xi)$, is bounded by

$$\sup\{|\widetilde{f}(\xi^1) - \widetilde{f}_L(\xi^1) - \widetilde{f}(\xi^2) + \widetilde{f}_L(\xi^2)| : \xi^{1/2} \in A; \xi^{1/2}_s = \xi_s, -n \leq s \leq n\}$$

(Note the appearance of A instead of the full set Σ!) because, if ξ^1 is not already an element of A, the value of $\widetilde{f}(\xi^1) - \widetilde{f}_L(\xi^1)$ is the limit of its values at some near (with respect to d_Σ) points $v \in A$. These v can be chosen so as to fulfill $v_s = \xi_s$, $-n \le s \le n$, so that they are contained in the set of ξ^1 in the supremum. A similar argument applies to ξ^2.

For $\xi^1 \in A$, the value $\widetilde{f}(\xi^1)$ is the limit of $\widetilde{f}_l(\xi^1)$. Thus, the above estimate leads to $\mathrm{osc}_n(\widetilde{f} - \widetilde{f}_L, \xi) \le \liminf_l \mathrm{osc}_n(\widetilde{f}_l - \widetilde{f}_L, \xi)$. Now we can apply Fatou's lemma to deduce Eqn. (3) from Eqn. (2) by interchanging $\liminf_l$ and integration $\int d\mu(\xi)$. □

Theorem 2 *The closed unit ball of $GH_{\alpha,m}$ is compact in $L^1(d\mu)$. Likewise, the closed unit ball of $GH^+_{\alpha,m}$ is compact in $L^1(d\mu^+)$.*

Proof. (Only the $GH_{\alpha,m}$-part.) We have to show: For every sequence $\{f_l\}_{l\in\mathbb{N}}$ in $GH_{\alpha,m}$ with $\|f_l\|_{\alpha,m} \le 1$ for all $l \in \mathbb{N}$ there exists some subsequence which converges in $L^1(d\mu)$ to some $f \in GH_{\alpha,m}$ with $\|f\|_{\alpha,m} \le 1$.

Due to Property 3, there exists $C < \infty$ such that for all $l \in \mathbb{N}$ we have $\|f_l\|_\infty \le C$, and hence $\|f_l\|_2 \le C$. Therefore there exists a subsequence of $\{f_l\}_{l\in\mathbb{N}}$ which converges weakly with respect to $L^2(d\mu)$ to some $f \in L^2(d\mu)$ with $\|f\|_2 \le C$. Call this converging subsequence again $\{f_l\}_{l\in\mathbb{N}}$.

Fix $\epsilon > 0$. By the definition of the unit ball in $GH_{\alpha,m}$ we can find for each f_l a $d\mu$-version $\widetilde{f}_l$ such that $|\widetilde{f}_l(\xi)| \le C$ for all ξ and

$$\int d\mu(\xi)\,\mathrm{osc}_n(\widetilde{f}_l, \xi) \le e^{-\alpha n}(1 - \|\widetilde{f}_l\|_1 + \epsilon) \tag{4}$$

for all $n \ge m$.

Next we show that $\widetilde{f}_l$ also converges strongly with respect to $\|\cdot\|_1$: Fix $M \ge m$. Choose a finite partition $\bigcup_i A_i$ of Σ into disjoint non-zero measurable sets A_i such that all symbol sequences from a specific set A_i coincide from the $-M$-th up to the M-th entry. Of $\widetilde{f}_l$ define a discrete approximation $F_l := \sum_i \mathbf{1}_i\, \mu(A_i)^{-1} \int_{A_i} d\mu \widetilde{f}_l$, where $\mathbf{1}_i$ is the indicator function of the set A_i. The weak convergence of $\widetilde{f}_l$ implies that F_l converges strongly with respect to $\|\cdot\|_1$ as $l \to \infty$. We can estimate the distance between F_l and $\widetilde{f}_l$:

$$\begin{aligned}\|F_l - \widetilde{f}_l\|_1 &\le \sum_i \int_{A_i} d\mu(\xi)\, \sup\{|\widetilde{f}_l(\xi^1) - \widetilde{f}_l(\xi)| : \xi^1 \in A_i\}\\ &\le (1 - \|\widetilde{f}_l\|_1 + \epsilon)\, e^{-\alpha M},\end{aligned}$$

where in the last step Eqn. (4) has been applied with $n = M$. Therefore, $\|\widetilde{f}_l - \widetilde{f}_k\|_1 \le 2(1+\epsilon)\, e^{-\alpha M} + \|F_l - F_k\|_1$. But the rhs. of this estimate tends to 0 as l, $k \to \infty$ because M can be chosen arbitrarily large and because F_l is strongly convergent in $L^1(d\mu)$. This shows that $\widetilde{f}_l$ is Cauchy with respect to $L^1(d\mu)$ and hence converges to f on a set A of full $d\mu$-measure. We construct a $d\mu$-version $\widetilde{f}$ of f similar to the proof of Theorem 1: Let $\widetilde{f}|_A := f|_A$. For ξ not in A, take some sequence $\{\xi^k(\xi)\}_{k\in\mathbb{N}}$ in A which converges to ξ. The sequence $\{\widetilde{f}(\xi^k(\xi))\}_{k\in\mathbb{N}}$ is bounded. Define $\widetilde{f}(\xi)$ to be one of its points of accumulation.

Using a similar argument as in the proof of Theorem 1 we find from Eqn. (4) that $\mathrm{var}_{\alpha,m}(f) \le 1 - \|f\|_1 + \epsilon$. But $\epsilon > 0$ has been arbitrary, and therefore $\|f\|_1 + \mathrm{var}_{\alpha,m}(f) \le 1$, so that f is an element of the unit ball of $GH_{\alpha,m}$. □

3 Decay of correlations

An operator P in $L^1(d\mu^+)$ is uniquely determined by demanding that for all $g \in L^\infty(d\mu^+)$:

$$\int d\mu^+ \, g \, P(f) = \int d\mu^+ (g \circ T) f. \tag{5}$$

P is a contractive mapping both of the space $L^1(d\mu^+)$ into itself and of the space $L^\infty(d\mu^+)$ into itself.

With the help of the function h which has been introduced in the beginning, $P(f)$ can also be defined pointwise:

$$P(f)(\xi) := \sum_{a=1}^{r} h(a\xi) f(a\xi), \tag{6}$$

where $a\xi$ is the symbol sequence with first entry a, second entry ξ_1, third ξ_2, and so on. Note that $a\xi$ is not necessarily an element of Σ^+, so that $f(a\xi)$ may not make sense. This problem is solved by the special form of h: It vanishes for symbol sequences which are no elements of Σ^+.

The following theorem shows that T has strong ergodic properties with respect to the new class of function spaces GH^+:

Theorem 3 *The theorem of Ionescu-Tulcea and Marinescu (see Appendix A) can be applied to P acting in $GH^+_{\alpha,m} \subset L^1(d\mu^+)$ if $\alpha \le \alpha_h$ and if $m \in \mathbb{N}_0$ is large enough.*

Proof. We have to show that for $\alpha \le \alpha_h$ and for large $m \in \mathbb{N}_0$ there exist $r_1 \in (0,1)$ and $r_2 \in \mathbb{R}_0^+$ such that $\mathrm{var}^+_{\alpha,m}(P(f)) \le r_1 \mathrm{var}^+_{\alpha,m}(f) + r_2\|f\|_1$ for all $f \in GH^+_{\alpha,m}$. To see this, fix some $\epsilon > 0$ and choose according to the definition of $GH^+_{\alpha,m}$ a $d\mu^+$-version $\tilde{f}$ of f which only depends on symbols ξ_s, $s \ge 0$, such that for all $n \ge m$ (where m is unknown at this stage):

$$\int d\mu^+(\xi)\, \mathrm{osc}^+_n(\tilde{f}, \xi) \le (\mathrm{var}^+_{\alpha,m}(f) + \epsilon)\, e^{-\alpha n}. \tag{7}$$

For $a = 1, \ldots, r$ the value $|h(a\xi^1)\tilde{f}(a\xi^1) - h(a\xi^2)\tilde{f}(a\xi^2)|$ is bounded by

$$h(a\xi)|\tilde{f}(a\xi^1) - \tilde{f}(a\xi^2)| + |\tilde{f}(a\xi^1)|\, |h(a\xi^1) - h(a\xi)| + |\tilde{f}(a\xi^2)|\, |h(a\xi^2) - h(a\xi)|,$$

where we define $\tilde{f}$ to be 0 for non-admissible sequences of symbols. Thus:

$$\int d\mu^+(\xi)\,\mathrm{osc}_n^+(h(a\cdot)\widetilde{f}(a\cdot),\xi) \tag{8}$$
$$\le \int d\mu^+(\xi)\,h(a\xi)\,\mathrm{osc}_n^+(\widetilde{f}(a\cdot),\xi)$$
$$+2\int d\mu^+(\xi)\ \sup\{|\widetilde{f}(a\xi^1)| : \xi^1 \in \Sigma^+; \xi_s^1 = \xi_s, 0 \le s \le n\}$$
$$\times \sup\{|h(a\xi^1) - h(a\xi)| : \xi^1 \in \Sigma^+; \xi_s^1 = \xi_s, 0 \le s \le n\}$$

The domain of the first integral on the rhs. can be reduced to those ξ with $a\xi \in \Sigma^+$ because $h(a\xi)$ is 0 otherwise. But then, also $a\xi^1$ and $a\xi^2$ in the definition of $\mathrm{osc}_n^+(\widetilde{f}(a\cdot),\xi)$ are elements of Σ^+, because of the definition of Σ^+ via the matrix $\mathscr{T}$ and because $\xi_0^1 = \xi_0 = \xi_0^2$. Likewise, in the second integral on the rhs. only those ξ have to be considered for which $a\xi \in \Sigma^+$ because otherwise the first supremum would evaluate $\widetilde{f}$ only at points $a\xi^1 \notin \Sigma^+$, where $\widetilde{f}$ is defined to be 0. This shows that expression (8) is bounded for all $n \in \mathbb{N}$ by

$$\int d\mu^+(\xi)\,h(a\xi)\,\mathrm{osc}_{n+1}^+(\widetilde{f},a\xi) \tag{9}$$
$$+2\int d\mu^+(\xi)\ \sup\{|\widetilde{f}(\xi^1)| : \xi^1 \in \Sigma^+; \xi_s^1 = (a\xi)_s, 0 \le s \le n+1\}C_h e^{-\alpha_h(n+1)},$$

where we have applied Eqn. (1).

Summing over a we get for the first term of this expression for all $n \ge m$:

$$\sum_{a=1}^{r}\int d\mu(\xi)\,h(a\xi)\,\mathrm{osc}_{n+1}^+(\widetilde{f},a\xi) \le \int d\mu^+(\xi)\,\mathrm{osc}_{n+1}^+(\widetilde{f},\xi)$$
$$\le (\mathrm{var}_{\alpha,m}^+(f)+\epsilon)\,e^{-\alpha(n+1)},$$

where the $L^1(d\mu)$-contractivity of P as defined by Eqn. (6) has been used. For the same reason, the sum over a for the second term of expression (9) is bounded by

$$2C_h e^{-\alpha_h(n+1)}\sum_{a=1}^{r}\int d\mu^+(\xi)H^{-1}h(a\xi)$$
$$\times \sup\{|\widetilde{f}(\xi^1)| : \xi^1 \in \Sigma; \xi_s^1 = (a\xi)_s, 0 \le s \le n+1\}$$
$$\le 2C_h H^{-1} e^{-\alpha_h(n+1)}\int d\mu^+(\xi)\ \sup\{|\widetilde{f}(\xi^1)| : \xi^1 \in \Sigma; \xi_s^1 = \xi_s, 0 \le s \le n+1\},$$

Let us estimate for $n \ge m$ the value of the integral appearing in the preceding expression:

$$\int d\mu^+(\xi)\ \sup\{|\widetilde{f}(\xi^1)| : \xi^1 \in \Sigma; \xi_s^1 = \xi_s, 0 \le s \le n+1\}$$
$$\le \int d\mu^+(\xi)\left(|\widetilde{f}(\xi)| + \mathrm{osc}_{n+1}^+(\widetilde{f},\xi)\right)$$
$$\le \|f\|_1 + \left(\mathrm{var}_{\alpha,m}^+(f)+\epsilon\right)e^{-\alpha(n+1)},$$

where in the last step Eqn. (7) has been used.

Assembling these pieces together and letting $\epsilon \downarrow 0$, we find

$$\begin{aligned}
&\operatorname{var}^+_{\alpha,m}(P(f)) \\
&\le \sum_{a=1}^{r} \operatorname{var}^+_{\alpha,m}(h(a\cdot)f(a\cdot)) \\
&\le \sup_{n\ge m} \left(\operatorname{var}^+_{\alpha,m}(f)\, e^{-\alpha} + 2C_h H^{-1} e^{\alpha n-\alpha_h(n+1)} \left(\|f\|_1 + \operatorname{var}^+_{\alpha,m}(f)\, e^{-\alpha(n+1)}\right)\right) \\
&\le \left(e^{-\alpha} + 2C_h H^{-1} e^{-\alpha_h(m+1)-\alpha}\right) \operatorname{var}^+_{\alpha,m}(f) + 2C_h H^{-1} e^{-(\alpha_h-\alpha)m-\alpha_h} \|f\|_1 .
\end{aligned}$$

The coefficient of $\operatorname{var}^+_{\alpha,m}(f)$ can be made smaller than 1 by choosing m large enough. □

Applying the theorem of Ionescu-Tulcea and Marinescu and using that $(T, d\mu)$ is weakly mixing we find that the n-th power of the operator P can be decomposed as:

$$P^n = \Pi + R^n \qquad \text{for all } n \in \mathbb{N}, \tag{10}$$

where $\Pi : f \mapsto \int d\mu^+ f$ is a one-dimensional projector. The spectral radius of R w.r.t. $GH^+_{\alpha,m}$ (α and m chosen according to the previous theorem) is strictly smaller than 1.

To make use of this result for the full space $GH_{\alpha,m}$ instead of $GH^+_{\alpha,m}$, we need two lemmas, which are standard for functions which are Hölder continuous w.r.t. d_Σ.

Lemma 4 *For $f \in GH_{\alpha,m}$ and all $N \in \mathbb{Z}$ also $f \circ T^N \in GH_{\alpha,m}$ with $\|f \circ T^N\|_{\alpha,m} \le Ce^{\alpha|N|}\|f\|_{\alpha,m}$, where $C < \infty$ only depends on α and m.*

Proof. (Only the case $N \ge 0$.) Choose a $d\mu$-version $\tilde{f}$ of $f \in GH_{\alpha,m}$ such that $|\tilde{f}|$ is bounded everywhere by $\|f\|_\infty$ and such that

$$\int d\mu(\xi)\, \operatorname{osc}_n(\tilde{f}, \xi) \le 2e^{-\alpha n} \operatorname{var}_{\alpha,m}(f) \tag{11}$$

for all $n \ge m$. Then $\operatorname{var}_{\alpha,m}(f \circ T^N)$ is bounded by

$$\sup_{n\ge m} e^{\alpha n} \int d\mu(\xi)\, \sup\{|\tilde{f}(\xi^1)-\tilde{f}(\xi^2)| : \xi^{1/2} \in \Sigma; \xi^{1/2}_s = \xi_s, -n-N \le s \le n-N\},$$

because of the invariance of $d\mu$ under T. For $n - N \ge m$ Eqn. (11) implies that the integral in the preceding expression is bounded by $2e^{-\alpha(n-N)} \operatorname{var}_{\alpha,m}(f)$. Additionally, for all n this integral is bounded by $2\|f\|_\infty$. So we find

$$\operatorname{var}_{\alpha,m}(f \circ T^N) \le 2(C'e^{\alpha m} + 1)e^{\alpha N} \|f\|_{\alpha,m},$$

where $C' < \infty$ is chosen according to property 3 of GH. Furthermore, $\|f \circ T^N\|_1 = \|f\|_1$. Therefore it is possible to take $C := 2C'e^{\alpha m} + 2$ in the statement of the lemma. □

Lemma 5 (Finite approximation) *For every* $f \in GH_{\alpha,m}$ *and for all* $N \geq m$ *there exist* $f_N \in GH_{\alpha,m}$ *such that the following is true: (i)* $\xi \mapsto f_N(\xi)$ *only depends on* $\xi_{-N}, \dots, \xi_N$. *(ii)* $\|f - f_N\|_1 \leq 2e^{-\alpha N}\|f\|_{\alpha,m}$. *(iii)* $|f_N|$ *is bounded everywhere by* $\|f\|_\infty$. *(iv)* $\|f_N\|_{\alpha,m} \leq 4\|f\|_{\alpha,m}$.

Proof. For every $a = 1, \dots, r$ choose some $\xi(a) \in \Sigma$ with $\xi(a)_0 = a$. Now define for $N \in \mathbb{N}_0$ a truncation $\mathcal{Q}_N : \Sigma \to \Sigma$ by

$$\mathcal{Q}_N(\xi) := (\dots, \xi(\xi_{-N})_{-2}, \xi(\xi_{-N})_{-1}, \xi_{-N}, \dots, \xi_0, \dots, \xi_N, \xi(\xi_N)_1, \xi(\xi_N)_2, \dots).$$

With $\tilde{f}$ as in the proof of Lemma 4 define $f_N := \tilde{f} \circ \mathcal{Q}_N$ for all $N \geq m$. The claimed properties of f_N follow easily from Eqn. (11). □

Theorem 6 (Exponential decay of correlations) *As* $N \to \infty$, *the correlation* $\int d\mu (g \circ T^N) f$ *decays exponentially to 0 for all* $f, g \in GH_{\alpha,m}$ *with* $\int d\mu f = 0$ *if* $0 < \alpha \leq \alpha_h$ *and if m is large enough for Theorem 3 to be valid.*

Proof. According to Lemma 5 choose finite approximations f_n and g_n, $n \geq m$, of f and g. Then $\left|\int d\mu (g \circ T^N) f\right|$ is bounded by

$$2e^{-\alpha n}\|g\|_{\alpha,m}\|f\|_\infty + 2e^{-\alpha n}\|g\|_\infty\|f\|_{\alpha,m} + \left|\int d\mu\, g_n \circ T^{n+N} f_n \circ T^n\right|$$

Now note that $f_n \circ T^n$ and $g_n \circ T^n$ are elements of $GH^+_{\alpha,m}$:

$$\begin{aligned}\left|\int d\mu\, g_n \circ T^{n+N} f_n \circ T^n\right| &= \left|\int d\mu^+ g_n \circ T^n\, P^N(f_n \circ T^n)\right| \\ &\leq \|g_n\|_\infty \left(\|R^N\|_{+,\alpha,m}\|f_n \circ T^n\|_{+,\alpha,m} + \left|\int d\mu f_n\right|\right) \\ &\leq \|g\|_\infty \left(C_1\kappa^N C_2 e^{\alpha n} 4\|f\|_{\alpha,m} + 2e^{-\alpha n}\|f\|_{\alpha,m}\right),\end{aligned}$$

where $C_1 < \infty$ and $\kappa \in (0,1)$ have been chosen according to Lemma 13 and $C_2 < \infty$ according to Lemma 4. Choose such a $k \in \mathbb{N}$ that $\kappa^k e^\alpha < 1$. Then take $n = \lfloor N/k \rfloor$ in the estimates above. Hence for $N \geq km$:

$$\begin{aligned}\left|\int d\mu (g \circ T^N) f\right| &\leq 2e^{-\alpha(N/k-1)}\|g\|_{\alpha,m}\|f\|_\infty + 4e^{-\alpha(N/k-1)}\|g\|_\infty\|f\|_{\alpha,m} \\ &\quad + \|g\|_\infty C_1 C_2 (\kappa^k e^\alpha)^{N/k} 4\|f\|_{\alpha,m},\end{aligned}$$

which decays exponentially as $N \to \infty$. □

4 Central limit theorem

Let k be a $\mathbb{R}^d$-valued ($d < \infty$) function with entries of type $GH_{\alpha,m}$ and vanishing mean: $\int d\mu\, k = 0$. We give conditions on which k fulfills a central limit theorem, i.e. conditions when the distribution of the random variable $N^{-1/2}\sum_{n=0}^{N-1} k \circ T^n$ tends to a non-degenerate Gaussian measure as $N \to \infty$. Again, the first step is to reduce the problem to the space $GH^+_{\alpha,m}$. Define $\delta := \ln\sup_{\Sigma^+} h - \ln\inf_{\Sigma^+} h$. We have to introduce a condition concerning δ:

Lemma 7 *Assume that the value range of h is so small or h is so smooth that* $2\delta < \alpha_h$. *If* α *and* α' *fulfill* $0 < 2\alpha' + 2\delta < \alpha \le \alpha_h$ *and m is large enough, then for every* $f \in GH_{\alpha,m}$ *there exists* $f^+ \in GH^+_{\alpha',m}$ *and* $g \in GH_{\alpha',m}$ *such that*

$$f = f^+ - g + g \circ T \qquad a.e.$$

(The proof of this lemma is given in Appendix B.)

Remark. Given that $f(\xi)$ does not depend too singularly (this can be made precise) on ξ_s, $s < 0$, the condition $2\delta < \alpha_h$ can be dropped, and the decomposition of f is true for $0 < 2\alpha' < \alpha \le \alpha_h$.

In the following we will tacitly assume $2\delta < \alpha_h$ and that there exists $\alpha \in (2\delta, \alpha_h]$ and $m \in \mathbb{N}$ such that every entry of the function k is an element of $GH_{\alpha,m}$. Choose some $\alpha' \in (0, \alpha/2 - \delta)$. If necessary, increase m until Theorem 3 is valid for $GH_{\alpha',m}$ and Lemma 7 is valid.

Let $\rho \in GH_{\alpha',m}$ be a normalized non-negative weight function on Σ with respect to $d\mu$, i.e. $\int \rho \, d\mu = 1$. We can calculate the characteristic function of $X_N := \sum_{n=0}^{N-1} k \circ T^n$ for all $p \in \mathbb{R}^d$ given the initial probability measure $\rho \, d\mu$:

$$E_\rho[e^{ip\cdot X_N}] = \int \rho \, d\mu \exp i \sum_{n=0}^{N-1} p \cdot k \circ T^n.$$

According to Lemma 7 there exists a real vector-valued function k^+ with entries of type $GH^+_{\alpha',m}$ and a real vector-valued function u with entries of type $GH_{\alpha',m}$ such that $k = k^+ + u \circ T - u$ a.e. This yields:

$$E_\rho[e^{ip\cdot X_N}] = \int \rho \, d\mu \exp ip \cdot \left(u \circ T^N - u + \sum_{n=0}^{N-1} k^+ \circ T^n \right). \tag{12}$$

Choose finite approximations $\rho_s \in GH_{\alpha',m}$, $s \ge m$, according to Lemma 5. We approximate the characteristic function (12) with their help ($s \ge m$):

$$\left| E_\rho[e^{ip\cdot X_N}] - \int \rho_s \, d\mu \exp i \sum_{n=0}^{N-1} p \cdot k^+ \circ T^n \right| \tag{13}$$

$$\le \int d\mu \left| e^{ip\cdot u\circ T^N} - 1 \right| \rho + \int d\mu \left| e^{-ip\cdot u} - 1 \right| \rho + \int d\mu \, |\rho - \rho_s|$$

$$\le C_1 \left(|p| + e^{-\alpha' s} \right), \tag{14}$$

with some $C_1 < \infty$, which depends on ρ.

Now note that for $s \ge m$ the approximation of the expectation value (12) in Eqn. (13) may be written

$$\int \rho_s \, d\mu \exp i \sum_{n=0}^{N-1} p \cdot k^+ \circ T^n = \int d\mu^+ P_p^N P^s(\rho_s \circ T^s), \tag{15}$$

where we have introduced the operator family $f \mapsto P_p(f) := P(e^{ip\cdot k^+}f)$, $p \in \mathbb{R}^d$, which acts in the space $L^1(d\mu^+)$. (This scheme follows [3].)

For small $|p|$, one can view P_p as an analytic perturbation [4] of P: There is an ϵ such that for $|p| \leq \epsilon$ we have a decomposition analogous to Eqn. (10):

$$P_p^n = K_p^n \Pi_p + R_p^n \qquad \text{for all } n \in \mathbb{N}, \tag{16}$$

where K_p is the leading, complex eigenvalue, Π_p a one-dimensional projector in $GH_{\alpha',m}$, R_p a bounded operator in $GH_{\alpha',m}$, all three of them analytic with respect to p, and the spectral radius of R_p in $GH_{\alpha',m}$ is strictly less than 1. Furthermore, for $p = 0$ the decomposition Eqn. (16) reduces to that of Eqn. (10): $K_0 = 1$, $\Pi_0 = \Pi$, and $R_0 = R$. The leading eigenvalue K_p may be written $e^{-p\cdot Dp/2+F(p)}$, where F is a complex-valued function of type C^∞, which is of order 3 at $p = 0$. The real symmetric matrix D is uniquely determined and non-negative, because for all $p \in \mathbb{R}^d$:

$$p \cdot Dp = \lim_{N\to\infty} N^{-1} \int d\mu^+ \left(\sum_{n=0}^{N-1} p \cdot k^+ \circ T^n \right)^2 .$$

Lemma 8 *The following statements are equivalent:*

(i) The matrix D is strictly positive.
(ii) There exists an open neighborhood of $0 \in \mathbb{R}^d$ such that for all $p \neq 0$ from this neighborhood, the spectral radius of P_p w.r.t. $GH^+_{\alpha',m}$ is strictly less than 1.
(iii) The following equation cannot be solved by any $\phi \in L^\infty(d\mu)$ and any $p \neq 0$:

$$p \cdot k = \phi \circ T - \phi \qquad a.e. \tag{17}$$

Proof. (i)⇒(ii): We will apply Theorem 12 for the spaces $GH^+_{\alpha',m} \subset L_1(d\mu^+)$, the operator P_p and the norm $\|\cdot\|' := \|\cdot\|_{L^\infty(d\mu^+)} = \|\cdot\|_\infty$. This is possible because of the following facts:

- There exists $C_1 < \infty$ such that $\|f\|_\infty \leq C_1\|f\|_{\alpha',m}$ for all $f \in GH^+_{+,\alpha',m}$.
- $\|P_p^n(f)\|_\infty \leq \|f\|_\infty \|P^n(1)\|_\infty \leq C_2\|f\|_\infty$ with some fixed $C_2 < \infty$ for all $f \in GH^+_{\alpha',m}$ and $n \in \mathbb{N}$.
- As a consequence of Theorem 3, there exist $r_1 \in (0, 1)$ and $r_2 > 0$ such that for all $f \in GH^+_{\alpha',m}$ and all $p \in \mathbb{R}^d$:

$$\begin{aligned} \operatorname{var}^+_{\alpha',m}(P_p(f)) &\leq r_1 \operatorname{var}^+_{\alpha',m}(e^{ip\cdot k^+}f) + r_2\|f\|_1 \\ &\leq r_1 \operatorname{var}^+_{\alpha',m}(f) + \left(r_2 + r_1 \operatorname{var}^+_{\alpha',m}(e^{ip\cdot k^+})\right)\|f\|_\infty . \end{aligned}$$

Now let $\mathscr{S}$ be the set of all $p \in \mathbb{R}^d$ for which the spectral radius in $GH^+_{\alpha,m}$ of P_p is not strictly less than 1. Let p be an element of this set. Then by Theorem 12 the spectral radius of P_p actually equals 1 and there exists an eigenvector $f \in GH^+_{\alpha',m}$ of P_p with eigenvalue $e^{i\theta}$, $\theta \in \mathbb{R}$. The operator P has the property that a.e. $P(|g|) \geq |P(g)|$ for all $g \in L^1(d\mu^+)$ and hence $P(|f|) \geq |f|$ for the eigenvector f. On the other hand, P is a contractive operator in $L^1(d\mu^+)$. Taken together, this implies $P(|f|) = |f|$ a.e. But the 1-eigenspace of P consists of the constant functions; so (perhaps after normalization) we have a.e. $|f| = 1$, so that a.e. $f = e^{i\psi}$ with some real-valued $\psi \in L^\infty(d\mu)$. Combining the eigenvalue equation with the definition Eqn. (5) of the operator P we obtain

$$\int d\mu\, g \circ T\, e^{ip\cdot k^+} e^{i\psi} = \int d\mu\, g\, e^{i\theta} e^{i\psi} \qquad \text{for all } g \in L^\infty(d\mu^+).$$

We choose $g := e^{-i\theta} e^{-i\psi}$ and examine the exponents, which yields that a.e. $p \cdot k^+ - \theta - \psi \circ T + \psi \in 2\pi\mathbb{Z}$. It is now obvious that for arbitrary $n \in \mathbb{Z}$, $e^{in\psi}$ is an eigenvector with eigenvalue of modulus 1. Thus, we have shown $\mathbb{Z}\mathscr{S} \subset \mathscr{S}$. Now assume that $\mathscr{S}$ contains non-zero elements q with arbitrarily small modulus $|q|$. Then from $\mathbb{Z}\mathscr{S} \subset \mathscr{S}$ follows that there exists a line $\mathbb{R}p$, $p \neq 0$, so that every point of this line is a limit point of $\mathscr{S}$. From the semicontinuity of the spectrum [4] follows that $\mathscr{S}$ is closed; hence, $\mathbb{R}p \subset \mathscr{S}$ and $1 = |K_q| = |e^{-q\cdot Dq/2+F(p)}|$ for all q, $|q| < \epsilon$, which are parallel to p. This yields $p \cdot Dp = 0$. □

(ii)⇒(iii): Let ϕ be a solution of Eqn. (17) for some fixed p. Then $p\cdot k^+ = \psi\circ T - \psi$, where $\psi := \phi - p \cdot u$. Thus, for all $N \in \mathbb{N}$:

$$e^{i\psi} = e^{i\psi\circ T^N} \exp -i \sum_{n=0}^{N-1} p \cdot k^+ \circ T^n \qquad \text{a.e.}$$

By Lemma 5, there exists functions $f_N \circ T^N \in L^1(d\mu^+)$ such that $\|\psi - f_N\|_1$ tends to zero as $N \to \infty$. The expression $e^{if_N\circ T^N} \exp -i \sum_{n=0}^{N-1} p \cdot k^+ \circ T^n$ defines an element of $L^1(d\mu^+)$. We can estimate:

$$\begin{aligned}\left\| e^{i\psi} - e^{if_N\circ T^N} \exp -i \sum_{n=0}^{N-1} p \cdot k^+ \circ T^n \right\|_1 &= \left\| e^{i\psi\circ T^N} - e^{if_N\circ T^N} \right\|_1 \\ &\leq \|\psi - f_N\|_1 ,\end{aligned}$$

which becomes arbitrarily small as $N \to \infty$. But $L^1(d\mu^+)$ is a closed subspace of $L^1(d\mu)$, so we see that $e^{i\psi}$ itself is an element of $L^1(d\mu^+) \subset L^1(d\mu)$. Therefore, in $L^1(d\mu^+)$, the operator P_p is conjugated to P, namely $P_p(f) = e^{i\psi} P(e^{-i\psi} f)$ for all $f \in L^1(d\mu^+)$. This implies that P_p possesses an $L^1(d\mu^+)$-eigenfunction with eigenvalue of modulus 1. But then also the spectral radius of P_p in $GH^+_{\alpha,m}$ must equal 1, because $GH^+_{\alpha,m}$ is dense in $L^1(d\mu^+)$. This remains true if we replace p by a real multiple of itself. Therefore, on the whole line $q \in \mathbb{R}p$ (which intersects any neighborhood of 0) the spectral radius of P_q in $GH^+_{\alpha,m}$ equals 1. □

(iii)⇒(i): Assume that D is not strictly positive, i.e. that for some $p \neq 0$ the variance of the random variable $N^{-1/2} \sum_{n=0}^{N-1} p \cdot k^+ \circ T^n$ tends to 0 as $N \to \infty$. Define $\psi := \sum_{n=1}^{\infty} P^n(p \cdot k^+)$, which converges exponentially in $GH_{\alpha',m}$ because $\int d\mu^+ k^+ = 0$. The variance of the telescoped random variable

$$N^{-1/2} \sum_{n=0}^{N-1} (p \cdot k^+ + \psi - \psi \circ T) \circ T^n ,$$

too, tends to 0 as $N \longrightarrow \infty$. Due to the decay of correlations, this can be expressed with the help of the Green-Kubo formula:

$$0 = \int d\mu^+ (p\cdot k^+ + \psi - \psi\circ T)^2 + 2\sum_{n=1}^{\infty} \int d\mu^+ (p\cdot k^+ + \psi - \psi\circ T) P^n (p\cdot k^+ + \psi - \psi\circ T).$$

But $P(\psi \circ T) = \psi$ and $\psi - P(\psi) = P(p \cdot k^+)$ so that in the above equation all expressions $P^n(\cdots)$ vanish. Thus, also $\int d\mu^+ (p\cdot k^+ + \psi - \psi\circ T)^2$ vanishes, which leads to Eqn. (17) for $\phi := \psi + p \cdot u$. □

Theorem 9 (Central limit theorem) *Assume that any one of the statements of the preceding lemma is true. Then the distribution of the random variable* $X_N/\sqrt{N}$ *on the probability space* $(\Sigma, \rho\, d\mu)$ *converges to a non-degenerate Gaussian distribution as* $N \longrightarrow \infty$.

Proof. We show that for every $p \in \mathbb{R}^d$ the expectation $E_\rho[e^{ip\cdot X_N/\sqrt{N}}]$ converges to $e^{-p\cdot Dp/2}$. Fix an arbitrary $p \in \mathbb{R}^d$ and choose M so large that $|p| < \epsilon/\sqrt{M}$. Then Eqn. (16) is valid for $P_{p/\sqrt{N}}$ if $N \geq M$. Thus, for $N \geq M$ and $s \geq m$ we find with the help of Eqn. (14) and (15):

$$\begin{aligned} &\left| e^{-p\cdot Dp/2} - E_\rho[e^{ip\cdot X_N/\sqrt{N}}] \right| \\ &\leq \left| e^{-p\cdot Dp/2} - \int d\mu^+ K^N_{p/\sqrt{N}} \Pi_{p/\sqrt{N}} P^s(\rho_s \circ T^s) \right| \\ &\quad + \int d\mu^+ \left| R^N_{p/\sqrt{N}} P^s(\rho_s \circ T^s) \right| + C_1(|p|N^{-1/2} + e^{-\alpha' s}). \end{aligned}$$

Given the operator R_q, $|q| \leq \epsilon$, choose $C_2 \leq \infty$ and $\kappa \in (0,1)$ according to Lemma 13. By the smoothness of $q \mapsto \Pi_q$ we find the existence of $C_3 < \infty$ such that

$$\left\| \Pi_q(f) - \int d\mu^+ f \right\|_{+,\alpha',m} \leq C_3 |q| \, \|f\|_{+,\alpha',m}$$

for all $f \in GH^+_{\alpha',m}$ and all q with $|q| \leq \epsilon$. The norm $\|P^s(\rho_s \circ T^s)\|_{+,\alpha',m}$ is bounded by some $C_4 < \infty$. Therefore, if $N \geq M$:

$$\begin{aligned} &\left| e^{-p\cdot Dp/2} - E_\rho[e^{ip\cdot X_N/\sqrt{N}}] \right| \\ &\leq \left| e^{-p\cdot Dp/2} - K^N_{p/\sqrt{N}} \right| + (C_1 + C_3C_4)|p|N^{-1/2} + C_2C_4\kappa^N + (C_1 + 2C_4)\, e^{-\alpha' s}. \end{aligned}$$

Consider only N which are larger than both of M and m. Take $s := N$ in the calculations above. Then it is easy to see that this expression decays to 0 as $N \longrightarrow \infty$. □

5 Application

We have introduced a space of generalized Hölder continuous functions and have shown that correlations of these functions decay exponentially. The multidimensional central limit theorem has been proved for a certain subset of this function space. (The condition $\ln\sup_{\Sigma^+} h - \ln\inf_{\Sigma^+} h < \alpha_h/2$ remains to be checked.)

Via symbolic dynamics, these results can be immediately carried over to Anosov maps: A C^2-diffeomorphism Φ of a compact finite-dimensional Riemannian C^∞-manifold $\mathcal{M}$ onto itself is called *Anosov* [1], if there exists a continuous invariant splitting of the tangent spaces of $\mathcal{M}$ at all $x \in \mathcal{M}$ into subspaces $E_x^+ \oplus E_x^-$ such that the following is true: There are $C > 0$, $\theta > 1$ such that for all $n \in \mathbb{N}$, $x \in \mathcal{M}$, $u \in E_x^+$, and $v \in E_x^-$:

$$|(D_x\Phi^n)(u)|_{\Phi^n(x)} \geq C\theta^n |u|_x \qquad \text{(expansion)},$$
$$|(D_x\Phi^n)(v)|_{\Phi^n(x)} \leq C^{-1}\theta^{-n} |v|_x \qquad \text{(contraction)},$$

where $|\cdot|_x$ denotes the Riemannian length in the tangent space at $x \in \mathcal{M}$.

Assume that $\Phi : \mathcal{M} \hookleftarrow$ is transitive and that an invariant measure $d\mu$ is known which is absolutely continuous with respect to the canonical Riemann-Lebesgue measure $d\lambda$ on $\mathcal{M}$. Then it is well-known [1] that such an Anosov diffeomorphism can be described by a Markov chain $(\Sigma, d\mu)$ of the type we have considered: Up to a set of measure 0 one can identify the manifold $\mathcal{M}$ with the space of allowed symbol sequences Σ. With this identification, the shift T is the representation of the Anosov map Φ. The operator P is related [3] to the Ruelle-Perron-Frobenius operator L by $P(f) = (ke)^{-1}L(ef)$, where k is the leading eigenvalue and e spans the corresponding eigenspace.

There exist [1] constants C, $\beta > 0$, such that if the symbol sequences $\xi^1 = \{\xi_s^1\}_{s\in\mathbb{Z}}$ and $\xi^2 = \{\xi_s^2\}_{s\in\mathbb{Z}} \in \Sigma$ corresponding to some points x_1, $x_2 \in \mathcal{M}$ coincide from place $-n$ to place n, then the Riemannian distance $d(x_1,x_2)$ is bounded from above by $Ce^{-\beta n} = Cd_\Sigma^\beta(\xi_1,\xi_2)$.

The following obvious theorem shows that our construction generalizes the notion of Hölder continuity w.r.t. Riemannian distance d. Recall that the upper capacity of a set D is defined by $\overline{C} := \limsup_{t\downarrow 0}(\log 1/t)^{-1}\log N(t)$, where $N(t)$ is the number of balls with Riemannian radius t needed to cover D.

Theorem 10 *For some bounded function f assume that there exists a number $C < \infty$ and a subset D which cuts $\mathcal{M}$ into a countable union $\mathcal{M} - D = \bigcup_i A_i$ of disjoint open sets A_i such that the restriction of f to each of these sets A_i fulfills*

$$|f(x_1) - f(x_2)| \leq C\, d(x_1,x_2)^{\alpha/\beta} \qquad \textit{for all } x_1,\, x_2 \in A_i.$$

Assume furthermore that the upper capacity $\overline{C}$ of D is smaller than $d + \dim\mathcal{M} - \alpha/\beta$. (This is the case e.g. if α is small enough and D is the union of a finite number of smooth hypersurfaces.) Then $f \in GH_{\alpha,m}$ for m large enough.

So from our considerations follow exponential decay of correlations and central limit theorem for a larger class than the usual class of Hölder continuous functions. A byproduct are *local* central limit theorems and renewal theorems, which can be proved by the methods of [3]. In addition to these basic probabilistic properties, the results of this work allow to study the periodic extensions of Anosov maps instead of piecewise expanding maps along the lines of [5].

A The theorem of Ionescu-Tulcea and Marinescu

Theorem 11 *(Ionescu-Tulcea and Marinescu [6]) Consider Banach spaces* $(\mathscr{L}, \|\cdot\|_{\mathscr{L}}) \subset (\mathscr{B}, \|\cdot\|_{\mathscr{B}})$ *with the property that the closed unit ball of* $\mathscr{L}$ *is* $\mathscr{B}$*-compact. Let* $P : (\mathscr{L}, \|\cdot\|_{\mathscr{L}}) \hookleftarrow$ *be a bounded operator which can be extended to a bounded operator in* $(\mathscr{B}, \|\cdot\|_{\mathscr{B}})$*. Suppose that* $\sup_{n\in\mathbb{N}_0} \|P^n\|_{\mathscr{B}} < \infty$ *and that there exist* $r_{\mathscr{L}} \in (0, 1)$ *and* $r_{\mathscr{B}} \in \mathbb{R}_0^+$ *such that* $\|P(f)\|_{\mathscr{L}} \leq r_{\mathscr{L}}\|f\|_{\mathscr{L}} + r_{\mathscr{B}}\|f\|_{\mathscr{B}}$ *for all* $f \in \mathscr{L}$.

Then P *can be decomposed as* $P^n = \sum_\gamma \gamma^n \Pi_\gamma + R^n$ *for all* $n \in \mathbb{N}$*, where the sum runs over all eigenvalues* γ *of modulus 1 of* P *which belong to eigenvectors in* $\mathscr{L}$*. The span of these is finite-dimensional, so that the sum is well-defined. The operators* Π_γ *are (some)* $\mathscr{L}$*-projectors onto the corresponding eigenspaces.* R *maps* $\mathscr{L}$ *into* $\mathscr{L}$*, and its* $\mathscr{L}$*-spectral radius is strictly smaller than 1. Furthermore,* $\Pi_\gamma \Pi_\delta = 0$ *and* $\Pi_\gamma R = 0 = R\Pi_\gamma$ *for all* $\gamma \neq \delta$ *which occur as eigenvalues of modulus 1.*

We also need a special form of this theorem with weakened assumptions:

Theorem 12 *Consider Banach spaces* $(\mathscr{L}, \|\cdot\|_{\mathscr{L}}) \subset (\mathscr{B}, \|\cdot\|_{\mathscr{B}})$ *with the property that the closed unit ball of* $\mathscr{L}$ *is* $\mathscr{B}$*-compact. On the space* $\mathscr{L}$ *let a seminorm* $\|\cdot\|'$ *be given, such that there exists a* $C < \infty$ *with* $\|f\|' \leq C\|f\|_{\mathscr{L}}$ *for all* $f \in \mathscr{L}$*. Let* $P : (\mathscr{L}, \|\cdot\|_{\mathscr{L}}) \hookleftarrow$ *be a bounded operator which can be extended to a bounded operator in* $(\mathscr{B}, \|\cdot\|_{\mathscr{B}})$*. Suppose that* $\sup_{n\in\mathbb{N}_0} \|P^n\|' < \infty$ *and that there exist* $r_{\mathscr{L}} \in (0, 1)$ *and* $r' \in \mathbb{R}_0^+$ *such that* $\|Pf\|_{\mathscr{L}} \leq r_{\mathscr{L}}\|f\|_{\mathscr{L}} + r'\|f\|'$ *for all* $f \in \mathscr{L}$.

Then the $\mathscr{L}$*-spectral radius of* P *is equal to or smaller than 1. It equals 1 iff there exists an eigenvector in* $\mathscr{L}$ *with eigenvalue of modulus 1.*

When applying the above theorems, the following is helpful:

Lemma 13 *Let* $K \subset \mathbb{R}^d$*,* $d < \infty$*, be a compact set. Assume* $A : p \mapsto A_p$ *maps* K *continuously (with respect to operator norm) to the bounded operators in some Banach space with norm* $\|\cdot\|$*. If the spectral radius of all* A_p*,* $p \in K$*, is strictly smaller than 1, then there exist* $C < \infty$ *and* $\kappa \in (0, 1)$ *such that* $\|A_p^n\| \leq C\kappa^n$ *for all* $p \in K$ *and all* $n \in \mathbb{N}$.

B Proof of Lemma 7

Construct $f_N \in GH_{\alpha,m}$ for all $N \geq m$ according to Lemma 5. For $N > m$ define

$$g_N := \sum_{n=0}^{N-1} \left(P^{N-n}(f_N \circ T^N) - f_N \circ T^n\right)$$

and

$$f_N^+ := f_N \circ T^N + \sum_{n=0}^{N-1} \left(P^{N-n}(f_N \circ T^N) - P^{N-n}(f_N \circ T^N) \circ T \right)$$

Obviously $f_N = f_N^+ - g_N + g_N \circ T$ with $f_N^+ \in GH^+_{\alpha,m}$ and $g_N \in GH_{\alpha,m}$ for all $N > m$.

g_N is Cauchy with respect to $L^1(d\mu)$, because $\|g_{N+1} - g_N\|_1$ decays exponentially fast as $N \to \infty$: For $N > m$ we can estimate

$$\|g_{N+1} - g_N\|_1 \leq 2 \sum_{n=0}^{N} \|f_{N+1} - f_N\|_1 \leq 8(N+1)e^{-\alpha N} \|f\|_{\alpha,m},$$

because $f_N \circ T^N = P(f_N \circ T^{N+1})$. Hence, as $N \to \infty$, the functions g_N converge to some $g \in L^1(d\mu)$ and f_N^+ tends to $f^+ := f + g - g \circ T \in L^1(d\mu^+)$, both with respect to $\|\cdot\|_1$.

Now assume we would know that $\|g_N\|_{\alpha',m}$ remains bounded as $N \to \infty$. Then also $\|f_N^+\|_{+,\alpha',m} = \|f_N + g_N - g_N \circ T\|_{+,\alpha',m}$ is bounded as $N \to \infty$. Thus, according to Theorem 2, g is an element of $GH_{\alpha',m}$ and f^+ is an element of $GH^+_{\alpha',m}$, which had to be shown.

So it is sufficient to prove that $\|g_N\|_{\alpha',m}$ remains bounded as $N \to \infty$. This will follow if we show that there exists $C_1 < \infty$ such that for all n and N with $0 \leq n < N > m$ the following inequality holds: $\|P^{N-n}(f_N \circ T^N) - f_N \circ T^n\|_{\alpha',m} \leq C_1 e^{-(\alpha - 2\alpha' - 2\delta)n}$. According to Lemma 4 applied to $GH_{\alpha',m}$ for that it is sufficient that there exists $C_2 < \infty$ with the property

$$\|P^{N-n}(f_N \circ T^N) \circ T^{-n} - f_N\|_{\alpha',m} \leq C_2 e^{-(\alpha - \alpha' - 2\delta)n}. \tag{18}$$

First, we estimate the $L^1(d\mu)$-part of the lhs.: f_N depends only on the symbols ξ_s with $-N \leq s \leq N$: $f_N(\xi) = f_N(\xi_{-N}, \dots, \xi_N)$. With the help of $\sum_{a=1}^r h(a\cdot) = 1$ we can calculate

$$\begin{aligned} &P^{N-n}(f_N \circ T^N) \circ T^{-n}(\xi) - f_N(\xi) \qquad (19)\\ &= \sum_{a_1=1}^{r} \cdots \sum_{a_{N-n}=1}^{r} h(a_1, \xi_{-n}, \xi_{-n+1}, \dots) \cdots h(a_{N-n}, \dots, a_1, \xi_{-n}, \xi_{-n+1}, \dots) \\ &\quad \times \left(f_N(a_{N-n}, \dots, a_1, \xi_{-n}, \dots, \xi_N) - f_N(\xi_{-N}, \dots, \xi_N) \right). \end{aligned}$$

If $(a_{N-n}, \dots, a_1, \xi_{-n}, \xi_{-n+1}, \dots)$ is not admissible, the former expression is 0 by the definition of h. If, on the other hand, $(a_{N-n}, \dots, a_1, \xi_{-n}, \xi_{-n+1}, \dots)$ is admissible, we have

$$\left| f_N(a_{N-n}, \dots, a_1, \xi_{-n}, \dots, \xi_N) - f_N(\xi_{-N}, \dots, \xi_N) \right| \leq \mathrm{osc}_n(f_N, \xi)$$

and thus

$$\|P^{N-n}(f_N \circ T^N) \circ T^{-n}(\xi) - f_N(\xi)\|_1 \leq \int d\mu(\xi) P^{N-n}(1)(T^{-n}(\xi)) \, \mathrm{osc}_n(f_N, \xi).$$

This expression is always bounded by $2\,\|f\|_\infty$. Additionally, due to Eqn. (11) it is bounded by $2e^{-\alpha n}\,\mathrm{var}_{\alpha,m}(f)$ for $n \geq m$. Hence, this expression is for all $n \in \mathbb{N}$ bounded by a finite constant times $e^{-(\alpha-\alpha'-2\delta)n}$. Therefore, of Eqn. (18) only the following inequality remains to be shown:

$$\mathrm{var}_{\alpha',m}\left(P^{N-n}(f_N \circ T^N) \circ T^{-n} - f_N\right) \leq C_3\, e^{-(\alpha-\alpha'-2\delta)n} \tag{20}$$

for all $0 \leq n < N > m$ with some fixed $C_3 < \infty$.

To estimate $\mathrm{osc}_t(P^{N-n}(f_N \circ T^N) \circ T^{-n} - f_N, \xi)$ which is imbedded in the lhs., we look at

$$\left|P^{N-n}(f_N \circ T^N) \circ T^{-n}(\xi^1) - f_N(\xi^1) - P^{N-n}(f_N \circ T^N) \circ T^{-n}(\xi^2) + f_N(\xi^2)\right| \tag{21}$$

for ξ, ξ^1, $\xi^2 \in \Sigma$ which all coincide from place $-t$ up to t, $t \geq m$. (All functions are declared to vanish for non-admissible sequences of symbols.) We consider two cases: first, $t \leq n$ (which can only happen if $n \geq m$) and second, $t > n$.

Case 1: $m \leq t \leq n$. Expression (21) is bounded by

$$\left|P^{N-n}(f_N \circ T^N) \circ T^{-n}(\xi^1) - f_N(\xi^1)\right| + \left|P^{N-n}(f_N \circ T^N) \circ T^{-n}(\xi^2) + f_N(\xi^2)\right|$$

Thus, considering Eqn. (19) we find that

$$\begin{aligned}
&\int d\mu(\xi)\,\mathrm{osc}_t(P^{N-n}(f_N \circ T^N) \circ T^{-n} - f_N, \xi) \\
&\leq 2 \int d\mu(\xi) \sup_{\substack{\xi^1 \in \Sigma \\ \xi^1_s = \xi_s, -t \leq s \leq t}} \sum_{a_1=1}^{r} \cdots \sum_{a_{N-n}=1}^{r} h(a_1, \xi^1_{-n}, \xi^1_{-n+1}, \ldots) \cdots \\
&\qquad \cdots h(a_{N-n}, \ldots, a_1, \xi^1_{-n}, \xi^1_{-n+1}, \ldots) \sup_{\substack{\xi^{2/3} \in \Sigma \\ \xi^{2/3}_s = \xi^1_s, -n \leq s \leq n}} |\tilde{f}(\xi^2) - \tilde{f}(\xi^3)| \\
&= 2 \int d\mu(\xi) \sup_{\substack{\xi^1 \in \Sigma \\ \xi^1_s = \xi_s, -t \leq s \leq t}} \sup_{\substack{\xi^{2/3} \in \Sigma \\ \xi^{2/3}_s = \xi^1_s, -n \leq s \leq n}} |\tilde{f}(\xi^2) - \tilde{f}(\xi^3)| \\
&\leq \sup_{\substack{a_{-n}, \ldots, a_n \in \{1, \ldots, r\} \\ \mathcal{T}_{a_j, a_{j+1}} = 1, -n \leq j \leq n-1}} \frac{\mu(\xi_{-t} = a_{-t}, \ldots, \xi_t = a_t)}{\mu(\xi_{-t} = a_{-n}, \ldots, \xi_t = a_n)} \\
&\qquad \times 2 \int d\mu(\xi) \sup_{\substack{\xi^{2/3} \in \Sigma \\ \xi^{2/3}_s = \xi_s, -n \leq s \leq n}} |\tilde{f}(\xi^2) - \tilde{f}(\xi^3)| \\
&\leq \frac{\sup_{\Sigma^+} h^{2t+1}}{\inf_{\Sigma^+} h^{2n+1}} 2 \cdot 2\, e^{-\alpha n}\, \mathrm{var}_{\alpha,m}(f) = 4\, e^{\delta}\, e^{2t\delta - \alpha n}\, \mathrm{var}_{\alpha,m}(f),
\end{aligned}$$

because

$$\begin{aligned}
&\mu(\xi_{-t} = a_{-t}, \ldots, \xi_t = a_t) \\
&= \int d\mu^+(\xi)\, h(a_t, \xi_0, \xi_1, \ldots) \cdots h(a_{-t}, \ldots, a_t, \xi_0, \xi_1, \ldots).
\end{aligned}$$

Case 2: $m \le t > n$. Now we use that expression (21) is bounded by

$$\begin{aligned}
&\left|P^{N-n}(f_N \circ T^N)\circ T^{-n}(\xi^1) - P^{N-n}(f_N \circ T^N)\circ T^{-n}(\xi^2)\right| + \left|f_N(\xi^1) - f_N(\xi^2)\right| \\
&\le \sum_{a_1=1}^{r} \cdots \sum_{a_{N-n}=1}^{r} h(a_1, \xi_{-n}, \xi_{-n+1}, \ldots) \cdots h(a_{N-n}, \ldots, a_1, \xi_{-n}, \xi_{-n+1}, \ldots) \\
&\quad \times \left| f_N(a_{N-n}, \ldots, a_1, \xi^1_{-n}, \ldots, \xi^1_N) - f_N(a_{N-n}, \ldots, a_1, \xi^2_{-n}, \ldots, \xi^2_N)\right| \\
&\quad + \sum_{a_1=1}^{r} \cdots \sum_{a_{N-n}=1}^{r} \Big| h(a_1, \xi^1_{-n}, \xi^1_{-n+1}, \ldots) \cdots h(a_{N-n}, \ldots, a_1, \xi^1_{-n}, \xi^1_{-n+1}, \ldots) \\
&\quad - h(a_1, \xi_{-n}, \xi_{-n+1}, \ldots) \cdots h(a_{N-n}, \ldots, a_1, \xi_{-n}, \xi_{-n+1}, \ldots)\Big| \sup_{\xi^3 \in \Sigma} 2|f_N(\xi^3)| \\
&\quad + \text{the same multiple sum with } \xi^1 \text{ replaced by } \xi^2 + \left|f_N(\xi^1) - f_N(\xi^2)\right| .
\end{aligned}$$

Assume that the sequence $(a_{N-n}, \ldots, a_1, \xi_{-n}, \xi_{-n+1}, \ldots)$ is admissible. Then so are $(a_k, \ldots, a_1, \xi^1_{-n}, \xi^1_{-n+1}, \ldots)$ and $(a_k, \ldots, a_1, \xi^2_{-n}, \xi^2_{-n+1}, \ldots)$ for all $1 \le k \le N-n$, and it easy to show that for m large enough there exists $C_4 < \infty$ such that

$$\left| \frac{h(a_1, \xi^1_{-n}, \xi^1_{-n+1}, \ldots) \cdots h(a_{N-n}, \ldots, a_1, \xi^1_{-n}, \xi^1_{-n+1}, \ldots)}{h(a_1, \xi_{-n}, \xi_{-n+1}, \ldots) \cdots h(a_{N-n}, \ldots, a_1, \xi_{-n}, \xi_{-n+1}, \ldots)} - 1 \right| \le C_4\, e^{-\alpha_h t}$$

if $(a_{N-n}, \ldots a_1, \xi_{-n}, \xi_{-n+1}, \ldots)$ is admissible. Hence,

$$\begin{aligned}
&\Big| h(a_1, \xi^1_{-n}, \xi^1_{-n+1}, \ldots) \cdots h(a_{N-n}, \ldots, a_1, \xi^1_{-n}, \xi^1_{-n+1}, \ldots) \\
&\quad - h(a_1, \xi_{-n}, \xi_{-n+1}, \ldots) \cdots h(a_{N-n}, \ldots, a_1, \xi_{-n}, \xi_{-n+1}, \ldots)\Big|
\end{aligned}$$

is always bounded by $h(a_1, \xi_{-n}, \xi_{-n+1}, \ldots) \cdots h(a_{N-n}, \ldots a_1, \xi_{-n}, \xi_{-n+1}, \ldots) \times C_4\, e^{-\alpha_h t}$. This remains true if ξ^1 is replaced by ξ^2.

So in this case ($m \le t > n$) we find

$$\begin{aligned}
&\int d\mu(\xi)\, \mathrm{osc}_t(P^{N-n}(f_n \circ T^N) \circ T^{-n} - f_N, \xi) \\
&\le \theta(N-t) \int d\mu\, P^{N-n} \left(\mathrm{osc}_t(f_N, T^N(\cdot))\right) \circ T^{-n} \\
&\quad + (2+2) C_4\, e^{-\alpha_h t}\, \|f\|_\infty \int d\mu\, P^{N-n}(1) + \theta(N-t)\, e^{-\alpha t} 2\, \mathrm{var}_{\alpha,m}(f_N) \\
&\le C_5\, e^{-\alpha t},
\end{aligned}$$

with $C_5 := 4\, \mathrm{var}_{\alpha,m}(f) + 4 C_4 \|f\|_\infty$ and where $\theta(N-t)$ equals 1 for $N > t$ and vanishes else.

Finally, we collect the estimates of case 1 and case 2 to achieve the following expression as a bound for the lhs. of Eqn. (20):

$$\max\left(\sup_{t:m\le t\le n} 4\,e^{\alpha' t} e^{\delta}\, e^{2t\delta-\alpha n}\operatorname{var}_{\alpha,m}(f),\ \sup_{t>n,t\ge m} e^{\alpha' t} C_5\, e^{-\alpha t}\right)$$
$$\le e^{-(\alpha-\alpha'-2\delta)n}\max\bigl(4\,e^{\delta}\operatorname{var}_{\alpha,m}(f), C_5\, e^{-2\delta m}\bigr). \quad \square$$

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