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Compound Poisson approximation for unbounded functions on a group, with application to large deviations

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Summary. A second order error bound is obtained for approximating $\int h\, d\tilde{Q}$ by $\int h\, dQ$, where $\tilde{Q}$ is a convolution of measures and Q a compound Poisson measure on a measurable abelian group, and the function h is not necessarily bounded. This error bound is more refined than the usual total variation bound in the sense that it contains the function h. The method used is inspired by Stein's method and hinges on bounding Radon–Nikodym derivatives related to $d\tilde{Q}/dQ$. The approximation theorem is then applied to obtain a large deviation result on groups, which in turn is applied to multivariate Poisson approximation.

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1 Introduction

The Poisson distribution provides a good approximation to the distribution of a sum of dependent and small indicators if the dependence is sufficiently weak. However, if the dependence is strong or the indicators are replaced by non-negative random variables, then in many cases the compound Poisson distribution provides a better approximation. Compound Poisson approximation in this direction has been considered by Arratia et al. [1], Barbour et al. [4], Barbour et al. [6] and Roos [12].

An interesting aspect of compound Poisson approximation is finding a 'correct' generalization of the 'magic' factor $\lambda^{-1} \wedge 1$, which appears in the error bound in the Poisson approximation. Here λ is the parameter of the approximating Poisson distribution. This factor is significant in that it gives the error bound the correct order for all values of λ in the range $(0, \infty)$ (see, for example, Barbour and Hall [5]).

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Michel [11] observed that in a special case of compound Poisson approximation, the same 'magic' factor is present, with λ being the parameter of the approximating compound Poisson distribution. Borovkov and Pfeifer [7] showed that in this special case, the second order error bound is also as good as for Poisson approximation. However, in general, no 'correct' generalization of the 'magic' factor seems to have been found. Although the approach of Barbour et al. [4], which applies Stein's method directly, holds promise for obtaining such a 'magic' factor, progress has been slow due to the difficulty in studying the smoothness of the solution of the Stein equation. Fortunately, in many applications, the parameter of the approximating compound Poisson distribution is bounded, and so in these cases, the presence of a 'magic' factor is not crucial.

In this paper, we consider compound Poisson approximation for unbounded functions on a measurable abelian group, but do not attempt to find a 'magic' factor in the error bound. A pair $(\mathscr{X},\mathscr{B})$ is a measurable abelian group if $\mathscr{X}$ is an abelian group and $\mathscr{B}$ a σ-algebra of subsets of $\mathscr{X}$ such that the mapping from $\mathscr{X}\times\mathscr{X}$ to $\mathscr{X}$ defined by the group operation is $(\mathscr{B}\times\mathscr{B},\mathscr{B})$-measurable. We let $(\mathscr{X},\mathscr{B})$ be a measurable abelian group such that $\mathscr{B}$ contains the singleton consisting of the identity element of $\mathscr{X}$. We also let $\mathscr{M}$ be the class of all finite signed measures on $\mathscr{B}$. With the usual operations of real scalar multiplication, addition and convolution, and with the norm defined to be the mass of total variation, $\mathscr{M}$ is a real commutative Banach algebra. We denote by I the Dirac measure at the identity element. For any two finite signed measures μ and ν, we denote their convolution by $\mu\nu$, the total variation of μ by $|\mu|$ and the norm of μ by $\|\mu\|$. If $\mu(B) \leqq \nu(B)$ for all $B \in \mathscr{B}$, we say $\mu \leqq \nu$.

Let $\mu_1,\dots,\mu_n$ be probability measures on $\mathscr{B}$ which do not have any atom at the identity element. Also let $p_1,\dots,p_n$ be numbers between 0 and 1, and let $\lambda = \Sigma_{i=1}^n p_i$ and $\mu = \lambda^{-1}\Sigma_{i=1}^n p_i\mu_i$. Define two probability measures $\tilde{Q}$ and Q on $\mathscr{B}$ by

$$\tilde{Q} = \prod_{i=1}^{n} [(1-p_i)I + p_i\mu_i], \tag{1.1}$$

$$Q = e^{\lambda(\mu - I)} = e^{-\lambda}\sum_{r=0}^{\infty}\frac{\lambda^r\mu^r}{r!}. \tag{1.2}$$

We call Q the compound Poisson measure generated by μ and with parameter λ. In the case $\mathscr{X}$ is an abelian group, $\tilde{Q}$ is the distribution of the sum of n independent random elements which equal the identity element with probabilities $1-p_1,\dots,1-p_n$ and have distributions $\mu_1,\dots,\mu_n$ with probabilities $p_1,\dots,p_n$ respectively, while Q is a Poisson random sum with parameter λ of random elements each with distribution μ.

It is well known that (see, for example, Le Cam [10])

$$\|\tilde{Q} - Q\| \leqq 2\sum_{i=1}^{n} p_i^2. \tag{1.3}$$

In this general setting, the order of the error bound in (1.3) is best possible. When specialized to Poisson approximation, the constant 2 is also best possible.

In Chen [8], an error bound was obtained on $\int h\,d|\tilde{Q}-Q|$ for any non-negative function h such that $\int h\,dQ\mu^2 < \infty$, assuming boundedness of the

Radon–Nikodym derivatives $d\mu_i/d\mu_j$, $i,j = 1,\ldots,n$. While the result of Chen [8] was inspired by the work of Simons and Johnson [13], the techniques used were inspired by Stein's method. A crucial step in Chen [8] is finding explicit bounds on Radon–Nikodym derivatives related to $d\tilde{Q}/dQ$. The method of using such bounds in the context of Poisson approximation was developed in Chen and Choi [9] and refined in Barbour et al. [3].

The objective of this paper is to continue the work in Chen [8] and obtain a second order error bound for approximating $\int h\,d\tilde{Q}$ by $\int h\,dQ$ in the spirit of Barbour et al. [3], where the real-valued function h is such that $\int |h|dQ\,\mu^4 < \infty$. This result and Theorem 2.1 of Chen [8] are then shown to be applicable to large deviations on groups and multivariate Poisson approximation. In particular, the main result of this paper is applied to obtain a large deviation result on a measurable abelian group, which in turn is applied to multivariate Poisson approximation.

The error bound obtained in this paper and that in Theorem 2.1 of Chen [8] are more refined than the usual total variation bounds in that they contain the function h. As a result the bounds are always relatively small compared to $\int h\,d\tilde{Q}$ or $\int h\,dQ$. This is not the case for total variation bounds.

2 Theorems for unbounded functions

We begin this section by stating Theorem 2.1 of Chen [8] in a form which is easy to apply.

Theorem 2.1 (Chen [8]) *Let $(\mathscr{X},\mathscr{B})$ be a measurable abelian group and let $\tilde{Q}$ and Q be given by* (1.1) *and* (1.2) *respectively. If there exists a constant K such that $\mu_i \leqq K\mu_j$ for $i,j = 1,\ldots,n$, then for every real-valued function h defined on $\mathscr{X}$ such that $\int |h|dQ\,\mu^2 < \infty$, we have*

$$\left|\int h\,d\tilde{Q} - \int h\,dQ\right| \leqq \sum_{k=0}^{2} C'_k \int |h|dQ\,\mu^k\,, \tag{2.1}$$

where

$$C'_0 = \left(\frac{3}{2} + \frac{1}{\lambda}\right) K^2 M \sum_{i=1}^{n} \frac{p_i^2}{1-p_i}\,; \qquad C'_1 = \frac{1}{\lambda}\sum_{i=1}^{n} p_i^2\,;$$

$$C'_2 = \frac{1}{2} K^2 M \sum_{i=1}^{n} \frac{p_i^2}{1-p_i}\,;$$

and

$$M = \left\{1 + \frac{K}{\lambda}\sum_{i=1}^{n}\frac{p_i^2}{1-p_i}\right\}^s, \tag{2.2}$$

with s being the largest integer not exceeding $K\{\lambda + \Sigma_{i=1}^n \frac{p_i^2}{1-p_i}\} + 1$.

Remark. 2.1 We note that for a fixed K, $M \leqq \exp\{c\,\Sigma_{i=1}^n \frac{p_i^2}{1-p_i}\}$ for some constant c.

The next theorem is the main result of this section.

Theorem 2.2 *Let* $(\mathscr{X},\mathscr{B})$ *be a measurable abelian group and let* $\tilde{Q}$ *and* Q *be given by* (1.1) *and* (1.2) *respectively. If there exists a constant* K *such that* $\mu_i \leqq K\mu_j$ *for* $i,j = 1,\dots,n$, *then for every real-valued function* h *defined on* $\mathscr{X}$ *such that* $\int |h| dQ\,\mu^4 < \infty$, *we have*

$$\left|\int h\,d\tilde{Q} - \int h\,dQ + \Delta(Q,h)\right| \leqq \sum_{k=0}^{4} C_k \int |h| dQ\,\mu^k\,, \tag{2.3}$$

where

$$\Delta(Q,h) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 \left[\int h\,dQ\,\mu_i^2 - 2\int h\,dQ\,\mu_i + \int h\,dQ\right]; \tag{2.4}$$

$$C_0 = \frac{1}{3}\sum_{i=1}^{n} \frac{p_i^3}{1-p_i}\,;$$

$$C_1 = \left\{\frac{K}{2}\sum_{i=1}^{n} p_i^2 \sum_{j=1}^{n} \frac{p_j^2}{1-p_j} + KM\sum_{i=1}^{n}\frac{p_i^3}{1-p_i}\right\};$$

$$C_2 = \left\{\left|1 - \frac{1}{2}\sum_{i=1}^{n} p_i^2\right| \frac{\left(\sum_{j=1}^{n} p_j^2\right)^2}{\lambda^2} + \frac{K}{\lambda}\left(\sum_{i=1}^{n} p_i^2\right)^2\right.$$

$$\left. + \frac{K^2(1+2M)}{4}\sum_{i=1}^{n} p_i^2 \sum_{j=1}^{n}\frac{p_j^2}{1-p_j} + K^2 M\sum_{i=1}^{n}\frac{p_i^3}{1-p_i}\right\};$$

$$C_3 = \left\{\frac{K^2}{2\lambda}\left(1+\frac{1}{n-1}\right)\left(\sum_{i=1}^{n} p_i^2\right)^2 + \frac{K^3 M}{2}\sum_{i=1}^{n} p_i^2 \sum_{j=1}^{n}\frac{p_j^2}{1-p_j}\right.$$

$$\left. + \frac{K^3 M}{3}\sum_{i=1}^{n}\frac{p_i^3}{1-p_i}\right\};$$

$$C_4 = \frac{K^4 M}{8}\sum_{i=1}^{n} p_i^2 \sum_{j=1}^{n}\frac{p_j^2}{1-p_j}\,;$$

and M *is given by* (2.2).

Remark. 2.2 (i) Theorems 2.1 and 2.2 allow a very wide choice of possible functions h. No smoothness or positivity condition is assumed, and in view of the proof of Lemma 2.6 below, the growth condition $\int |h|\,dQ\,\mu^k < \infty$ ($k = 2$ or 4) is hardly restrictive at all.

(ii) If h is such that $\int |h|\,dQ$ is small (for example, $h = I_A$ where $Q(A)$ is small), then the smallness is also reflected in the error bounds.

The proof of Theorem 2.2 consists of the following lemmas.

Lemma 2.1 *We have*

$$\int h\,d\tilde{Q} - \int h\,dQ = -\Delta(Q,h) + S_1 + S_2 + R_1 + R_2 + R_3\,, \tag{2.5}$$

where

$$S_1 = -\left(1 - \frac{1}{2}\sum_{i=1}^{n} p_i^2\right) \sum_{r=n+1}^{\infty} e^{-\lambda} \frac{\lambda^r}{r!} \int h\,d\mu^r - \sum_{i=1}^{n} p_i^2 \sum_{r=n}^{\infty} e^{-\lambda} \frac{\lambda^r}{r!} \int h\,d\mu_i\,\mu^r$$
$$+ \frac{1}{2}\sum_{i=1}^{n} p_i^2 \sum_{r=n-1}^{\infty} e^{-\lambda}\frac{\lambda^r}{r!} \int h\,d\mu_i^2\,\mu^r\,;$$

$$S_2 = -\sum_{r=0}^{n} \frac{\lambda^r}{r!}\left[e^{-\lambda}\left(1 - \frac{1}{2}\sum_{i=1}^{n} p_i^2\right) - \prod_{i=1}^{n}(1-p_i)\right] \int h\,d\mu^r$$
$$- \sum_{i=1}^{n} p_i^2 \sum_{r=0}^{n-1} \frac{\lambda^r}{r!}\left[e^{-\lambda} - \prod_{j=1}^{n}(1-p_j)\right] \int h\,d\mu_i\,\mu^r$$
$$+ \frac{1}{2}\sum_{i=1}^{n} p_i^2 \sum_{r=0}^{n-2} \frac{\lambda^r}{r!}\left[e^{-\lambda} - \prod_{j=1}^{n}(1-p_j)\right] \int h\,d\mu_i^2\,\mu^r\,;$$

$$R_1 = \sum_{i=1}^{n} p_i^2 \sum_{j=1}^{n} p_j^2 \sum_{r=1}^{n}\sum_{l=1}^{r}\sum_{s=1}^{r-l} \frac{\lambda^{l+s-2}(r-l-s)!}{r!}\left(\int h\,d\mu_i\,\mu_j\,\nu_{r-l-s}^{(j)}\,\mu^{l+s-2}\right.$$
$$\left.-\int h\,d\mu_i\,\mu_j^2\,\nu_{r-l-s-1}^{(j)}\,\mu^{l+s-2}\right);$$

$$R_2 = -\sum_{i=1}^{n} p_i^2 \sum_{j=1}^{n} p_j^2 \sum_{r=1}^{n}\sum_{l=1}^{r}\sum_{s=1}^{r-l-1} \frac{\lambda^{l+s-2}(r-l)(r-l-s-1)!}{r!}$$
$$\times\left(\int h\,d\mu_i^2\,\mu_j\,\nu_{r-l-s-1}^{(j)}\,\mu^{l+s-2} - \int h\,d\mu_i^2\,\mu_j^2\,\nu_{r-l-s-2}^{(j)}\,\mu^{l+s-2}\right);$$

$$R_3 = \sum_{i=1}^{n} p_i^3 \sum_{r=1}^{n}\sum_{l=1}^{r} \frac{\lambda^{l-1}(r-l)!}{r!}\left(\int h\,d\mu_i\,\nu_{r-l}^{(i)}\,\mu^{l-1} - 2\int h\,d\mu_i^2\,\nu_{r-l-1}^{(i)}\,\mu^{l-1}\right.$$
$$\left.+\int h\,d\mu_i^3\,\nu_{r-l-2}^{(i)}\,\mu^{l-1}\right).$$

Proof. We shall use arguments similar to those of Chen [8], making use of the fact that $\mathscr{M}$ is a real commutative Banach algebra. We write

$$\tilde{Q} = \prod_{i=1}^{n}(1-p_i)\prod_{i=1}^{n}[I + q_i\mu_i] = \sum_{r=0}^{n} \nu_r\,,$$

where $q_i = \frac{p_i}{1-p_i}$, $\nu_0 = \left[\Pi_{i=1}^{n}(1-p_i)\right] I$, and for $r \geqq 1$,

$$\nu_r = \prod_{i=1}^{n}(1-p_i)\sum_{i_1<\cdots<i_r}\prod_{k=1}^{r} q_{i_k}\,\mu_{i_k}\,. \tag{2.6}$$

We also write $v_0^{(i)} = \left[\Pi_{j \neq i}(1-p_j)\right] I$, and for $r \geqq 1$,

$$v_r^{(i)} = \prod_{j \neq i} (1-p_j) \sum_{\substack{j_1 < \cdots < j_r: \\ j_1, \dots, j_r \neq i}} \prod_{k=1}^{r} q_{j_k} \mu_{j_k} .$$

As in [8], for $r \geqq 0$, we consider the identities

$$r v_r = \sum_{i=1}^{n} p_i \mu_i v_{r-1}^{(i)} \tag{2.7}$$

and

$$v_{r-1} = p_i \mu_i v_{r-2}^{(i)} + (1-p_i) v_{r-1}^{(i)} , \tag{2.8}$$

where v_r and $v_r^{(i)}$ are both taken to be the zero measure if r is negative. Combining (2.7) and (2.8) we have

$$r v_r = \lambda \mu v_{r-1} + \sum_{i=1}^{n} p_i^2 \mu_i (v_{r-1}^{(i)} - \mu_i v_{r-2}^{(i)}) .$$

From this we obtain

$$v_r = \frac{\lambda^r \mu^r}{r!} v_0 + \sum_{i=1}^{n} p_i^2 \mu_i \sum_{l=1}^{r} \frac{\lambda^{l-1} \mu^{l-1} (r-l)!}{r!} (v_{r-l}^{(i)} - \mu_i v_{r-l-1}^{(i)}) . \tag{2.9}$$

In order to obtain a second order expansion we apply (2.8) to (2.9) to get

$$\begin{aligned} v_r = {} & \frac{\lambda^r \mu^r}{r!} v_0 + \sum_{i=1}^{n} p_i^2 \mu_i \sum_{l=1}^{r} \frac{\lambda^{l-1} \mu^{l-1} (r-l)!}{r!} (v_{r-l} - \mu_i v_{r-l-1}) \\ & + \sum_{i=1}^{n} p_i^3 \mu_i \sum_{l=1}^{r} \frac{\lambda^{l-1} \mu^{l-1} (r-l)!}{r!} (v_{r-l}^{(i)} - \mu_i v_{r-l-1}^{(i)} - \mu_i (v_{r-l-1}^{(i)} - \mu_i v_{r-l-2}^{(i)})) . \end{aligned} \tag{2.10}$$

Next we apply (2.9) to v_{r-l} and v_{r-l-1} in (2.10) to obtain

$$\begin{aligned} v_r = {} & \frac{\lambda^r \mu^r}{r!} v_0 + \sum_{i=1}^{n} p_i^2 \mu_i \frac{\lambda^{r-1} \mu^{r-1}}{(r-1)!} v_0 - \frac{1}{2} \sum_{i=1}^{n} p_i^2 \mu_i^2 \frac{\lambda^{r-2} \mu^{r-2}}{(r-2)!} v_0 \\ & + \sum_{i=1}^{n} p_i^2 \mu_i \sum_{j=1}^{n} p_j^2 \mu_j \sum_{l=1}^{r} \frac{\lambda^{l-1} \mu^{l-1} (r-l)!}{r!} \\ & \times \sum_{s=1}^{r-l} \frac{\lambda^{s-1} \mu^{s-1} (r-l-s)!}{(r-l)!} (v_{r-l-s}^{(j)} - \mu_j v_{r-l-s-1}^{(j)}) \\ & - \sum_{i=1}^{n} p_i^2 \mu_i^2 \sum_{j=1}^{n} p_j^2 \mu_j \sum_{l=1}^{r} \frac{\lambda^{l-1} \mu^{l-1} (r-l)!}{r!} \\ & \times \sum_{s=1}^{r-l-1} \frac{\lambda^{s-1} \mu^{s-1} (r-l-s-1)!}{(r-l-1)!} (v_{r-l-s-1}^{(j)} - \mu_j v_{r-l-s-2}^{(j)}) \\ & + \sum_{i=1}^{n} p_i^3 \mu_i \sum_{l=1}^{r} \frac{\lambda^{l-1} \mu^{l-1} (r-l)!}{r!} (v_{r-l}^{(i)} - \mu_i v_{r-l-1}^{(i)} - \mu_i (v_{r-l-1}^{(i)} - \mu_i v_{r-l-2}^{(i)})) . \end{aligned} \tag{2.11}$$

Recall that $v_0 = [\Pi_{j=1}^{n}(1 - p_j)]I$. After summing the v_r given in (2.11) over r and integrating h with respect to the sum $\tilde{Q}$, we obtain (2.5). This proves the lemma. □

The next lemma was proved in [8].

Lemma 2.2 (Chen [8]) *Let v_r be as given in* (2.6). *We have for $r = 0, 1, \ldots, n$,*

$$v_r \leqq M e^{-\lambda} \frac{\lambda^r \mu^r}{r!}, \tag{2.12}$$

where M is given by (2.2).

Lemma 2.3 *For $i = 1, \ldots, n$ and $r = 0, 1, \ldots, n$, we have*

$$\text{(a)} \quad \frac{\mu}{K} \leqq \mu_i \leqq K\mu; \qquad \text{(b)} \quad v_{r-1}^{(i)} \leqq \frac{v_{r-1}}{1 - p_i}.$$

Proof. Part (a) follows from the condition that $\mu_i \leqq K\mu_j$ for $i, j = 1, \ldots, n$ and part (b) follows from (2.8). □

The next lemma is proved by using the identity $1 - x = \exp\{-\Sigma_{k=1}^{\infty} \frac{x^k}{k}\}$ for $|x| < 1$.

Lemma 2.4 *We have*

$$\text{(a)} \quad 0 \leqq e^{-\lambda} - \prod_{i=1}^{n} (1 - p_i) \leqq \frac{1}{2} e^{-\lambda} \sum_{i=1}^{n} \frac{p_i^2}{1 - p_i};$$

$$\text{(b)} \quad 0 \leqq e^{-\lambda} \left(1 - \frac{1}{2} \sum_{i=1}^{n} p_i^2\right) - \prod_{i=1}^{n} (1 - p_i) \leqq \frac{1}{3} e^{-\lambda} \sum_{i=1}^{n} \frac{p_i^3}{1 - p_i}.$$

The next lemma is proved by applying the Chebyshev inequality and using the inequality $\lambda^2 \leqq n \Sigma_{i=1}^{n} p_i^2$.

Lemma 2.5 *We have, for $h \geqq 0$,*

$$\text{(a)} \quad \sum_{r=n+1}^{\infty} e^{-\lambda} \frac{\lambda^r}{r!} \int h \, d\mu^r \leqq \frac{\left(\sum_{i=1}^{n} p_i^2\right)^2}{\lambda^2} \int h \, dQ \, \mu^2;$$

$$\text{(b)} \quad \sum_{r=n}^{\infty} e^{-\lambda} \frac{\lambda^r}{r!} \int h \, d\mu^r \leqq \frac{\sum_{i=1}^{n} p_i^2}{\lambda} \int h \, dQ \, \mu;$$

$$\text{(c)} \quad \sum_{r=n-1}^{\infty} e^{-\lambda} \frac{\lambda^r}{r!} \int h \, d\mu^r \leqq \left(1 + \frac{1}{n-1}\right) \frac{\sum_{i=1}^{n} p_i^2}{\lambda} \int h \, dQ \, \mu.$$

Lemma 2.6 *The condition $\int |h| dQ \, \mu^4 < \infty$ implies that $\int |h| dQ \, \mu^k < \infty$ for $k = 0, 1, 2, 3$.*

Proof. It suffices to show that $\int |h| dQ\,\mu^{k+1} < \infty$ implies that $\int |h| dQ\,\mu^{k} < \infty$ for $k = 0, 1, 2, 3$. First, we observe that $\int |h| dQ\,\mu^{k} < \infty$ for any $k \geqq 0$ implies $\int |h| d\mu^{r} < \infty$ for $r = 0, 1, 2, \ldots$. Next,

$$\int |h| dQ\,\mu^{k+1} = e^{-\lambda} \sum_{r=0}^{\infty} \frac{\lambda^r}{r!} \int |h| d\mu^{r+k+1} = e^{-\lambda} \sum_{s=1}^{\infty} \frac{\lambda^{s-1}}{(s-1)!} \int |h| d\mu^{s+k}$$

$$\geqq \frac{e^{-\lambda}}{\lambda} \sum_{s=1}^{\infty} \frac{\lambda^s}{s!} \int |h| d\mu^{s+k} = \frac{1}{\lambda} \int |h| dQ\,\mu^{k} - \frac{e^{-\lambda}}{\lambda} \int |h| d\mu^{k}.$$

This proves the lemma. □

Combining Lemmas 2.1–2.6, we prove Theorem 2.2.

3 Large deviations

From now on let $\mathscr{K}_1$ be a coset of a subgroup $\mathscr{K}_0$ of $\mathscr{X}$ such that $\mathscr{K}_1$ has infinite order in the quotient group $\mathscr{X}/\mathscr{K}_0$. For convenience, assume $\mathscr{X}$ to be an additive group. Define $\mathscr{K}_r = \mathscr{K}_{r-1} + \mathscr{K}_1$ for $r = 1, 2, \ldots$. Then $\mathscr{K}_1, \mathscr{K}_2, \ldots$ are distinct cosets of $\mathscr{K}_0$. Let $\mathscr{K} = \bigcup_{r=1}^{\infty} \mathscr{K}_r$. Since $\mu_i \leqq K\mu_j$ for $i, j = 1, \ldots, n$, the measures μ and $\mu_i, i = 1, \ldots, n$, have the same support. Assume that $\mathrm{supp}(\mu) \subset \mathscr{K}$ and that there is a positive integer l such that $\mathrm{supp}(\mu) \cap \mathscr{K}_r = \emptyset$ for $1 \leqq r < l$ and $\mathrm{supp}(\mu) \cap \mathscr{K}_l \neq \emptyset$. Then for $z = 1, 2, \ldots$, $\mathrm{supp}(\mu^z) \cap \mathscr{K}_r = \emptyset$ for $1 \leqq r < zl$ and $\mathrm{supp}(\mu^z) \cap \mathscr{K}_{zl} \neq \emptyset$.

Here is an example of $\mathscr{X}$ with subgroup $\mathscr{K}_0$ and distinct cosets $\mathscr{K}_1, \mathscr{K}_2, \ldots$ such that $\mathscr{K}_r = \mathscr{K}_{r-1} + \mathscr{K}_1$. Take $\mathscr{X} = \mathbb{Z}^d$, where $\mathbb{Z}$ is the set of integers and $d = 1, 2, \ldots$. Let $k_1, \ldots, k_d$ be integers such that $k_1x_1 + \cdots + k_dx_d = 1$ has an integral solution. Define $\mathscr{K}_r = \{(x_1, \ldots, x_d) \in \mathbb{Z}^d : k_1x_1 + \cdots + k_dx_d = r\}$, where $r = 0, 1, 2, \ldots$. Then $\mathscr{K}_0$ is a subgroup of $\mathbb{Z}^d$ and $\mathscr{K}_1, \mathscr{K}_2, \ldots$ are distinct cosets of $\mathscr{K}_0$ with $\mathscr{K}_r = \mathscr{K}_{r-1} + \mathscr{K}_1$.

Define $\mathscr{A}_\mu$ to be the class of real-valued functions h defined on $\mathscr{K}$ which satisfy the following property: there exist a positive integer r_0 and a positive function α defined on $\mathrm{supp}(\mu)$ (both r_0 and α depending on h) such that $\int \alpha^k\, d\mu < \infty$ for $k = 1, 2, \ldots$, and $0 \leqq h(x + \xi) \leqq \alpha(\xi)h(x)$ for $x \in \bigcup_{r=r_0}^{\infty} \mathrm{supp}(\mu^r)$ and $\xi \in \mathrm{supp}(\mu)$.

Proposition 3.1 (a) *The class $\mathscr{A}_\mu$ is closed under addition, nonnegative scalar multiplication, and multiplication of functions* (*in the sense that if h_1 and $h_2 \in \mathscr{A}_\mu$, then $h_1h_2 \in \mathscr{A}_\mu$*).

(b) *It contains all eventually decreasing functions, that is, functions h with the property*: $0 \leqq h(x + \xi) \leqq h(x)$ *for* $x \in \bigcup_{r=r_0}^{\infty} \mathrm{supp}(\mu^r)$ *and* $\xi \in \mathrm{supp}(\mu)$. *In particular, it contains indicators of sets from the class $\mathscr{C}$, where*

$$\mathscr{C} = \{A \subset \mathscr{K} : \textit{ if } x \notin A, \textit{then } x + \xi \notin A \textit{ where } x \in \bigcup_{r=r_0}^{\infty} \mathrm{supp}(\mu^r) \textit{ and } \xi \in \mathrm{supp}(\mu)\}.$$

(c) *Suppose* $\mathscr{X} = \mathbb{Z}^d$, $d = 1, 2, \ldots$, *and suppose* $\mathscr{K}_r = \{(x_1, \ldots, x_d) \in \mathbb{Z}^d : k_1x_1 + \cdots + k_dx_d = r\}$ *such that* $\mathscr{K}_1 \neq \emptyset$, *where* $r = 0, 1, 2, \ldots$ *and* $k_1, \ldots, k_d$ *are* *integers. If*

$$\int \exp\left\{\sum_{i=1}^{d} t_i x_i\right\} d\mu(x_1,\ldots,x_d) < \infty$$

for all real numbers $t_1,\ldots,t_d$ *(in particular, if* $\operatorname{supp}(\mu)$ *is finite), then* $\mathscr{A}_\mu$ *contains* (i) h *where* $h(x)$ *is a polynomial in* $a_1x_1+\cdots+a_dx_d$ *which is positive and bounded away from* 0 *on* $\bigcup_{r=r_0}^{\infty}\operatorname{supp}(\mu^r)$ *for sufficiently large* r_0, *and* (ii) h *where* $h(x)=\exp\{\Sigma_{i=1}^d a_ix_i\}$, *where* $a_1,\ldots,a_d$ *are real numbers and* $x=(x_1,\ldots,x_d)\in\mathbb{Z}^d$.

Proof. We omit the proofs of (a), (b) and (c) (ii) as they are easy. For (c) (i), we first note that $h(x+\xi)$ is a polynomial in $a_1\xi_1+\cdots+a_d\xi_d$ whose coefficients are polynomials in $a_1x_1+\cdots+a_dx_d$, where $\xi=(\xi_1,\ldots,\xi_d)\in\operatorname{supp}(\mu)$. So there exists $B>0$ such that for $|a_1x_1+\cdots+a_dx_d|>B$, $h(x+\xi)/h(x)=1+(h(x+\xi)-h(x))/h(x)$ where $(h(x+\xi)-h(x))/h(x)$ is a polynomial in $a_1\xi_1+\cdots+a_d\xi_d$ with bounded coefficients. On the other hand, for $|a_1x_1+\cdots+a_dx_d|\leqq B$, $h(x+\xi)\leqq\varepsilon^{-1}h(x+\xi)h(x)$ on $\bigcup_{r=r_0}^{\infty}\operatorname{supp}(\mu^r)$, where ε is a lower bound of h and $\varepsilon^{-1}h(x+\xi)$ is also a polynomial in $a_1\xi_1+\cdots+a_d\xi_d$ with bounded coefficients. This proves (c) (i). □

The following two theorems are the main results in this section.

Theorem 3.1 *Let* $(\mathscr{X},\mathscr{B})$ *be a measurable abelian group and let* $\tilde{Q}$ *and* Q *be given by* (1.1) *and* (1.2) *respectively. Assume that there exists a constant* K *such that* $\mu_i\leqq K\mu_j$ *for* $i,j=1,\ldots,n$. *Let* $h\in\mathscr{A}_\mu$ *be such that* $\int|h|dQ\,\mu^2<\infty$ *and let* $h_z=hI\left[\bigcup_{r=zl}^{\infty}\mathscr{K}_r\right]$ *for* $z=1,2,\ldots$. *Then, for* $\lambda>0$ *bounded and* $z+1>\max\{c\lambda,r_0\}$,

$$\left|\frac{\int h_z\,d\tilde{Q}}{\int h_z\,dQ}-1\right|\leqq\frac{1}{2}K^2M\left(\frac{z+1}{z+1-c\lambda}\right)\left\{\frac{z(z+1)}{\lambda^2}+3+\frac{2}{\lambda}\right\}\sum_{i=1}^{n}\frac{p_i^2}{1-p_i}, \quad (3.1)$$

where $c=\max\{1,\int\alpha\,d\mu\}$ *and* M *is given by* (2.2), *provided* $\int h_z\,dQ>0$.

For the next theorem, we regard $p_i,\mu_i,i=1,\ldots,n,\lambda$ and μ as being dependent on n. Note that λ is not required to be bounded or bounded away from zero.

Theorem 3.2 *Let* $(\mathscr{X},\mathscr{B})$ *be a measurable abelian group and let* $\tilde{Q}$ *and* Q *be given by* (1.1) *and* (1.2) *respectively. Assume that there exists a constant* K *such that* $\mu_i\leqq K\mu_j$ *for* $i,j=1,\ldots,n$. *Let* $h\in\mathscr{A}_\mu$ *be such that* $\int|h|dQ\,\mu^4<\infty$ *and let* $h_z=hI\left[\bigcup_{r=zl}^{\infty}\mathscr{K}_r\right]$ *for* $z=1,2,\ldots$. *Suppose* $\Sigma_{i=1}^n p_i^2\to0$, $z/\lambda=o((\Sigma_{i=1}^n p_i^2)^{-1/2})$ *and* $z/\lambda\to\infty$ *as* $n\to\infty$. *Then, as* $n,z\to\infty$,

$$\frac{\int h_z\,d\tilde{Q}}{\int h_z\,dQ}-1\sim-\frac{z^2}{2\lambda^2}\sum_{i=1}^{n}p_i^2\theta_i(z,h_z), \quad (3.2)$$

where $\theta_i(z,h_z)=\int h_z\,d\mu^{z-2}\,\mu_i^2/\int h_z\,d\mu^z$, *provided* $\int h_z\,dQ>0$ *for sufficiently large* z.

Remark. 3.1 (i) By Lemma 2.3 (a), $K^{-2} \leqq \theta_i(z,h_z) \leqq K^2$ for $i = 1,\dots,n$.

(ii) $\theta_i(z,h_z) = \int h_z \psi_{i,z}\, d\mu^z / \int h_z\, d\mu^z$ for $i = 1,\dots,n$, where $\psi_{i,z} = d\mu^{z-2}\mu_i^2 / d\mu^z$ and is given by

$$\psi_{i,z}(x) = E\left\{ \frac{d\mu_i}{d\mu}(U)\frac{d\mu_i}{d\mu}(V) | Y_1 + \cdots + Y_{z-2} + U + V = x \right\},$$

where $Y_1,\dots,Y_{z-2},U,V$ are i.i.d. random elements taking values in $\mathscr{X}$ with distribution μ.

(iii) If $\mu_1 = \cdots = \mu_n$, then $\theta_i(z,h_z) = 1$ for $i = 1,\dots,n$.

(iv) The support of h is allowed to vary with z.

(v) Theorem 3.2 generalizes Theorem 4.2 of Barbour et al. [3] in the case when λ satisfies the stated conditions.

We shall give the proof of Theorem 3.2 only, since the proof of Theorem 3.1 is easier and uses the same argument. We need a few lemmas. The first lemma is proved in [3] (see Lemma 4.5 (b) and its proof).

Lemma 3.1 (Barbour et al. [3])*Suppose g is a real-valued function defined on the set of nonnegative integers such that $0 \leqq g(r+1) \leqq cg(r)$ for $r \geqq r_0$, where c is a constant $\geqq 1$ and r_0 a positive integer (r_0 and c both depending on g). Let N be a Poisson random variable with parameter λ. Assume $E\{g(N)I[N \geqq r_0]\} < \infty$. Then for $\lambda > 0$ and $z+1 > \max\{c\lambda, r_0\}$,*

$$1 \leqq \frac{Eg(N)I[N \geqq z]}{g(z)P[N = z]} \leqq \frac{z+1}{z+1-c\lambda}, \tag{3.3}$$

provided $g(z) > 0$.

Lemma 3.2 *Let $h \in \mathscr{A}_\mu$ and let $\tilde{h}_{i,k}(r) = \int h d\mu^{r-k}\mu_i^k$ for $i = 1,\dots,n$, $r = 0,1,2,\dots$ and $k = 0,1,\dots,r$ (assuming integrability). Then $\tilde{h}_{i,k}$ satisfies condition on g in Lemma 3.1.*

Proof. We have for $r \geqq r_0$,

$$\begin{aligned}\tilde{h}_{i,k}(r+1) &= \int\int h(x+\xi) d\mu(\xi) d\mu^{r-k}\mu_i^k(x) \\ &\leqq \int\int \alpha(\xi) h(x) d\mu(\xi) d\mu^{r-k}\mu_i^k(x) = \left(\int \alpha d\mu\right)\tilde{h}_{i,k}(r).\end{aligned}$$

This proves Lemma 3.2. □

We state the next lemma without proof.

Lemma 3.3 *We have*

$$\sum_{i=1}^n \frac{p_i^3}{1-p_i} \leqq \sum_{i=1}^n p_i^2 \sqrt{\sum_{i=1}^n \frac{p_i^2}{(1-p_i)^2}}.$$

Proof of Theorem 3.2 We use Theorem 2.2. It suffices to prove the following:

$$\text{(i)}\qquad \frac{\sum_{k=0}^{4} C_k \int h_z dQ\mu^k}{\Delta(Q, h_z)} \to 0\,, \tag{3.4}$$

$$\text{(ii)}\qquad \frac{\Delta(Q, h_z)}{\int h_z\, dQ} \sim \frac{z^2}{2\lambda^2} \sum_{i=1}^{n} p_i^2 \theta_i(z, h_z)\,, \tag{3.5}$$

as $n, z \to \infty$.

Here we note that $h_z \geqq 0$ for sufficiently large z. Let N be a Poisson random variable with parameter λ and let $\tilde{h}_z(r) = \int h_z\, d\mu^r$ for $r = 0, 1, 2, \ldots$. Then for $k = 0, 1, \ldots, 4$,

$$\begin{aligned} \int h_z\, dQ\mu^k &= \sum_{r=z-k}^{\infty} e^{-\lambda} \frac{\lambda^r}{r!} \int h_z d\mu^{r+k} = E\tilde{h}_z(N+k) I[N+k \geqq z] \\ &= \frac{EN(N-1)\cdots(N-k+1)\tilde{h}_z(N) I[N \geqq z]}{\lambda^k}\,. \end{aligned} \tag{3.6}$$

By Lemmas 3.1 and 3.2, we have for $k = 0, 1, \ldots, 4$, and sufficiently large z,

$$1 \leqq \frac{\lambda^k \int h_z\ dQ\mu^k}{z(z-1)\cdots(z-k+1)\tilde{h}_z(z) P[N=z]} \leqq \frac{z+1}{z+1-c\lambda}\,,$$

where $c = \max\{1, \int \alpha\, d\mu\}$.

Therefore, for $k = 0, 1, \ldots, 4$,

$$\int h_z\, dQ\mu^k \sim \frac{z^k}{\lambda^k} \left(\int h_z\, d\mu^z\right) P[N = z]\,, \tag{3.7}$$

as $n, z \to \infty$.

Similarly, by defining $\tilde{h}_{z,i,k}(r) = \int h_z\, d\mu^{r-k} \mu_i^k$, we have for $i = 1, \ldots n$ and $k = 0, 1, 2$,

$$\begin{aligned} \int h_z\, dQ\mu_i^k &= \sum_{r=z-k}^{\infty} e^{-\lambda} \frac{\lambda^r}{r!} \int h_z\ d\mu^r\ \mu_i^k = E\tilde{h}_{z,i,k}(N+k) I[N+k \geqq z] \\ &= \frac{EN(N-1)\cdots(N-k+1)\tilde{h}_{z,i,k}(N) I[N \geqq z]}{\lambda^k}\,. \end{aligned}$$

By Lemmas 3.1 and 3.2 again, we have for $i = 1, \ldots, n, k = 0, 1, 2$, and sufficiently large z,

$$1 \leqq \frac{\lambda^k \int h_z\, dQ\mu_i^k}{z(z-1)\cdots(z-k+1)\tilde{h}_{z,i,k}(z) P[N=z]} \leqq \frac{z+1}{z+1-c\lambda}\,,$$

where $c = \max\{1, \int \alpha\, d\mu\}$.

This implies that for $k = 0, 1, 2,$

$$\int h_z \, dQ\mu_i^k \sim \frac{z^k}{\lambda^k} \left(\int h_z \, d\mu^{z-k}\mu_i^k\right) P[N = z] \,. \tag{3.8}$$

uniformly in i for $i = 1, \ldots, n$, as $n, z \to \infty$.

By Lemma 2.3(a), (3.8) implies that for $k = 0, 1, 2,$

$$\int h_z \, dQ\mu_i^k \asymp \frac{z^k}{\lambda^k} \left(\int h_z \, d\mu^z\right) P[N = z] \,. \tag{3.9}$$

uniformly in i for $i = 1, \ldots, n$, as $n, z \to \infty$.

In view of (3.9), $\frac{1}{2}\Sigma_{i=1}^n p_i^2 \int h_z \, dQ\mu_i^2$ is the dominant term in $\Delta(Q, h_z)$. So, to prove (3.4), it suffices to show that

$$\frac{\lambda^2 \sum_{k=0}^4 C_k \int h_z \, dQ\mu^k}{z^2(\sum_{i=1}^n p_i^2)(\int h_z \, d\mu^z) P[N = z]} \to 0 \,,$$

as $n, z \to \infty$.

Indeed, by (3.7), for $k = 0, \ldots, 4$,

$$\frac{\lambda^2 C_k \int h_z \, dQ\mu^k}{z^2(\sum_{i=1}^n p_i^2)(\int h_z \, d\mu^z) P[N = z]} \sim \frac{z^{k-2} C_k}{\lambda^{k-2} \sum_{i=1}^n p_i^2}$$

which tends to 0 as $n, z \to \infty$, using Lemma 3.3 where applicable. This proves (3.4).

To prove (3.5), it suffices to show that

$$\frac{\int h_z \, dQ\mu_i^2}{\int h_z \, dQ} \sim \frac{z^2}{\lambda^2} \theta_i(z, h_z)$$

uniformly in i for $i = 1, \ldots, n$, as $n, z \to \infty$.

Indeed, this follows from (3.7) and (3.8). This proves (3.5) and completes the proof of Theorem 3.2. □

4 Multivariate Poisson approximation

Let $e(j)$ be the basis vector in $\mathbb{Z}^d$ with 1 in the jth position, where $j = 1, \ldots, d$ and $d = 1, 2, \ldots$. Consider independent random vectors $X_1, \ldots, X_n$ which take values in $\mathbb{Z}^d$ with $P[X_i = e(j)] = p_{ij} = p_i \alpha_{ij}$ and $P[X_i = 0] = 1 - p_i$, where $\Sigma_{j=1}^d \alpha_{ij} = 1$, for $i = 1, \ldots, n$. Let $W = \Sigma_{i=1}^n X_i, \lambda_j = \Sigma_{i=1}^n p_{ij} = \Sigma_{i=1}^n p_i \alpha_{ij}$ and $\lambda = \Sigma_{j=1}^d \lambda_j = \Sigma_{i=1}^n p_i$.

Define $Z_1, \ldots, Z_d$ to be independent Poisson random variables with parameters $\lambda_1, \ldots, \lambda_d$ respectively. Let $Z = (Z_1, \ldots, Z_d)$. The problem of approximating $\mathscr{L}(W)$ by $\mathscr{L}(Z)$ has been considered by Barbour [2] who obtained an error bound on the total variation distance between $\mathscr{L}(W)$ and $\mathscr{L}(Z)$ using a probabilistic approach to Stein's method.

In this section we show that multivariate Poisson approximation can be considered as a special case of compound Poisson approximation on a group. However, instead of considering total variation bounds, we consider unbounded function approximation and large deviations. To this end, we consider $\mathbb{Z}^d$

to be an additive group. Take $\mathscr{X} = \mathbb{Z}^d$ and define $\mathscr{K}_r = \{(x_1,\dots,x_d) \in \mathbb{Z}^d : x_1 + \cdots + x_d = r\}$, for $r = 0,1,2,\dots$. Clearly $\mathscr{K}_0$ is a subgroup of $\mathscr{X}$ and $\mathscr{K}_1, \mathscr{K}_2, \dots$ are distinct cosets of $\mathscr{K}_0$ with $\mathscr{K}_r = K_{r-1} + \mathscr{K}_1, r = 1,2,\dots$. Define μ_i by $\mu_i(\{e(j)\}) = \alpha_{ij}$. Then $\mathscr{L}(W) = \Pi_{i=1}^n[(1-p_i)I + p_i\mu_i] = \tilde{Q}$. Now $\mu = \lambda^{-1}\Sigma_{i=1}^n p_i\mu_i$, so $\mu(\{e(j)\}) = \lambda_j/\lambda$. Therefore μ^r is the multinomial distribution $MN(r, \frac{\lambda_1}{\lambda}, \dots, \frac{\lambda_d}{\lambda})$. This implies that $\mathscr{L}(Z) = e^{-\lambda}\Sigma_{r=0}^\infty \frac{\lambda^r\mu^r}{r!} = Q$. The condition that $\mu_i \leqq K\mu_j$ for $i,j = 1,\dots,n$ is equivalent to

$$\max_{1\leqq i\leqq n} \alpha_{ij} \leqq K \min_{1\leqq i\leqq n} \alpha_{ij} \quad \text{for } j = 1,\dots,d. \tag{4.1}$$

Hence, if we assume (4.1), then Theorems 2.1 and 2.2 can be applied to approximate $Eh(W)$ by $Eh(Z)$ for unbounded functions h defined on $\mathbb{Z}^d$.

Next we observe that under (4.1), $\text{supp}(\mu_i) = \text{supp}(\mu)$ for $i = 1,\dots,n$, and $\text{supp}(\mu^r) \subset \mathscr{K}_r$ for $r = 1,2,\dots$. Hence Theorems 3.1 and 3.2 are also applicable. The following is a corollary of Theorem 3.2.

Theorem 4.1 *Let h be a real-valued function defined on $\mathbb{Z}^{+d}$ such that $0 \leqq h(x + e(j)) \leqq ch(x)$ for all $x = (x_1,\dots,x_d)$ with $x_1 + \cdots + x_d \geqq r_0$ and $j = 1,\dots,d$, where c is a constant and r_0 a positive integer (c and r_0 both depending on h). Assume that $E(Ze^{\mathrm{T}})^4|h(Z)| < \infty$ where e^{T} is the transpose of $e = (1,\dots,1)$. Suppose that condition (4.1) holds and that $\Sigma_{i=1}^n p_i^2 \to 0$, $z/\lambda = o((\Sigma_{i=1}^n p_i^2)^{-1/2})$ and $z/\lambda \to \infty$ as $n \to \infty$. Then, as $n, z, \to \infty$,*

$$\frac{Eh(W)I[We^{\mathrm{T}} \geq z]}{Eh(Z)I[Ze^{\mathrm{T}} \geqq z]} - 1 \sim -\frac{1}{2Eh(U_z)}\sum_{i=1}^n p_i^2 Eh(U_z)\phi_i(U_z), \tag{4.2}$$

where $U_z \sim MN\left(z, \frac{\lambda_1}{\lambda}, \dots, \frac{\lambda_d}{\lambda}\right)$ and

$$\phi_i(x_1,\dots,x_d) = \sum_{j=1}^d \sum_{\substack{k=1\\k\neq j}}^d \frac{\alpha_{ij}\alpha_{ik}x_jx_k}{\lambda_j\lambda_k} + \sum_{j=1}^d \frac{\alpha_{ij}^2 x_j(x_j-1)}{\lambda_j^2},$$

for $i = 1,\dots,n$, provided $Eh(Z)I[Ze^{\mathrm{T}} \geqq z] > 0$ for sufficiently large z.

Remark. 4.1 (i) Since $\mathscr{X} = \mathbb{Z}^d$ with $\mathscr{K}_r = \{(x_1,\dots,x_d) \in \mathbb{Z}^d\colon x_1 + \cdots + x_d = r\}$, and $\text{supp}(\mu) = \{e(1),\dots,e(d)\}$ which is finite, the class of functions h satisfying the condition of Theorem 4.1 has all the properties stated in Proposition 3.1.

(ii) The unbounded function approximations deduced from Theorems 2.1 and 2.2 yield total variation bounds. While the second total variation bound is of second order, the first is not as good as that obtained by Barbour [2] in the case of multivariate Poisson approximation, due to the requirement of the condition (4.1) and the absence of a 'magic' factor.

Proof of Theorem 4.1 From the proof of Lemma 2.6 and the fact that $\text{supp}(\mu^r) = \text{supp}(\mathscr{L}(Z \mid Ze^{\mathrm{T}} = r))$, it is clear that the condition $\int |h| dQ\mu^4 < \infty$ is equivalent to $E(Ze^{\mathrm{T}})^4|h(Z)| < \infty$. It remains to show that $\psi_{i,z}(x_1,\dots,x_d)$

given in Remark 3.1(ii) is equal to $\lambda^2\phi_i(x_1,\ldots,x_d)/z(z-1)$, where $z = x_1 + \cdots + x_d$. Indeed, $\frac{d\mu_i}{d\mu}(\{e(j)\}) = \lambda\alpha_{ij}/\lambda_j$, so, with $x = (x_1,\ldots,x_d)$,

$$\begin{aligned}
\psi_{i,z}(x) &= \frac{1}{P[U_z = x]}\left\{\sum_{j=1}^{d}\sum_{\substack{k=1\\k\neq j}}^{d}\frac{\lambda\alpha_{ij}}{\lambda_j}\frac{\lambda\alpha_{ik}}{\lambda_k}\frac{\lambda_j\lambda_k}{\lambda^2}P[U_{z-2} = x - e(j) - e(k)]\right.\\
&\qquad\left.+\sum_{j=1}^{d}\frac{\lambda^2\alpha_{ij}^2}{\lambda_j^2}\frac{\lambda_j}{\lambda^2}P[U_{z-2} = x - 2e(j)]\right\}\\
&= \frac{1}{P[U_z = x]}\left\{\sum_{j=1}^{d}\sum_{\substack{k=1\\k\neq j}}^{d}\frac{\lambda^2}{z(z-1)}\frac{\alpha_{ij}\alpha_{ik}x_jx_k}{\lambda_j\lambda_k}P[U_z = x]\right.\\
&\qquad\left.+\sum_{j=1}^{d}\frac{\lambda^2}{z(z-1)}\frac{\alpha_{ij}^2x_j(x_j-1)}{\lambda_j^2}P[U_z = x]\right\}\\
&= \frac{\lambda^2}{z(z-1)}\phi_i(x_1,\ldots,x_d)\,.
\end{aligned}$$

This completes the proof of Theorem 4.1. □

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