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Probability Theory and Related Fields

The behavior of solutions of stochastic differential inequalities

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Summary. Let X and Z be $\mathbb{R}^d$-valued solutions of the stochastic differential inequalities $\mathrm{d}X_t \leq a(t,X_t)\mathrm{d}t + \sigma(t,X_t)\mathrm{d}W_t$ and $b(t,Z_t)\mathrm{d}t + \sigma(t,Z_t)\mathrm{d}W_t \leq \mathrm{d}Z_t$, respectively, with a fixed $\mathbb{R}^m$-valued Wiener process W. In this paper we give conditions on a, b and σ under which the relation $X_0 \leq Z_0$ of the initial values leads to the same relation between the solutions with probability one. Further we discuss whether in general our conditions can be weakened or not. Then we deal with notions like 'maximal/minimal solution' of a stochastic differential inequality. Using the comparison result we derive a sufficient condition for the existence of such 'solutions' as well as some Gronwall-type estimates.

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1 Introduction

One of the important and effective techniques in the theory of differential equations is the comparison method. It can be applied in proving a priori estimates for existence and uniqueness theorems as well as in stability investigations and in many other directions (cf. [9] or [14]). From this point of view it is quite clear that comparison theorems also appear in the theory of stochastic differential equations (cf. [6] and the references therein). The aim of the present paper is to derive a comparison theorem for systems of stochastic differential *inequalities.* In this situation the famous machinery of the Itô-formula does not work. Instead of it we develop a method which is based on a simple idea. As far as we know Anderson [1] was the first who used it. Let us shortly sketch this idea.

Anderson considered $\mathbb{R}^d$-valued solutions X and Z of the homogeneous stochastic differential equations

$$\mathrm{d}X_t = a(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t$$

$$dZ_t = b(Z_t)dt + \sigma(Z_t)dW_t$$

with the same initial value. Then he proved that for $j = 1, ..., d$

$$\lim_{t \searrow 0} \frac{Z_j(t) - X_j(t)}{t} = b_j(X(0)) - a_j(X(0)) \tag{1.1}$$

holds a.s. if a, b are continuous and σ is Hölder continuous with exponent greater than $\frac{1}{2}$. Under the condition

$$a_j(x) < b_j(x), \qquad x \in \mathbb{R}^d,$$

this implies

$$\mathbf{P}(\{X_j(t) < Z_j(t) \text{ for all sufficiently small } t > 0\}) = 1. \tag{1.2}$$

Using the homogeneity of the considered equations one can extend this result to

$$\mathbf{P}(\{X_j(t) \leq Z_j(t),\ t \geq 0\}) = 1$$

if an additional condition on σ is required. But in case of stochastic differential inequalities

$$dX_t \leq a(X_t)dt + \sigma(X_t)dW_t$$

and

$$dZ_t \geq b(Z_t)dt + \sigma(Z_t)dW_t$$

one can derive only

$$\limsup_{t \searrow 0} \frac{Z_j(t) - X_j(t)}{t} \leq b_j(X(0)) - a_j(X(0)) \qquad \text{a.s.}$$

instead of (1.1) which does not allow to conclude (1.2).

In the one-dimensional case the authors of [3] attempted to solve this problem with a complicated stopping procedure which we shall briefly discuss now: For two processes X, Z with the same initial value and for $q \in \mathbb{Q}_+$ they consider the stopping times

$$\begin{aligned} \tau^q &= \inf\{t > q : X_t = Z_t\}, \\ \tau_1^q &= \inf\{t > \tau^q : X_t > Z_t\}, \\ \overline{\tau}^q &= \inf\{t > \tau_1^q : X_t < Z_t\} \end{aligned}$$

and conclude (cf. [3], page 18)

If $\mathbf{P}(\{\tau_1^q = \overline{\tau}^q \text{ for all } q \in \mathbb{Q}_+\}) = 1$ **then** $\mathbf{P}(\{X_t \leq Z_t \text{ for all } t \geq 0\}) = 1.$

But the last implication is not true in general. Indeed, if we take for example $X \equiv 0$ as well as for $Z = W$ (the standard Wiener process) then from the implication would be follow that the Wiener process is nonnegative.

In higher but finite dimensions we found an analogous mistake in [4]. This has stimulated us to write the present paper. With our technique we obtain for systems of inhomogeneous stochastic differential inequalities with possible explosions a better result than Anderson in [1] for homogeneous stochastic differential equations. It should be noted here that the authors in [5], [6], [10] proved our result in the special case of stochastic differential equations but we do not know how one could use their techniques with respect to inequalities. Further we also admit different initial values and this turns out to be very useful which the proofs below will show. Our results are in a certain sense stronger than those stated in [3], [5], [6] or [10]: We shall give conditions guaranteeing that the solutions cannot come in contact with each other.

At first look the conditions in our Theorem 2.3 seem to be very restrictive. Therefore we shall give several arguments showing that this is not really true.

Inspired by the deterministic theory we introduce the notions 'maximal/minimal solution' of a stochastic differential inequality and give a sufficient condition for the existence of such 'solutions'. If there does not exist the 'maximal/minimal solution' then we look for upper/lower boundaries for the solutions of the stochastic differential inequality, respectively. In dimension one this leads to some Gronwall-type estimates.

2 A comparison theorem

We shall always work with a complete probability space $(\Omega, \mathscr{F}, \mathbf{P})$ equipped with a right continuous filtration $\mathbb{F} = (\mathscr{F}_t)_{t \geq 0}$ such that $\mathscr{F}_0$ contains all $\mathbf{P}$-null sets of $\mathscr{F}$. In order to deal with processes with a possible explosion we introduce the one-point compactification $\overline{\mathbb{R}}^d = \mathbb{R}^d \cup \{\triangle\}$ of $\mathbb{R}^d$ together with the corresponding Borel σ-algebra. In our framework it is technically useful to define $\|\triangle\| = 0$. In the sequel we shall deal with the following two systems of stochastic integral inequalities:

$$(\leq) \qquad X_j(t) \leq X_j(s) + \int_s^t a_j(r, X(r))\,\mathrm{d}r + \sum_{k=1}^m \int_s^t \sigma_{jk}(r, X(r))\,\mathrm{d}W_k(r)$$

$$(\geq) \qquad Z_j(t) \geq Z_j(s) + \int_s^t b_j(r, Z(r))\,\mathrm{d}r + \sum_{k=1}^m \int_s^t \sigma_{jk}(r, Z(r))\,\mathrm{d}W_k(r)$$

where $j = 1, \dots, d$ and $W = (W_1, \dots, W_m)$ is a given m-dimensional $\mathbb{F}$-adapted Wiener process. The initial values $X(0)$ and $Z(0)$ are supposed to be $\mathbb{F}_0$-measurable. Moreover they satisfy the condition

$$(C_0) \qquad X_j(0) \leq Z_j(0)$$

for $j = 1, \dots, d$ $\mathbf{P}$-a.s. The mappings $a_j, b_j, \sigma_{jk} : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$, $j = 1, \dots, d$, $k = 1, \dots, m$, are always assumed to be continuous. For a later use we need the following condition on σ.

$$(C_\sigma)\quad \begin{cases} \text{For some } \alpha \in (\frac{1}{2}, \infty) \text{ and each } T, N > 0 \text{ there exists a constant } K_N(T) \\ \text{such that for every } j = 1, ..., d \\ \qquad \sum_{k=1}^{m} \mid \sigma_{jk}(t,x) - \sigma_{jk}(t,z) \mid \leq K_N(T) \cdot \mid x_j - z_j \mid^\alpha \\ \text{holds for all } t \in [0,T] \text{ and any } x, z \in \mathbb{R}^d \text{ with } \|x\|, \|z\| \leq N. \end{cases}$$

We shall always use the sum-norm, i.e.

$$\|x\| = \sum_{k=1}^{d} |x_k|.$$

Let θ be a strictly positive predictable stopping time and let $(\theta_n)_{n\in\mathbb{N}}$ be an announcing sequence for θ.

Definition 2.1 *Let X be an $\mathbb{F}$-adapted stochastic process with values in $\overline{\mathbb{R}}^d$ which is continuous on $[0, \theta)$. X is called a solution to $(\leq)$ up to θ if in replacing s and t by $s \wedge \theta_n$ and $t \wedge \theta_n$, respectively, the inequality $(\leq)$ holds for all $t \geq 0$ and $s \in [0, t]$ $\mathbf{P}$-a.s., $n \in \mathbb{N}$.*

Remark 2.2 (i) In particular the assertion that $(\leq)$ is fulfilled means that the integrals on the right hand side of $(\leq)$ are well-defined. This includes the assumption $X(t \wedge \theta_n) \notin \Delta$ for all $t \geq 0$ $\mathbf{P}$-a.s. in the definition above.
(ii) Analogously, one introduces the notion of a *solution to* $(\geq)$ *up to* θ.
(iii) For $\theta = \infty$ we obtain the usual notion of a *global solution.*
(iv) If θ is the explosion time of X, i.e. $\theta = \lim_{N\to\infty} \inf\{t \geq 0 : \|X(t)\| > N\}$, then X is said to be a *local solution.*

In order to formulate the comparison theorem we introduce the main condition on the 'drift' coefficients:

$$(C_{a,b})\quad \begin{cases} \text{For any } t \geq 0, j = 1, ..., d, \text{ it holds} \\ \qquad a_j(t,x) < b_j(t,z) \\ \text{if } x_j = z_j,\ x_l \leq z_l,\ l \neq j. \end{cases}$$

The following assertion is the main result of this section.

Theorem 2.3 *Let X and Z be arbitrary solutions to $(\leq)$ and $(\geq)$ up to a strictly positive predictable stopping time θ, respectively. Then the conditions (C_0), (C_σ) and $(C_{a,b})$ imply*

$$\mathbf{P}(\{X(t) \leq Z(t),\ t \in [0, \theta)\}) = 1.$$

Moreover, the sign "$<$" in (C_0) leads to "$<$" in the above assertion.

Remark 2.4 For two local solutions to $(\leq)$ and $(\geq)$ with the explosion times θ_X and θ_Z, respectively, the natural choice for θ is $\theta_X \wedge \theta_Z$.

Proof. (i) First we prove the strict inequality. Let

$$X_j(0) < Z_j(0), \qquad \mathbf{P}\text{-a.s.}, j = 1, ..., d.$$

Put $\tau_j = \inf\{t \geq 0 : X_j(t) = Z_j(t)\}$, where $\inf\emptyset = \infty$, and $\tau = \tau_1 \wedge ... \wedge \tau_d$. In this case we trivially have $\tau > 0$ **P**-a.s. For a later use we introduce

$$\mathfrak{A}(t) = \sum_{j=1}^{d}\sum_{k=1}^{m}\int_0^t [\sigma_{jk}(r,X(r)) - \sigma_{jk}(r,Z(r))]^2\,\mathrm{d}r, \qquad t < \theta,$$

and the stopping times

$$T_N = \inf\{0 \leq t < \theta : \|X(t)\| \vee \|Z(t)\| \vee \mathfrak{A}(t) > N\} \wedge \theta_N$$

where $(\theta_N)_{N\in\mathbb{N}}$ is an announcing sequence for θ. By Remark 2.2(i) it holds $T_N \uparrow \theta$. Assume for a moment we would have proved

$$\text{(2.5)} \qquad \mathbf{P}(\{\tau_j = \tau < T_N\}) = 0$$

for $j = 1, ..., d$ and all $N \in \mathbb{N}$. Taking $N \to \infty$ it would follow $\mathbf{P}(\{\tau < \theta\}) = 0$ which is equivalent to

$$\mathbf{P}(\{X(t) < Z(t),\ t \in [0,\theta)\}) = 1$$

proving the theorem in the considered case. Therefore it remains to verify (2.5). To do this we assume the opposite, namely, that there exists a $j \in \{1, ..., d\}$ and an $N \in \mathbb{N}$ such that $\mathbf{P}(\{\tau_j = \tau < T_N\}) > 0$. After fixing these two numbers we introduce the notation

$$M_j(t) = \sum_{k=1}^{m}\int_0^{t\wedge T_N} [\sigma_{jk}(r,X(r)) - \sigma_{jk}(r,Z(r))]\,\mathrm{d}W_k(r), \qquad t \geq 0,$$

and note that M_j is a continuous martingale with the quadratic variation process

$$A_j(t) = \sum_{k=1}^{m}\int_0^{t\wedge T_N} [\sigma_{jk}(r,X(r)) - \sigma_{jk}(r,Z(r))]^2\,\mathrm{d}r, \qquad t \geq 0, \mathbf{P}\text{-a.s.}$$

It is well-known (cf. [6], Th. II.7.2') that there exists a Wiener process B_j on a possibly extended probability space with the property

$$M_j(t) = B_j(A_j(t)), \qquad t \geq 0, \mathbf{P}\text{-a.s.}$$

If it is necessary to extend the basic probability space we also define all random variables on the extended probability space in the canonical way using the same notations. Fix a real

$$\gamma \in \left(\frac{1}{2\alpha+1}, \frac{1}{2\alpha}\right) \cap \left(0, \frac{1}{2}\right).$$

Let Ω^γ be the measurable subset of Ω which is defined by the following six requirements:

1^0 $X(\cdot,\omega), Z(\cdot,\omega)$ and $A_j(\cdot,\omega)$ are continuous on $[0,\theta(\omega))$.

2^0 Replacing s and t by $s \wedge \theta_n(\omega)$ and $t \wedge \theta_n(\omega)$, respectively, the inequalities ($\leq$) and ($\geq$) hold for all $t \geq 0$, $s \in [0,t]$ and $n \in \mathbb{N}$ where again $(\theta_n)_{n\in\mathbb{N}}$ is an announcing sequence for θ.

3^0 $\tau(\omega) > 0$.

4^0 It holds $M_j(t,\omega) = B_j(A_j(t,\omega),\omega)$, $t \geq 0$.

5^0 There exists an $h_j(\omega)$ such that

$$\sup_{\substack{0 \leq t-s \leq h_j(\omega) \\ t,s \in [0,N]}} \frac{|B_j(t,\omega) - B_j(s,\omega)|}{(t-s)^\gamma} \leq \delta = \frac{2}{1-2^{-\gamma}}.$$

6^0 $0 \leq A_j(t,\omega) \leq N$, $t \geq 0$, holds.

Since B_j for the chosen δ possesses the property 5^0 with probability one (cf. [8], sect. 2.2) and 6^0 holds for $\mathbf{P}$-a.a. $\omega \in \Omega$ by the definition of T_N we get $\mathbf{P}(\Omega^\gamma) = 1$. Consequently we observe $\mathbf{P}(\{\tau_j = \tau < T_N\} \cap \Omega^\gamma) > 0$. Choose an $\omega_0 \in \{\tau_j = \tau < T_N\} \cap \Omega^\gamma$. Then from the continuity of A_j for the fixed ω_0 we can conclude that there exists a $t_0(\omega_0) \in [0,\tau(\omega_0))$ such that

$$|A_j(\tau(\omega_0),\omega_0) - A_j(t,\omega_0)| \leq h_j(\omega_0) \tag{2.6}$$

for all $t \in [t_0(\omega_0),\tau(\omega_0))$. Moreover we have

$$a_j(\tau(\omega_0), X(\tau(\omega_0),\omega_0)) < b_j(\tau(\omega_0), Z(\tau(\omega_0),\omega_0))$$

by combining the definition of τ and condition $(C_{a,b})$. But a_j and b_j are continuous why we find neighbourhoods U_x and U_z of the points $(\tau(\omega_0), X(\tau(\omega_0),\omega_0))$ and $(\tau(\omega_0), Z(\tau(\omega_0),\omega_0))$, respectively, such that

$$a_j(t,x) < b_j(t',z)$$

for all $(t,x) \in U_x$ and $(t',z) \in U_z$. Clearly, if $t_0(\omega_0)$ chosen above is sufficiently close to $\tau(\omega_0)$ then it holds

$$(t, X(t,\omega_0)) \in U_x \quad \text{and} \quad (t, Z(t,\omega_0)) \in U_z$$

for all $t \in [t_0(\omega_0),\tau(\omega_0))$ by the continuity of X and Z for the fixed ω_0. All in all this yields

$$a_j(t, X(t,\omega_0)) < b_j(t, Z(t,\omega_0)) \tag{2.7}$$

for all $t \in [t_0(\omega_0),\tau(\omega_0))$. For $0 \leq s < \tau$ the basic inequalities ($\leq$) and ($\geq$) lead to

$$\begin{aligned} & X_j(\tau) - \int_s^\tau a_j(r,X(r))\,\mathrm{d}r - \sum_{k=1}^m \int_s^\tau \sigma_{jk}(r,X(r))\,\mathrm{d}W_k(r) \\ \leq\ & X_j(s) \ \leq \\ \leq\ & Z_j(s) \ \leq \\ \leq\ & Z_j(\tau) - \int_s^\tau b_j(r,Z(r))\,\mathrm{d}r - \sum_{k=1}^m \int_s^\tau \sigma_{jk}(r,Z(r))\,\mathrm{d}W_k(r) \end{aligned}$$

where for simplicity of the notation the dependence on ω_0 is here and from now on omitted. We shall always deal with the paths determined by the chosen ω_0. The last inequalities give finally

$$(2.8)\ 0 \le Z_j(s) - X_j(s) \le \int_s^\tau [a_j(r,X(r)) - b_j(r,Z(r))]\,\mathrm{d}r + M_j(\tau) - M_j(s)$$

for $s \in [0,\tau)$. Because of (2.7) the integral in (2.8) is negative for $s \in [t_0,\tau)$. Therefore it holds

$$(2.9)\qquad 0 \le Z_j(s) - X_j(s) \le M_j(\tau) - M_j(s)$$

for $s \in [t_0,\tau)$. Further relation (2.8) implies

$$(2.10)\quad \frac{1}{\tau - s}\int_s^\tau [b_j(r,Z(r)) - a_j(r,X(r))]\,\mathrm{d}r \le \frac{1}{\tau - s}(M_j(\tau) - M_j(s))$$

if $s \in [t_0,\tau)$.

For the right hand side of (2.10) we get

$$\frac{1}{\tau - s}(M_j(\tau) - M_j(s)) = \frac{B_j(A_j(\tau)) - B_j(A_j(s))}{|A_j(\tau) - A_j(s)|^\gamma} \cdot \frac{|A_j(\tau) - A_j(s)|^\gamma}{\tau - s}$$

$$(2.11)\qquad \le \delta \cdot \frac{|A_j(\tau) - A_j(s)|^\gamma}{\tau - s}$$

if $s \in [t_0,\tau)$ where the last inequality follows from (2.6) and the properties 5^0, 6^0. From the definition of A_j, $\tau < T_N$ and condition (C_σ) we derive for $s \in [t_0,\tau)$

$$\begin{aligned} |A_j(\tau) - A_j(s)|^\gamma &= |\sum_{k=1}^m \int_s^\tau [\sigma_{jk}(r,X(r)) - \sigma_{jk}(r,Z(r))]^2\,\mathrm{d}r|^\gamma \\ &\le (\tau - s)^\gamma \left[\sup_{r\in[s,\tau]} \sum_{k=1}^m |\sigma_{jk}(r,X(r)) - \sigma_{jk}(r,Z(r))|^2\right]^\gamma \\ &\le (\tau - s)^\gamma \left[\sup_{r\in[s,\tau]} \sum_{k=1}^m |\sigma_{jk}(r,X(r)) - \sigma_{jk}(r,Z(r))|\right]^{2\gamma} \\ &\le K_N^{2\gamma}(\tau - s)^\gamma \left[\sup_{r\in[s,\tau]} |X_j(r) - Z_j(r)|^\alpha\right]^{2\gamma} \\ (2.12)\qquad &\le K_N^{2\gamma}(\tau - s)^\gamma \left[\sup_{r\in[s,\tau]} |X_j(r) - Z_j(r)|\right]^{2\alpha\gamma} \end{aligned}$$

where $K_N = K_N(\tau)$. The fact that $Z_j(r) \ge X_j(r)$ for $r \in [s,\tau]$,(2.11) and (2.12) lead to

$$(2.13)\quad \frac{1}{\tau - s}(M_j(\tau) - M_j(s)) \le \delta \cdot \frac{K_N^{2\gamma}(\tau - s)^\gamma}{\tau - s}\left[\sup_{r\in[s,\tau]} (Z_j(r) - X_j(r))\right]^{2\alpha\gamma}$$

if $s \in [t_0, \tau)$. Moreover from (2.9) and (2.13) we get

$$\frac{1}{\tau - s}(M_j(\tau) - M_j(s)) \leq \delta \cdot \frac{K_N^{2\gamma}(\tau - s)^\gamma}{\tau - s}\left[\sup_{r\in[s,\tau]}(M_j(\tau) - M_j(r))\right]^{2\alpha\gamma} \tag{2.14}$$

if $s \in [t_0, \tau)$. The continuity of the chosen trajectories of X and Z, respectively, and $X_j(\tau) = Z_j(\tau)$ imply the existence of a time $t_1 \in [t_0, \tau)$ such that for $n = 1$

$$\sup_{r\in[t_n,\tau]}(Z_j(r) - X_j(r)) < n^{-(2\alpha\gamma)^{-n}}. \tag{2.15}$$

From (2.13) and (2.15) we can conclude for $n = 1$

$$\frac{1}{\tau - s}(M_j(\tau) - M_j(s)) \leq \delta \cdot \frac{K_N^{2\gamma}(\tau - s)^\gamma}{\tau - s} \cdot n^{-1}, \qquad s \in [t_n, \tau).$$

To obtain the next iteration step we multiply (2.13) with $(\tau - s)$ and put the result into (2.14). This gives

$$\begin{aligned}\frac{1}{\tau - s}(M_j(\tau) - M_j(s)) &\leq \frac{\delta \cdot K_N^{2\gamma}(\tau - s)^\gamma \delta^{2\alpha\gamma} K_N^{4\alpha\gamma^2}(\tau - s)^{2\alpha\gamma^2}}{\tau - s} \\ &\quad \times \left[\sup_{r\in[s,\tau]}(Z_j(r) - X_j(r))\right]^{(2\alpha\gamma)^2} \\ &= \frac{\delta^{(1+2\alpha\gamma)} K_N^{2\gamma(1+2\alpha\gamma)}(\tau - s)^{\gamma(1+2\alpha\gamma)}}{\tau - s} \\ &\quad \times \left[\sup_{r\in[s,\tau]}(Z_j(r) - X_j(r))\right]^{(2\alpha\gamma)^2}\end{aligned} \tag{2.16}$$

if $s \in [t_1, \tau)$ (here one could also take t_0 instead of t_1). By the same arguments as above we can find a $t_2 \in [t_1, \tau)$ such that for $n = 2$

$$\sup_{r\in[t_n,\tau]}(Z_j(r) - X_j(r)) < n^{-(2\alpha\gamma)^{-n}}. \tag{2.17}$$

Putting (2.17) into (2.16) we arrive at

$$\frac{1}{\tau - s}(M_j(\tau) - M_j(s)) \leq \frac{\delta^{(1+2\alpha\gamma)} K_N^{2\gamma(1+2\alpha\gamma)}(\tau - s)^{\gamma(1+2\alpha\gamma)}}{\tau - s} \cdot n^{-1}$$

for $s \in [t_n, \tau)$ and $n = 2$. A continuation of this procedure leads to

$$\begin{aligned}\frac{1}{\tau - s}(M_j(\tau) - M_j(s)) &\leq (1 \vee \delta)^{\frac{1}{1-2\alpha\gamma}}(1 \vee K_N)^{\frac{2\gamma}{1-2\alpha\gamma}} \\ &\quad \times \frac{(\tau - s)^{\gamma \sum_{k=0}^{n-1}(2\alpha\gamma)^k}}{\tau - s} \cdot n^{-1}\end{aligned} \tag{2.18}$$

if $s \in [t_n, \tau)$. Note that because of the choice of γ we obtain $2\alpha\gamma < 1$ which ensures the convergence of the geometric series above. On the other hand $n^{-(2\alpha\gamma)^{-n}}$

tends to zero as $n \to \infty$ and this together with the continuity of X_j and Z_j implies $t_n \to \tau$, $n \to \infty$.

The relations (2.10) and (2.18) lead finally to

$$\begin{aligned} &\frac{1}{\tau - t_n} \int_{t_n}^{\tau} [b_j(r, Z(r)) - a_j(r, X(r))] \, dr \\ (2.19) \qquad &\le \text{constant} \cdot \frac{(\tau - t_n)^{\gamma} \sum_{k=0}^{n-1} (2\alpha\gamma)^k}{\tau - t_n} \cdot n^{-1}. \end{aligned}$$

The left hand side of (2.19) converges to

$$b_j(\tau, Z(\tau)) - a_j(\tau, X(\tau))$$

which is *strictly positive* by $(C_{a,b})$ since we have

$$X_j(\tau) = Z_j(\tau) \text{ and } X_l(\tau) \le Z_l(\tau), j \ne l.$$

But by the choice of γ we obtain $\frac{\gamma}{1-2\alpha\gamma} > 1$ and consequently for sufficiently large n the right hand side of (2.19) can be estimated from above by

$$\text{constant} \cdot n^{-1} \to 0, \qquad n \to \infty.$$

This contradiction proves the theorem in case of

$$X_j(0) < Z_j(0), \qquad \mathbf{P}\text{-a.s.}, j = 1, ..., d.$$

(ii) In the remaining case we admit the equality of the initial values:

$$X_j(0) \le Z_j(0), \qquad \mathbf{P}\text{-a.s.}, j = 1, ..., d.$$

Instead of τ_j and τ we now introduce the stopping times $\overline{\tau}_j = \inf\{t \ge 0 : X_j(t) > Z_j(t)\}$ and $\overline{\tau} = \overline{\tau}_1 \wedge ... \wedge \overline{\tau}_d$, respectively. Clearly if we could prove

$$\mathbf{P}(\{\overline{\tau}_j = \overline{\tau} < T_N\}) = 0$$

for $j = 1, ..., d$ and all $N \in \mathbb{N}$ then the theorem would be shown. And if $\overline{\tau} > 0$ would be held $\mathbf{P}$-a.s. then we could prove the last equality in the same way as (2.5) in part (i) above. Therefore let us show $\overline{\tau} > 0$ $\mathbf{P}$-a.s. finishing the proof of the theorem.

At first we shall establish that there exists a locally Lipschitz continuous mapping $c : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ such that for a fixed N and a sufficiently small $\varepsilon > 0$

$$(2.20) \qquad a_j(t, x) < c_j(t, y) \pm \varepsilon < b_j(t, z)$$

for $t \in [0, N]$, $\|x\|, \|z\| \le N$ and $x_j = y_j = z_j$ and $x_l \le y_l \le z_l$, $l \ne j$, $j = 1, ..., d$. To make this clear we fix $j \in \{1, ..., d\}$ as well as N and introduce the sets

$$\mathfrak{M}_y = \{x \in \mathbb{R}^d : \|x\| \le N, x_j = y_j, x_l \le y_l, l \ne j\}, \qquad y \in \mathbb{R}^d,$$

and

$$\widetilde{\mathfrak{M}}_y = \{z \in \mathbb{R}^d : \|z\| \le N, z_j = y_j, z_l \ge y_l, l \ne j\}, \qquad y \in \mathbb{R}^d.$$

Since $\mathfrak{M}_y$ and $\widetilde{\mathfrak{M}}_y$ are compact we observe

$$\sup_{x\in\mathfrak{M}_y} a_j(t,x) =: a_j^*(t,y) < b_j^*(t,y) := \inf_{z\in\widetilde{\mathfrak{M}}_y} b_j(t,z)$$

for each $t \geq 0$ and $y \in \mathbb{R}^d$ with $\|y\| \leq N$. The continuity of a_j^* and b_j^* on $[0,N] \times \{y \in \mathbb{R}^d : \|y\| \leq N\}$ implies the existence of a $\delta > 0$ such that

$$\inf_{(t,y)\in[0,N]\times\{y\in\mathbb{R}^d:\|y\|\leq N\}} (b_j^*(t,y) - a_j^*(t,y)) \geq \delta.$$

Now any sufficiently smooth mapping $c_j : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ with

$$\sup_{(t,y)\in[0,N]\times\{y\in\mathbb{R}^d:\|y\|\leq N\}} |c_j(t,y) - \frac{1}{2}(a_j^*(t,y) + b_j^*(t,y))| < \frac{\delta}{4}$$

and $\varepsilon \in (0, \frac{\delta}{4})$ are good candidates for c_j and ε as desired.

Now we define $Y_j(0) = X_j(0)$, $j = 1, ..., d$. Because of $X_j(0) \leq Z_j(0)$ we obtain $X_j(0) \leq Y_j(0) \leq Z_j(0)$, $j = 1, ..., d$. Let us consider the systems of stochastic integral equations

$$Y_j^{(\pm\varepsilon)}(t) = Y_j^{(\pm\varepsilon)}(0) + \int_0^t (c_j(r, Y^{(\pm\varepsilon)}(r)) \pm \varepsilon)\,\mathrm{d}r + \sum_{k=1}^m \int_0^t \sigma_{jk}(r, Y^{(\pm\varepsilon)}(r))\,\mathrm{d}W_k(r)$$

(2.21)

where in case of $\varepsilon > 0$

$$Y_j^{(+\varepsilon)}(0) := Y_j(0) + \varepsilon > Y_j(0) \geq X_j(0)$$

and

$$Y_j^{(-\varepsilon)}(0) := Y_j(0) - \varepsilon < Y_j(0) \leq Z_j(0).$$

By Corollary 5.4 of the appendix there exist pathwise unique local solutions $Y^{(\pm\varepsilon)}$, $\varepsilon \geq 0$, to (2.21) with the explosion times $\theta^{(\pm\varepsilon)}$. We will show that there exists a **P**-a.s. strictly positive stopping time $\widetilde{\theta}$ such that the unique local solution $Y = Y^{(\pm 0)}$ with the initial value $Y(0)$ coincides on $[0, \widetilde{\theta})$ with the limit of both sequences of solutions $(Y^{(-\varepsilon)})$ and $(Y^{(+\varepsilon)})$ for $\varepsilon \searrow 0$. This will help us to show $X \leq Y \leq Z$ up to $\theta \wedge \widetilde{\theta}$ proving the desired relation $\overline{\tau} \geq \theta \wedge \widetilde{\theta} > 0$ **P**-a.s.

For $0 < \varepsilon_1 < \varepsilon_2$ the comparison in the strict case gives the following useful lemma which will be shown after the end of the present proof.

Lemma 2.22 *It holds*

(i) $Y^{(-\varepsilon_2)}(t) < Y^{(-\varepsilon_1)}(t) < Y^{(+\varepsilon_1)}(t) < Y^{(+\varepsilon_2)}(t)$, $t \in [0, \theta^{(-\varepsilon_2)} \wedge \theta^{(+\varepsilon_2)})$, **P**-a.s.
(ii) $\theta^{(-\varepsilon_2)} \wedge \theta^{(+\varepsilon_2)} \leq \theta^{(-\varepsilon_1)} \wedge \theta^{(+\varepsilon_1)}$ **P**-a.s.

This lemma has two consequences. First for each $\delta > 0$ there exists

$$Y^{(\pm)}(t) = \lim_{\varepsilon\searrow 0} Y^{(\pm\varepsilon)}(t), \qquad t \in [0, \theta^{(-\delta)} \wedge \theta^{(+\delta)}),\ \mathbf{P}\text{-a.s.},$$

and second the limit

$$\widetilde{\theta} = \lim_{\delta \searrow 0} (\theta^{(-\delta)} \wedge \theta^{(+\delta)})$$

is a predictable stopping time. Hence $Y^{(\pm)}$ can be defined **P**-a.s. on the stochastic interval $[0, \widetilde{\theta})$ where they take values in $\mathbb{R}^d$. By setting

$$Y^{(\pm)} = 0 \quad \text{on} \quad [\widetilde{\theta}, \infty)$$

we so obtain two adapted processes $Y^{(+)}$ and $Y^{(-)}$. Further each explosion time $\theta^{(\pm\delta)}$ is **P**-a.s. strictly positive providing

$$\widetilde{\theta} > 0 \qquad \mathbf{P}\text{-a.s.}$$

The next step is to show that $Y^{(+)}$ and $Y^{(-)}$ solve (2.21) with $\varepsilon = 0$ up to $\widetilde{\theta}$. Then we can identify $Y^{(+)}$ and $Y^{(-)}$ with the above mentioned process Y up to $\widetilde{\theta}$. Let us fix δ as well as N and define

$$T_N^\delta = \inf\{t \geq 0 : \|Y^{(+\delta)}(t)\| \vee \|Y^{(-\delta)}(t)\| > N\} \wedge N.$$

If $0 < \varepsilon < \delta$ from (2.21) it follows

$$\begin{aligned} Y_j^{(\pm\varepsilon)}(t \wedge T_N^\delta) &= Y_j^{(\pm\varepsilon)}(0) + \int_0^{t \wedge T_N^\delta} (c_j(r, Y^{(\pm\varepsilon)}(r)) \pm \varepsilon)\, \mathrm{d}r \\ &\quad + \sum_{k=1}^m \int_0^{t \wedge T_N^\delta} \sigma_{jk}(r, Y^{(\pm\varepsilon)}(r))\, \mathrm{d}W_k(r) \end{aligned}$$

for all $t \geq 0$ **P**-a.s., $j = 1, \ldots, d$. Fixing t and letting $\varepsilon \searrow 0$ we observe that the left hand side converges to $Y_j^{(\pm)}(t \wedge T_N^\delta)$ **P**-a.s. The first term on the right hand side tends **P**-a.s. to $Y_j(0)$. Because of the boundedness of both c and σ on $[0, t] \times \{y \in \mathbb{R}^d : \|y\| \leq N\}$ we can use Lebesgue's theorem on dominated convergence to take the limit into the integrals where the stochastic integral is handled in $L_2(\Omega)$. Finally the continuity of both coefficients implies

$$Y_j^{(\pm)}(t \wedge T_N^\delta) = Y_j(0) + \int_0^{t \wedge T_N^\delta} c_j(r, Y^{(\pm)}(r))\, \mathrm{d}r + \sum_{k=1}^m \int_0^{t \wedge T_N^\delta} \sigma_{jk}(r, Y^{(\pm)}(r))\, \mathrm{d}W_k(r)$$

for all $t \geq 0$ **P**-a.s., $j = 1, \ldots, d$. Since (T_N^δ) with $\delta \searrow 0$ and $N \to \infty$ is an announcing sequence for $\widetilde{\theta}$ the processes $Y^{(\pm)}$ are solutions of (2.21) with $\varepsilon = 0$ up to $\widetilde{\theta}$. But by Proposition 5.1 and Remark 5.3 of the appendix there is only one solution of (2.21) with $\varepsilon = 0$ up to $\widetilde{\theta}$. Therefore we can identify $Y^{(+)}$ and $Y^{(-)}$ with Y on $[0, \widetilde{\theta})$.

Now we shall study the connection between X, Y and Z. In addition to the stopping times T_N^δ we use the stopping times T_N introduced at the beginning of part (i) of this proof (but we only need $\|X(t)\|, \|Z(t)\| \leq N$ on $[0, T_N]$) which converge to θ for $N \to \infty$. Then for a fixed N Lemma 2.22(ii), (2.20) and part (i) of the present proof give

$$X(t) < Y^{(+\varepsilon)}(t), \qquad t \in [0, T_N \wedge T_N^\delta),\ \mathbf{P}\text{-a.s.},$$

and

$$Z(t) > Y^{(-\varepsilon)}(t), \qquad t \in [0, T_N \wedge T_N^\delta), \ \mathbf{P}\text{-a.s.},$$

for all $0 < \varepsilon < \delta$, δ sufficiently small. Here we can apply (2.20) in place of $(C_{a,c+\varepsilon})$ and $(C_{b,c-\varepsilon})$, respectively, since we only compare the processes on $[0, T_N \wedge T_N^\delta)$. If ε converges to zero we obtain

$$X(t) \le Y(t), \qquad t \in [0, T_N \wedge T_N^\delta), \ \mathbf{P}\text{-a.s.},$$

and

$$Z(t) \ge Y(t), \qquad t \in [0, T_N \wedge T_N^\delta), \ \mathbf{P}\text{-a.s.},$$

which leads to

$$X(t) \le Y(t) \le Z(t), \qquad t \in [0, \theta \wedge \widetilde{\theta}), \ \mathbf{P}\text{-a.s.},$$

if δ tends to zero and N to infinity. But this means $\overline{\tau} \ge \theta \wedge \widetilde{\theta} > 0$ **P**-a.s. □

Proof of Lemma 2.22. Reminding of the proof of (2.20) the function c_j can also be chosen with the property[1] such that for each $t \ge 0$

$$c_j(t,x) \le c_j(t,z)$$

if $x_j = z_j$ and $x_l \le z_l$, $l \ne j$. This carries over from the corresponding property of the function $\frac{1}{2}(a_j^*(t,\cdot) + b_j^*(t,\cdot))$ introduced there. As a consequence we may compare the solutions of (2.21) for the fixed $\varepsilon_1, \varepsilon_2$ by applying the already proved part (i) of the theorem.

It suffices to prove (ii) because statement (i) then immediately follows from the assertion which has been shown in part (i) of the previous proof. But instead of (ii) we shall prove

$$\theta^{(-\varepsilon_2)} \wedge \theta^{(+\varepsilon_2)} \le \theta^{(\delta)} \tag{2.23}$$

P-a.s. for an arbitrary $\delta \in (-\varepsilon_2, +\varepsilon_2)$. Substituting δ by $-\varepsilon_1$ and $+\varepsilon_1$, respectively, we get (ii).

In order to show (2.23) let us fix $\delta \in (-\varepsilon_2, +\varepsilon_2)$. Applying the result from part (i) of the previous proof we obtain

$$\begin{cases} Y^{(-\varepsilon_2)}(t) < Y^{(\delta)}(t), & t \in [0, \theta^{(-\varepsilon_2)} \wedge \theta^{(\delta)}), \quad \mathbf{P}\text{-a.s.}, \\ Y^{(\delta)}(t) < Y^{(+\varepsilon_2)}(t), & t \in [0, \theta^{(\delta)} \wedge \theta^{(+\varepsilon_2)}), \quad \mathbf{P}\text{-a.s.} \end{cases} \tag{2.24}$$

If we now assume the opposite of (2.23) that is

$$\mathbf{P}(\{\theta^{(\delta)} < \theta^{(-\varepsilon_2)} \wedge \theta^{(+\varepsilon_2)}\}) > 0$$

then we can easily derive a contradiction proving (2.23). Indeed, using (2.24) under the assumption above there exists an $\omega \in \{\theta^{(\delta)} < \theta^{(-\varepsilon_2)} \wedge \theta^{(+\varepsilon_2)}\}$ such that

$$Y^{(-\varepsilon_2)}(t,\omega) < Y^{(\delta)}(t,\omega) < Y^{(+\varepsilon_2)}(t,\omega), \qquad t \in [0, \theta^{(\delta)}(\omega)).$$

This is a contradiction to the definition of the explosion time in Remark 2.2(iv).

□

[1] See also Definition 3.1

3 Discussion of the conditions

Let us deal with the question wether the conditions of Theorem 2.3 can be weakened. Clearly condition (C_0) should remain untouched. But what about (C_σ) and $(C_{a,b})$?

Let us first consider (C_σ). Having the one-dimensional theory in mind one could have expected that Theorem 2.3 could be proved under the usual condition of local Hölder continuity with exponent $\alpha > \frac{1}{2}$ which follows from (C_σ). However the example below shows that (C_σ) cannot be weakened in this sense. Intuitively a reason for that is the following: In the critical situation when X_j and Z_j are close to each other the remaining coordinates should have no influence on the noise intensity of the jth coordinate. Our example works already for systems of equations in dimension 2. Consider

$$
\begin{aligned}
X_1(t) &= 1 + \int_0^t X_2(r)\,\mathrm{d}W(r) \\
X_2(t) &= \int_0^t X_1(r)\,\mathrm{d}W(r)
\end{aligned}
$$

and

$$
\begin{aligned}
Z_1(t) &= 1 + \frac{1}{2}t + \int_0^t Z_2(r)\,\mathrm{d}W(r) \\
Z_2(t) &= 1 + \frac{1}{2}t + \int_0^t Z_1(r)\,\mathrm{d}W(r)
\end{aligned}
$$

where W is a given one-dimensional Wiener process. In both systems the drift and diffusion coefficients are continuous. Moreover we observe $X(0) \leq Z(0)$. Condition $(C_{a,b})$ is obviously fulfilled. Because of

$$|\sigma_1(x) - \sigma_1(z)| = |x_2 - z_2|$$

and

$$|\sigma_2(x) - \sigma_2(z)| = |x_1 - z_1|$$

the diffusion is Lipschitz continuous but condition (C_σ) is violated. Both systems can be explicitly solved. Adding and subtracting the corresponding equations we easily obtain

$$
\begin{aligned}
X_1(t) &= \frac{1}{2}\left[\exp\{W(t) - \frac{1}{2}t\} + \exp\{-W(t) - \frac{1}{2}t\}\right] \\
X_2(t) &= \frac{1}{2}\left[\exp\{W(t) - \frac{1}{2}t\} - \exp\{-W(t) - \frac{1}{2}t\}\right]
\end{aligned}
$$

as well as

$$Z_1(t) = Z_2(t) = \frac{1}{2}\left[2 + \int_0^t \exp\{\frac{1}{2}r - W(r)\}\,\mathrm{d}r\right] \cdot \exp\{W(t) - \frac{1}{2}t\}.$$

If the comparison principle $X(t) \leq Z(t)$ would hold we would have in particular $X_1(t) \leq Z_1(t)$. Putting the explicite solutions into this inequality and dividing the result by $\exp\{W(t) - \frac{1}{2}t\}$ we establish after a trivial calculation

$$\exp\{-2W(t)\} \leq 1 + \int_0^t \exp\{\frac{1}{2}r - W(r)\}\,\mathrm{d}r.$$

Taking the expectations on both sides this gives

$$\exp\{2t\} \leq 1 + \int_0^t \exp\{r\}\,\mathrm{d}r = \exp\{t\}$$

which is a contradiction. Hence, already in this simple case the comparison principle does not hold.

A further problem is the following: Does the assertion of Theorem 2.3 hold if we admit $\alpha \leq \frac{1}{2}$ in (C_σ)? We believe that the answer is negative.

Now let us investigate condition $(C_{a,b})$. Here one would prefer to take the condition

$$(C^*_{a,b}) \qquad a_j(t,x) \leq b_j(t,x), \qquad t \geq 0,\ x \in \mathbb{R}^d,\ j = 1, \dots, d,$$

instead of $(C_{a,b})$. Unfortunately it turns out that if we replace $(C_{a,b})$ by $(C^*_{a,b})$ in Theorem 2.3 then we do not obtain a true assertion. This has already been well-known in the deterministic theory (cf. [14]). The following counterexample even works if in $(C^*_{a,b})$ the strict inequality between a_j and b_j holds as well as σ does not vanish. Consider the systems

$$\begin{aligned} X_1(t) &= \int_0^t [-X_2(r)]\,\mathrm{d}r + W(t) \\ X_2(t) &= \int_0^t [-X_1(r)]\,\mathrm{d}r + W(t) \end{aligned}$$

and

$$\begin{aligned} Z_1(t) &= \int_0^t [-Z_2(r) + \frac{1}{2}]\,\mathrm{d}r + W(t) \\ Z_2(t) &= \int_0^t [-Z_1(r) + \frac{1}{2}]\,\mathrm{d}r + W(t) \end{aligned}$$

where W is again a given one-dimensional Wiener process. That means that the corresponding drift coefficients $a, b : \mathbb{R}^2 \to \mathbb{R}^2$ are defined by

$$\begin{matrix} a_1(x_1,x_2) = -x_2 \\ a_2(x_1,x_2) = -x_1 \end{matrix} \quad \text{and} \quad \begin{matrix} b_1 = a_1 + \frac{1}{2} \\ b_2 = a_2 + \frac{1}{2} \end{matrix} .$$

The diffusion coefficient equals one. All in all the conditions (C_0), (C_σ) and $(C^*_{a,b})$ are satisfied. But it is easy to verify that $(C_{a,b})$ is not fulfilled. Solving the equations above we get the unique global solutions

$$X_1(t) = X_2(t) = \exp\{-t\} \int_0^t \exp\{r\}\, \mathrm{d}W(r)$$

and

$$\begin{aligned} Z_1(t) &= \frac{1}{2} + \exp\{-t\} \int_0^t \exp\{r\}\, \mathrm{d}W(r) - \frac{1}{2}\exp\{t\} \\ Z_2(t) &= \frac{1}{2} + \exp\{-t\} \int_0^t \exp\{r\}\, \mathrm{d}W(r) + \frac{1}{2}\exp\{t\}. \end{aligned}$$

This gives

$$Z_1(t) - X_1(t) = \frac{1}{2}(1 - \exp\{t\}) < 0, \qquad t > 0,\ \mathbf{P}\text{-a.s.},$$

which is in contradiction to the assertion of Theorem 2.3.

The counterexample above implies that if we replace $(C_{a,b})$ by $(C^*_{a,b})$ in Theorem 2.3 then we need further conditions in order to derive the conclusion of this theorem. In the next proposition we shall give such conditions. Among them is the so-called *quasi-monotonicity* which is defined as follows:

Definition 3.1 *A mapping $f : \mathbb{R}^d \to \mathbb{R}^d$ is said to be quasi-monotonously increasing if for each $j = 1, ..., d$*

$$f_j(x) \leq f_j(y)$$

provided $x_j = y_j$ and $x_l \leq y_l$, $l \neq j$.

Remark 3.2 The property which is called quasi-monotonicity here is well-known in the deterministic theory of differential equations. It was first recognized by Müller [11], [12] and Kamke [7] in carrying over fundamental theorems from *one* differential equation to *systems* of differential equations.

Probably this has motivated Mel'nikov in [10] to generalize a comparison result for stochastic differential *equations* from the one-dimensional case to the higher dimensional case without proof using an additional quasi-monotonicity of one of the drift coefficients. However, he arrives at a better condition as the authors in the next proposition only with respect to σ. Indeed he can employ the condition (C^*_σ) which will be introduced in the appendix. But already in the one-dimensional case we do not know how to apply his technique to stochastic differential *inequalities*.

Proposition 3.3 *Let one of the inequalities $(\leq)$ or $(\geq)$ be an equation which possesses a pathwise unique solution*[2] *up to a strictly positive predictable stopping time θ. Suppose that the remaining inequality has a solution up to θ. Further we assume that the drift coefficient in the equation is quasi-monotonously increasing in the second variable. If we denote the solutions of $(\leq)$ and $(\geq)$ by X and Z, respectively, then (C_0), (C_σ) and $(C^*_{a,b})$ imply*

$$\mathbf{P}(\{X(t) \leq Z(t),\ t \in [0, \theta)\}) = 1.$$

Applying Theorem 2.3 in place of the comparison theorem in [5] the proof is exactly the same as the proof of Th. 1.2 in [5]. Therefore we omit it.

[2] For the definition of this notion see Proposition 5.1 of the appendix

4 Existence of solutions of stochastic differential inequalities

Although we introduced the notion of a solution of a stochastic differential inequality in Definition 2.1 we have not yet discussed questions like existence or uniqueness of such solutions. It is a useful consequence of our comparison results in section 2 that we are now able to deal with these questions. In this framework we shall derive some Gronwall-type estimates, too.

At first let us consider the following simple example in dimension one,

$$X(t) \leq X(s) + \int_s^t X(r)\,\mathrm{d}W(r) + (t-s), \tag{4.1}$$

where W is a given one-dimensional Wiener process. Clearly, for each initial value $X(0)$ there exists a pathwise unique solution Z of the equation

$$Z(t) = X(0) + \int_0^t Z(r)\,\mathrm{d}W(r) + t$$

and this is also a solution of (4.1). But for each fixed initial value $X(0)$ we can give a whole class of other solutions to the inequality, namely $(X^\alpha)_{\alpha\in(-\infty,1)}$ where X^α is the solution to

$$X^\alpha(t) = X(0) + \int_0^t X^\alpha(r)\,\mathrm{d}W(r) + \alpha t.$$

Already this example demonstrates what is intuitively clear: It is not reasonable to search for 'unique solutions' of stochastic differential inequalities in the usual sense of pathwise uniqueness.

By applying Proposition 3.3 to

$$\begin{aligned} X(t) &\leq X(s) + \int_s^t X(r)\,\mathrm{d}W(r) + (t-s) \\ Z(t) &= Z(s) + \int_s^t Z(r)\,\mathrm{d}W(r) + (t-s) \end{aligned}$$

we observe that the solution Z of (4.1) mentioned above is maximal in the following sense: If Y is any other solution of (4.1) with $X(0) = Y(0)$ **P**-a.s. then it holds

$$Y(t) \leq Z(t), \qquad t \geq 0,\ \mathbf{P}\text{-a.s.}$$

It is evident that such a maximal solution is pathwise unique in the usual sense. Moreover it turns out to be natural that in case of $(\leq)$ one should only search for upper boundaries for the solutions of the inequality. Indeed, already in our simple example there does not exist a lower boundary for the class of solutions $(X^\alpha)_{\alpha\in(-\infty,1)}$. This has inspired us to the following definition. Let θ be a strictly positive predictable stopping time.

Definition 4.2 *(i) A solution X of $(\leq)$ up to θ with initial value $X(0)$ satisfying the property that if Y is any other solution of $(\leq)$ up to θ with $Y(0) = X(0)$ $\mathbf{P}$-a.s. then*

$$\mathbf{P}(\{Y(t) \leq X(t),\ t \in [0, \theta)\}) = 1$$

is said to be the maximal solution of $(\leq)$ up to θ.

(ii) A solution X of $(\geq)$ up to θ with initial value $X(0)$ satisfying the property that if Y is any other solution of $(\geq)$ up to θ with $Y(0) = X(0)$ $\mathbf{P}$-a.s. then

$$\mathbf{P}(\{Y(t) \geq X(t),\ t \in [0, \theta)\}) = 1$$

is said to be the minimal solution of $(\geq)$ up to θ.

Remark 4.3 (i) In the particular case that we have an equation in place of an inequality a solution X is pathwise unique if and only if it is the maximal as well as the minimal solution.

(ii) With respect to deterministic differential *equations* these concepts are well-known (cf. [14]).

(iii) Finally, in [2] analogous concepts have already been introduced for describing certain decompositions of semimartingales which are related to the so-called 'reflection problem'.

The following theorem gives a sufficient condition for the existence of the maximal solution of $(\leq)$ up to the explosion time. An analogous result can be proved for $(\geq)$.

Theorem 4.4 *If the coefficient a is quasi-monotonously increasing in the second variable as well as satisfies*

$$\begin{cases} \text{For each } T, N > 0 \text{ there exists a constant } L_N(T) \text{ such that} \\ \qquad \|a(t,x) - a(t,y)\| \leq L_N(T)\|x - y\| \\ \text{for all } t \leq T \text{ and } x, y \in \mathbb{R}^d \text{ with } x, y \leq N. \end{cases}$$

and it holds (C_σ) then for each initial value $X(0)$ there exists the maximal solution X of $(\leq)$ up to the explosion time θ_X.

Proof. Let $X(0)$ be a fixed initial value. Then Corollary 5.4 of the appendix gives the existence of a pathwise unique solution X of the corresponding equation up to the explosion time θ_X. If now Y is a solution of $(\leq)$ up to θ_X with $Y(0) = X(0)$ then Proposition 3.3 implies

$$\mathbf{P}(\{Y(t) \leq X(t),\ t \in [0, \theta_X)\}) = 1. \qquad \square$$

All the next considerations concern the inequality $(\leq)$ but one could also consider $(\geq)$ in a similar way. We want to deal with the situation where the maximal solution does not exist. Here we should be interested in finding a process whose trajectories are upper boundaries for the trajectories of all solutions of $(\leq)$. This idea comes from the deterministic differential equations where in case of non-uniqueness Perron [13] called a majorant for all solutions a 'superfunction'. Now for solutions of stochastic differential inequalities we are able to give such a 'superfunction' by applying our comparison theorem.

Let $X(0)$ be a fixed initial value. We assume (C_σ) and search for a continuous function $c : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ which possesses the following two properties:

- $$a_j(t,x) < c_j(t,z)$$
  for all $t \geq 0$ and $x, z \in \mathbb{R}^d$ with $x_j = z_j$, $x_l \leq z_l$, $l \neq j$.
- For the given m-dimensional Wiener process W there exists a solution Z, $Z(0) = X(0)$, of the inequality
  $$Z(t) \geq Z(s) + \int_s^t c(r, Z(r))\,\mathrm{d}r + \int_s^t \sigma(r, Z(r))\,\mathrm{d}W(r)$$
  up to a strictly positive predictable stopping time θ_Z.

From Theorem 2.3 (after changing the assumptions correspondingly one could also apply Proposition 3.3) we get then that Z is an upper boundary on $[0, \theta_X \wedge \theta_Z)$ for each solution X of $(\leq)$ up to a strictly positive pedictable stopping time θ_X:

$$\mathbf{P}(\{X(t) \leq Z(t),\ t \in [0, \theta_X \wedge \theta_Z)\}) = 1.$$

In dimension one this method leads to some Gronwall-type estimates which follow finishing this section. Here certain coefficients will be assumed to be locally bounded. That is not the best condition and it is left to the reader to make it better in special cases.

Proposition 4.5 *For locally bounded coefficients* $\alpha, \beta, \gamma, \delta : \mathbb{R}_+ \to \mathbb{R}$ *let* X *be a (global) solution to*

$$X(t) \leq X(s) + \int_s^t a(r, X(r))\,\mathrm{d}r + \int_s^t (\gamma(r) + \delta(r)X(r))\,\mathrm{d}W(r)$$

where the continuous function a *is assumed to satisfy*

$$a(r,x) \leq \alpha(r) + \beta(r)x, \qquad r \geq 0,\ x \in \mathbb{R}.$$

We set

$$\zeta(r) = \exp\left\{\int_0^r [\beta(u) - \frac{1}{2}\delta^2(u)]\,\mathrm{d}u + \int_0^r \delta(u)\,\mathrm{d}W(u)\right\}.$$

Then it holds $\mathbf{P}$*-a.s. for all* $t \geq 0$

$$X(t) \leq \zeta(t)\left[X(0) + \int_0^t \frac{1}{\zeta(r)}\{\alpha(r) - \gamma(r)\delta(r)\}\,\mathrm{d}r + \int_0^t \frac{\gamma(r)}{\zeta(r)}\,\mathrm{d}W(r)\right].$$

With the preceding considerations in mind the proof is clear and therefore omitted. Let us emphasize the particular case $\alpha = \gamma = 0$ and $\beta = \delta =$ constant. Then the proposition above gives

$$X(t) \le X(0) \exp\left\{ [\beta - \frac{1}{2}\delta^2]t + \delta W(t) \right\}.$$

Moreover, for $\delta = 0$ we arrive at the simplest form of Gronwall's lemma.

Proposition 4.6 *For locally bounded coefficients $g, h, l : \mathbb{R}_+ \to \mathbb{R}$ let X be a (global) solution to*

$$X(t) \le X(s) + \int_s^t a(r, X(r))\,\mathrm{d}r + \int_s^t l(r)X(r)\,\mathrm{d}W(r)$$

with a strictly positive initial value $X(0)$ where the continuous function a is assumed to satisfy

$$a(r,x) \le g(r)x + h(r)x^p, \qquad r \ge 0,\ x \in \mathbb{R},$$

for an exponent $p \ge 2$. We set $q = 1 - p$ and define

$$\zeta(r) = \exp\left\{ q \cdot \int_0^r [g(u) - \frac{1}{2}l^2(u)]\,\mathrm{d}u + q \cdot \int_0^r l(u)\,\mathrm{d}W(u) \right\}.$$

Then it holds **P**-*a.s.*

$$X(t) \le \left(\zeta(t) \left[X(0)^q + q \cdot \int_0^t \frac{h(r)}{\zeta(r)}\,\mathrm{d}r \right] \right)^{\frac{1}{q}}, \qquad t \in [0, \theta),$$

where

$$\theta = \inf\{t \ge 0 : X(0)^q + q \cdot \int_0^t \frac{h(r)}{\zeta(r)}\,\mathrm{d}r < 0\}.$$

Proof. From Corollary 5.4 of the appendix we know that the equation

$$Z(t) = X(0) + \int_0^t [g(r)Z(r) + h(r)Z(r)^p]\,\mathrm{d}r + \int_0^t l(r)Z(r)\,\mathrm{d}W(r)$$

has a pathwise unique solution Z up to the explosion time θ_Z. Applying our comparison method and computing the explicit structure of Z^q by the Itô-formula we arrive with $V(t) = Z(t)^q$ at

$$V(t) = V(0) + \int_0^t [\alpha(r) + \beta(r)V(r)]\,\mathrm{d}r + \int_0^t \delta(r)V(r)\,\mathrm{d}W(r)$$

where

$$\begin{aligned} \alpha(r) &= q \cdot h(r), \\ \beta(r) &= q \cdot \left(g(r) - \tfrac{1}{2}p \cdot l(r)^2\right), \\ \delta(r) &= q \cdot l(r). \end{aligned}$$

The solution V of this linear equation is exactly

$$V(t) = \zeta(t)\left[X(0)^q + q \cdot \int_0^t \frac{h(r)}{\zeta(r)}\,\mathrm{d}r\right].$$

Now the rest of the proof follows by taking V to the power of $\frac{1}{q}$ which also gives $\theta = \theta_Z$. □

Remark 4.7 (i) We have to demand $X(0) > 0$ in the proposition because in our proof we have to be able to compute $X(0)^q$.
(ii) Note that $h \leq 0$ implies $\theta = \infty$.

5 Appendix

In this section we shall prove the existence and uniqueness results which we have used in the previous sections. For this purpose let us consider the system

$$(*) \qquad Y_j(t) = Y_j(0) + \int_0^t c_j(r, Y(r))\,\mathrm{d}r + \sum_{k=1}^m \int_0^t \sigma_{jk}(r, Y(r))\,\mathrm{d}W_k(r)$$

where $j = 1, ..., d$ and W is a given m-dimensional Wiener process. The mapping $c : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is supposed to satisfy

$$\|c(t,x) - c(t,y)\| \leq L_N(T)\|x - y\|$$

whenever $t \leq T$ and $\|x\|, \|y\| \leq N$ for each $T, N > 0$ while σ is the same as at the beginning of section 2. We need the following condition.

(C_σ^*) There exists a strictly increasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ with $\rho(0) = 0$ and

$$\int_{0+} \rho^{-2}(u)\,\mathrm{d}u = \infty$$

such that for each $j = 1, ..., d$

$$\sum_{k=1}^m |\, \sigma_{jk}(t,x) - \sigma_{jk}(t,z) \,| \leq \rho(|\, x_j - z_j \,|)$$

for all $t \geq 0$, $x, z \in \mathbb{R}^d$.

Proposition 5.1 *If σ satisfies (C_σ^*) then there exists at most one solution of $(*)$ up to a strictly positive predictable stopping time θ in the following sense: If $Y^{(1)}$ and $Y^{(2)}$ are two solutions of $(*)$ up to θ it holds*

$$\mathbf{P}(\{Y^{(1)}(t) = Y^{(2)}(t),\ t < \theta\}) = 1.$$

Remark 5.2 This result is well-known in the one-dimensional case for global solutions. It was proved by T. Yamada [15] improving an idea of H. Tanaka. Using the condition (C_σ^*) in higher dimensions the proof is nearly a copy of the proof in the one-dimensional case (compare with the proof of Theorem IV.3.2 in [6]). Therefore we omit it.

Remark 5.3 Proposition 5.1 remains true if (C_σ^*) is replaced by (C_σ). In this case we can use $\rho(u) = u^\alpha$, $\alpha > \frac{1}{2}$.

Corollary 5.4 *Let σ satisfy (C_σ^*) or (C_σ). Then there exists one and only one local solution Y of $(*)$ up to the explosion time θ_Y.*

Proof. We shall use the usual truncation and prolongation method. Let $\psi : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ be continuous. We define $\psi_j^{(N)}$, $j = 1, ..., d$, as follows. For $t \vee \|x\| \leq N$ it coincides with ψ_j. If $t \vee \|x\| > N$ we take for $\psi_j^{(N)}(t,x)$ the value of $\psi_j^{(N)}(\widetilde{(t,x)})$ where $\widetilde{(t,x)}$ is the point of intersection of the straight line connecting (t,x) with $(0,0)$ and the boundary of $\{(t,x) : t \vee \|x\| \leq N\}$. Obviously, $\psi^{(N)}$ is again continuous and it holds $|\psi_j^{(N)}(t,x)| \leq$ constant. By this way we can introduce the mappings $c^{(N)}$ and $\sigma^{(N)}$. With them we obtain a system of stochastic differential equations with continuous and bounded coefficients for which a weak solution (cf. [6], Th. IV.2.2) exists. On the other hand we observe the pathwise uniqueness by Proposition 5.1 and Remark 5.3. Consequently the Yamada-Watanabe theorem (cf. [8], Cor. 5.3.23) gives us the existence of a strong solution $Y^{(N)}$. If we now introduce

$$\eta_N = \inf\{t \geq 0 : \|Y^{(N)}(t)\| > N\}$$

then the uniqueness implies

$$Y^{(N)} = Y^{(N+1)} \quad \text{on} \quad [0, \eta_N \wedge N) \qquad \mathbf{P}\text{-a.s.}$$

because $c^{(N)}, \sigma^{(N)}$ and $c^{(N+1)}, \sigma^{(N+1)}$ coincide on $\{(t,x) : t \vee \|x\| \leq N\}$. Therefore, up to indistinguishability we can define a process Y by setting

$$Y(t) = Y^{(N)}(t), \qquad t \in [0, \eta_N \wedge N),\ N = 1, 2, ...,$$

and

$$Y(t) = \triangle, \qquad t \in [\lim_{N\to\infty} \eta_N, \infty).$$

From this definition it immediately follows that Y is a local solution of $(*)$ up to the explosion time $\theta_Y = \lim_{N\to\infty} \eta_N$. □

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