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Integral Equations
and Operator Theory

ERRATA
"HARMONICALLY WEIGHTED DIRICHLET SPACES ASSOCIATED WITH FINITELY ATOMIC MEASURES"

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1. CONFESSIONAL

There are two serious errors, uncovered by Anatolii Grinshpan, in the paper [1]. First, Theorem 4 is incorrect. The mistake in the proof occurs at the very end, where it is asserted that a certain constant c must be positive. As Grinshpan points out, the case $c < 0$ can also arise. A corrected version of Theorem 4 is presented below, in Section 2.

Second, Lemma 5 is also incorrect. The proof has two mistakes. Toward the end, it is asserted unjustifiably that two polynomials F and G are constant multiples of each other. Subsequently, the oversight made in the proof of Theorem 4 is repeated.

Lemma 5 was used in the proof of Theorem 5. The proof of Theorem 5 in [1] is thus incomplete. The additions needed to complete the proof are given below, in Section 3. The needed tools are already in Section 4 of [1].

All notations below are as in [1].

2. THEOREM 4 CORRECTED

Theorem 4 from [1] should be replaced by the following statement.

Theorem 4. *Let $\nu = \sum_{l=1}^{N} \nu_l \delta_{\beta_l}$ be a positive measure on $\partial\mathbf{D}$, a sum of N atoms. Let the sesquilinear functional W_ν on $\mathcal{H}(v_\mu)$ be defined by*

$$W_\nu(f,g) = \int \overline{\lambda}\, f(\lambda)\, \overline{g'(\lambda)}\, d\nu(\lambda)\,.$$

If W_ν is positive definite, then ν is a positive multiple of μ.

The correction thus consists in the replacement of the condition that W_ν be Hermitian with the condition that it be positive definite.

The proof of the corrected Theorem 4 follows the argument in [1] up to the point where the mistake was made. Namely, from the assumption that W_ν is Hermitian one concludes that there is a nonzero constant c such that

$$(1) \qquad 1 - c \sum_{l=1}^{N} \nu_l K(\overline{\beta}_l z) = \frac{\prod_{j=1}^{N} \left(1 - \frac{z}{w_j}\right)(1 - \overline{w}_j z)}{\prod_{l=1}^{N} (1 - \overline{\beta}_l z)^2} ;$$

this is equality (2.10) from [1]. As noted in [1], on $\partial\mathbf{D}$, the function on the right side of (1) has a constant argument, the argument of $\prod_{l=1}^{N} \beta_l / \prod_{j=1}^{N} w_j$, while the range of the function on the left side lies in the ray $\{1 + tc : t > 0\}$. This leaves only two possibilities: $c > 0$ and $c < 0$ (the latter being overlooked in [1]). As in [1], the case $c > 0$ leads to the conclusion $c\nu = \mu$. It will be shown that, if $c < 0$, then W_ν, although Hermitian, is not positive definite.

We can reduce the case $c < 0$ to the case $c = -1$ by replacing ν by a suitable multiple of itself. We thus assume that (1) holds with $c = -1$. The function on the left side of (1) then becomes

$$1 + \sum_{l=1}^{N} \nu_l K(\overline{\beta}_l z) ;$$

we denote this function by $\mathcal{K}_{-\nu}$. We know that $\mathcal{K}_{-\nu}$ has a constant argument on $\partial\mathbf{D}$, so, because it tends to $-\infty$ at each point β_l, it must be negative on $\partial\mathbf{D}$. The zeros of $\mathcal{K}_{-\nu}$ are the points $w_1, \ldots, w_N$ and their reflections with respect to $\partial\mathbf{D}$.

We define an indefinite inner product $[\cdot, \cdot]_\nu$ on $D(\nu)$ by setting

$$[f, g]_\nu = \langle f, g \rangle - D_\nu(f, g) .$$

Some of the arguments from Section 2 of [1] can be employed. Define the function χ_0 by

$$\chi_0(z) = \omega^{1/2} \prod_{l=1}^{N} (1 - \overline{\beta}_l z) / q_\mu(z) .$$

Then $\mathcal{K}_{-\nu} = -|\chi_0|^{-2}$ on $\partial\mathbf{D}$, and the proof of Theorem 1 from [1] shows that, for f in H^2,

$$[\chi_0 f, \chi_0 f]_\nu = -\|f\|_2^2 .$$

Thus $\chi_0 H^2$, a subspace of $D(\nu)$ of codimension N, is an anti-Hilbert space under the inner product $[\cdot, \cdot]_\nu$. However, $D(\nu)$ itself is not an anti-Hilbert space under $[\cdot, \cdot]_\nu$, because $[1, 1]_\nu = 1$.

Next, the proof of Theorem 2 of [1] shows that, for f in $D(\nu)$ and g in $\mathcal{H}(v_\mu)$,

$$[f, g]_\nu = -\int \overline{\lambda} f(\lambda) \overline{g'(\lambda)} d\nu(\lambda) .$$

In particular, $W_\nu(f, g) = -[f, g]_\nu$ for f and g in $\mathcal{H}(v_\mu)$. Moreover, the subspaces $\mathcal{H}(v_\mu)$ and $\chi_0 H^2$, whose algebraic direct sum is $D(\nu)$, are orthogonal relative to $[\cdot, \cdot]_\nu$. Since $[\cdot, \cdot]_\nu$

is negative definite on $\chi_0 H^2$ but not on all of $D(\nu)$, it is not negative definite on $\mathcal{H}(v_\mu)$. From the relation above between $[\cdot,\cdot]_\nu$ and W_ν, it follows that W_ν is not positive definite. This completes the proof of the corrected Theorem 4.

Example. One can make explicit calculations for the case $\nu = a\delta_i + a\delta_{-i}$. One finds that $\mathcal{K}_{-\nu}$ is negative on $\partial\mathbf{D}$ provided $a > 1/2$, and then its zeros in $\mathbf{D}$ are the points $w_1 = \rho$, $w_2 = -\rho$, where

$$\rho = \left[2a - \left(4a^2 - 1\right)^{1/2}\right]^{1/2}.$$

The sesquilinear functional W_ν defined as in Theorem 4 on the span of the kernel functions k_{w_1} and k_{w_2} is Hermitian. Calculations give

$$W_\nu(k_{w_1}, k_{w_1}) = W_\nu(k_{w_2}, k_{w_2}) = \frac{-2a\rho^2}{(1+\rho^2)^2}$$

$$W_\nu(k_{w_1}, k_{w_2}) = W_\nu(k_{w_2}, k_{w_1}) = \frac{-2a\rho^2(3-\rho^2)}{(1+\rho^2)^3}.$$

The matrix $\left(W_\nu(k_{w_i}, k_{w_j})\right)_{i,j=1}^2$ has a negative determinant, implying that W_ν is indefinite.

3. Proof of Theorem 5 Completed

The theorem will be restated.

Theorem 5. *Let h be a wandering vector of S_μ, with inner-outer factorization $h = uh_0$. Let Λ' be the set of points in Λ at which h vanishes, and let $\nu = \|h\|_\mu^{-2}|h|^2\mu$. Then*

$$h_0(z) = cq_\nu(z) \prod_{\lambda_l \in \Lambda'} (1 - \overline{\lambda}_l z) \Big/ q_\mu(z),$$

where c is a constant. The equality

$$|v_\nu'(\lambda_l)| = 2|u'(\lambda_l)| + |v_\mu'(\lambda_l)|$$

holds for every l such that $h(\lambda_l) \neq 0$.

Much of the proof in [1] can be retained. As explained in [1], the case $\Lambda' = \Lambda$ is easily disposed of, so we assume $\Lambda' \neq \Lambda$. Let $L = \operatorname{card}(\Lambda \backslash \Lambda')$. We may suppose without loss of generality that $\Lambda' = \{\lambda_{L+1}, \dots, \lambda_N\}$ (interpreted as the empty set if $L = N$).

By Lemma 3 of [1], the function h_0 has the form

$$h_0(z) = \frac{c \prod_{k=1}^{M} (1 - \overline{\alpha}_k z) \prod_{l=L+1}^{N} (1 - \overline{\lambda}_l z)}{q_\mu(z)},$$

where c is a constant, $0 \leq M \leq L$, and $\alpha_1, \ldots, \alpha_M$ are points of $\overline{\mathbf{D}} \backslash (\Lambda \backslash \Lambda')$. As was done at the start of the "proof"of the abortive Lemma 5 from [1], we introduce two polynomials,

$$G(z) = z^{L-M} \prod_{k=1}^{M} \left(1 - \frac{z}{\alpha_k}\right)(1 - \overline{\alpha}_k z)$$

$$F(z) = \prod_{l=1}^{L}(1 - \overline{\lambda}_l z)^2 - \sum_{l=1}^{L} \eta_l \overline{\lambda}_l z \prod_{\substack{m \neq l \\ m \leq L}} (1 - \overline{\lambda}_m z)^2,$$

where the numbers $\eta_1, \ldots, \eta_L$ are so chosen that $F(\lambda_l) = G(\lambda_l)$ for $l = 1, \ldots, L$. A calculation gives

$$\eta_l = \frac{(-1)^{L-M} \prod_{m=1}^{L} \lambda_m \prod_{k=1}^{M} |\lambda_l - \alpha_k|^2}{\prod_{k=1}^{M} \alpha_k \prod_{\substack{m \neq l \\ m \leq L}} |\lambda_l - \lambda_m|^2},$$

showing that the numbers $\eta_1, \ldots, \eta_L$ have the same argument.

By Lemma 4 of [1] we have the equalities

$$\sum_{k=1}^{M} \frac{\overline{\lambda}_l \alpha_k - \lambda_l \overline{\alpha}_k}{|\lambda_l - \alpha_k|^2} = \sum_{\substack{m \neq l \\ m \leq L}} \frac{\overline{\lambda}_l \lambda_m - \lambda_l \overline{\lambda}_m}{|\lambda_l - \lambda_m|^2} \quad (l = 1, \ldots, L).$$

As explained in [1], these imply that the functions F and G have the same logarithmic derivatives at $\lambda_1, \ldots, \lambda_L$. (Detailed calculations can be found in [2].) As F and G are polynomials of at most degree $2L$ that coincide along with their derivatives at $\lambda_1, \ldots, \lambda_L$, we must have

$$F(z) - G(z) = b \prod_{l=1}^{L}(1 - \overline{\lambda}_l z)^2 \tag{2}$$

for some constant b.

The case $M = L$ will be dealt with first. In that case $G(0) = 1$, and since also $F(0) = 1$, it follows that $b = 0$, and $F = G$. The last equality can be rewritten as

$$1 - \sum_{l=1}^{L} \eta_l K(\overline{\lambda}_l z) = \frac{\prod_{k=1}^{L} \left(1 - \frac{z}{\alpha_k}\right)(1 - \overline{\alpha}_k z)}{\prod_{l=1}^{L}(1 - \overline{\lambda}_l z)^2}. \tag{3}$$

Because $\eta_1, \ldots, \eta_L$ have the same argument, the reasoning above in the proof of the corrected Theorem 4 applies to show that there are only two possibilities: either $\eta_l > 0$ for every l, or $\eta_l < 0$ for every l. If $\eta_l > 0$ for every l, the reasoning in [1] applies and leads to the desired conclusions. It will be shown that the case $\eta_l < 0$ does not arise for a wandering vector.

Assume every η_l is negative, and let η denote the measure $-\sum_{l=1}^{L}\eta_l\delta_{\lambda_l}$. Let $\mathcal{K}_{-\eta}$ denote the function on either side of the equality (3), a function that is nonpositive on $\partial\mathbf{D}$. Let $v_{-\eta}$ be the Blaschke product for the zero set of $\mathcal{K}_{-\eta}$ in $\mathbf{D}$. According to Lemma 4 in [1], if h is a wandering vector of S_μ then $|v'_{-\eta}(\lambda_l)| \geq |v_\mu(\lambda_l)|$ for $l = 1,\ldots,L$. It will be shown that the last condition fails.

Along with the measure $-\eta$ we consider the measures $t\eta$ and $-t\eta$ for $t > 1$, and the corresponding Blaschke products $v_{t\eta}$ and $v_{-t\eta}$. Lemma 8 from [1] states that, for $l = 1,\ldots,L$, the numbers $|v'_{t\eta}(\lambda_l)|$ decrease as t increases. The proof of that lemma shows that the numbers $|v'_{-t\eta}(\lambda_l)|$ increase as t increases ($l = 1,\ldots,L$). It is asserted that the numbers $|v'_{t\eta}(\lambda_l)|$ and $|v'_{-t\eta}(\lambda_l)|$ have the same limit as $t \to \infty$.

To prove the assertion we introduce the function

$$\mathcal{K}_{\infty\eta}(z) = \sum_{l=1}^{L}\eta_l K(\overline{\lambda}_l z),$$

and we let $v_{\infty\eta}$ be the Blaschke product for its zero set in $\mathbf{D}$. The zero set of $\mathcal{K}_{t\eta}$ is the set of points where $\mathcal{K}_{\infty\eta}$ takes the value $-1/t$. Fix $\epsilon > 0$, and construct open disks of radius ϵ centered at the zeros of $\mathcal{K}_{\infty\eta}$ in $\mathbf{D}$. Assume ϵ is small enough so that those disks are contained in $\mathbf{D}$ and mutually disjoint. The function $\mathcal{K}_{\infty\eta}$, being positive on $\partial\mathbf{D}$, is bounded away from 0 on the complement in $\overline{\mathbf{D}}$ of the union of those disks. Hence, for t sufficiently large, the solutions in $\mathbf{D}$ of the equation $\mathcal{K}_{\infty\eta} = -1/t$ lie in those disks, and each disk contains as many solutions as the multiplicity of its center as a zero of $\mathcal{K}_{\infty\eta}$. It follows that $v_{t\eta} \to v_{\infty\eta}$ as $t \to \infty$, the convergence being locally uniform off the set of poles of $v_{\infty\eta}$. By the same reasoning, $v_{-t\eta} \to v_{\infty\eta}$ as $t \to \infty$, in the same manner. The assertion follows.

From the results in the last two paragraphs we can conclude that $|v'_{-\eta}(\lambda_l)| < |v'_{t\eta}(\lambda_l)|$ for all $t > 0$ ($l = 1,\ldots,L$). Now let t be chosen large enough so that the measure $t\eta$ is not dominated by μ. Then, according to Theorem 8 of [1], there is an l such that $|v'_{t\eta}(\lambda_l)| < |v'_\mu(\lambda_l)|$, and hence also $|v'_{-\eta}(\lambda_l)| < |v'_\mu(\lambda_l)|$. By the necessary condition mentioned earlier, it follows that h cannot be a wandering vector in this case.

It remains to treat the case $M < L$. In that case $G(0) = 0$, so the constant b in (2) must equal 1. In place of (3) we now have

$$-\sum_{l=1}^{L}\eta_l K(\overline{\lambda}_l z) = \frac{z^{L-M}\prod_{k=1}^{M}\left(1-\frac{z}{\alpha_k}\right)(1-\overline{\alpha}_k z)}{\prod_{l=1}^{L}(1-\overline{\lambda}_l z)^2}. \tag{4}$$

As noted earlier, the numbers $\eta_1,\ldots,\eta_L$ have the same argument. Let $\eta = \sum_{l=1}^{L}|\eta_l|\delta_{\lambda_l}$. The function on either side of (4) is then a multiple of $\mathcal{K}_{\infty\eta}$. According to Lemma 4 in

[1], the status of h as a wandering vector would entail the inequalities $|v'_{\infty\eta}(\lambda_l)| \geq |v'_\mu(\lambda_l)|$ $(l = 1, \dots, L)$. The latter condition is not fulfilled, by the reasoning just given for the case $M = L$. Thus, the case $M < L$ does not arise for a wandering vector. The proof of Theorem 5 is now complete.

4. Misprints

The following minor corrections should be made in [1].

Page 193, Line 4−. Replace w^m by w^{m-n}.
Page 193, Line 2−. Replace $\overline{w}^m$ by $\overline{w}^{m-n}$.
Page 197, Line 9−. Replace β_l on the left side of the equality by β_m.
Page 201, Line 10. Replace ν by v.

References

1. D. Sarason, Harmonically weighted Dirichlet spaces associated with finitely atomic measures. *Integral Equations and Operator Theory* 31 (1998), 186–213.
2. D. Sarason and D. Suarez, Inverse problem for zeros of certain Koebe-related functions. *Journal d'Analyse Mathématique* 71 (1997), 149–158.

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