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Erratum

Approximation of biholomorphic mappings by automorphisms of $\mathbf{C}^n$

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There is a missing hypothesis in Theorem 1.1, p. 325. In the part of the statement concerning *volume preserving* holomorphic automorphisms of $\mathbf{C}^n$ it has been overlooked that one needs an additional cohomological assumption to be able to approximate holomorphic vector fields with divergence zero by entire divergence zero fields. Here is the correct result.

Theorem. *Let Ω be a domain of holomorphy in $\mathbf{C}^n$, $n \geqq 2$, satisfying $H^{n-1}(\Omega; \mathbf{C}) = 0$. For every $t \in [0,1]$, let Φ_t be a biholomorphic map from Ω into $\mathbf{C}^n$, of class $\mathscr{C}^2$ in $(t,z) \in [0,1] \times \Omega$. Assume that each domain $\Omega_t = \Phi_t(\Omega)$ is Runge in $\mathbf{C}^n$. If every map Φ_t is volume preserving (i.e., its Jacobian determinant equals one), and if Φ_0 can be approximated on Ω by volume preserving automorphisms of $\mathbf{C}^n$, then every Φ_t can be approximated on Ω by volume preserving automorphisms of $\mathbf{C}^n$.*

The mistake does not affect any other result. We indicate briefly the proof. The family Φ_t, $0 \leqq t \leqq 1$, is the flow of a time dependent, divergence zero vector field X_t, defined on Ω_t. The statement of Theorem 1.1 holds whenever we can approximate X_t for every $t \in [0,1]$, uniformly on compacts in Ω_t, by divergence zero vector fields X_t', defined on all of $\mathbf{C}^n$ (see Lemma 1.4).

Assuming the cohomological condition in the theorem above one gets the approximation as follows. We associate to X_t in a standard way a holomorphic $(n-1)$-form α_t on Ω_t such that div $X_t = 0$ is equivalent to $d\alpha_t = 0$. The cohomological condition implies $\alpha_t = d\beta_t$ for some holomorphic $(n-2)$-form β_t on Ω_t. Approximating β_t by a global form β_t' on $\mathbf{C}^n$ (which is possible since Ω_t is Runge) and taking $\alpha_t' = d\beta_t'$ we obtain approximation of α_t by holomorphic, d-closed, $(n-1)$-form on $\mathbf{C}^n$. The vector field X_t' corresponding to α_t' is divergence free on $\mathbf{C}^n$ and approximates X_t on Ω_t.

Example. The map $\Omega_t(z,w) = (z, w + t/z)$ is a volume preserving automorphism of $\mathbf{C}_* \times \mathbf{C}$ for all $t \in \mathbf{C}$. The circle $T = \{(z,\bar{z}) \in \mathbf{C}^2 : |z| = 1\}$ is polynomially convex, hence it has pseudoconvex tubular neighborhoods Ω which are Runge in $\mathbf{C}^2$. All conditions in the theorem, except the cohomological one, are satisfied. However, Φ_t for $t \neq 0$ is not the limit of volume preserving automorphisms of $\mathbf{C}^2$ in any neighborhood of T. This can be seen by calculating the 'action integral' of the 1-form $\theta = w dz$ on $T_t = \Phi_t(T)$: The integral equals $2\pi i(1+t)$ and so it depends on t, while Stokes' theorem shows that this integral is preserved by any volume preserving automorphism of $\mathbf{C}^2$ (and therefore by limits of such maps).

An example of this type exists for every $n \geqq 2$. Let S be the $(n-1)$-dimensional sphere, embedded as a hypersurface in $\mathbf{R}^n \subset \mathbf{C}^n$. There exists a closed, but non-exact holomorphic $(n-1)$-form α in a tubular neighborhood of S such that $\int_S \alpha \neq 0$. The flow Φ_t of the divergence zero vector field associated to α as above is a family of volume preserving mappings near S such that Φ_t cannot be approximated by volume preserving automorphisms of $\mathbf{C}^n$ when t is small but not 0.